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On equilibrium solutions of a bistable reaction-diffusion equation with a nonlocal convection

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ABSTRACT. We are concerned with an ecological model described by a bistable reaction-diffusion equation with a nonlocal convection. We prove that there exist two symmetric stationary solutions.

1. Introduction

We are concerned with a bistable reaction-diffusion equation with a nonlocal convection.

(1) $u_t = du_{xx} - [(K * u)u]_x + ku(1-u)(u-a), \qquad x \in \boldsymbol{R},\ t > 0,\ (0 < a < 1)$

where $K * u = \int_{\boldsymbol{R}} K(x-y)u(y,t)dy$ and d, k are positive constants. Here $u = u(x,t)$ denotes the population density at time t and the position x. The convection term $[(K * u)u]_x$ corresponds to an aggregating mechanism of the population. The function $f(u) = ku(1-u)(u-a)$ represents the growth rate of the population. Several spatially aggregating population models were discussed in [1], [2], [3] and [4]. In this paper we specify the kernel $K(x)$ to be the following function:

(2) $$K(x) = \begin{cases} b & (x < 0) \\ -b & (x > 0), \end{cases}$$

where b is a positive constant. With this choice of the kernel, one can see that $u(x,t)$ moves in the right (resp. left) direction when

$$\int_{-\infty}^{x} u(y,t)dy - \int_{x}^{\infty} u(y,t)dy < 0. \qquad (\text{resp. } >0)$$

This means that the individuals move in the direction of higher distribution. When the aggregative convection term $[(K * u)u]_x$ is absent, we have a well-known bistable reaction-diffusion equation.

$$u_t = du_{xx} + ku(1-u)(u-a).$$

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It is known that $u \equiv 0$, $u \equiv 1$ are stable stationary solutions, while the solution $u \equiv a$ is unstable. When $0 < a < 1/2$, there exists one pulse-like solution, which plays a role of a "separator". We expect that due to the interplay of the effects of "diffusion" and "aggregation" equation (1) has a stable pulse-like equilibrium.

We study the existence of stationary solutions of (1). Hence let us consider the stationary equation of (1):

$$(3)\qquad 0 = du'' + \left[\left(\int_{-\infty}^{x} u(y)dy - \int_{x}^{\infty} u(y)dy\right)u\right]' + ku(1-u)(u-a), \qquad x \in \boldsymbol{R},$$

where $'$ denotes d/dx and where we chose $b = 1$ without loss of generality. First we define a solution of (3) as a nonnegative function such that

(i) $u \in C^2(\boldsymbol{R}) \cap L^1(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R})$.

(ii) u satisfies (3).

Our main result is as follows.

THEOREM. *Let $d > 0$, $k > 0$ be arbitrarily fixed. Then there exists $a_0 > 0$ such that for $0 < a < a_0$ the equation* (3) *has two positive symmetric solutions.*

Here we call a solution of (3) symmetric if $u(x) \equiv u(2x_0 - x)$ for some x_0. A similar result was obtained in [1]. They use singular perturbation techniques to find stationary solutions. They treat the case when the diffusion constant d is sufficiently small. We employ a "shooting method" as used in [2]. Our method applies to arbitrary $d > 0$. We have also obtained a necessary condition on the parameter a to guarantee the existence of (not necessarily symmetric) solutions of (3).

2. Proofs

From Lemma 1 in [2] it follows that a solution of (3) satisfies

$$u(\pm\infty) = 0 \qquad \text{and} \qquad u'(\pm\infty) = 0.$$

We introduce a new function $p(x)$ defined by

$$p(x) = \int_{-\infty}^{x} u(y)dy.$$

Then (3) becomes

$$(4)\qquad du'' + \{(2p - I)u\}' + ku(1-u)(u-a) = 0,$$

where $I = \int_{-\infty}^{\infty} u(y)dy$, the total population number to be determined. We transform (4) into the three dimensional system of differential equations.

$$(5)\qquad \begin{cases} p' = u \\ u' = v \\ v' = (1/d)\{(I-2p)v + ku^3 - (ka+k+2)u^2 + kau\}. \end{cases}$$

The boundary conditions of (5) at $x = \pm\infty$ are

$$(6)\qquad \begin{cases} (p(-\infty), u(-\infty), v(-\infty)) = (0,0,0) \\ (p(\infty), u(\infty), v(\infty)) = (I,0,0). \end{cases}$$

Note that the constant I is unknown *a priori.* In order to show the existence of a symmetric solution of (3), it is sufficient to show that there exists a solutuion of (3) such that

$$(7)\qquad \begin{cases} (p(-\infty), u(-\infty), v(-\infty)) = (0,0,0) \\ (p(x_0), u(x_0), v(x_0)) = (I/2, u_0, 0) \end{cases}$$

for some I, x_0 and u_0 (For details, see [2]). Thus we will look for a solution of (5) connecting $O = (0,0,0)$ and $Q = (I/2, u_0, 0)$. We consider a trajectory originating from O. All points on the p-axis are critical points of (5). To investigate the behavior of the flow of (5) near O, we linearize (5) about the critical point O. The Jacobian matrix of the linearized system at O is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & ka/d & I/d. \end{pmatrix}$$

The eigenvalues of A are

$$0 \qquad \text{and} \qquad \{I \pm (I^2 + 4dka)^{1/2}\}/2d = \mu\pm.$$

An eigenvector corresponding to μ_+ is

$$\boldsymbol{e} = (1, \mu_+, \mu_+^2).$$

There exists a one-dimensional unstable manifold U near the point O, which is tangent to $\boldsymbol{e}$.

Let α, β $(0 < \alpha < \beta)$ be the two solution of the quadratic equation

$$ku^2 - (ka + k + 2)u + ka = 0.$$

Define the two lines m_1, m_2 by $m_1 = \{(p,u,v) | u = \alpha, v = 0\}$ and $m_2 = \{(p,u,v) | u = \beta, v = 0\}$. Then note that the two lines themselves are trajectories of (5).

In the following $(p(x), u(x), v(x))$ denotes a solution of (5). We outline the proof of our main Theorem. Define the regions D, $\tilde{E}$ and G by $D =$

$\{(p,u,v)|0<p<I/2, \alpha<u<\beta, v=0\}$, $\tilde{E}=\{(p,u,v)|p=I/2,\ 0<u<\beta, v>0\}$ $\cup\{(p,u,v)|0<p\le I/2,\quad u=\beta, v>0\}$ and $G=\{(p,u,v)|p=I/2,\quad \alpha<u<\beta,$ $v=0\}$. When I is sufficiently large or sufficiently small, we will show that the trajectory on the unstable manifold U reaches the region $\tilde{E}$ (Lemmas 3 and 4). Next, when I takes an appropriate value, we will show that the trajectory on U reaches the region D (Lemma 6). By a continuity argument, it can be shown that there exist two values I_1, I_2 such that when $I=I_1$ or I_2, the trajectory reaches the segment G. This completes the proof.

LEMMA 1. *Choose* m *such that* $m>\mu_+$. *Suppose* $0<u(0)<mp(0)$, $0<v(0)<mu(0)$. *Then for any* $x>0$ *as long as* $0<p(x)<I/2, 0<u(x)<\beta$, *we have* $u(x)<mp(x)$, $v(x)<mu(x)$.

PROOF. We consider the following two cases.

(i) Suppose that there exists the smallest $x_0>0$ such that $u(x_0)=mp(x_0)$ and $v(x_0)<mu(x_0)$. Then it follows that $mp'(x_0)-u'(x_0)\le 0$. On the other hand, we have $mp'(x_0)-u'(x_0)=mu(x_0)-v(x_0)>0$, which is a contradiction.

(ii) Suppose that there exists the smallest $x_0>0$ such that $u(x_0)\le mp(x_0)$, $v(x_0)=mu(x_0)$. Then it follows that $v'(x_0)-mu'(x_0)\ge 0$. On the other hand, we have

$$\begin{aligned} v'-mu' &= (1/d)\{(I-2p)mu+ku^3-(ka+k+2)u^2+kau\}-m^2u \\ &\le (u/d)\{ku^2-(ka+k+2)u+ka+Im-dm^2\} \\ &\le (u/d)\{ka+Im-dm^2\} \\ &<0 \end{aligned}$$

at $x=x_0$, which is again a contradiction. □

LEMMA 2. *Let* $I>0$ *be sufficiently small. Suppose* $u(0)>(I/2d)p(0)$, $v(0)>(I/2d)u(0)$. *Then for any* $x>0$ *as long as* $0<p(x)\le I/2$, $0<u(x)<2\mu_+p(x)$, *we have* $u(x)>(I/2d)p(x)$, $v(x)>(I/2d)u(x)$.

PROOF. (i) Suppose that there exists the smallest $x_0>0$ such that $u(x_0)=(I/2d)p(x_0)$, $v(x_0)>(I/2d)u(x_0)$. It then follows that $u'(x_0)-(I/2d)p'(x_0)\le 0$. On the other hand we have $u'(x_0)-(I/2d)p'(x_0)=v(x_0)-(I/2d)u(x_0)>0$, which is a contradiction.

(ii) Suppose that there exists the smallest $x_0>0$ such that $u(x_0)\ge (I/2d)p(x_0)$, $v(x_0)=(I/2d)u(x_0)$. It follows that $v'(x_0)-(I/2d)u'(x_0)\le 0$. On the other hand we have at $x=x_0$,

$$\begin{aligned} v'-(I/2d)u' &= (1/d)\{(I-2p)v+ku^3-(ka+k+2)u^2+kau\}-(I/2d)v \\ &= (u/d)\{(I-2p)(I/2d)+ku^2-(ka+k+2)u+ka-I^2/4d\} \end{aligned}$$

$$\geq (u/d)\{I^2/4d + ku^2 - (ka + k + 4)u + ka\}$$
$$\geq (u/d)\{I^2/4d - (ka + k + 4)\mu_+ I + ka\}.$$

The last expression is positive when I is sufficiently small. This leads to a contradiction. □

REMARK. Lemma 2 is also valid when I (>0) is sufficiently large.

There is a portion of the one-dimensional unstable manifold U which lies in the octant $\{(p, u, v) | p > 0, u > 0, v > 0\}$. we denote this portion by U_+.

LEMMA 3. *Let $d > 0$, $k > 0$ be fixed and choose $I > 0$ sufficiently small. Then the trajectory on U_+ reaches the region $E = \{(p, u, v) | p = I/2, 0 < u < \beta, v > 0\}$.*

PROOF. Define a cone W by $W = \{(p, u, v) | 0 \leq p \leq I/2,\ (I/2d)p \leq u \leq mp,\ (I/2d)u \leq v \leq mu\}$, where m satisfies $\mu_+ < m < 2\mu_+$. It then follows from Lemmas 1 and 2 that the trajectory on U_+ is trapped in W as long as $0 \leq p \leq I/2$. Hence the trajectory must reach the region E. □

LEMMA 4. *Let $d > 0$, $k > 0$ be fixed and choose $I > 0$ sufficiently large. Then the trajectory on U_+ reaches the region $\tilde{E} = E \cup \{(p, u, v) | 0 < p \leq I/2, u = \beta, v > 0\}$.*

PROOF. The proof is similar to that of Lemma 3. □

LEMMA 5. *Let s be a number such that $(k/2d)^{1/2} I \leq s \leq k(a+1)/2$. Suppose $v(0) < \dfrac{1}{(1+s)d}\{-ku^2(0) + (ka + k + 2)u(0) - ka\}(I/2 - p(0))$. Then for any $x > 0$ as long as $\alpha < u(x) < \beta$, we have $v(x) < \dfrac{1}{(1+s)d}\{-ku^2(x) + (ka + k + 2)u(x) - ka\}(I/2 - p(x))$.*

PROOF. Suppose that there exists the smallest $x_0 > 0$ such that $v(x_0) = \dfrac{1}{(1+s)d}\{-ku^2(x_0) + (ka + k + 2)u(x_0) - ka\}(I/2 - p(x_0))$. Then we have at $x = x_0$,

$$v' - \left[\frac{1}{(1+s)d}\{-ku^2 + (ka + k + 2)u - ka\}(I/2 - p)\right]'$$
$$= (1/d)\{(I - 2p)v + ku^3 - (ka + k + 2)u^2 + kau\}$$
$$- \frac{1}{(1+s)d}\{-2kuv + (ka + k + 2)v\}(I/2 - p)$$
$$+ \frac{1}{(1+s)d}\{-ku^2 + (ka + k + 2)u - ka\}u$$

$$
\begin{aligned}
&= \frac{v}{d}(I/2-p)\left\{2-\frac{1}{1+s}(-2ku+ka+k+2)\right\} \\
&\quad + \frac{s}{(1+s)d}\{ku^3-(ka+k+2)u^2+kau\} \\
&= \frac{1}{(1+s)d}\{ku^2-(ka+k+2)u+ka\} \\
&\quad \times \frac{1}{1+s}\left[\left\{s(1+s)-\frac{2k}{d}(I/2-p)^2\right\}u+\frac{1}{d}(I/2-p)^2(ka+k-2s)\right] \\
&< 0.
\end{aligned}
$$

This contradicts the choice of x_0. □

LEMMA 6. *Let $d>0$, $k>0$ be fixed and put $I=(dk/8)^{1/2}$. Then there exists a_0 such that for $0<a<a_0$ the trajectory on U_+ reaches the region $D=\{(p,u,v)|0<p<I/2, \alpha<u<\beta, v=0\}$.*

PROOF. Put $m=(k/8d)^{1/2}+(ka/d)^{1/2}$. It is easy to see that $\mu_+<m<2\mu_+$. We define a cone V by $V=\{(p,u,v)|(I/2d)p\le u\le mp, (I/2d)u\le v\le mu, 0\le u\le 3a\}$. Note that $\alpha<3a<\beta$ when $0<a<1/3$. Since the eigenvector e is contained in V near the origin, U_+ lies in V near the origin. When a is sufficiently small, it is easily verified that Lemma 2 is also valid for $I=(dk/8)^{1/2}$, $0\le u\le 3a$. It then follows form Lemmas 1 and 2 that the trajectory on U_+ is trapped in V as long as $0\le u\le 3a$. Hence the trajectory reaches the cross section V_c of V, where $V_c=\{(p,u,v)|(I/2d)p\le u\le mp, (I/2d)u\le v\le mu, u=3a\}$. Define a surface S by

$$
v=\frac{1}{(1+s)d}\{-ku^2+(ka+k+2)u-ka\}(I/2-p).
$$

From Lemma 5 we know that no trajectories starting below the surface S can reach the surface S. We will show that the cross section V_c is on the underside of the surface S if a is sufficiently small. Put $s=k/4$. Then s satisfies the assumption of Lemma 5. V_c is below the surface S if the following inequality is valid.

$$
\begin{aligned}
(8)\quad &\{(k/8d)^{1/2}+(ka/d)^{1/2}\}3a \\
&< \frac{1}{(1+k/4)d}(-6ka^2+2ka+6a)\{(dk/8)^{1/2}/2-12a(2d/k)^{1/2}\}.
\end{aligned}
$$

It is easy to show that the inequality (8) is valid when a is small. Since $p'=u>0$ and $u'=v>0$, the trajectory starting from V_c reaches the region D. This completes the proof. □

We can now prove our main theorem using Lemmas 3, 4 and 6. Let $d > 0$, $k > 0$ be arbitrarily fixed and choose a so small that inequality (8) is satisfied. By Lemma 1 the trajectory on U_+ reaches one of the following three regions, that is, D, $\tilde{E}$ or $G = \{(p,u,v) | p = I/2, \alpha < u < \beta, v = 0\}$. Define T by $T = \{I > 0 |$ The trajectory on U_+ reaches the region $D\}$. Note that U_+ depends continuously on I. By Lemmas 3, 4 and 6 it follows that $T \neq \emptyset$ and T is bounded. Hence $\inf T = I_1 > 0$ and $\sup T = I_2 > 0$ exist. By Lemma 6 it is clear that $I_1 \neq I_2$. We will show that the trajectory on U_+ reaches G when $I = I_1$ or $I = I_2$. Let us consider the case $I = I_1$. By the continuous dependence of the unstable manifold U_+ on the parameters and the continuous dependence of the solutions of (5) on initial values, it is easy to see that T is open. Therefore U_+ intersects ∂D when $I = I_1$. In fact, one can see that U_+ must intersect the segment G when $I = I_1$. (Recall that the two lines m_1, m_2 are trajectories of (5).). The case $I = I_2$ can be treated analogously. This completes the proof of the Theorem. □

The following lemma shows the shape of the solutions of (3).

LEMMA 7. *Let $u \not\equiv 0$ be a (not necessarily symmetric) solution of* (3). *Then u has only one local maximum point.*

PROOF. Assume that u has more than one local maximum point. Then there exist three points $x_0 < x_1 < x_2$ such that $u(x_0) = u(x_1) = u(x_2)$ and $u'(x_1) = 0$.

Case (i). $\int_{-\infty}^{x_1} u(y)dy \leq (1/2)\int_{-\infty}^{\infty} u(y)dy.$

Multiply (4) by u' and integrate on the interval $[x_0, x_1]$. Then we have

$$\frac{d}{2}(u')^2\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1}\left(2\int_{-\infty}^{x} u\,dy - I\right)(u')^2 dx + F(u)\Big|_{x_0}^{x_1} = 0, \tag{9}$$

where $F(u)(x) = \int_0^{u(x)} \{2u^2 + ku(1-u)(u-a)\}du$. Since $u'(x_1) = 0$ and $F(u)(x_0) = F(u)(x_1)$, (9) becomes

$$-\frac{d}{2}(u')^2(x_0) + \int_{x_0}^{x_1}\left(2\int_{-\infty}^{x} u\,dy - I\right)(u')^2 dx = 0.$$

Since $2\int_{-\infty}^{x} u\,dy - I < 0$ on the interval $[x_0, x_1)$, this leads to a contradiction.

Case (ii). $\int_{-\infty}^{x_1} u(y)dy \geq (1/2)\int_{-\infty}^{\infty} u(y)dy.$

This case can be treated similarly. □

The next proposition shows a condition on a with respect to the existence of a (not necessarily symmetric) solution of (3).

PROPOSITION. *If* (3) *has a* (*not necessarily symmetric*) *solution* $u \not\equiv 0$, *then the parameter* a *must satisfy*

$$2\{k(a+1)+2\}^2 - 9k^2a > 0. \tag{10}$$

PROOF. We may assume that $u'(0) = 0$.

Case (i). $\int_{-\infty}^{0} u(y)dy \leq (1/2)\int_{-\infty}^{\infty} u(y)dy.$

Multiply (3) by u' and integrate on the interval $(-\infty, 0)$. Then we have

$$\int_{-\infty}^{0}\left(2\int_{-\infty}^{x} u(y)dy - I\right)(u')^2 dx + F(u)\Big|_{-\infty}^{0} = 0.$$

Since $2\int_{-\infty}^{x} u(y)dy - I < 0$ on the interval $(-\infty, 0)$, we have

$$F(u)|_{-\infty}^{0} = \int_{0}^{u(0)} \{2u^2 + ku(1-u)(u-a)\}du > 0.$$

Hence

$$(k/4)u^2(0) - \{k(a+1)+2\}u(0)/3 + ka/2 < 0. \tag{11}$$

Since there must exists $u(0)$ which satisfies inequality (11), a needs to satisfy

$$\{k(a+1)+2\}^2/9 - k^2a/2 > 0.$$

Case (ii). $\int_{-\infty}^{0} u(y)dy \geq (1/2)\int_{-\infty}^{\infty} u(y)dy.$

This case can be treated similarly. □

REMARK. From the inequality (10) we know that a satisfies

$$0 < a < \{5k - 8 - (9k^2 - 144k)^{1/2}\}/4k \qquad (k \geq 16).$$

When $k \geq 8 + 6\sqrt{2}$, this becomes an essential restriction on a.

3. Concluding remark

We have shown that equation (3) has two positive symmetric solutions. These solutions have different L^1 norm. We conjecture that the large pulse-like stationary solution is stable and the small one is unstable. The small pulse may play a role of a "separator". These facts are numerically confirmed in [1].

Numerical computations in [1] suggest that there exist *exactly* two symmetric solutions of (3). However, whether equation (3) has more than two symmetric solutions or not is unknown. The question about the existence of nonsymmetric solutions is still open.

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