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Analytic functionals and entire functionals on the complex light cone

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Introduction

Let $\tilde{M} = \{z \in \mathbf{C}^{n+1}; z^2 = 0\}$ be the *complex light cone*, where $z^2 = z_1^2 + z_2^2 + \cdots + z_{n+1}^2$ for $z = (z_1, z_2, \ldots, z_{n+1})$, $n \in \mathbf{N}$, and let S^n be the n-dimensional real sphere. We are concerned with holomorphic functions, analytic functionals, entire functions, and entire functionals on $\tilde{M}$. $\tilde{M} \setminus \{0\}$ can be identified with the cotangent bundle to S^n minus its zero section. We call $M = \{z \in \tilde{M}; \|z\| = 1/\sqrt{2}\}$ the *spherical sphere* and identify it with the cotangential sphere bundle to S^n. Holomorphic functions on the complex light cone were discussed by several authors ([2], [10] and [11]) and are related to hyperfunctions on the sphere ([4]).

Let $\mathcal{O}(\mathbf{C}^{n+1})$ and $\mathcal{O}(\tilde{M})$ be the spaces of entire functions on $\mathbf{C}^{n+1}$ and $\tilde{M}$, respectively. We denote by $\mathrm{Exp}(\mathbf{C}^{n+1})$ and $\mathrm{Exp}(\tilde{M})$ the spaces of entire functions of exponential type on $\mathbf{C}^{n+1}$ and $\tilde{M}$, respectively, and by $\mathcal{O}'(\tilde{M})$ and $\mathrm{Exp}'(\tilde{M})$ the dual spaces of $\mathcal{O}(\tilde{M})$ and $\mathrm{Exp}(\tilde{M})$, respectively. Put $\mathrm{Exp}_\Delta(\mathbf{C}^{n+1}) = \mathcal{O}_\Delta(\mathbf{C}^{n+1}) \cap \mathrm{Exp}(\mathbf{C}^{n+1})$, where $\mathcal{O}_\Delta(\mathbf{C}^{n+1})$ is the space of complex harmonic functions on $\mathbf{C}^{n+1}$.

We call the function $\mathscr{F}_\lambda T(\zeta) = \langle T_z, \exp(i\lambda z \cdot \zeta)\rangle$ the *Fourier-Borel transform* of T.

First, we give an integral representation of holomorphic functions on $\tilde{M}$ and by using the integral kernel we define the *Cauchy transform* of analytic functionals on $\tilde{M}$. The integral representation gives the inverse mapping of the restriction mapping $\mathcal{O}_\Delta(\mathbf{C}^{n+1}) \xrightarrow{\sim} \mathcal{O}(\tilde{M})$, and the Cauchy transformation gives isomorphisms such as $\mathcal{O}'(\tilde{M}) \xrightarrow{\sim} \mathcal{O}_\Delta(\{0\})$ (Theorem 9).

Second, we prove such topological isomorphisms as

$$\mathscr{F}_\lambda\colon \mathcal{O}'(\tilde{M}) \xrightarrow{\sim} \mathrm{Exp}_\Delta(\mathbf{C}^{n+1}) \text{ and } \mathscr{F}_\lambda\colon \mathrm{Exp}'(\tilde{M}) \xrightarrow{\sim} \mathcal{O}_\Delta(\mathbf{C}^{n+1})$$

by using the growth behavior of homogeneous expansions (Theorems 18, 19 and 20). These results were announced in [7], and the first author gave them a different proof in [8] by means of exact sequences and Martineau's theorem. The fact that the Fourier-Borel transformation is a topological isomorphism on $\mathrm{Exp}'(\tilde{M})$ generalizes a Kowata-Okamoto theorem (Theorem 2 in [3]).

1. Preliminaries

Lie ball and complex light cone

Let $\|x\|$ be the Euclidean norm on $\mathbf{R}^{n+1}$. We denote the open and the closed balls of radius r with center at 0 in $\mathbf{R}^{n+1}$ by

$$B(r) = \{x \in \mathbf{R}^{n+1};\ \|x\| < r\}, \qquad 0 < r \le \infty$$

and by

$$B[r] = \{x \in \mathbf{R}^{n+1};\ \|x\| \le r\}, \qquad 0 \le r < \infty,$$

respectively. Note that $B(\infty) = \mathbf{R}^{n+1}$ and $B[0] = \{0\}$. The cross norm $L(z)$ on $\mathbf{C}^{n+1}$ corresponding to $\|x\|$ is the Lie norm given by

$$L(z) = L(x+iy) = [\|x\|^2 + \|y\|^2 + 2\sqrt{\|x\|^2\|y\|^2 - (x\cdot y)^2}]^{\frac{1}{2}},$$

where $x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1}$. Its dual norm $L^*(z)$ is given by

$$\begin{aligned} L^*(z) &= \sup\{|z\cdot\zeta|;\ L(\zeta) \le 1\} \\ &= 1/\sqrt{2}\,[\|x\|^2 + \|y\|^2 + \sqrt{(\|x\|^2 - \|y\|^2)^2 + 4(x\cdot y)^2}]^{\frac{1}{2}} \end{aligned}$$

(see [1]).

The open and the closed Lie balls of radius r with center at 0 are defined by

$$\tilde{B}(r) = \{z \in \mathbf{C}^{n+1};\ L(z) < r\}, \qquad 0 < r \le \infty$$

and by

$$\tilde{B}[r] = \{z \in \mathbf{C}^{n+1};\ L(z) \le r\}, \qquad 0 \le r < \infty,$$

respectively. It is clear that $B(r) = \tilde{B}(r) \cap \mathbf{R}^{n+1}$ and $B[r] = \tilde{B}[r] \cap \mathbf{R}^{n+1}$.

Since $L(z)$ is a norm, $\tilde{B}(r)$ is an open convex and balanced subset of $\mathbf{C}^{n+1}$. In particular, $\tilde{B}(r)$ is a domain of holomorphy in $\mathbf{C}^{n+1}$.

We define the *spherical sphere* M by $M = \{z \in \tilde{M};\ L(z) = 1\}$, where $\tilde{M}$ is the complex light cone. If $z = x + iy \in M$, then $z^2 = x^2 - y^2 + 2ix\cdot y = 0$, and hence, $x^2 = y^2$ and $x \cdot y = 0$. We have $L(x+iy) = 2\|x\|$, $L^*(x+iy) = \|x\| = L(x+iy)/2$, and $\|x+iy\| = \sqrt{2}\,\|x\|$. Therefore, we can write

$$\begin{aligned} M &= \{x+iy;\ \|x\| = \|y\| = 1/2,\ x\cdot y = 0\} \\ &= \{z \in \tilde{M};\ L^*(z) = 1/2\} = \{z \in \tilde{M};\ \|z\| = 1/\sqrt{2}\}. \end{aligned}$$

Further, put

$$\tilde{M}(r) = \{z \in \tilde{M}; L(z) < r\} = \tilde{M} \cap \tilde{B}(r), \qquad 0 < r \leq \infty,$$
$$\tilde{M}[r] = \{z \in \tilde{M}; L(z) \leq r\} = \tilde{M} \cap \tilde{B}[r], \qquad 0 \leq r < \infty.$$

Note that we can also write $M = \{(1/2, i/2, 0, \cdots, 0)g;\, g \in SO(n+1)\}$ and it is isomorphic with the Stiefel manifold $O(n+1)/O(n-1)$. So, there is the unique $O(n+1)$-invariant measure $d\mu$ on M with $\int_M d\mu(z) = 1$.

Complex harmonic polynomials

We denote by $\mathscr{P}^k(\mathbf{C}^{n+1})$ the space of k-homogeneous polynomials with complex coefficients of $n+1$ variables and by $\mathscr{P}^k(\tilde{M})$ the restriction to $\tilde{M}$ of $\mathscr{P}^k(\mathbf{C}^{n+1})$. We call $\mathscr{P}^k(\tilde{M})$ the space of k-homogeneous polynomials on $\tilde{M}$ and $\mathscr{P}(\tilde{M}) = \sum_{k=0}^{\infty} \mathscr{P}^k(\tilde{M})$ the space of polynomials on $\tilde{M}$. Put

$$\mathscr{P}^k_\Delta(\mathbf{C}^{n+1}) = \{F \in \mathscr{P}^k(\mathbf{C}^{n+1});\, \Delta_z F = 0\},$$

where $\Delta_z = \partial^2/\partial z_1^2 + \partial^2/\partial z_2^2 + \cdots + \partial^2/\partial z_{n+1}^2$ is the complex Laplacian. It is the space of k-homogeneous complex harmonic polynomials. Put

$$N(k, n) = \dim \mathscr{P}^k_\Delta(\mathbf{C}^{n+1}).$$

We know $N(k, n) = (2k+n-1)(k+n-2)!/(k!(n-1)!) = O(k^{n-1})$, $(k, n) \neq (0, 1)$ and $N(0, 1) = 1$. (See [5] or [9].)

Lemma 1 ([10, Lemma 1.3]).
(*i*) (*Reproducing property*) *For any* $f_k \in \mathscr{P}^k(\tilde{M})$, *we have*

$$f_k(w) = 2^k N(k, n) \int_M f_k(z)(\bar{z} \cdot w)^k d\mu(z), \qquad w \in \tilde{M}.$$

(*ii*) (*Orthogonality*) *If* $f_k \in \mathscr{P}^k(\tilde{M})$, $f_l \in \mathscr{P}^l(\tilde{M})$, $k \neq l$, *then we have*

$$\int_M f_k(z) f_l(\bar{z}) d\mu(z) = 0.$$

Lemma 2. *The restriction mapping* $\beta: F \mapsto F|_{\tilde{M}}$ *is a linear topological isomorphism of* $\mathscr{P}^k_\Delta(\mathbf{C}^{n+1})$ *onto* $\mathscr{P}^k(\tilde{M})$:

$$\beta: \mathscr{P}^k_\Delta(\mathbf{C}^{n+1}) \xrightarrow{\sim} \mathscr{P}^k(\tilde{M}).$$

Moreover, we have

$$\| f_k \|_{C(M)} \leq \| F_k \|_{C(B)} \leq N(k, n) \| f_k \|_{C(M)}, \tag{1}$$

where $\| f \|_{C(B)} \equiv \sup\{|f(z)|;\, z \in \tilde{B}[1]\}$ *and* $\| f \|_{C(M)} \equiv \sup\{|f(z)|;\, z \in \tilde{M}[1]\}$.

PROOF. Let $f_k \in \mathscr{P}^k(\tilde{M})$. We define F_k by

$$F_k(w) = 2^k N(k, n) \int_M f_k(z)(\bar{z} \cdot w)^k d\mu(z), \qquad w \in \mathbf{C}^{n+1}. \tag{2}$$

Then $F_k \in \mathscr{P}^k_\Delta(\mathbf{C}^{n+1})$ and by Lemma 1, $F_k|_{\tilde{M}} = f_k$. Thus β is surjective. Further, (2) implies

$$\begin{aligned}\| F_k \|_{C(B)} &\le 2^k N(k, n) \| f_k \|_{C(M)} \sup \{ |\bar{z} \cdot w|^k ; z \in M, w \in \tilde{B}[1] \} \\ &= N(k, n) \| f_k \|_{C(M)}.\end{aligned}$$

Thus β is injective. q.e.d.

2. Holomorphic functions on $\tilde{M}$

We denote by $\mathcal{O}(\tilde{B}(r))$ (resp., $\mathcal{O}(\tilde{M}(r))$) the space of holomorphic functions on $\tilde{B}(r)$ (resp., $\tilde{M}(r)$) equipped with the topology of uniform convergence on compact sets. A fundamental system of seminorms is given by

$$\| F \|_{\rho, L} = \sup \{ |F(z)| ; L(z) \le \rho \}, \qquad 0 \le \rho < r.$$

We call $\mathcal{O}(\mathbf{C}^{n+1})$ (resp., $\mathcal{O}(\tilde{M})$) the space of entire functions on $\mathbf{C}^{n+1}$ (resp., $\tilde{M}$). $\mathcal{O}(\tilde{B}(r))$ and $\mathcal{O}(\tilde{M}(r))$ are FS spaces (Fréchet-Schwartz spaces).

For $0 \le r < \infty$, we define the space $\mathcal{O}(\tilde{B}[r])$ (resp., $\mathcal{O}(\tilde{M}[r])$) of germs of holomorphic functions on $\tilde{B}[r]$ (resp., $\tilde{M}[r]$) by

$$\mathcal{O}(\tilde{B}[r]) = \operatorname{ind\,lim} \{ \mathcal{O}(\tilde{B}(r')) ; r' > r \}$$
$$(\text{resp., } \mathcal{O}(\tilde{M}[r]) = \operatorname{ind\,lim} \{ \mathcal{O}(\tilde{M}(r')) ; r' > r \}).$$

$\mathcal{O}(\tilde{B}[r])$ and $\mathcal{O}(\tilde{M}[r])$ are DFS spaces (dual Fréchet-Schwartz spaces).

By restriction mappings, we have the following inclusion relations:

$$\mathcal{O}(\tilde{M}) \subset \mathcal{O}(\tilde{M}[r]) \subset \mathcal{O}(\tilde{M}(r)) \subset \mathcal{O}(\tilde{M}[0]), \qquad 0 < r < \infty. \tag{3}$$

LEMMA 3. *The following sequences are exact:*

$$0 \longrightarrow \mathscr{I}_{\tilde{M}}(\tilde{B}(r)) \xrightarrow{\iota} \mathcal{O}(\tilde{B}(r)) \xrightarrow{\beta} \mathcal{O}(\tilde{M}(r)) \longrightarrow 0, \qquad 0 < r \le \infty, \tag{4}$$

$$0 \longrightarrow \mathscr{I}_{\tilde{M}}(\tilde{B}[r]) \xrightarrow{\iota} \mathcal{O}(\tilde{B}[r]) \xrightarrow{\beta} \mathcal{O}(\tilde{M}[r]) \longrightarrow 0, \qquad 0 \le r < \infty, \tag{5}$$

where

$$\mathscr{I}_{\tilde{M}}(\tilde{B}(r)) = \{ f \in \mathcal{O}(\tilde{B}(r)) ; f(z) = 0, z \in \tilde{M}(r) \},$$
$$\mathscr{I}_{\tilde{M}}(\tilde{B}[r]) = \operatorname{ind\,lim} \{ \mathscr{I}_{\tilde{M}}(\tilde{B}(r')) ; r' > r \},$$

ι is the canonical injection and β is the restriction mapping. Further,

$$\mathscr{I}_{\tilde{M}}(\tilde{B}(r)) = z^2 \mathcal{O}(\tilde{B}(r)) \text{ and } \mathscr{I}_{\tilde{M}}(\tilde{B}[r]) = z^2 \mathcal{O}(\tilde{B}[r]). \tag{6}$$

PROOF. Since $\tilde{B}(r)$ is a domain of holomorphy, the exact sequence (4) is a consequence of the Oka-Cartan Theorem B. The exactness of (5) follows from that of (4) by taking inductive limits. q.e.d.

Suppose $f \in \mathcal{O}(\tilde{M}(r))$, $0 < r \leq \infty$. For ε with $0 < \varepsilon < 1$ we define the k-homogeneous component $f_k \in \mathscr{P}^k(\tilde{M})$ of f by

$$f_k(z) = \frac{1}{2\pi i} \oint_{|t|=1-\varepsilon} \frac{f(tz)}{t^{k+1}} dt, \qquad z \in \tilde{M}(r). \tag{7}$$

The right-hand side of (7) is independent of ε with $0 < \varepsilon < 1$. In [6], we defined the k-homogeneous component $F_k \in \mathscr{P}^k(\mathbf{C}^{n+1})$ of $F \in \mathcal{O}(\tilde{B}(r))$ for $z \in \mathbf{C}^{n+1}$ by (7).

Put

$$\mathcal{O}_\Delta(\tilde{B}(r)) = \{F \in \mathcal{O}(\tilde{B}(r));\ \Delta_z F(z) = 0\}, \qquad 0 < r \leq \infty,$$

$$\mathcal{O}_\Delta(\tilde{B}[r]) = \text{ind lim}\ \{\mathcal{O}_\Delta(\tilde{B}(r'));\ r' > r\}, \qquad 0 \leq r < \infty.$$

We call an element of $\mathcal{O}_\Delta(\tilde{B}(r))$ (resp., $\mathcal{O}_\Delta(\tilde{B}[r])$) a complex harmonic function on $\tilde{B}(r)$ (resp., $\tilde{B}[r]$).

Since $\mathcal{O}_\Delta(\tilde{B}(r))$ is a closed subspace of the FS space $\mathcal{O}(\tilde{B}(r))$, it is an FS space. Since $\mathcal{O}_\Delta(\tilde{B}[r])$ is a closed subspace of the DFS space $\mathcal{O}(\tilde{B}[r])$, it is a DFS space.

Using the inequality (1) in Lemma 2, we can deduce the following theorem from Lemma 4.1 in [6].

THEOREM 4. *Let $0<r<\infty$. Suppose $F \in \mathcal{O}_\Delta(\{0\})$. Then the k-homogeneous component $F_k(z)$ is complex harmonic and we have the following relations:*

(i) $$F \in \mathcal{O}_\Delta(\mathbf{C}^{n+1}) \Longleftrightarrow \limsup_{k\to\infty} [\|F_k\|_{C(M)}]^{\frac{1}{k}} = 0,$$

(ii) $$F \in \mathcal{O}_\Delta(\tilde{B}[r]) \Longleftrightarrow \limsup_{k\to\infty} [\|F_k\|_{C(M)}]^{\frac{1}{k}} < \frac{1}{r},$$

(iii) $$F \in \mathcal{O}_\Delta(\tilde{B}(r)) \Longleftrightarrow \limsup_{k\to\infty} [\|F_k\|_{C(M)}]^{\frac{1}{k}} \leq \frac{1}{r},$$

(iv) $$F \in \mathcal{O}_\Delta(\{0\}) \Longleftrightarrow \limsup_{k\to\infty} [\|F_k\|_{C(M)}]^{\frac{1}{k}} < \infty.$$

Further, the expansion $F(z) = \sum_{k=0}^\infty F_k(z)$ converges in the topology of respective spaces.

Similarly, our spaces of holomorphic functions on $\tilde{M}$ can be characterized by the growth behavior of their homogeneous expansions:

THEOREM 5 ([7, Theorem 3.3]). *Let* $0 < r < \infty$, $f \in \mathcal{O}(\tilde{M}[0])$ *and* $f_k \in \mathscr{P}^k(\tilde{M})$ *the k-homogeneous component of f. Then we have the following relations*:

$$\text{(i)} \qquad f \in \mathcal{O}(\tilde{M}) \iff \limsup_{k \to \infty} [\| f_k \|_{C(M)}]^{\frac{1}{k}} = 0,$$

$$\text{(ii)} \qquad f \in \mathcal{O}(\tilde{M}[r]) \iff \limsup_{k \to \infty} [\| f_k \|_{C(M)}]^{\frac{1}{k}} < \frac{1}{r},$$

$$\text{(iii)} \qquad f \in \mathcal{O}(\tilde{M}(r)) \iff \limsup_{k \to \infty} [\| f_k \|_{C(M)}]^{\frac{1}{k}} \leq \frac{1}{r},$$

$$\text{(iv)} \qquad f \in \mathcal{O}(\tilde{M}[0]) \iff \limsup_{k \to \infty} [\| f_k \|_{C(M)}]^{\frac{1}{k}} < \infty.$$

Further, $f(z)$ *is expanded by homogeneous components as follows*:

$$f(z) = \sum_{k=0}^{\infty} f_k(z),$$

where the convergence is in the topology of respective spaces.

PROOF. We prove only (iii). Let $f \in \mathcal{O}(\tilde{M}(r))$ and $0 < \rho < r$. Then the k-homogeneous component f_k of f is given by (7) in a neighborhood of $\tilde{M}[1]$. Therefore, for any ρ with $0 < \rho < r$, $\| f_k \|_{C(M)} \leq 1/\rho^k \sup \{ |f(z)|; z \in \tilde{M}[\rho] \}$, and hence,

$$\limsup_{k \to \infty} [\| f_k \|_{C(M)}]^{\frac{1}{k}} \leq \frac{1}{r}. \tag{8}$$

Conversely, suppose a sequence $\{ f_k \in \mathscr{P}^k(\tilde{M}); k = 0, 1, 2, \cdots \}$ satisfies (8). For any r' with $0 < r' < r$ there is $C \geq 0$ such that $\| f_k \|_{C(M)} \leq C(1/r')^k$. Then we have

$$\sum_{k=0}^{\infty} |f_k(z)| \leq \sum_{k=0}^{\infty} L(z)^k \| f_k \|_{C(M)} \leq \sum_{k=0}^{\infty} C \left(\frac{L(z)}{r'} \right)^k,$$

which converges uniformly for $z \in \tilde{M}(r'')$, where $r'' < r'$. Since r'' and r' can be arbitrarily close to r, $f(z) = \sum f_k(z)$ is holomorphic on $\tilde{M}(r)$. q.e.d.

Since the restriction mapping is continuous, the following theorem is clear from Theorems 4, 5, Lemma 2 and the closed graph theorem:

THEOREM 6 ([11, Theorem 2.4]). *Let* $0 < r < \infty$. The restriction mapping establishes the following linear topological isomorphisms:

(i) $$\beta\colon \mathcal{O}_\Delta(\mathbf{C}^{n+1}) \xrightarrow{\sim} \mathcal{O}(\tilde{M}),$$

(ii) $$\beta\colon \mathcal{O}_\Delta(\tilde{B}[r]) \xrightarrow{\sim} \mathcal{O}(\tilde{M}[r]),$$

(iii) $$\beta\colon \mathcal{O}_\Delta(\tilde{B}(r)) \xrightarrow{\sim} \mathcal{O}(\tilde{M}(r)),$$

(iv) $$\beta\colon \mathcal{O}_\Delta(\{0\}) \xrightarrow{\sim} \mathcal{O}(\tilde{M}[0]).$$

Now we consider the integral representation of holomorphic functions on $\tilde{M}$. Let $r > 0$ and $f \in \mathcal{O}(\tilde{M}(r))$. We denote by f_k the k-homogeneous component of f. Fix $0 < \rho < r$. Then,

$$f(\rho z) = \sum_{k=0}^{\infty} f_k(\rho z) = \sum_{k=0}^{\infty} \rho^k f_k(z)$$

converges uniformly in a neighborhood of $\tilde{M}[1]$. Now suppose $w \in \tilde{M}$. Then,

$$\int_M f(\rho z)(\bar{z} \cdot w)^k d\mu(z) = \int_M \sum_{j=0}^{\infty} \rho^j f_j(z)(\bar{z} \cdot w)^k d\mu(z)$$

$$= \sum_{j=0}^{\infty} \rho^j \int_M f_j(z)(\bar{z} \cdot w)^k d\mu(z) = \frac{\rho^k}{2^k N(k, n)} f_k(w),$$

where the last equality is implied by Lemma 1. Put

$$K_0(t) = \sum_{k=0}^{\infty} N(k, n) t^k = \frac{1+t}{(1-t)^n}.$$

Let $0 < \rho' < \rho$ and $L(w) \le \rho'$. Then for $z \in M$, we have

$$2|\bar{z} \cdot w|/\rho \le 2L^*(z)L(w)/\rho \le \rho'/\rho < 1.$$

Therefore, for $w \in \tilde{M}[\rho']$, we have

$$f(w) = \sum_{k=0}^{\infty} f_k(w)$$

$$= \int_M f(\rho z) \sum_{k=0}^{\infty} N(k, n)\left(\frac{2}{\rho}\right)^k (\bar{z} \cdot w)^k d\mu(z)$$

$$= \int_M f(\rho z) K_0\left(\frac{2}{\rho} \bar{z} \cdot w\right) d\mu(z). \tag{9}$$

This is an integral representation of holomorphic function $f \in \mathcal{O}(\tilde{M}(r))$ which is valid on $\tilde{M}(\rho)$.

Note that

$$F(w)=\int_M f(\rho z)K_0\left(\frac{2}{\rho}\bar{z}\cdot w\right)d\mu(z), \qquad w\in\tilde{B}(\rho), \tag{10}$$

is a holomorphic function on $\tilde{B}(\rho)$ and satisfies $\Delta F(w)=0$. Because the right-hand side of (10) is independent of ρ with $L(w)<\rho<r$, $F(w)$ defined by (10) belongs to $\mathcal{O}_\Delta(\tilde{B}(r))$ and satisfies $F|_{\tilde{M}(r)}=f$ by (9).

Let $K_0: \mathcal{O}(\tilde{M}(r))\to\mathcal{O}_\Delta(\tilde{B}(r))$ be the integral operator defined by (10). The operator K_0 is the inverse of the restriction mapping β. Thus by Lemma 3 we have the following:

COROLLARY 7. *The exact sequences* (4) *and* (5) *are split. More precisely, we have*

$$\mathcal{O}(\tilde{B}(r))=z^2\mathcal{O}(\tilde{B}(r))\oplus\mathcal{O}_\Delta(\tilde{B}(r)), \qquad 0<r\le\infty,$$
$$\mathcal{O}(\tilde{B}[r])=z^2\mathcal{O}(\tilde{B}[r])\oplus\mathcal{O}_\Delta(\tilde{B}[r]), \qquad 0\le r<\infty.$$

3. Analytic functionals on $\tilde{M}$

For $0<r\le\infty$, $\mathcal{O}'(\tilde{M}(r))$ denotes the dual space of $\mathcal{O}(\tilde{M}(r))$. If $r=\infty$, we denote it by $\mathcal{O}'(\tilde{M})$. Similarly, for $0\le r<\infty$, $\mathcal{O}'(\tilde{M}[r])$ denotes the dual space of $\mathcal{O}(\tilde{M}[r])$. An element of $\mathcal{O}'(\tilde{M}(r))$ or $\mathcal{O}'(\tilde{M}[r])$ is generally called an *analytic functional* on $\tilde{M}$.

Theorem 5 implies that $\mathcal{P}(\tilde{M})$ is dense in each of the spaces in (3). There fore, by taking dual spaces, (3) gives rise to the following relations:

$$\mathcal{O}'(\tilde{M})\supset\mathcal{O}'(\tilde{M}[r])\supset\mathcal{O}'(\tilde{M}(r))\supset\mathcal{O}'([0]), \qquad 0<r<\infty.$$

LEMMA 8. *The following sequnces are exact and split*:

$$0\longrightarrow\mathcal{O}'(\tilde{M}(r))\xrightarrow{\beta^*}\mathcal{O}'(\tilde{B}(r))\xrightarrow{\iota^*}\mathcal{I}'_{\tilde{M}}(\tilde{B}(r))\longrightarrow 0, \qquad 0<r\le\infty,$$
$$0\longrightarrow\mathcal{O}'(\tilde{M}[r])\xrightarrow{\beta^*}\mathcal{O}'(\tilde{B}[r])\xrightarrow{\iota^*}\mathcal{I}'_{\tilde{M}}(\tilde{B}[r])\longrightarrow 0, \qquad 0\le r<\infty,$$

PROOF. This is the dual statement of Lemma 3 and Corollary 7. q.e.d.

By the mapping β^* we will regard $\mathcal{O}'(\tilde{M}(r))$ and $\mathcal{O}'(\tilde{M}[r])$ as subspaces of $\mathcal{O}'(\tilde{B}(r))$ and $\mathcal{O}'(\tilde{B}[r])$, respectively.

Let $T\in\mathcal{O}'(\tilde{M}[r])$ and $f\in\mathcal{O}(\tilde{M}[r])$. Then there is $r'>r$ such that $f\in\mathcal{O}(\tilde{M}(r'))$. For ρ with $r<\rho<r'$, we have

$$\langle T,f\rangle=\sum_{k=0}^{\infty}\langle T_z,f_k(z)\rangle=\sum_{k=0}^{\infty}\left\langle T_z,\frac{2^kN(k,n)}{\rho^k}\int_M f(\rho w)(z\cdot\bar{w})^k d\mu(w)\right\rangle$$

$$= \sum_{k=0}^{\infty} \int_M f(\rho w) S_k\left(T; \frac{\bar{w}}{\rho}\right) d\mu(w),$$

where $S_k(T; w) = 2^k N(k, n)\langle T_z, (z \cdot w)^k \rangle$. It is clear that $S_k(T; w) \in \mathscr{P}^k(\tilde{M})$. We call $S_k(T; w)$ the *k-homogeneous component* of T. The harmonic extension $\tilde{S}_k(T; w)$ of $S_k(T; w)$ is given by the same formula:

$$\tilde{S}_k(T; w) = 2^k N(k, n)\langle T_z, (z \cdot w)^k \rangle, \qquad w \in \mathbf{C}^{n+1}. \tag{11}$$

We define the *Cauchy transform* of T by

$$\check{T}(z) = \langle T_w, K_0(2z \cdot w) \rangle.$$

If $L(z) \le 1/r' < 1/r$, then we have $2|z \cdot w| \le 2L^*(z)L(w) \le L(w)/r'$. Thus $w \mapsto K_0(2z \cdot w) \in \mathcal{O}(\tilde{M}(r'))$. Therefore, $\check{T}(z)$ is defined for $L(z) \le 1/r'$ and holomorphic and complex harmonic in a neighborhood of $\tilde{B}[1/r']$. Since $r' > r$ is arbitrary, $\check{T}(z)$ belongs to $\mathcal{O}_\Delta(\tilde{B}(1/r))$.

THEOREM 9. *Let $0 < r < \infty$. The Cauchy transformation and the restriction mapping establish the following linear topological isomorphisms*:

(i) $\mathcal{O}'(\tilde{M}[0]) \xrightarrow{\sim} \mathcal{O}_\Delta(\mathbf{C}^{n+1}) \xrightarrow{\sim} \mathcal{O}(\tilde{M})$,

(ii) $\mathcal{O}'(\tilde{M}(r)) \xrightarrow{\sim} \mathcal{O}_\Delta(\tilde{B}[1/r]) \xrightarrow{\sim} \mathcal{O}(\tilde{M}[1/r])$,

(iii) $\mathcal{O}'(\tilde{M}[r]) \xrightarrow{\sim} \mathcal{O}_\Delta(\tilde{B}(1/r)) \xrightarrow{\sim} \mathcal{O}(\tilde{M}(1/r))$,

(iv) $\mathcal{O}'(\tilde{M}) \xrightarrow{\sim} \mathcal{O}_\Delta(\{0\}) \xrightarrow{\sim} \mathcal{O}(\tilde{M}[0])$.

PROOF. We prove only (iii). By Theorem 6 (iii), $\mathcal{O}_\Delta(\tilde{B}(1/r)) \cong \mathcal{O}(\tilde{M}(1/r))$. Let $T \in \mathcal{O}'(\tilde{M}[r])$. We shall prove that the Cauchy transformation $\mathscr{C}: \mathcal{O}'(\tilde{M}[r]) \to \mathcal{O}_\Delta(\tilde{B}(1/r))$ is a linear topological isomorphism. For any $f \in \mathcal{O}(\tilde{M}[r])$ there is r' with $r < r'$ such that $f \in \mathcal{O}(\tilde{M}(r'))$. Therefore, for $r < \rho < r'$ we have

$$\langle T_w, f(w) \rangle = \int_M f(\rho z) \left\langle T_w, K_0\left(\frac{2}{\rho}\bar{z} \cdot w\right)\right\rangle d\mu(z).$$

This can be rewritten as follows:

$$\langle T, f \rangle = \int_M f(\rho z) \check{T}\left(\frac{\bar{z}}{\rho}\right) d\mu(z).$$

Thus $\mathscr{C}$ is injective.

Conversely, let $\varphi \in \mathcal{O}(\tilde{M}(1/r))$. For any $f \in \mathcal{O}(\tilde{M}[r])$ there is $r' > r$ such that $f \in \mathcal{O}(\tilde{M}(r'))$. Take ρ with $r < \rho < r'$ and form

$$\langle T_\varphi, f \rangle = \int_M f(\rho z)\varphi\left(\frac{\bar{z}}{\rho}\right) d\mu(z).$$

This does not depend on ρ with $r < \rho < r'$ and $T_\varphi \in \mathcal{O}'(\tilde{M}[r])$. Further,

$$\begin{aligned} \check{T}_\varphi(z) &= \langle (T_\varphi)_w, K_0(2z \cdot w) \rangle \\ &= \int_M K_0(2z \cdot \rho w)\varphi\left(\frac{\bar{w}}{\rho}\right) d\mu(w) \\ &= \varphi(z). \end{aligned}$$

Thus $\mathscr{C}$ is surjective.

Since $\mathscr{C}$ is continuous and linear, its inverse mapping is also continuous by the closed graph theorem.

Since $\check{T}_\varphi = \varphi$, we have $\varphi_k = (\check{T}_\varphi)_k = S_k(T_\varphi; w)$. Therefore, from Theorems 5 and 9, we obtain the following theorem:

THEOREM 10 ([7, Theorem 3.4]). *Let* $0 < r < \infty$, $T \in \mathcal{O}'(\tilde{M})$ *and* $S_k(w) = S_k(T; w)$ *the k-homogeneous component of T. Then we have the following relations*:

(i) $$T \in \mathcal{O}'(\tilde{M}[0]) \Longleftrightarrow \limsup_{k \to \infty} [\| S_k(w) \|_{C(M)}]^{\frac{1}{k}} = 0,$$

(ii) $$T \in \mathcal{O}'(\tilde{M}(r)) \Longleftrightarrow \limsup_{k \to \infty} [\| S_k(w) \|_{C(M)}]^{\frac{1}{k}} < r,$$

(iii) $$T \in \mathcal{O}'(\tilde{M}[r]) \Longleftrightarrow \limsup_{k \to \infty} [\| S_k(w) \|_{C(M)}]^{\frac{1}{k}} \leq r,$$

(iv) $$T \in \mathcal{O}'(\tilde{M}) \Longleftrightarrow \limsup_{k \to \infty} [\| S_k(w) \|_{C(M)}]^{\frac{1}{k}} < \infty.$$

Further, we have

$$\langle T, f \rangle = \sum_{k=0}^{\infty} \int_M S_k(\bar{w}) f(w) d\mu(w),$$

where f is a test function in respective spaces.

4. Entire functions of exponential type on $\tilde{M}$

Let $N(z)$ be a norm on $\mathbf{C}^{n+1}$ and $A > 0$. For an entire function f, we put

$$\|f\|_{X(A,N)} = \sup\{|f(z)|\exp(-AN(z));\ z\in\mathbf{C}^{n+1}\},$$
$$X(A, N) = \{f\in\mathcal{O}(\mathbf{C}^{n+1});\ \|f\|_{X(A,N)} < \infty\}.$$

Then $X(A, N)$ is a Banach space with respect to the norm $\|f\|_{X(A,N)}$. Define

$$\mathrm{Exp}\,(\mathbf{C}^{n+1}; (A, N)) = \mathrm{proj\,lim}\,\{X(B, N);\ B > A\}, \qquad 0 \le A < \infty,$$
$$\mathrm{Exp}\,(\mathbf{C}^{n+1}; [A, N]) = \mathrm{ind\,lim}\,\{X(B, N);\ B < A\}, \qquad 0 < A \le \infty.$$

It is clear that $\mathrm{Exp}\,(\mathbf{C}^{n+1}; (A, N))$ is an FS space and that $\mathrm{Exp}\,(\mathbf{C}^{n+1}; [A, N])$ is a DFS space.

Note that $\mathrm{Exp}\,(\mathbf{C}^{n+1}; (0)) = \mathrm{Exp}\,(\mathbf{C}^{n+1}; (0, N))$ is the space of entire functions of minimal exponential type and that $\mathrm{Exp}\,(\mathbf{C}^{n+1}) = \mathrm{Exp}\,(\mathbf{C}^{n+1}; [\infty, N])$ is the space of entire functions of exponential type. Define

$$\mathrm{Exp}_\Delta\,(\mathbf{C}^{n+1}; (A, N)) = \mathcal{O}_\Delta(\mathbf{C}^{n+1})\cap \mathrm{Exp}\,(\mathbf{C}^{n+1}; (A, N)),$$
$$\mathrm{Exp}_\Delta\,(\mathbf{C}^{n+1}; [A, N]) = \mathcal{O}_\Delta(\mathbf{C}^{n+1})\cap \mathrm{Exp}\,(\mathbf{C}^{n+1}; [A, N]).$$

Similarly, for $f\in\mathcal{O}(\tilde{M})$, we put

$$\|f\|_{Z(A,N)} = \sup\{|f(z)|\exp(-AN(z));\ z\in\tilde{M}\},$$
$$Z(A, N) = \{f\in\mathcal{O}(\tilde{M});\ \|f\|_{Z(A,N)} < \infty\}.$$

Then $Z(A, N)$ is a Banach space with respect to the norm $\|f\|_{Z(A,N)}$. Define

$$\mathrm{Exp}\,(\tilde{M}; (A, N)) = \mathrm{proj\,lim}\,\{Z(B, N);\ B > A\}, \qquad 0 \le A < \infty,$$
$$\mathrm{Exp}\,(\tilde{M}; [A, N]) = \mathrm{ind\,lim}\,\{Z(B, N);\ B < A\}, \qquad 0 < A \le \infty.$$

It is clear that $\mathrm{Exp}\,(\tilde{M}; (A, N))$ is an FS space and that $\mathrm{Exp}\,(\tilde{M}; [A, N])$ is a DFS space. $\mathrm{Exp}\,(\tilde{M}; (0)) = \mathrm{Exp}\,(\tilde{M}; (0, N))$ and $\mathrm{Exp}\,(\tilde{M}) = \mathrm{Exp}\,(\tilde{M}; [\infty, N])$ are independent of the norm $N(z)$. In the sequel, the norm $N(z)$ will be the Lie norm $L(z)$ or the dual Lie norm $L^*(z)$. Because $2L^*(z) = L(z)$ on $\tilde{M}$, we have

$$\mathrm{Exp}\,(\tilde{M}; (A, L^*)) = \mathrm{Exp}\left(\tilde{M}; \left(\frac{A}{2}, L\right)\right).$$

We are mainly concerned with the following spaces:

$$\mathrm{Exp}\,(\tilde{M}; (0)) \subset \mathrm{Exp}\,(\tilde{M}; [A, L^*]) \subset \mathrm{Exp}\,(\tilde{M}; (A, L^*)) \subset \mathrm{Exp}\,(\tilde{M})$$
$$\subset \mathcal{O}(\tilde{M}) \subset \mathcal{O}(\tilde{M}[r]) \subset \mathcal{O}(\tilde{M}(r)) \subset \mathcal{O}(\tilde{M}[0]), \tag{12}$$

where $0 < A < \infty$ and $0 < r < \infty$.

Using the inequality (1) in Lemma 2 we can deduce the following theorem from Lemma 4.2 in [6].

THEOREM 11. *Let $A > 0$. Suppose $F(z) \in \mathcal{O}_\Delta(\mathbf{C}^{n+1})$ and $F_k(z) \in \mathscr{P}^k_\Delta(\mathbf{C}^{n+1})$ is the k-homogeneous component of F. Then we have the following relations*:

$$\text{(i)} \qquad F \in \mathrm{Exp}_\Delta(\mathbf{C}^{n+1}; (0)) \iff \limsup_{k\to\infty} [k! \, \|F_k\|_{C(M)}]^{\frac{1}{k}} = 0,$$

$$\text{(ii)} \qquad F \in \mathrm{Exp}_\Delta(\mathbf{C}^{n+1}; [A, L^*]) \iff \limsup_{k\to\infty} [k! \, \|F_k\|_{C(M)}]^{\frac{1}{k}} < \frac{A}{2},$$

$$\text{(iii)} \qquad F \in \mathrm{Exp}_\Delta(\mathbf{C}^{n+1}; (A, L^*)) \iff \limsup_{k\to\infty} [k! \, \|F_k\|_{C(M)}]^{\frac{1}{k}} \leq \frac{A}{2},$$

$$\text{(iv)} \qquad F \in \mathrm{Exp}_\Delta(\mathbf{C}^{n+1}) \iff \limsup_{k\to\infty} [k! \, \|F_k\|_{C(M)}]^{\frac{1}{k}} < \infty.$$

Further, the expansion

$$F(z) = \sum_{k=0}^{\infty} F_k(z), \qquad z \in \mathbf{C}^{n+1},$$

converges in the topology of respective spaces.

Similarly, our spaces of entire functions of exponential type on $\tilde{M}$ can be characterized by the growth behavior of their homogeneous components:

THEOREM 12 ([7, Theorem 3.3]). *Let $0 < A < \infty$. Suppose $f \in \mathcal{O}(\tilde{M})$ and $f_k \in \mathscr{P}^k(\tilde{M})$ is the k-homogeneous component of f. Then we have the following relations*:

$$\text{(i)} \qquad f \in \mathrm{Exp}(\tilde{M}; (0)) \iff \limsup_{k\to\infty} [k! \, \|f_k\|_{C(M)}]^{\frac{1}{k}} = 0,$$

$$\text{(ii)} \qquad f \in \mathrm{Exp}(\tilde{M}; [A, L^*]) \iff \limsup_{k\to\infty} [k! \, \|f_k\|_{C(M)}]^{\frac{1}{k}} < \frac{A}{2},$$

$$\text{(iii)} \qquad f \in \mathrm{Exp}(\tilde{M}; (A, L^*)) \iff \limsup_{k\to\infty} [k! \, \|f_k\|_{C(M)}]^{\frac{1}{k}} \leq \frac{A}{2},$$

$$\text{(iv)} \qquad f \in \mathrm{Exp}(\tilde{M}) \iff \limsup_{k\to\infty} [k! \, \|f_k\|_{C(M)}]^{\frac{1}{k}} < \infty.$$

Further, the expansion

$$f(z) = \sum_{k=0}^{\infty} f_k(z), \qquad z \in \tilde{M},$$

converges in the topology of respective spaces.

PROOF. We prove only (iii). Let $f \in \mathrm{Exp}(\tilde{M}; (A, L^*))$ and $B > A$. The k-homogeneous component f_k of f is defined by (7) and hence satisfies $\|f_k\|_{C(M)} \leq \|f\|_{Z(B,L^*)}/\rho^k \exp(B\rho/2)$ for any $\rho > 0$. Here, we used $2L^*(z) = L(z)$

for $z \in \tilde{M}$. Putting $\rho = 2k/B$, we get

$$\| f_k \|_{C(M)} \le \| f \|_{Z(B,L^*)} (B/2k)^k \exp(k).$$

Therefore, by the Stirling formula $\limsup_{k\to\infty} [k! \| f_k \|_{C(M)}]^{1/k} \le B/2$. Since $B > A$ is arbitrary,

$$\limsup_{k\to\infty} [k! \| f_k \|_{C(M)}]^{\frac{1}{k}} \le \frac{A}{2}. \tag{13}$$

Conversely, suppose a sequence $\{f_k \in \mathscr{P}^k(\tilde{M});\ k = 0, 1, 2, \cdots\}$ satisfies (13). Then, for any $B > A$ there is $C \ge 0$ such that $\| f_k \|_{C(M)} \le C(B/2)^k/k!$. Therefore, we have

$$|f(z)| \le \sum_{k=0}^{\infty} |f_k(z)| \le C \sum_{k=0}^{\infty} \left(\frac{B}{2}\right)^k L(z)^k \frac{1}{k!}$$
$$\le C \exp\left(\frac{B}{2} L(z)\right) = C \exp(BL^*(z))$$

for $z \in \tilde{M}$. Because $B > A$ is arbitrary, $f(z) = \sum_{k=0}^{\infty} f_k(z)$ belongs to $\mathrm{Exp}(\tilde{M}; (A, L^*))$. q.e.d.

Since the restriction mapping is continuous, the following theorem is clear from Theorems 11, 12, Lemma 2 and the closed graph theorem:

THEOREM 13 ([7, Theorem 3.2]). *Let* $0 < r < \infty$. *The restriction mapping establishes the following linear topological isomorphisms*:

(i) $\beta\colon \mathrm{Exp}_\Delta(\mathbf{C}^{n+1}; (0)) \xrightarrow{\sim} \mathrm{Exp}(\tilde{M}; (0))$,

(ii) $\beta\colon \mathrm{Exp}_\Delta(\mathbf{C}^{n+1}; [A, L^*]) \xrightarrow{\sim} \mathrm{Exp}(\tilde{M}; [A, L^*])$,

(iii) $\beta\colon \mathrm{Exp}_\Delta(\mathbf{C}^{n+1}; (A, L^*)) \xrightarrow{\sim} \mathrm{Exp}(\tilde{M}; (A, L^*))$,

(iv) $\beta\colon \mathrm{Exp}_\Delta(\mathbf{C}^{n+1}) \xrightarrow{\sim} \mathrm{Exp}(\tilde{M})$.

We have the following corollary, which generalizes Lemma 3.

COROLLARY 14. *The following sequences are exact*:

$$0 \longrightarrow \mathscr{I}_{\mathrm{Exp}}(\mathbf{C}^{n+1}; (A, L^*)) \xrightarrow{\iota} \mathrm{Exp}(\mathbf{C}^{n+1}; (A, L^*)) \xrightarrow{\beta} \mathrm{Exp}(\tilde{M}; (A, L^*)) \longrightarrow 0$$

for $0 \le A < \infty$ *and*

$$0 \longrightarrow \mathscr{I}_{\mathrm{Exp}}(\mathbf{C}^{n+1}; [A, L^*]) \xrightarrow{\iota} \mathrm{Exp}(\mathbf{C}^{n+1}; [A, L^*]) \xrightarrow{\beta} \mathrm{Exp}(\tilde{M}; [A, L^*]) \longrightarrow 0$$

for $0 < A \le \infty$, *where*

$$\mathscr{I}_{\mathrm{Exp}}(\mathbf{C}^{n+1}; (A, L^*)) = \mathscr{I}_{\tilde{M}}(\mathbf{C}^{n+1}) \cap \mathrm{Exp}(\mathbf{C}^{n+1}; (A, L^*)),$$
$$\mathscr{I}_{\mathrm{Exp}}(\mathbf{C}^{n+1}; [A, L^*]) = \mathscr{I}_{\tilde{M}}(\mathbf{C}^{n+1}) \cap \mathrm{Exp}(\mathbf{C}^{n+1}; [A, L^*]).$$

5. Entire functionals on $\tilde{M}$

Let $0 < A < \infty$. $\mathrm{Exp}'(\tilde{M}; (A, L^*))$ and $\mathrm{Exp}'(\tilde{M}; [A, L^*])$ denote the dual spaces of $\mathrm{Exp}(\tilde{M}; (A, L^*))$ and $\mathrm{Exp}(\tilde{M}; [A, L^*])$, respectively. An element of $\mathrm{Exp}'(\tilde{M}; (A, L^*))$ or $\mathrm{Exp}'(\tilde{M}; [A, L^*])$ is called an *entire functional* on $\tilde{M}$.

Since $\mathscr{P}(M)$ is dense in each of the spaces in (12), by duality, the relations in (12) imply

$$\begin{aligned}\mathrm{Exp}'(\tilde{M}; (0)) &\supset \mathrm{Exp}'(\tilde{M}; [A, L^*]) \supset \mathrm{Exp}'(\tilde{M}; (A, L^*)) \supset \mathrm{Exp}'(\tilde{M}) \\ &\supset \mathscr{O}'(\tilde{M}) \supset \mathscr{O}'(\tilde{M}[r]) \supset \mathscr{O}'(\tilde{M}(r)) \supset \mathscr{O}'(\tilde{M}[0]),\end{aligned} \tag{14}$$

where $0 < A < \infty$ and $0 < r < \infty$.

LEMMA 15. *The following sequences are exact:*

$$0 \longrightarrow \mathrm{Exp}'(\tilde{M}; (A, L^*)) \xrightarrow{\beta^*} \mathrm{Exp}'(\mathbf{C}^{n+1}; (A, L^*)) \xrightarrow{\iota^*} \mathscr{I}'_{\mathrm{Exp}}(\mathbf{C}^{n+1}; (A, L^*)) \longrightarrow 0$$

for $0 \leq A < \infty$ *and*

$$0 \longrightarrow \mathrm{Exp}'(\tilde{M}; [A, L^*]) \xrightarrow{\beta^*} \mathrm{Exp}'(\mathbf{C}^{n+1}; [A, L^*]) \xrightarrow{\iota^*} \mathscr{I}'_{\mathrm{Exp}}(\mathbf{C}^{n+1}; [A, L^*]) \longrightarrow 0$$

for $0 < A \leq \infty$.

PROOF. This is the dual statement of Corollary 14. q.e.d.

Thanks to this lemma, we can regard $\mathrm{Exp}'(\tilde{M}; (A, L^*))$ and $\mathrm{Exp}'(\tilde{M}; [A, L^*])$ as subspaces of $\mathrm{Exp}'(\mathbf{C}^{n+1}; (A, L^*))$ and $\mathrm{Exp}'(\mathbf{C}^{n+1}; [A, L^*])$, respectively.

The following theorem extends Theorem 10 in case of entire functionals. Because we have no Cauchy transformation of entire functionals, we cannot prove it by Theorem 12 as we did for Theorem 10.

THEOREM 16 ([7, Theorem 3.4]). *Let* $0 < A < \infty$. *Let* $S_k(w) = S_k(T; w)$ *be the k-homogeneous component of* $T \in \mathrm{Exp}'(\tilde{M}; (0))$. *Then we have the following relations:*

$$\text{(i)} \qquad T \in \mathrm{Exp}'(\tilde{M}) \iff \limsup_{k \to \infty} \left[\frac{\| S_k(w) \|_{C(M)}}{k!} \right]^{\frac{1}{k}} = 0,$$

$$\text{(ii)} \qquad T \in \mathrm{Exp}'(\tilde{M}; (A, L^*)) \iff \limsup_{k \to \infty} \left[\frac{\| S_k(w) \|_{C(M)}}{k!} \right]^{\frac{1}{k}} < \frac{2}{A},$$

(iii) $$T \in \operatorname{Exp}'(\tilde{M}; [A, L^*]) \Longleftrightarrow \limsup_{k\to\infty} \left[\frac{\| S_k(w) \|_{C(M)}}{k!} \right]^{\frac{1}{k}} \leq \frac{2}{A},$$

(iv) $$T \in \operatorname{Exp}'(\tilde{M}; (0)) \Longleftrightarrow \limsup_{k\to\infty} \left[\frac{\| S_k(w) \|_{C(M)}}{k!} \right]^{\frac{1}{k}} < \infty.$$

Further, we have

$$\langle T, f \rangle = \sum_{k=0}^{\infty} \int_M S_k(\bar{w}) f(w) d\mu(w), \tag{15}$$

where f is a test function in respective spaces.

Proof. We prove only (iii). Suppose $T \in \operatorname{Exp}'(\tilde{M}; [A, L^*])$. Then by the continuity of T, for any B with $0 \leq B < A$ there is $C_B \geq 0$ such that

$$|\langle T, f \rangle| \leq C_B \sup \{ |f(z)| \exp(-BL^*(z)); z \in \tilde{M} \}$$

for any $f \in Z(B, L^*)$. Therefore, for $w \in M$,

$$\begin{aligned} |S_k(w)| &= |S_k(T; w)| \\ &\leq C_B 2^k N(k, n) \sup \{ |z \cdot w|^k \exp(-BL^*(z)); z \in \tilde{M} \} \\ &\leq C_B 2^k N(k, n) \sup \{ L^*(z)^k \exp(-BL^*(z)); z \in \tilde{M} \} \\ &\leq C_B 2^k N(k, n) (k/B)^k e^{-k}. \end{aligned}$$

By the Stirling formula, we get $\limsup_{k\to\infty} [\| S_k \|_{C(M)}/k!]^{1/k} \leq 2/B$. Since $B < A$ is arbitrary, we have $\limsup_{k\to\infty} [\| S_k \|_{C(M)}/k!]^{1/k} \leq 2/A$.

Conversely, suppose a sequence $\{S_k \in \mathscr{P}^k(\tilde{M}); k = 0, 1, 2, \cdots\}$ satisfies the condition $\limsup_{k\to\infty} [\| S_k \|_{C(M)}/k!]^{1/k} \leq 2/A$. Then, for any B with $0 \leq B < A$ there is $C_B \geq 0$ such that $|S_k(w)| \leq C_B k! (2/B)^k$ for $k = 0, 1, 2, \cdots$. Let $f \in \operatorname{Exp}(\tilde{M}; [A, L^*])$. Then there is B' with $0 \leq B' < A$ such that

$$\| f \|_{Z(B'; L^*)} = \sup \{ |f(z)| \exp(-B'L^*(z)); z \in \tilde{M} \} < \infty.$$

Since the k-homogeneous component of f is defined by (7), we have

$$|f_k(w)| = \left| \frac{1}{2\pi i} \oint_{|t|=\rho} \frac{f(tw)}{t^{k+1}} dt \right| \leq \frac{1}{\rho^k} \exp(B'\rho/2) \| f \|_{Z(B', L^*)}$$

for $w \in \tilde{M}$, and hence

$$\| f_k(w) \|_{C(M)} \leq (B'/(2k))^k e^k \| f \|_{Z(B', L^*)}.$$

For $B' < B$, the Stirling formula implies

$$\left|\sum_{k=0}^{\infty}\int_M S_k(\bar{w})f_k(w)d\mu(w)\right| \le \sum_{k=0}^{\infty} C_B k!(1/k)^k e^k (B'/B)^k \|f\|_{Z(B',L^*)}$$
$$\le C' \|f\|_{Z(B',L^*)}$$

for $f\in Z(B', L^*)$. Therefore,

$$\langle T, f\rangle = \sum_{k=0}^{\infty}\int_M S_k(\bar{w})f(w)d\mu(w) = \sum_{k=0}^{\infty}\int_M S_k(\bar{w})f_k(w)d\mu(w)$$

converges and defines a linear functional T on $\mathrm{Exp}(\tilde{M}; [A, L^*])$. It is clear that $S_k(w) = S_k(T; w)$. q.e.d.

6. Fourier-Borel transformation

For $\zeta\in\mathbf{C}^{n+1}$ fixed, we consider the exponential function $f(z) = \exp(i\lambda z\cdot\zeta)$. The Taylor expansion $f(z) = \exp(i\lambda z\cdot\zeta) = \sum_{k=0}^{\infty}(i\lambda)^k(z\cdot\zeta)^k/k!$ coincides with the expansion by homogeneous polynomials:

$$f_k(z) = \frac{(i\lambda)^k}{k!}(z\cdot\zeta)^k, \qquad z\in\tilde{M}. \tag{16}$$

For $T\in\mathcal{O}'(\tilde{M})$, the function

$$\mathscr{F}_\lambda T(\zeta) = \langle T_z, \exp(i\lambda z\cdot\zeta)\rangle$$

is defined for $\zeta\in\mathbf{C}^{n+1}$ and is called the *Fourier-Borel transform* of T. If $T\in\mathrm{Exp}'(\tilde{M}; (0))$, then $\mathscr{F}_\lambda T(\zeta)$ is defined only for ζ in a neighborhood of 0 in $\mathbf{C}^{n+1}$.

LEMMA 17. *If* $T\in\mathrm{Exp}'(\tilde{M}; (0))$, *then we have*

$$\mathscr{F}_\lambda T(\zeta) = \sum_{k=0}^{\infty}\frac{(i\lambda)^k}{k!}\frac{1}{2^k N(k, n)}\tilde{S}_k(T; \zeta) \tag{17}$$

where $\tilde{S}_k(T; \zeta)$ *is the k-homogeneous polynomial defined by* (11). *In particular,*

$$\tilde{S}_k(\mathscr{F}_\lambda T; \zeta) = \frac{(i\lambda)^k}{k!}\frac{1}{2^k N(k, n)}\tilde{S}_k(T; \zeta). \tag{18}$$

The function $F(z) = \mathscr{F}_\lambda T(z)$ *satisfies the differential equation* $\Delta F(z) = 0$.

PROOF. By (15) and Lemma 1,

$$\mathscr{F}_\lambda T(\zeta) = \sum_{k=0}^{\infty}\int_M \tilde{S}_k(T; \bar{w})\frac{(i\lambda)^k}{k!}(w\cdot\zeta)^k d\mu(w)$$

$$= \sum_{k=0}^{\infty} \frac{(i\lambda)^k}{k!} \frac{1}{2^k N(k, n)} \tilde{S}_k(T; \zeta)$$

for $\zeta \in \mathbf{C}^{n+1}$ and (18) is clear. Because (17) is a uniformly convergent series of complex harmonic functions, the limit function is also complex harmonic.
q.e.d.

We now prove theorems on the Fourier-Borel transformation analogous to Martineau's theorem (for example see [6]):

THEOREM 18 ([7, Theorem 4.4]). *Let $\lambda \neq 0$ and $0 < r < \infty$. The Fourier-Borel transformation $\mathscr{F}_\lambda$ establishes the following linear topological isomorphisms*:

(i) $$\mathscr{F}_\lambda : \mathcal{O}'(\tilde{M}[0]) \xrightarrow{\sim} \mathrm{Exp}_\Delta(\mathbf{C}^{n+1}; (0)),$$

(ii) $$\mathscr{F}_\lambda : \mathcal{O}'(\tilde{M}(r)) \xrightarrow{\sim} \mathrm{Exp}_\Delta(\mathbf{C}^{n+1}; [|\lambda| r, L^*]),$$

(iii) $$\mathscr{F}_\lambda : \mathcal{O}'(\tilde{M}[r]) \xrightarrow{\sim} \mathrm{Exp}_\Delta(\mathbf{C}^{n+1}; (|\lambda| r, L^*)),$$

(iv) $$\mathscr{F}_\lambda : \mathcal{O}'(\tilde{M}) \xrightarrow{\sim} \mathrm{Exp}_\Delta(\mathbf{C}^{n+1}).$$

PROOF. We prove only (iii). Let $T \in \mathcal{O}'(\tilde{M}[r])$. By Lemma 17, $F(z) = \mathscr{F}_\lambda T(z)$ is holomorphic in a neighborhood of 0 in $\mathbf{C}^{n+1}$ and satisfies $\Delta F(z) = 0$. By (18) and Theorem 10 (iii), we have

$$\limsup_{k\to\infty} [k! \, \| S_k(\mathscr{F}_\lambda T; \zeta) \|_{C(M)}]^{\frac{1}{k}} = \frac{|\lambda|}{2} \limsup_{k\to\infty} [\| S_k(T; \zeta) \|_{C(M)}]^{\frac{1}{k}} \leq \frac{|\lambda| r}{2}.$$

By Theorem 11 (iii), $\mathscr{F}_\lambda T$ belongs to $\mathrm{Exp}_\Delta(\mathbf{C}^{n+1}; (|\lambda| r, L^*))$.

Conversely, let $F \in \mathrm{Exp}_\Delta(\mathbf{C}^{n+1}; (|\lambda| r, L^*))$. Expand F into homogeneous polynomials: $F(z) = \sum_{k=0}^{\infty} F_k(z)$. Since F is complex harmonic, F_k belongs to $\mathscr{P}_\Delta^k(\mathbf{C}^{n+1})$. Theorem 11 (iii), implies

$$\limsup_{k\to\infty} [k! \, \| F_k \|_{C(M)}]^{\frac{1}{k}} \leq \frac{|\lambda| r}{2}. \tag{19}$$

Define

$$S_k(\zeta) = \left(\frac{2}{i\lambda} \right)^k k! N(k, n) F_k(\zeta). \tag{20}$$

Then (19) implies $\limsup_{k\to\infty} [\| S_k(\zeta) \|_{C(M)}]^{1/k} \leq r$. Since $r > 0$, by Theorem 10 (iii), the formal series $T(\zeta) = \sum_{k=0}^{\infty} S_k(\zeta)$ defines an analytic functional and belongs to $\mathcal{O}'(\tilde{M}[r])$. By the construction of T and Lemma 17, we have $\mathscr{F}_\lambda T = F$. Since $\mathscr{F}_\lambda$ is continuous, $\mathscr{F}_\lambda^{-1}$ is also continuous by the closed graph theorem. q.e.d.

The following theorem generalizes Kowata-Okamoto's theorem:

THEOREM 19 ([7, Theorem 4.4]). *Let* $\lambda \neq 0$ *and* $0 < A < \infty$. *The Fourier-Borel transformation* $\mathscr{F}_\lambda$ *establishes the following linear topological isomorphisms*:

(i) $\mathscr{F}_\lambda\colon \operatorname{Exp}'(\tilde{M}) \xrightarrow{\sim} \mathcal{O}_\Delta(\mathbf{C}^{n+1}),$

(ii) $\mathscr{F}_\lambda\colon \operatorname{Exp}'(\tilde{M}; (A, L^*)) \xrightarrow{\sim} \mathcal{O}_\Delta(\tilde{B}[A/|\lambda|]),$

(iii) $\mathscr{F}_\lambda\colon \operatorname{Exp}'(\tilde{M}; [A, L^*]) \xrightarrow{\sim} \mathcal{O}_\Delta(\tilde{B}(A/|\lambda|)),$

(iv) $\mathscr{F}_\lambda\colon \operatorname{Exp}'(\tilde{M}; (0)) \xrightarrow{\sim} \mathcal{O}_\Delta(\{0\}).$

PROOF. We prove only (iii). Let $T \in \operatorname{Exp}'(\tilde{M}; [A, L^*])$. If $L(\zeta) < A/|\lambda|$, then there is $B < A$ such that $L(\zeta) < B/|\lambda| < A/|\lambda|$ and we have

$$|\exp(i\lambda\zeta \cdot z)| \leq \exp(|\lambda| L(\zeta) L^*(z)) \leq \exp(BL^*(z)).$$

Therefore, $\exp(i\lambda\zeta \cdot z) \in \operatorname{Exp}(\tilde{M}; [A, L^*])$ and we can define the Fourier-Borel transformation:

$$\mathscr{F}_\lambda T(\zeta) = \langle T_z, \exp(i\lambda\zeta \cdot z)\rangle, \qquad L(\zeta) < \frac{A}{|\lambda|}.$$

By Lemma 17, Theorems 16 (iii) and 4 (iii), $F(z) = \mathscr{F}_\lambda T(z)$ belongs to $\mathcal{O}_\Delta(\tilde{B}(A/|\lambda|))$.

Conversely, let $F \in \mathcal{O}_\Delta(\tilde{B}(A/|\lambda|))$. Expand F into homogeneous polynomials: $F(z) = \sum_{k=0}^\infty F_k(z)$. Since F is complex harmonic, $F_k \in \mathscr{P}_\Delta^k(\mathbf{C}^{n+1})$. By Theorem 4 (iii),

$$\limsup_{k\to\infty} [\| F_k \|_{C(M)}] \leq \frac{|\lambda|}{A}. \tag{21}$$

Define $S_k(\zeta)$ by (20). Then (21) implies

$$\limsup_{k\to\infty} [\| S_k(\zeta) \|_{C(M)}/k!]^{1/k} \leq 2/A.$$

Because $A > 0$, by Theorem 16, the formal series $T(\zeta) = \sum_{k=0}^\infty S_k(\zeta)$ defines an entire functional and belongs to $\operatorname{Exp}'(\tilde{M}; [A, L^*])$. By the construction of T and Lemma 17, we have $\mathscr{F}_\lambda T = F$. Since $\mathscr{F}_\lambda$ is continuous, $\mathscr{F}_\lambda^{-1}$ is also continuous by the closed graph theorem. q.e.d.

Thanks to Theorem 13, Theorems 18 and 19 may be stated as follows:

THEOREM 20 ([7, Theorem 4.5]). *Let* $\lambda \neq 0$, $0 < r < \infty$ *and* $0 < A < \infty$. *The conical Fourier-Borel transformation* $\mathscr{Q}_\lambda = \beta \circ \mathscr{F}_\lambda$ *establishes the following linear topological isomorphisms*:

(i) $\mathscr{Q}_\lambda\colon \mathcal{O}'(\tilde{M}[0]) \xrightarrow{\sim} \mathrm{Exp}(\tilde{M}; (0))$,

(ii) $\mathscr{Q}_\lambda\colon \mathcal{O}'(\tilde{M}(r)) \xrightarrow{\sim} \mathrm{Exp}(\tilde{M}; [|\lambda| r, L^*])$,

(iii) $\mathscr{Q}_\lambda\colon \mathcal{O}'(\tilde{M}[r]) \xrightarrow{\sim} \mathrm{Exp}(\tilde{M}; (|\lambda| r, L^*))$,

(iv) $\mathscr{Q}_\lambda\colon \mathcal{O}'(\tilde{M}) \xrightarrow{\sim} \mathrm{Exp}(\tilde{M})$,

(v) $\mathscr{Q}_\lambda\colon \mathrm{Exp}'(\tilde{M}) \xrightarrow{\sim} \mathcal{O}(\tilde{M})$,

(vi) $\mathscr{Q}_\lambda\colon \mathrm{Exp}'(\tilde{M}; (A, L^*)) \xrightarrow{\sim} \mathcal{O}(\tilde{M}[A/|\lambda|])$,

(vii) $\mathscr{Q}_\lambda\colon \mathrm{Exp}'(\tilde{M}; [A, L^*]) \xrightarrow{\sim} \mathcal{O}(\tilde{M}(A/|\lambda|))$,

(viii) $\mathscr{Q}_\lambda\colon \mathrm{Exp}'(\tilde{M}; (0)) \xrightarrow{\sim} \mathcal{O}(\tilde{M}[0])$.

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