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The polynomials on w_1, w_2 and w_3 in the universal Wu classes

Dedicated to Professor Teiichi Kobayashi on his 60th birthday

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ABSTRACT. The cohomology ring $H^*(BO; Z_2)$ is the polynomial algebra $Z_2[w_1, w_2, w_3, \cdots]$, where w_i is the i-dimensional universal Stiefel–Whitney class. The i-dimensional universal Wu class v_i is defined inductively as follows: $v_0 = w_0 = 1$ and $w_i = v_i + \sum_{j=1}^{i} Sq^j v_{i-j} (i \geq 1)$, where Sq^j is the Steenrod squaring operation. We can describe explicitly the polynomials on w_1, w_2 and w_3 in v_i.

1. Introduction

Let BO be the space which classifies stable real vector bundles. Then its mod 2 cohomology $H^*(BO; Z_2)$ is the polynomial algebra over Z_2 on the universal Stiefel–Whitney classes $w_i \in H^i(BO; Z_2)$ for $i \geq 1$ (cf. [4], [10]).

The i-dimensional universal Wu class v_i $(i \geq 0)$ is the element of $H^i(BO; Z_2)$, and this is defined inductively by using the Steenrod squaring operations Sq^j in the following way (cf. [3], [6], [7], [8]):

(1.1) $\quad v_0 = w_0 = 1$ and $w_i = v_i + Sq^1 v_{i-1} + \cdots + Sq^i v_0 \quad$ if $\ i \geq 1$.

The i-dimensional Wu class $v_i(M)$ of a closed n-dimensional manifold M is the unique element of $H^i(M; Z_2)$ such that

$$Sq^i x = x v_i(M) \qquad \text{for all} \quad x \in H^{n-i}(M; Z_2),$$

and the following relations between the Stiefel–Whitney classes and the Wu classes of M hold (cf. [4], [9]):

(1.2) $\quad v_0(M) = 1$ and $w_i(M) = v_i(M) + Sq^1 v_{i-1}(M) + \cdots + Sq^i v_0(M) \quad$ if $i \geq 1$.

So if f denotes the classifying map for the stable tangent bundle of M, then

$$f^* w_i = w_i(M) \text{ and } f^* v_i = v_i(M) \qquad \text{if} \quad i \geq 0 .$$

Let J be the ideal of $H^*(BO; Z_2)$ generated by the squares $w_1^2, w_2^2, w_3^2, \cdots$.

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Then the total universal Wu class $v = 1 + v_1 + v_2 + \cdots \mod J$ is as follows (cf. [13]):

$$v \equiv 1 + \sum w_{i_1} \cdots w_{i_l} \mod J,$$

where Σ is taken over all sequences $1 \leq i_1 < \cdots < i_l$ $(l \geq 1)$ satisfying $\{i_1, \cdots, i_l\} = \{\alpha_1, \beta_1, \cdots, \alpha_m, \beta_m, \gamma_1, \cdots, \gamma_n\}$ $(l = 2m + n, m \geq 0, n \geq 0)$ such that $\alpha_j + \beta_j$ and γ_j are all powers of 2.

So the final goal is to describe all monomials in v which belong to J. It is known that w_j^2 appears in v_i $(i = 2j > 0)$ if and only if $\alpha(i) = 1$ or 2, and also $w_j w_1^{i-j} (i \geq j \geq 2)$ appears in v_i if and only if $\alpha(i) = 1$ and $i/2 < j \leq i$, or $\alpha(i) = 2$ and $2^b < j \leq 2^a$ with $i = 2^a + 2^b$ $(a > b \geq 0)$, where $\alpha(i)$ denotes the number of 1's in the dyadic expansion of i (cf. [2]).

Now applying the Wu formula (cf. [5], [11])

$$(1.3) \qquad Sq^j w_i = \sum_{t=0}^{j} \binom{i-j-1+t}{t} w_{j-t} w_{i+t} \qquad (0 \leq j < i),$$

we can suppose the following (cf. Theorem 4.9, [12]):

$$(1.4) \quad v_j \equiv \begin{cases} (w_2 + w_1^2)^{j/2} & \mod I_3 \quad \text{if } j = 2^a \ (a \geq 1), \\ (\sum_{s=0}^{a-b-1} w_2^{2^s} w_1^{j/2^b - 2^{s+1}})^{2^b} & \mod I_3 \quad \text{if } j = 2^a + 2^b \ (a > b \geq 0), \\ 0 & \mod I_3 \quad \text{if } \alpha(j) \geq 3, \end{cases}$$

where I_3, generally I_k denotes the ideal of $H^*(BO; Z_2)$ generated by w_k, w_{k+1}, $\cdots$.

But the Wu classes modulo I_4 seem very complicated. So we study these classes in this paper.

Let P_t be the element of $H^{2^{t+1}+1}(BO; Z_2)$ defined by

$$P_t = Sq^{2^t} Sq^{2^{t-1}} \cdots Sq^1 w_2 \qquad \text{if } t \geq 0; \text{ and } P_t = 0 \quad \text{if } t = -1.$$

Then the Wu classes modulo I_4 are given by the following theorems.

THEOREM 1.5.

(i) $v_j \equiv (v_4)^{j/4} \equiv (w_3 w_1 + w_2^2 + w_1^4)^{j/4} \mod I_4$ *if* $j = 2^a$ $(a \geq 2)$.

(ii) $v_i \equiv P_{a-2} w_1^{(i-1)/2} + w_2^{(i-1)/2} w_1 \mod I_4$ *if* $i = 2^a + 1$ $(a \geq 1)$.

(iii) $v_i \equiv P_{a-1} w_1 + (P_{a-2} + P_{a-3} w_1^{(i-2)/4} + w_2^{(i-2)/4} w_1)^2 + w_2 w_1^{i-2} \mod I_4$ *if* $i = 2^a + 2$ $(a \geq 2)$; *and* $v_j \equiv (v_i)^{j/i} \mod I_4$ *if* $j = 2^a + 2^b$ $(a > b \geq 1)$ *and* $i = j/2^{b-1}$.

(iv) $v_i \equiv P_{a-1} w_1^{2^b} + P_{a-b-1}^{2^b} w_1 + P_{b-1} w_1^{2^a} \mod I_4$ *if* $i = 2^a + 2^b + 1$ $(a > b \geq 1)$; *and* $v_j \equiv (v_i)^{j/i} \mod I_4$ *if* $j = 2^a + 2^b + 2^c$ $(a > b > c \geq 0)$ *and* $i = j/2^c$.

(v) $v_j \equiv 0 \mod I_4$ *if* $\alpha(j) \geq 4$.

The above theorem will be proved in §3.

Let $F_{m,n}$ be the element of $H^{2^m-2^{n+1}}(BO; Z_2)$ defined by

$$F_{m,n} = \sum_{i=n}^{m-1} w_2^{2^i-2^n} w_1^{2^m-2^{i+1}} \quad \text{if } m > n \geq 0\text{; and } F_{m,n} = 0 \quad \text{if } n \geq m \geq 0\,.$$

And if $p = 2^{p_1} + 2^{p_2} + \cdots + 2^{p_s}$ with $s \geq 1$ and $p_1 > p_2 > \cdots > p_s \geq 0$, then set

$$G_p = \begin{cases} F_{p_1,p_2+1}F_{p_2,p_3+1}\cdots F_{p_{s-1},p_s+1} & \text{if } s \geq 2\,, \\ 1 & \text{if } s = 1\,; \end{cases}$$

and also set $h(p) = p_1$ and $l(p) = p_s$.

Let $P_t(p)$ be the sum consisting of all monomials on w_3, w_2 and w_1 in P_t such that each power of w_3 for such monomials is p. If there are no such monomials, then also set $P_t(p) = 0$ in $H^{2^{t+1}+1}(BO; Z_2)$.

Then we have the following, which will be proved in §4.

THEOREM 1.6. *Let $p \geq 2$ be an even integer. Then*

(i) $P_t(1) = w_3 F_{t+1,0}$.

(ii) $P_t(p) = w_3^p F_{t+1,h(p)+1} G_p F_{l(p),0} w_2 w_1$.

(iii) $P_t(p+1) = w_3^{p+1} F_{t+1,h(p)+1} G_p F_{l(p),0}$.

If we apply Theorem 1.6 to Theorem 1.5, then common monomials will appear and they will cancel each other. Explicit descriptions by the distinct monomials of the Wu classes modulo I_4 will be obtained in §5.

2. Iterated Steenrod operations on the Stiefel–Whitney classes

Let θ^i be the elements of the mod 2 Steenrod algebra defined inductively by

(2.1)
$$\theta^0 = Sq^0 = 1\,,\ \theta^1 = Sq^1 \text{ and}$$
$$\theta^i = Sq^i + Sq^{i-1}\theta^1 + Sq^{i-2}\theta^2 + \cdots + Sq^1\theta^{i-1} \qquad \text{if } i \geq 2 \text{ (cf. [12])}\,.$$

Then $\theta^i = \Sigma Sq^{j_1}\cdots Sq^{j_s}$, where Σ is taken over all sequences $(j_1, \cdots, j_s)$ consisting of positive integers such that $j_1 + \cdots + j_s = i$ for $i \geq 1$; and this implies the following equality:

(2.2)
$$\theta^i = Sq^i + \theta^1 Sq^{i-1} + \theta^2 Sq^{i-2} + \cdots + \theta^{i-1}Sq^1 \qquad \text{if } i \geq 2\,.$$

From (2.1) and (2.2), the following equalities hold:

(2.3)
$$(Sq^0 + Sq^1 + Sq^2 + \cdots)(\theta^0 + \theta^1 + \theta^2 + \cdots) = 1\,,$$
$$(\theta^0 + \theta^1 + \theta^2 + \cdots)(Sq^0 + Sq^1 + Sq^2 + \cdots) = 1\,.$$

Thus the inverse Sq^{-1} of $Sq = Sq^0 + Sq^1 + Sq^2 + \cdots$ is given by

$$Sq^{-1} = \theta^0 + \theta^1 + \theta^2 + \cdots . \tag{2.4}$$

PROPOSITION 2.5. *Let* $i \geq 0$. *Then*

$$v_i = \theta^i w_0 + \theta^{i-1} w_1 + \cdots + \theta^0 w_i .$$

PROOF. Set $w = w_0 + w_1 + w_2 + \cdots$ and $v = v_0 + v_1 + v_2 + \cdots$. Then using (1.1) we see $w = Sq\ v$, and so $v = Sq^{-1}w$. Thus (2.4) implies the conclusion. □

The following lemma is well-known (cf. [12]).

LEMMA 2.6. (i) *Let x be a one dimensional cohomology class. Then*

$$\theta^i x = \begin{cases} x^{i+1} & \text{if } i+1 \text{ is a power of } 2 , \\ 0 & \text{otherwise} . \end{cases}$$

(ii) *Let x and y be cohomology classes. Then*

$$\theta^i(xy) = \sum_{j+k=i} (\theta^j x)(\theta^k y) .$$

PROOF. (i) For a sequence $(j_1, j_2, \cdots, j_s)$ consisting of positive integers, we see

$$Sq^{j_1} Sq^{j_2} \cdots Sq^{j_s} x = \begin{cases} x^{2^s} & \text{if } (j_1, j_2, \cdots, j_s) = (2^{s-1}, 2^{s-2}, \cdots, 1) , \\ 0 & \text{otherwise} . \end{cases}$$

Thus we obtain (i).

(ii) It holds that $Sq\{Sq^{-1}(xy)\} = Sq\{(Sq^{-1}x)(Sq^{-1}y)\}$ since the left side is $(SqSq^{-1})(xy) = xy$ and the right side is $(SqSq^{-1}x)(SqSq^{-1}y) = xy$. Applying Sq^{-1} on both sides of this equality, we see $Sq^{-1}(xy) = (Sq^{-1}x)(Sq^{-1}y)$. Thus (ii) follows from (2.4). □

PROPOSITION 2.7. *Let i and j be positive integers such that $\alpha(i) > j$. Then* $\theta^{i-j} w_j = 0$.

PROOF. Let $BO(n)$ be the space which classifies real n-plane bundles. Then $H^*(BO(n), Z_2)$ is the polynomial algebra over Z_2 on the Stiefel–Whitney classes $w_m(\gamma^n) \in H^m(BO(n); Z_2)$ $(1 \leq m \leq n)$ of the universal bundle γ^n over $BO(n)$. And if $g: BO(n) \to BO$ denotes the natural inclusion map, then $g^* w_m = w_m(\gamma^n)$ and

$$g^*: H^k(BO; Z_2) \to H^k(BO(n); Z_2)$$

is an isomorphism for all $k \leq n$ (cf. [4], [10]).

Now by the splitting principle, there exists a space X and a map $f: X \to BO(n)$ such that the induced bundle $f^*\gamma^n$ is isomorphic to the Whitney sum $\xi_1 \oplus \cdots \oplus \xi_n$ of suitable real line bundles ξ_s $(1 \leq s \leq n)$ over X, and also

$$f^*: H^k(BO(n); Z_2) \to H^k(X; Z_2)$$

is a monomorphism for all k (cf. [5]).

Let $x_s = w_1(\xi_s)$ $(1 \leq s \leq n)$. Then

$$(f^*g^*)w_j = f^*w_j(\gamma^n) = w_j(f^*\gamma^n) = w_j(\xi_1 \oplus \cdots \oplus \xi_n) = \sum x_{t_1} \cdots x_{t_j},$$

where Σ is taken over all sequences $(t_1, \cdots, t_j)$ such that $1 \leq t_1 < \cdots < t_j \leq n$. So we see

$$(f^*g^*)\theta^{i-j}w_j = \theta^{i-j}((f^*g^*)w_j) = \sum \theta^{i-j}(x_{t_1} \cdots x_{t_j}).$$

From Lemma 2.6 we have

$$\theta^{i-j}(x_{t_1} \cdots x_{t_j}) = \sum x_{t_1}^{p_1} \cdots x_{t_j}^{p_j},$$

where Σ is taken over all sequences $(p_1, \cdots, p_j)$ consisting of powers of 2 such that $p_1 + \cdots + p_j = i$. But such a sequence does not exist since $\alpha(p_1 + \cdots + p_j) \leq j < \alpha(i)$. Thus $(f^*g^*)\theta^{i-j}w_j = 0$, and so $\theta^{i-j}w_j = 0$ by choosing n such that $i \leq n$. □

Next we consider the case $\alpha(i) \leq j$, and obtain the following.

PROPOSITION 2.8. *Let i and j be positive integers such that $\alpha(i) \leq j \leq \alpha(i) + l(i)$. Then*

$$\theta^{i-j}w_j = (\theta^{i/2^m - j}w_j)^{2^m},$$

where $m = \alpha(i) + l(i) - j$.

PROOF. We use the same notations as the proof of Proposition 2.7. Then

$$(f^*g^*)\theta^{i-j}w_j = \sum \sum x_{t_1}^{p_1} \cdots x_{t_j}^{p_j}.$$

Here we have $p_k \geq 2^m$ for all k $(1 \leq k \leq j)$. To show this equality, if $j = 1$, then $\alpha(i) = 1$, and so $p_1 = i = 2^{l(i)} = 2^m$. Next let $j \geq 2$. If $p_1 = 2^a < 2^m$ for example, then $\alpha(p_2 + \cdots + p_j) = \alpha(i - p_1) = (\alpha(i) - 1) + (l(i) - a) > (\alpha(i) - 1) + (j - \alpha(i)) = j - 1$, which is incompatible with $\alpha(p_2 + \cdots + p_j) \leq j - 1$. Thus we obtain

$$(f^*g^*)\theta^{i-j}w_j = (\sum \sum x_{t_1}^{q_1} \cdots x_{t_j}^{q_j})^{2^m},$$

where $\Sigma\Sigma$ are taken over all sequences $(t_1, \cdots, t_j)$ and $(q_1, \cdots, q_j)$ such that $1 \leq t_1 < \cdots < t_j \leq n$ and $q_1 + \cdots + q_j = i/2^m$ with q_s $(1 \leq s \leq j)$ powers of 2.

Thus $(f^*g^*)\theta^{i-j}w_j = (f^*g^*)(\theta^{i/2^m-j}w_j)^{2^m}$, and so $\theta^{i-j}w_j = (\theta^{i/2^m-j}w_j)^{2^m}$ by choosing n such that $i \le n$. □

We obtain the following by using the above propositions.

THEOREM 2.9. *Let j and k be positive integers such that $\alpha(j) + 1 \le k \le \alpha(j) + l(j) + 1$. Then*

$$v_j \equiv (v_i)^{2^m} \mod I_k ,$$

where $i = j/2^m$ and $m = \alpha(j) + l(j) + 1 - k$.

PROOF. Since $l(j) - m = k - \alpha(j) - 1 \ge 0$, i is a positive integer, and also $i \ge k - 1$ holds because of $i \ge 2^{k-2} + 2^{k-3} + \cdots + 2^{k-\alpha(j)-1} \ge 2^{k-2} \ge k - 1$. Now from Proposition 2.7 and (1.3), we see $\theta^{j-s}w_s = 0$ $(1 \le s < \alpha(j))$ and $\theta^{j-s}w_s \equiv 0 \mod I_k$ $(k \le s \le j)$. Thus by Proposition 2.5 with $\theta^j w_0 = 0$ $(j \ge 1)$, we have

$$v_j \equiv \sum_{s=\alpha(j)}^{k-1} \theta^{j-s}w_s \mod I_k .$$

Using Proposition 2.8 because of $\alpha(j) \le s \le k - 1 \le \alpha(j) + l(j)$, we see

$$\theta^{j-s}w_s = (\theta^{j/2^p-s}w_s)^{2^p} \text{ for } p = \alpha(j) + l(j) - s .$$

Similarly noting $\alpha(i) = \alpha(j) \le k - 1$, we have

$$v_i \equiv \sum_{s=\alpha(i)}^{k-1} \theta^{i-s}w_s \mod I_k .$$

Since $\alpha(i) + l(i) = \alpha(j) + l(j) - m = k - 1$, we see

$$\theta^{i-s}w_s = (\theta^{i/2^q-s}w_s)^{2^q} \text{ for } q = \alpha(i) + l(i) - s .$$

Since $i/2^q = j/2^{m+q} = j/2^p$, we obtain

$$(\theta^{i-s}w_s)^{2^m} = (\theta^{i/2^q-s}w_s)^{2^{m+q}} = (\theta^{j/2^p-s}w_s)^{2^p} .$$

Therefore $v_j \equiv (v_i)^{2^m} \mod I_k$. □

REMARK 2.10. Let j and k be positive integers such that $\alpha(j) \ge k$. Then in the proof of Theorm 2.9, it is shown that $v_j \equiv 0 \mod I_k$.

3. The Wu classes modulo I_4

In this section we will study the Wu classes modulo I_4 and prove Theorem 1.5.

PROPOSITION 3.1.
(i) $v_j \equiv (v_4)^{j/4} \mod I_4$ *if* $\alpha(j) = 1$ $(j \geq 4)$.
(ii) $v_j \equiv (v_i)^{j/i} \mod I_4$ *if* $\alpha(j) = 2$ *and* $i = j/2^{l(j)-1}$, $l(j) \geq 1$.
(iii) $v_j \equiv (v_i)^{j/i} \mod I_4$ *if* $\alpha(j) = 3$ *and* $i = j/2^{l(j)}$.
(iv) $v_j \equiv 0 \mod I_4$ *if* $\alpha(j) \geq 4$.

PROOF. Let $1 \leq \alpha(j) \leq 3$. Then Theorem 2.9 implies

$$v_j \equiv (v_i)^{2^m} \mod I_4 \quad \text{if } l(j) \geq 3 - \alpha(j)\,,$$

where $i = j/2^m$ and $m = \alpha(j) + l(j) - 3$. Thus (i), (ii) and (iii) follow from this. Also (iv) follows from Remark 2.10. □

Thus we have the following remark from the above proposition.

REMARK 3.2. To describe the Wu classes v_i modulo I_4 $(i \geq 3)$, it is sufficient only to describe v_i modulo I_4, where $i = 4$, $2^a + 1$ $(a \geq 1)$, $2^a + 2$ $(a \geq 2)$ and $2^a + 2^b + 1$ $(a > b \geq 1)$.

The following lemma is known (cf. [1], [12]).

LEMMA 3.3. *Let i be a power of 2. Then*
(i) $\theta^{i-1} = Sq^{i/2}Sq^{i/4} \cdots Sq^1$ *if* $i \geq 2$.
(ii) $\theta^{i-1-j} = Sq^{i/2-s}\theta^{i/2-1-(j-s)} + Sq^{i/2}\theta^{i/2-1-j}$ *if* $1 \leq j < i/2$ *and* $s = 2^{h(j)}$.
(iii) $\theta^{i-j} = Sq^{i/2}Sq^{i/4} \cdots Sq^s\theta^{s-j}$ *if* $1 \leq j \leq h(i)$ *and* $s = 2^{j-1}$.
(iv) $\theta^{2k+1} = \theta^{2k}Sq^1$ *if* $k \geq 0$.

The Wu classes modulo I_4 of the dimenions in Remark 3.2 are as follows:

PROPOSITION 3.4. *Let i be $2^a + 1$ $(a \geq 1)$, $2^a + 2$ $(a \geq 2)$, $2^a + 2^b + 1$ $(a > b \geq 1)$. Then*

$$v_i \equiv \begin{cases} (\theta^{i/2-2}w_2)^2 + \theta^{i-3}(w_2w_1) & \mod I_4 \quad \text{if } i = 2^a + 2\,, \\ \theta^{i-3}(w_2w_1) & \mod I_4 \quad \text{otherwise}\,. \end{cases}$$

PROOF. We use Propositions 2.5 and 2.7. For $i \geq 1$, $\theta^i w_0 = 0$; and using (1.3), $\theta^{i-s}w_s \equiv 0 \mod I_4$ for $4 \leq s \leq i$.

Let $i = 2^a + 1$ $(a \geq 1)$. Then since $w_3 = Sq^1w_2 + w_2w_1$ by (1.3), and $\theta^{i-3}Sq^1 = \theta^{i-2}$ by Lemma 3.3 (iv), we see

$$v_i \equiv \theta^{i-2}w_2 + \theta^{i-3}w_3 = \theta^{i-2}w_2 + \theta^{i-3}(Sq^1w_2 + w_2w_1) = \theta^{i-3}(w_2w_1) \mod I_4\,.$$

Let $i = 2^a + 2^b + 1$ $(a > b \geq 1)$. Then similarly we see

$$v_i \equiv \theta^{i-3}w_3 = \theta^{i-3}(Sq^1w_2 + w_2w_1) = \theta^{i-2}w_2 + \theta^{i-3}(w_2w_1) = \theta^{i-3}(w_2w_1) \mod I_4$$

since $\theta^{i-2}w_2 = 0$ also by Proposition 2.7.

Let $i = 2^a + 2$ $(a \geq 2)$. Then

$$v_i \equiv \theta^{i-2} w_2 + \theta^{i-3} w_3 = \theta^{i-2} w_2 + \theta^{i-3} Sq^1 w_2 + \theta^{i-3}(w_2 w_1) \mod I_4 .$$

Here $\theta^{i-2} w_2 = (\theta^{i/2-2} w_2)^2$ by Proposition 2.8, and $\theta^{i-3} Sq^1 = \theta^{i-4} Sq^1 Sq^1 = 0$ by Lemma 3.3 (iv) and the Adem relation $Sq^1 Sq^1 = 0$. □

In the following proposition, we consider $\theta^{i-3}(w_2 w_1)$ in the above proposition.

PROPOSITION 3.5. *Let i be $2^a + 1$ $(a \geq 2)$, $2^a + 2$ $(a \geq 2)$, $2^a + 2^b + 1$ $(a > b \geq 1)$. Then*

$$\theta^{i-3}(w_2 w_1) = \begin{cases} w_2^m w_1 + P_{a-2} w_1^m & \text{if } i = 2^a + 1 \text{ and } m = 2^{a-1}, \\ P_{a-1} w_1 + w_2^m w_1^2 + P_{a-3}^2 w_1^m + w_2 w_1^{2m} & \text{if } i = 2^a + 2 \text{ and } m = 2^{a-1}, \\ P_{a-b-1}^n w_1 + P_{a-1} w_1^n + P_{b-1} w_1^{2m} & \text{if } i = 2^a + 2^b + 1 \text{ and } m = 2^{a-1},\ n = 2^b . \end{cases}$$

PROOF. Let $i = 2^a + 1$ $(a \geq 2)$ and $m = 2^{a-1}$. Then using Lemma 2.6 and Proposition 2.7, we see

$$\begin{aligned} \theta^{i-3}(w_2 w_1) &= (\theta^{2m-2} w_2) w_1 + \sum_{p=1}^{a-1} \{\theta^{(2m-2^p+1)-2} w_2\}(\theta^{2^p-1} w_1) \\ &= (\theta^{2m-2} w_2) w_1 + (\theta^{m-1} w_2) w_1^m \end{aligned}$$

since $\alpha(2m - 2^p + 1) = a - p + 1 > 2$ for $1 \leq p < a - 1$.
Here from Proposition 2.8 and Lemma 3.3 (i), we see

$$\theta^{2m-2} w_2 = (\theta^0 w_2)^m = w_2^m \quad \textit{and} \quad \theta^{m-1} w_2 = Sq^{m/2} Sq^{m/4} \cdots Sq^1 w_2 = P_{a-2} .$$

Let $i = 2^a + 2$ $(a \geq 2)$ and $m = 2^{a-1}$. Then similarly we see

$$\begin{aligned} \theta^{i-3}(w_2 w_1) &= (\theta^{2m-1} w_2) w_1 + (\theta^{2m-2} w_2) w_1^2 + \sum_{p=2}^{a-1} \{\theta^{(2m-2^p+2)-2} w_2\} w_1^{2^p} + w_2 w_1^{2m} \\ &= (\theta^{2m-1} w_2) w_1 + (\theta^{2m-2} w_2) w_1^2 + (\theta^m w_2) w_1^m + w_2 w_1^{2m} \end{aligned}$$

since $\alpha(2m - 2^p + 2) = a - p + 1 > 2$ for $2 \leq p < a - 1$.
Here $\theta^{2m-1} w_2 = P_{a-1}$, $\theta^{2m-2} w_2 = w_2^m$ and $\theta^m w_2 = (\theta^{m/2-1} w_2)^2 = P_{a-3}^2$ for $a \geq 3$. Thus noting $P_{a-3}^2 = 0$ for $a = 2$, we obtain the conclusion.

Let $i = 2^a + 2^b + 1$ $(a > b \geq 1)$ and $m = 2^{a-1}$, $n = 2^b$. Then similarly we see

$$\theta^{i-3}(w_2w_1) = (\theta^{2m+n-2}w_2)w_1 + \sum_{p=1}^{b-1} \{\theta^{(2m+n-2^p+1)-2}w_2\}w_1^{2^p} + (\theta^{2m-1}w_2)w_1^n$$

$$+ \sum_{p=b+1}^{a-1} \{\theta^{(2m-2^p+n+1)-2}w_2\}w_1^{2^p} + (\theta^{n-1}w_2)w_1^{2m}$$

$$= (\theta^{2m+n-2}w_2)w_1 + (\theta^{2m-1}w_2)w_1^n + (\theta^{n-1}w_2)w_1^{2m}$$

since $\alpha(2m+n-2^p+1) = b-p+2 > 2$ for $1 \le p \le b-1$, and also $\alpha(2m - 2^p + n + 1) = a - p + 2 > 2$ for $b+1 \le p \le a-1$.
Here $\theta^{2m+n-2}w_2 = (\theta^{2m/n-1}w_2)^n = P_{a-b-1}^n$, $\theta^{2m-1}w_2 = P_{a-1}$ and $\theta^{n-1}w_2 = P_{b-1}$. □

We are now in a position to prove Theorem 1.5.

Proof of Theorem 1.5. (i) $v_4 \equiv \theta^3 w_1 + \theta^2 w_2 + \theta^1 w_3 \mod I_4$. Here by Proposition 2.8 and (1.3), we see $\theta^3 w_1 = (\theta^0 w_1)^4 = w_1^4$, $\theta^2 w_2 = (\theta^0 w_2)^2 = w_2^2$ and $\theta^1 w_3 = Sq^1 w_3 = w_3 w_1$. Hence (i) follows from Proposition 3.1 (i).

(ii) $v_3 = w_2 w_1$ by Proposition 3.4. So (ii) holds for $a = 1$ since $P_{-1} = 0$ by the definition. Next let $i = 2^a + 1$ $(a \ge 2)$. Then Propositions 3.4 and 3.5 imply (ii).

(iii) Using Propositions 3.4, 3.5 and letting $m = 2^{a-1}$, we see

$$v_i \equiv (\theta^{m-1}w_2)^2 + \theta^{i-3}(w_2w_1)$$

$$= P_{a-2}^2 + P_{a-1}w_1 + w_2^m w_1^2 + P_{a-3}^2 w_1^m + w_2 w_1^{2m}$$

$$= P_{a-1}w_1 + (P_{a-2} + P_{a-3}w_1^{m/2} + w_2^{m/2}w_1)^2 + w_2 w_1^{2m} \mod I_4 ,$$

which is the result for v_i. And the one for v_j follows from Proposition 3.1 (ii).

(iv) Propositions 3.4 and 3.5 imply the result for v_i. And the one for v_j follows from Proposition 3.1 (iii).

(v) This follows from Proposition 3.1 (iv). □

4. $Sq^{2^t}Sq^{2^{t-1}} \cdots Sq^1 w_2$ modulo I_4

In this section we will study the terms $P_t = Sq^{2^t}Sq^{2^{t-1}} \cdots Sq^1 w_2$ which remain to be known in Theorem 1.5.

The following lemma will be used often.

Lemma 4.1 ([6]). *Let* $a = \Sigma_i a_i 2^i$ *and* $b = \Sigma_i b_i 2^i$ $(0 \le a_i, b_i \le 1)$. *Then*

$$\binom{a}{b} \equiv \prod_i \binom{a_i}{b_i} \mod 2 .$$

We have the following formula on P_t.

PROPOSITION 4.2. *Let* $x = w_3 + w_2w_1$ *and* $y = w_2 + w_1^2$. *Then*

$$P_t \equiv \sum_{0 \le 3i < n} \binom{3i}{i} x^{2i+1} y^{n-3i-1} \mod I_4 , \tag{4.3}$$

where $n = 2^t$ $(t \ge 0)$.

PROOF. We notice the following (4.4):

$$Sq^1x = 0 , \quad Sq^2x \equiv xy \mod I_4 , \quad Sq^3x = x^2 , \quad Sq^1y = x , \quad Sq^2y = y^2 . \tag{4.4}$$

In fact using (1.3) and the Cartan formula, we see

$$Sq^1x = w_3w_1 + w_2w_1^2 + (w_3 + w_2w_1)w_1 = 0 ,$$

$$Sq^2x = w_5 + w_4w_1 + w_3w_2 + (w_3 + w_2w_1)w_1^2 + w_2^2w_1 \equiv xy \mod I_4 ,$$

$$Sq^1y = Sq^1w_2 = w_3 + w_2w_1 = x .$$

And since $\dim x = 3$ and $\dim y = 2$, $Sq^3x = x^2$ and $Sq^2y = y^2$ hold.

We prove (4.3) by induction on t. Since $P_0 = Sq^1w_2 = x$ and $P_1 = Sq^2P_0 = Sq^2x \equiv xy \mod I_4$, (4.3) holds for $t = 0, 1$. Assume that (4.3) holds for $t-1$ $(t \ge 2)$. Then since $Sq^iI_4 \subseteq I_4$ and $n = 2^t = \dim P_{t-1} - 1$, the left side of (4.3) for t is as follows:

$$\begin{aligned}
P_t = Sq^nP_{t-1} &\equiv Sq^n \sum_{0 \le 3i < n/2} \binom{3i}{i} x^{2i+1} y^{n/2-3i-1} \\
&= \sum_{0 \le 3i < n/2} \binom{3i}{i} \{(2i+1)(Sq^2x)x^{4i}y^{n-6i-2} \\
&\quad + (n/2 - 3i - 1)x^{4i+2}(Sq^1y)y^{n-6i-4}\} \\
&= \sum_{0 \le 6j < n/2} \binom{6j}{2j} \{(Sq^2x)x^{8j}y^{n-12j-2} + x^{8j+2}(Sq^1y)y^{n-12j-4}\} \\
&\quad + \sum_{0 \le 6j+3 < n/2} \binom{6j+3}{2j+1} \{(Sq^2x)x^{8j+4}y^{n-12j-8}\} \\
&\equiv \sum_{0 \le 6j < n/2} \binom{6j}{2j} (x^{8j+1}y^{n-12j-1} + x^{8j+3}y^{n-12j-4}) \\
&\quad + \sum_{0 \le 6j+3 < n/2} \binom{6j+3}{2j+1} (x^{8j+5}y^{n-12j-7}) \mod I_4 .
\end{aligned}$$

On the other hand, using Lemma 4.1, we see

$$\sum_{0 \le 3i < n} \binom{3i}{i} x^{2i+1} y^{n-3i-1} = S_1 + S_2 + S_3 + S_4 ,$$

where

$$S_1 = \sum_{0 \le 12j < n} \binom{12j}{4j} x^{8j+1} y^{n-12j-1} = \sum_{0 \le 6j < n/2} \binom{6j}{2j} x^{8j+1} y^{n-12j-1},$$

$$S_2 = \sum_{0 \le 12j+3 < n} \binom{12j+3}{4j+1} x^{8j+3} y^{n-12j-4} = \sum_{0 \le 6j < n/2} \binom{6j}{2j} x^{8j+3} y^{n-12j-4},$$

$$S_3 = \sum_{0 \le 12j+6 < n} \binom{12j+6}{4j+2} x^{8j+5} y^{n-12j-7} = \sum_{0 \le 6j+3 < n/2} \binom{6j+3}{2j+1} x^{8j+5} y^{n-12j-7},$$

and since $\binom{12j+9}{4j+3}$ is even

$$S_4 = \sum_{0 \le 12j+9 < n} \binom{12j+9}{4j+3} x^{8j+7} y^{n-12j-10} = 0.$$

Therefore (4.3) for t holds. □

Next we have another inductive formula on P_t.

PROPOSITION 4.5. *Let* $x = w_3 + w_2 w_1$ *and* $y = w_2 + w_1^2$. *Then*

$$P_t \equiv y^n P_{t-1} + x^n P_{t-2} \mod I_4, \tag{4.6}$$

where $n = 2^{t-1}$ $(t \ge 1)$.

PROOF. We prove (4.6) by induction on t. Since $P_{-1} = 0$, $P_0 = x$, $P_1 \equiv xy \bmod I_4$, $P_2 = Sq^4 P_1 \equiv Sq^4(xy) = (Sq^2 x)y^2 + x^2(Sq^1 y) \equiv xy^3 + x^3 \equiv y^2 P_1 + x^2 P_0 \bmod I_4$, (4.6) holds for $t = 1, 2$. Assume that (4.6) holds for $t-1$ $(t \ge 3)$. Then the left side of (4.6) for t is as follows:

$$\begin{aligned} P_t = Sq^{2n} P_{t-1} &\equiv Sq^{2n}(y^{n/2} P_{t-2} + x^{n/2} P_{t-3}) \\ &= (Sq^{n-1} y^{n/2}) P_{t-2}^2 + y^n (Sq^n P_{t-2}) \\ &\quad + (Sq^{n+n/2-1} x^{n/2}) P_{t-3}^2 + x^n (Sq^{n/2} P_{t-3}) \mod I_4. \end{aligned}$$

Here $Sq^{n-1} y^{n/2} = 0$ and $Sq^{n+n/2-1} x^{n/2} = 0$ since $n-1$ and $n + n/2 - 1$ are both odd and $n/2$ is even; $Sq^n P_{t-2} = P_{t-1}$ and $Sq^{n/2} P_{t-3} = P_{t-2}$. Thus $P_t \equiv y^n P_{t-1} + x^n P_{t-2} \bmod I_4$, which completes induction on t. □

COROLLARY 4.7. *Let* $t \ge 0$. *Then*

$$P_t \equiv \sum_{i=0}^{t} w_2^{2^i} w_1^{m-2^{i+1}} \mod I_3, \tag{4.8}$$

where $m = 2^{t+1} + 1$.

PROOF. We prove (4.8) by induction on t. Since $P_0 = Sq^1 w_2 \equiv w_2 w_1 \bmod I_3$ and $P_1 = Sq^2 P_0 \equiv Sq^2(w_2 w_1) = (Sq^1 w_2)w_1^2 + w_2^2 w_1 \equiv w_2 w_1^3 + w_2^2 w_1 \bmod I_3$, (4.8) holds for $t = 0, 1$. Assume that (4.8) holds for less than $t-1$ $(t \geq 2)$. Then the left side of (4.8) for t is as follows by using Proposition 4.5 and letting $n = 2^{t-1}$, $m = 2^{t+1}+1$:

$$\begin{aligned} P_t &\equiv (w_2 + w_1^2)^n P_{t-1} + (w_2 w_1)^n P_{t-2} \\ &\equiv (w_2^n + w_1^{2n}) \sum_{i=0}^{t-1} w_2^{2^i} w_1^{2n+1-2^{i+1}} + (w_2^n w_1^n) \sum_{i=0}^{t-2} w_2^{2^i} w_1^{n+1-2^{i+1}} \\ &= \sum_{i=0}^{t-1} w_2^{2^i+n} w_1^{2n+1-2^{i+1}} + \sum_{i=0}^{t-1} w_2^{2^i} w_1^{4n+1-2^{i+1}} + \sum_{i=0}^{t-2} w_2^{2^i+n} w_1^{2n+1-2^{i+1}} \\ &= \sum_{i=0}^{t} w_2^{2^i} w_1^{m-2^{i+1}} \mod I_3\,, \end{aligned}$$

which completes induction on t. □

We obtain the following theorem which is (1.4) in §1 (cf. [12]).

THEOREM 4.9.

(i) $v_j \equiv (v_2)^{j/2} = (w_2 + w_1^2)^{j/2} \mod I_3$ *if* $j = 2^a$ $(a \geq 1)$.

(ii) $v_i \equiv \sum_{s=0}^{a-1} w_2^{2^s} w_1^{i-2^{s+1}} \mod I_3$ *if* $i = 2^a + 1$ $(a \geq 1)$; *and* $v_j \equiv (v_i)^{j/i} \bmod I_3$ *if* $j = 2^a + 2^b$ $(a > b \geq 0)$ *and* $i = j/2^b$.

(iii) $v_j \equiv 0 \mod I_3$ *if* $\alpha(j) \geq 3$.

PROOF. (iii) follows from Remark 2.10. So assume $\alpha(j) \leq 2$. Then by Theorem 2.9, we have

$$v_j \equiv (v_i)^{2^m} \mod I_3 \quad \text{if } l(j) \geq 2 - \alpha(j)\,,$$

where $i = j/2^m$ and $m = \alpha(j) + l(j) - 2$.

(i) If $j = 2^a$ $(a \geq 1)$, then $v_j \equiv (v_2)^{j/2} \bmod I_3$. Here we see that $v_2 = \theta^1 w_1 + \theta^0 w_2 = w_1^2 + w_2$.

(ii) If $j = 2^a + 2^b$ $(a > b \geq 0)$, then $v_j \equiv (v_i)^{j/i} \bmod I_3$ *where* $i = j/2^b$. Also if $i = 2^a + 1$ $(a \geq 1)$, then using Corollary 4.7, we see

$$v_i \equiv \theta^{i-2} w_2 = P_{a-1} \equiv \sum_{s=0}^{a-1} w_2^{2^s} w_1^{i-2^{s+1}} \mod I_3\,. \quad \square$$

REMARK 4.10. Using Theorem 1.5, we can also prove Theorem 4.9 in the following way:

(i) Let $j = 2^a$ $(a \geq 2)$. Then Theorem 1.5 (i) implies that $v_j \equiv (w_2^2 + w_1^4)^{j/4} = (w_2 + w_1^2)^{j/2} \bmod I_3$.

(ii) Let $i = 2^a + 1$ ($a \geq 1$) and $m = 2^{a-1} + 1$. Then Theorem 1.5(ii) and Corollary 4.7 imply

$$v_i \equiv \left(\sum_{s=0}^{a-2} w_2^{2^s} w_1^{m-2^{s+1}}\right) w_1^{m-1} + w_2^{m-1} w_1 = \sum_{s=0}^{a-1} w_2^{2^s} w_1^{i-2^{s+1}} \mod I_3 .$$

Let $j = 2^a + 2$ ($a \geq 2$), $i = j/2$ and $m = 2^a + 1$. Then Theorem 1.5(iii), Corollary 4.7 and the above result for $i = j/2$ imply

$$\begin{aligned} v_j &\equiv \left(\sum_{s=0}^{a-1} w_2^{2^s} w_1^{m-2^{s+1}}\right) w_1 \\ &+ \left\{\sum_{s=0}^{a-2} w_2^{2^s} w_1^{(m-1)/2+1-2^{s+1}} + \left(\sum_{s=0}^{a-3} w_2^{2^s} w_1^{(m-1)/4+1-2^{s+1}}\right) w_1^{(m-1)/4} + w_2^{(m-1)/4} w_1\right\}^2 \\ &+ w_2 w_1^{m-1} = \sum_{s=0}^{a-1} w_2^{2^s} w_1^{j-2^{s+1}} + w_2 w_1^{m-1} = \sum_{s=1}^{a-1} w_2^{2^s} w_1^{j-2^{s+1}} \\ &= \left(\sum_{t=0}^{a-2} w_2^{2^t} w_1^{i-2^{t+1}}\right)^2 \equiv (v_i)^{j/i} \mod I_3 . \end{aligned}$$

Let $j = 2^a + 2^b$ ($a > b \geq 1$) and $i = j/2^b$. Then Theorem 1.5(iii) and the above result for v_{2i} imply that $v_j \equiv (v_{2i})^{j/2i} \equiv (v_i)^{j/i} \mod I_3$.

(iii) Let $i = 2^a + 2^b + 1$ ($a > b \geq 1$), $m = 2^a + 1$ and $n = 2^b + 1$. Then Theorem 1.5(iv) and Corollary 4.7 imply

$$\begin{aligned} v_i &\equiv \left(\sum_{s=0}^{a-1} w_2^{2^s} w_1^{m-2^{s+1}}\right) w_1^{n-1} + \left(\sum_{s=0}^{a-b-1} w_2^{2^s} w_1^{(m-1)/(n-1)+1-2^{s+1}}\right)^{n-1} w_1 \\ &+ \left(\sum_{s=0}^{b-1} w_2^{2^s} w_1^{n-2^{s+1}}\right) w_1^{m-1} \\ &= \sum_{s=0}^{a-1} w_2^{2^s} w_1^{i-2^{s+1}} + \sum_{s=b}^{a-1} w_2^{2^s} w_1^{i-2^{s+1}} + \sum_{s=0}^{b-1} w_2^{2^s} w_1^{i-2^{s+1}} = 0 \mod I_3 . \end{aligned}$$

Let $j = 2^a + 2^b + 2^c$ ($a > b > c \geq 0$) and $i = j/2^c$. Then Theorem 1.5(iv) and the above result for $j/2^c$ imply that $v_j \equiv (v_i)^{2^c} \equiv 0 \mod I_3$.

Let $\alpha(j) \geq 4$. Then $v_j \equiv 0 \mod I_3$ by Theorem 1.5(v). □

We are now in a position to prove Theorem 1.6.

Proof of Theorem 1.6 (i). By the definition of $F_{t+1,0}$, (i) means the following:

(4.11) $$P_t(1) = w_3 \sum_{i=0}^{t} w_2^{2^i-1} w_1^{m-2^{i+1}} ,$$

where $m = 2^{t+1}$.

We prove (4.11) by induction on t. Since $P_0 = w_3 + w_2 w_1$ and $P_1 \equiv (w_3 + w_2 w_1)(w_2 + w_1^2) \bmod I_4$, we see that $P_0(1) = w_3$ and $P_1(1) = w_3(w_2 + w_1^2)$. So (4.11) holds for $t = 0, 1$. Assume that (4.11) holds for less than $t-1$ ($t \geq 2$). Then the left side of (4.11) for t is as follows by using Proposition 4.5 and letting $m = 2^{t+1}$:

$$\begin{aligned} P_t(1) &= (w_2 + w_1^2)^{m/4} P_{t-1}(1) + (w_2 w_1)^{m/4} P_{t-2}(1) \\ &= (w_2 + w_1^2)^{m/4} w_3 \sum_{i=0}^{t-1} w_2^{2^i - 1} w_1^{m/2 - 2^{i+1}} + (w_2 w_1)^{m/4} w_3 \sum_{i=0}^{t-2} w_2^{2^i - 1} w_1^{m/4 - 2^{i+1}} \\ &= w_3 \sum_{i=0}^{t} w_2^{2^i - 1} w_1^{m - 2^{i+1}}, \end{aligned}$$

which completes induction on t. □

PROOF OF THEOREM 1.6 (ii). We prove (ii) by induction on t. We see that $P_0(p) = 0$ and $P_1(p) = 0$ since $\dim P_0 = 3 < 3p$ and $\dim P_1 = 5 < 3p$. On the other hand, $F_{1,h(p)+1} = 0$ and $F_{2,h(p)+1} = 0$ since $h(p) + 1 \geq 2$. Thus (ii) holds for $t = 0, 1$. Also using Propositions 4.2 or 4.5, we see that $P_2 \equiv xy^3 + x^3 \bmod I_4$, where $x = w_3 + w_2 w_1$ and $y = w_2 + w_1^2$. So $P_2(2) = w_3^2 w_2 w_1$ and $P_2(p) = 0$ for $p \geq 4$. On the other hand, $w_3^2 F_{3,2} G_2 F_{1,0} w_2 w_1 = w_3^2 w_2 w_1$ and $F_{3,h(p)+1} = 0$ for $p \geq 4$ since $h(p) \geq 2$ for $p \geq 4$. Hence (ii) also holds for $t = 2$. Assume that (ii) holds for less than $t-1$ ($t \geq 3$). Then the left side of (ii) for t is as follows by using Proposition 4.5 and letting $n = 2^{t-1}$:

Case 1. $p < n$.
In this case $t \geq h(p) + 2$ holds, and we see

$$\begin{aligned} P_t(p) &= (w_2 + w_1^2)^n P_{t-1}(p) + (w_2 w_1)^n P_{t-2}(p) \\ &= w_3^p G_p F_{l(p),0} w_2 w_1 \{(w_2^n + w_1^{2n}) F_{t,h(p)+1} + w_2^n w_1^n F_{t-1,h(p)+1}\}. \end{aligned}$$

Here by letting $m = h(p) + 1$, we see

$$(w_2^n + w_1^{2n}) F_{t,m} + w_2^n w_1^n F_{t-1,m} = \sum_{i=m}^{t} w_2^{2^i - 2^m} w_1^{4n - 2^{i+1}} = F_{t+1,m}.$$

Thus $P_t(p) = w_3^p F_{t+1,h(p)+1} G_p F_{l(p),0} w_2 w_1$, which completes induction on t.

Case 2. $p = n$.
In this case, using Corollary 4.7 and noting $P_{t-1}(p) = 0$, $P_{t-2}(p) = 0$, we see

$$P_t(p) = w_3^n \sum_{i=0}^{t-2} w_2^{2^i} w_1^{n - 2^{i+1} + 1}.$$

On the other hand, since $h(p) = l(p) = t - 1$, $F_{t+1,t} = 1$ and $G_p = 1$, we see

$$w_3^p F_{t+1,h(p)+1} G_p F_{l(p),0} w_2 w_1 = w_3^n F_{t-1,0} w_2 w_1 = w_3^n \sum_{i=0}^{t-2} w_2^{2^i} w_1^{n-2^{i+1}+1} ,$$

which completes induction on t.

Case 3. $n < p < 2n$.

In this case $h(p) = t - 1$ and $p - n$ (> 0) is even. So noting $P_{t-1}(p) = 0$, $P_{t-2}(p) = 0$, $F_{t-1,h(p-n)+1} G_{p-n} = G_p$ and $l(p-n) = l(p)$, we see

$$\begin{aligned} P_t(p) &= w_3^n P_{t-2}(p-n) \\ &= w_3^n w_3^{p-n} F_{t-1,h(p-n)+1} G_{p-n} F_{l(p-n),0} w_2 w_1 = w_3^p G_p F_{l(p),0} w_2 w_1 . \end{aligned}$$

On the other hand, since $F_{t+1,h(p)+1} = F_{t+1,t} = 1$, we see

$$w_3^p F_{t+1,h(p)+1} G_p F_{l(p),0} w_2 w_1 = w_3^p G_p F_{l(p),0} w_2 w_1 ,$$

which completes induction on t.

Case 4. $p \geq 2n$.

Since $\dim P_t = 4n + 1$ and $3p \geq 6n > 4n + 1$, we see $P_t(p) = 0$. On the other hand, since $h(p) + 1 \geq t + 1$, we see $F_{t+1,h(p)+1} = 0$, which completes induction on t. □

Proof of Theorem 1.6 (iii). We prove (iii) in the same way as the proof of (ii). Then we see that (iii) holds for $t = 0, 1$. And since $P_2(3) = w_3^3$ and $P_2(p+1) = 0$ for $p \geq 4$, (iii) also holds for $t = 2$. Assume that (iii) holds for less than $t - 1$ ($t \geq 3$). Then the left side of (iii) for t is as follows by using Proposition 4.5 and letting $n = 2^{t-1}$:

Case 1. $p + 1 < n$.

In this case $t \geq h(p+1) + 2 = h(p) + 2$ holds, and we see

$$\begin{aligned} P_t(p+1) &= (w_2 + w_1^2)^n P_{t-1}(p+1) + (w_2 w_1)^n P_{t-2}(p+1) \\ &= w_3^{p+1} G_p F_{l(p),0} F_{t+1,h(p)+1} . \end{aligned}$$

Case 2. $p + 1 = n + 1$.

In this case, using Theorem 1.6(i) and noting $P_{t-1}(p+1) = 0$, $P_{t-2}(p+1) = 0$, $h(p) = l(p) = t - 1$, $F_{t+1,h(p)+1} = F_{t+1,t} = 1$ and $G_p = 1$, we see

$$P_t(p+1) = w_3^n P_{t-2}(1) = w_3^{n+1} F_{t-1,0} = w_3^{p+1} F_{t+1,h(p)+1} G_p F_{l(p),0} .$$

Case 3. $n + 1 < p + 1 < 2n$.

In the same way as the Case 3 in the proof of Theorem 1.6(ii), we see

$$\begin{aligned} P_t(p+1) &= w_3^n P_{t-2}(p-n+1) = w_3^{p+1} F_{t-1,h(p-n)+1} G_{p-n} F_{l(p-n),0} \\ &= w_3^{p+1} F_{t+1,h(p)+1} G_p F_{l(p),0} . \end{aligned}$$

Case 4. $p+1>2n$.

Since $3(p+1)>\dim P_t$, we have $P_t(p+1)=0$. Also since $h(p)+1\geq t+1$, we have $F_{t+1,h(p)+1}=0$. □

5. Some explicit descriptions of the Wu classes modulo I_4

In this section, using Theorems 1.5 and 1.6, we will describe v_i modulo I_4 by the distinct monomials on w_3, w_2 *and* w_1. We will do it only for $i=2^a+1$ ($a\geq 1$), 2^a+2 ($a\geq 2$) and 2^a+2^b+1 ($a>b\geq 1$), but this is sufficient by Theorem 1.5.

In the following theorems, $p\equiv q(n)$ denotes $p\equiv q \bmod n$.

THEOREM 5.1. *Let* $i=2^a+1$ ($a\geq 1$). *Then*

$$v_i \equiv F_{a,0}w_2w_1 + w_3F_{a-1,0}w_1^m + \sum_{p\equiv 0(2)} w_3^pF_{a-1,h(p)+1}G_pF_{l(p),0}(w_3+w_2w_1)w_1^m \mod I_4 ,$$

where $p>0$ *and* $m=2^{a-1}$.

PROOF. By Theorem 4.9(ii) or Remark 4.10(ii), we have

$$v_i \equiv \sum_{s=0}^{a-1} w_2^{2^s}w_1^{i-2^{s+1}} = F_{a,0}w_2w_1 \mod I_3 .$$

Thus the conclusion follows from Theorems 1.5 and 1.6. □

THEOREM 5.2. *Let* $i=2^a+2$ ($a\geq 2$). *Then*

$$\begin{aligned} v_i \equiv\ & (F_{a-1,0}w_2w_1)^2 + w_3F_{a,0}w_1 + w_3^2(w_2^{m-2}+F_{a,2}w_2w_1^2) \\ &+ \sum_{p\equiv 0(2)} w_3^{p+1}F_{a,h(p)+1}G_pF_{l(p),0}w_1 \\ &+ \sum_{p\equiv 0(4)} w_3^p(F_{a,h(p)+1}G_pw_2w_1^{2l(p)} + F_{a-1,h(p)+1}G_pF_{l(p),1}w_2^2w_1^{m+2}) \\ &+ \sum_{p\equiv 0(4)} w_3^{p+2}(F_{a,h(p)+1}G_pF_{l(p),2}w_2w_1^2 + w_2^{m-2^{h(p)+1}}G_pF_{l(p),1}) \mod I_4 , \end{aligned}$$

where $p>0$, $m-2^{h(p)+1}\geq 0$ *and* $m=2^{a-1}$.

PROOF. By Remark 4.10(ii), we have

$$v_i \equiv (v_{i/2})^2 \equiv (F_{a-1,0}w_2w_1)^2 \mod I_3 .$$

Let $v_j(p)$ be the sum consisting of all monomials on w_3, w_2 and w_1 in v_j such that each power of w_3 for such monomials is p. If there are no such monomials, then also set $v_j(p)=0$ in $H^j(BO;Z_2)$.

Then using Theorem 1.5(iii) and Theorem 1.6(i), (ii) and noting $(F_{m,n})^{2^t} = F_{m+t,n+t}$, we see

$$v_i(1) = P_{a-1}(1)w_1 = w_3F_{a,0}w_1 .$$

$$\begin{aligned} v_i(2) &= P_{a-1}(2)w_1 + (P_{a-2}(1) + P_{a-3}(1)w_1^{m/2})^2 \\ &= w_3^2F_{a,2}G_2F_{1,0}w_2w_1^2 + (w_3F_{a-1,0} + w_3F_{a-2,0}w_1^{m/2})^2 \\ &= w_3^2(F_{a,2}w_2w_1^2 + F_{a,1} + F_{a-1,1}w_1^m) = w_3^2(F_{a,2}w_2w_1^2 + w_2^{m-2}) . \end{aligned}$$

Let $p \equiv 0(2)$ and $p > 0$. Then from Theorems 1.5(iii) and 1.6(iii), we see

$$v_i(p+1) = P_{a-1}(p+1)w_1 = w_3^{p+1}F_{a,h(p)+1}G_pF_{l(p),0}w_1 .$$

Let $p \equiv 0(4)$ and $p > 0$. Then using Theorems 1.5(iii), 1.6(ii) and noting $(G_{p/2})^2 = G_p$, we see

$$\begin{aligned} v_i(p) &= P_{a-1}(p)w_1 + (P_{a-2}(p/2) + P_{a-3}(p/2)w_1^{m/2})^2 \\ &= w_3^p\{F_{a,h(p)+1}G_p(F_{l(p),0}w_2w_1^2 + F_{l(p),1}w_2^2w_1^2) + F_{a-1,h(p)+1}G_pF_{l(p),1}w_2^2w_1^{m+2}\} . \end{aligned}$$

Here we see that $F_{l(p),0}w_2w_1^2 + F_{l(p),1}w_2^2w_1^2 = w_2w_1^{2^{l(p)}}$.

Let $p \equiv 0(4)$ and $p > 0$. Then using Theorems 1.5(iii) and 1.6(ii), (iii) and noting $G_{p+2} = G_pF_{l(p),2}$, we see

$$\begin{aligned} v_i(p+2) &= P_{a-1}(p+2)w_1 + (P_{a-2}(p/2+1) + P_{a-3}(p/2+1)w_1^{m/2})^2 \\ &= w_3^{p+2}\{F_{a,h(p)+1}G_pF_{l(p),2}w_2w_1^2 + (F_{a,h(p)+1} + F_{a-1,h(p)+1}w_1^m)G_pF_{l(p),1}\} . \end{aligned}$$

Here we see that $F_{a,h(p)+1} + F_{a-1,h(p)+1}w_1^m = w_2^{m-2^{h(p)+1}}$ if $h(p) \le a-2$; 0 otherwise. □

THEOREM 5.3. *Let* $i = 2^a + 2^b + 1$ $(a > b \ge 1)$. *Then*

$$\begin{aligned} v_i \equiv\ & w_3\left(\sum_{s=b}^{a-1} w_2^{2^s-1}w_1^{i-2^{s+1}-1}\right) \\ &+ \sum_{p\equiv 0(2),\, h(p)\le a-2} w_3^{p+1}\left(\sum_{s=\max(b,h(p)+1)}^{a-1} w_2^{2^s-2^{h(p)+1}}w_1^{i-2^{s+1}-1}\right)G_pF_{l(p),0} \\ &+ \sum_{p\equiv 0(2),\, p\not\equiv 0(m),\, h(p)\le a-2} w_3^p\left(\sum_{s=\max(b,h(p)+1)}^{a-1} w_2^{2^s-2^{h(p)+1}}w_1^{i-2^{s+1}-1}\right)G_pF_{l(p),0}w_2w_1 \\ &+ w_3^m(F_{a,b+1}F_{b,0}w_2w_1^{m+1} + F_{a,b}w_1) \\ &+ \sum_{p\equiv 0(2m)} w_3^{p+m}F_{a,h(p)+1}G_p(F_{l(p),b+1}F_{b,0}w_2w_1^{m+1} + F_{l(p),b}w_1) \\ &+ \sum_{p\equiv 0(2m)} w_3^pF_{a,h(p)+1}G_p\left(\sum_{s=0}^{b-1} w_2^{2^s}w_1^{2^{l(p)}+m-2^{s+1}+1}\right) \mod I_4 , \end{aligned}$$

where $p > 0$ *and* $m = 2^b$.

PROOF. We have $v_i \equiv 0 \bmod I_3$ by Theorem 4.9(iii) or Remark 4.10(iii). We also use the notation $v_j(p)$ used in the proof of Theorem 5.2.

From Theorems 1.5(iv) and 1.6(i), we see

$$v_i(1) = P_{a-1}(1)w_1^{2^b} + P_{b-1}(1)w_1^{2^a} = w_3\left(\sum_{s=b}^{a-1} w_2^{2^s-1}w_1^{i-2^{s+1}-1}\right).$$

Let $p \equiv 0(2)$ and $p > 0$. Then using Theorems 1.5(iv), 1.6(iii) and letting $m = 2^b$, $n = 2^a$, we see

$$\begin{aligned} v_i(p+1) &= P_{a-1}(p+1)w_1^m + P_{b-1}(p+1)w_1^n \\ &= w_3^{p+1}(F_{a,h(p)+1}w_1^m + F_{b,h(p)+1}w_1^n)G_pF_{l(p),0}. \end{aligned}$$

Here we see that $F_{a,h(p)+1}w_1^m + F_{b,h(p)+1}w_1^n = \sum_{s=\max(b,h(p)+1)}^{a-1} w_2^{2^s-2^{h(p)+1}}w_1^{i-2^{s+1}-1}$ if $h(p) \leq a-2$; 0 if $h(p) \geq a-1$.

Let $p \equiv 0(2)$, $p > 0$ and $p \not\equiv 0(m)$. Then since $v_i(p) = P_{a-1}(p)w_1^m + P_{b-1}(p)w_1^n$ ($m = 2^b$, $n = 2^a$), we obtain the conclusion in the same way as the above case.

From Theorems 1.5(iv), 1.6(i) and (ii), we have

$$v_i(m) = P_{a-1}(m)w_1^m + (P_{a-b-1}(1))^m w_1 = w_3^m(F_{a,b+1}F_{b,0}w_2w_1^{m+1} + F_{a,b}w_1).$$

Let $p \equiv 0(2m)$ and $p > 0$. Then using Theorems 1.5(iv), 1.6(ii) and (iii), we see

$$\begin{aligned} v_i(p+m) &= P_{a-1}(p+m)w_1^m + (P_{a-b-1}(p/m+1))^m w_1 \\ &= w_3^{p+m}F_{a,h(p)+1}G_p(F_{l(p),b+1}F_{b,0}w_2w_1^{m+1} + F_{l(p),b}w_1). \end{aligned}$$

Let $p \equiv 0(2m)$ and $p > 0$. Then using Theorems 1.5(iv) and 1.6(ii), we see

$$\begin{aligned} v_i(p) &= P_{a-1}(p)w_1^m + (P_{a-b-1}(p/m))^m w_1 \\ &= w_3^pF_{a,h(p)+1}G_p(F_{l(p),0}w_2w_1^{m+1} + F_{l(p),b}w_2^mw_1^{m+1}). \end{aligned}$$

Here we see that $F_{l(p),0}w_2w_1^{m+1} + F_{l(p),b}w_2^mw_1^{m+1} = \sum_{s=0}^{b-1} w_2^{2^s}w_1^{2^{l(p)}+m-2^{s+1}+1}$. □

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