HIROSHIMA MATH. J.
**28** (1998), 139–147

# On the union of 1-convex open sets

Makoto ABE

(Received January 28, 1997)

ABSTRACT. A complex space $X$ is 1-convex if it satisfies the conditions that there exists a locally finite 1-convex open covering of $X$ of order $\leq 2$, that the dimension of $H^1(X, \mathcal{O}_X)$ is at most countably infinite and that $X$ is K-separable outside a compact set.

## 0. Introduction

It is well-known that the union of two Stein open sets in a complex space is not necessarily Stein. For example the union of two Stein open sets $\{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1| < 1, 0 < |z_2| < 1\}$ and $\{(z_1, z_2) \in \mathbb{C}^2 \,|\, 0 < |z_1| < 1, |z_2| < 1\}$ in $\mathbb{C}^2$ is not Stein. Tovar [22] proved that if $X$ is a union of two relatively compact Stein open sets $D_1$ and $D_2$ in a reduced Stein space $S$ such that $\dim H^1(X, \mathcal{O}_X) < +\infty$, then $X$ is also a Stein open set in $S$ (Theorem 3 of Tovar [22] or Theorem 1.1 of Cho-Shon [4]).

We prove the following theorem which is a generalization of Theorem 3 of Tovar [22]. It also gives a generalization of Proposition 3.4 of Cho-Shon [4] on the finite simple chain Stein open covering. In the proof we use the theorem of Nguyen-Nguyen [20].

*Let $X$ be a second countable (not necessarily reduced) complex space. Then $X$ is 1-convex if it satisfies the following three conditions.*

i) *There exists a locally finite 1-convex open covering of $X$ of order $\leq 2$.*
ii) *The dimension of $H^1(X, \mathcal{O}_X)$ is at most countably infinite.*
iii) *$X$ is K-separable outside a compact set.*

We also give a 1-convex version of the theorem of Markoe [16] and Silva [21] on the union of the monotone increasing sequence of Stein open sets. The results in this paper were announced in the author's articles [1, 2].

## 1. Preliminaries

Throughout this paper all complex spaces are supposed to be second countable. Let $X$ be a (not necessarily reduced) complex space. We always

---

1991 *Mathematics Subject Classification*. 32F10, 32E10, 32E05.
*Key words and phrases*. 1-convex space, Stein space, K-separability, Remmert reduction.

denote by $\mathcal{O}_X$ the complex structure sheaf of $X$ and by $\mathcal{N}_X$ the nilradical of $\mathcal{O}_X$. A compact analytic set $C$ in $X$ is said to be the *maximal compact analytic set* of $X$ if every nowhere discrete compact analytic set of $X$ is contained in $C$ and $\dim_x C > 0$ for every $x \in C$ (cf. Grauert [11, p. 339]). A complex space $X$ is said to be 1-*convex* if $X$ is holomorphically convex and contains the maximal compact analytic set. This definition is equivalent to the usual one (cf. Nguyen-Nguyen [20] or Coltoiu [5]).

Every closed complex subspace of a 1-convex complex space is 1-convex. Especially if a complex space $X$ is 1-convex, then $\operatorname{red} X$ is also 1-convex. If conversely $\operatorname{red} X$ is 1-convex, then $X$ is 1-convex. For the completeness we give a proof of this fact (Lemma 3 below). Here we remark that there exists a complex space which are not holomorphically convex and the holomorphic reduction of which is holomorphically convex (cf. [6, p. 33]).

LEMMA 1. *Let $X$ be a complex space. Assume that* $\operatorname{red} X$ *is* 1-*convex and that there exists $m \geq 1$ such that $\mathcal{N}_X^m = 0$.* Then we have that $\dim H^q(X, \mathcal{S}) < +\infty$ *for every coherent analytic sheaf $\mathcal{S}$ on $X$ and for every $q \geq 1$.*

PROOF. Let $q \geq 1$. Since $\mathcal{N}_X(\mathcal{N}_X^j\mathcal{S}/\mathcal{N}_X^{j+1}\mathcal{S}) = 0$, the sheaf $\mathcal{N}_X^j\mathcal{S}/\mathcal{N}_X^{j+1}\mathcal{S}$ is $(\mathcal{O}_X/\mathcal{N}_X)$-coherent for every $j \geq 0$ by the extension principle [13, p. 239]. Since $\operatorname{red} X$ is 1-convex, it holds that $\dim H^q(X, \mathcal{N}_X^j\mathcal{S}/\mathcal{N}_X^{j+1}\mathcal{S}) < +\infty$ for every $j \geq 0$ by Theorem V of Narasimhan [19]. The exact sequence $0 \to \mathcal{N}_X^j\mathcal{S}/\mathcal{N}_X^{j+1}\mathcal{S} \to \mathcal{S}/\mathcal{N}_X^{j+1}\mathcal{S} \to \mathcal{S}/\mathcal{N}_X^j\mathcal{S} \to 0$ induces the exact sequence of cohomology groups $\cdots \to H^q(X, \mathcal{N}_X^j\mathcal{S}/\mathcal{N}_X^{j+1}\mathcal{S}) \to H^q(X, \mathcal{S}/\mathcal{N}_X^{j+1}\mathcal{S}) \to H^q(X, \mathcal{S}/\mathcal{N}_X^j\mathcal{S}) \to \cdots$. Therefore by induction on $j$ it holds that $\dim H^q(X, \mathcal{S}/\mathcal{N}_X^j\mathcal{S}) < +\infty$ for every $j \geq 1$. Since $\mathcal{N}_X^m = 0$, it holds that $\dim H^q(X, \mathcal{S}) < +\infty$. □

We need the following theorem of Narasimhan (Theorem V of [19]).

LEMMA 2. *Let $X$ be a complex space. Then the following three conditions are equivalent.*
1) *$X$ is* 1-*convex.*
2) $\dim H^q(X, \mathcal{S}) < +\infty$ *for every coherent analytic sheaf $\mathcal{S}$ on $X$ and for every $q \geq 1$.*
3) $\dim H^1(X, \mathcal{J}) < +\infty$ *for every coherent ideal $\mathcal{J}$ of $\mathcal{O}_X$.*

PROOF. 1) $\to$ 2). Let $C$ be the maximal compact analytic set of $X$. There exists a strongly pseudoconvex open set $D$ of $X$ with globally defined boundary such that $C \subset D \subset\subset X$. The argument of the proof of Theorem V of Narasimhan [19, p. 214] is valid for not necessarily reduced complex space $X$. Therefore the natural homomorphism $H^q(X, \mathcal{S}) \to H^q(D, \mathcal{S})$ is injective. By theorem I of Narasimhan [19] $\operatorname{red} D$ is 1-convex.

Since $\bar{D}$ is compact, there exists $m \geq 1$ such that $\mathcal{N}_D^m = 0$. By Lemma 1 it holds that $\dim H^q(X, \mathcal{S}) \leq \dim H^q(D, \mathcal{S}) < +\infty$ for every $q \geq 1$.

2) $\to$ 3) $\to$ 1). The argument of the proof of Theorem V of Narasimhan [19, p. 215] is valid for not necessarily reduced complex space $X$. □

LEMMA 3. *Let $X$ be a complex space. If* red $X$ is 1-convex, *then $X$ is also 1-convex.*

PROOF. Let $C$ be the maximal compact analytic set of red $X$. Let $\psi' : \operatorname{red} X \to Y'$ be the Remmert reduction of red $X$ (cf. [13, p. 221]). Take a Stein open set $E_1$ of $Y'$ such that $\psi'(C) \subset E_1 \subset\subset Y'$. Then $D_1 := \psi'^{-1}(E_1)$ is a 1-convex open set of red $X$. Since $\bar{D}_1$ is compact, there exists $m \geq 1$ such that $\mathcal{N}_X^m = 0$ on $D_1$. Therefore $D_1$ as an open set of $X$ is also 1-convex by Lemmas 1 and 2. Let $(\varphi, \tilde{\varphi}) : (D_1, \mathcal{O}_{D_1}) \to (Y_1, \mathcal{O}_{Y_1})$ be the Remmert reduction of $(D_1, \mathcal{O}_{D_1})$, where $\mathcal{O}_{D_1} = \mathcal{O}_X|_{D_1}$ (cf. [13, p. 221] or Wiegmann [24]). Then $P_1 := \varphi(C)$ is a finite set of $Y_1$ and the induced map $D_1 - C \to Y_1 - P_1$ is biholomorphic. Let $Z$ be the direct sum of $Y_0 := X - C$ and $Y_1$. Identifying such $z_0 \in D_1 - C$ and $z_1 \in Y_1 - P_1$ that $\varphi(z_0) = z_1$, we obtain the quotient space $Y$ of $Z$. Then $Y$ is a Hausdorff space and the natural projection $p : Z \to Y$ is continuous and open. Let $U_i := p(Y_i)$, $p_i := p_{Y_i, U_i} : Y_i \to U_i$ and $\mathcal{O}_i := (p_i)_*(\mathcal{O}_{Y_i})$ for $i = 0, 1$. The map $p_i$ is homeomorphic and $(U_i, \mathcal{O}_i)$ is a complex space for each $i = 0, 1$. The homomorphism $\tilde{\varphi}$ induces the isomorphism $\theta : \mathcal{O}_1|_{U_0 \cap U_1} \to \mathcal{O}_0|_{U_0 \cap U_1}$. By the gluing lemma [13, p. 10] there exist a complex structure sheaf $\mathcal{O}_Y$ of $Y$ and $\mathbf{C}$-algebra isomorphisms $\tilde{p}_i : \mathcal{O}_Y|_{U_i} \to \mathcal{O}_i$ such that $\theta = \tilde{p}_0 \circ \tilde{p}_1^{-1}$ on $U_0 \cap U_1$. $(p_i, \tilde{p}_i) : (Y_i, \mathcal{O}_{Y_i}) \to (U_i, \mathcal{O}_Y|_{U_i})$ is a biholomorphic map for each $i = 0, 1$. Let $\psi : X \to Y$ be the continuous map defined by $\psi(x) = p(x)$ for $x \in Y_0$ and $\psi(x) = p(\varphi(x))$ for $x \in D_1$. The map $\psi : X \to Y$ is surjective and proper. For every open set $W \subset Y$ and for every $h \in \mathcal{O}_Y(W)$ there exists a unique $k \in \mathcal{O}_X(\psi^{-1}(W))$ such that $k|_{p_0^{-1}(W \cap U_0)} = \tilde{p}_0(h|_{W \cap U_0})$ and $k|_{\varphi^{-1}(p_1^{-1}(W \cap U_1))} = \tilde{\varphi}(\tilde{p}_1(h|_{W \cap U_1}))$. These local homomorphisms glue together to determine an isomorphism $\tilde{\psi} : \mathcal{O}_Y \to \psi_*(\mathcal{O}_X)$. Then $(\psi, \tilde{\psi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a holomorphic map. $P := \psi(C)$ is a finite set of $Y$. We have that $\psi^{-1}(P) = C$. The induced map $\psi_{X-C, Y-P} : X - C \to Y - P$ is biholomorphic. Using the properties of the map $\psi$, we can verify that $Y$ contains no positive dimensional compact analytic set. Since red $\psi$ : red $X \to$ red $Y$ is proper and surjective, red $Y$ is Stein by E.73b of [15, p. 314]. Therefore $(Y, \mathcal{O}_Y)$ is Stein by 52.19 of [15, p. 236]. Since $\psi$ is proper, $X$ is holomorphically convex. It follows that $X$ is 1-convex. □

Let $X$ be a complex space and $L$ a compact set of $X$. Then $X$ is said to be *K-separable* outside $L$ if for every $x \in X - L$ the analytic set $\{y \in X | [f](y) = [f](x) \text{for every} f \in \mathcal{O}_X(X)\}$ is of dimension 0. Here $[f]$ denotes the valuation

$x \mapsto f_x + \mathfrak{m}_{X,x} \in \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = \mathbf{C}, x \in X$. If $X$ is K-separable outside $L$, then every closed complex subspace $Y$ of $X$ is K-separable outside $L \cap Y$. A complex space $X$ is 1-convex if and only if $X$ is holomorphically convex and K-separable outside a compact set. A complex space $X$ is K-complete if and only if $X$ is K-separable outside the empty set $\emptyset$ (cf. E.51c of [15, p. 225]).

Nguyen-Nguyen [20] obtained the following characterization of the finite dimensional reduced 1-convex complex space.

LEMMA 4. *Let $X$ be a reduced complex space of finite dimension. Then $X$ is* 1*-convex if and only if it satisfies the following three conditions.*

i) *If the function $f \in \mathcal{O}_X(X)$ is not constant on any non-compact irreducible component of $X$, then the analytic set $\{x \in X | f(x) = 0\}$ is* 1*-convex.*
ii) $\dim H^1(X, \mathcal{O}_X) < +\infty$.
iii) *$X$ is K-separable outside a compact set.*

An open covering $\{D_i\}_{i \in I}$ of a complex space $X$ is said to be *of order* $\leq 2$ if for all pairwise different three indices $i_0, i_1$ and $i_2$ the intersections $D_{i_0} \cap D_{i_1} \cap D_{i_2}$ are empty (cf. [17, p. 18]). A finite open covering $\{D_i\}_{i=1}^N$ of a complex space $X$ is said to be a finite simple chain covering of $X$ if $D_{i_0} \cap D_{i_1} = \emptyset$ for $|i_0 - i_1| \geq 2$ (Definition 3.3 of Cho-Shon [4]). Then every finite simple chain covering of $X$ is of order $\leq 2$.

## 2. 1-convex open covering of order $\leq 2$

LEMMA 5. *Let $X$ be a complex space. Assume that there exists a finite* 1*-convex open covering of $X$ of order $\leq 2$. Then it holds that* $\dim H^q(X, \mathcal{S}) < +\infty$ *for every coherent analytic sheaf $\mathcal{S}$ on $X$ and $q \geq 2$.*

PROOF. There exists a finite 1-convex open covering $\{D_i\}_{i=1}^N$ of $X$ of order $\leq 2$. Let $Y_k := \bigcup_{i=1}^k D_i$ for $1 \leq k \leq N$. By induction on $k$ we prove that $\dim H^q(Y_k, \mathcal{S}) < +\infty$. The case $k = 1$ is by Lemma 2. Assume that $2 \leq k \leq N$. $Y_{k-1} \cup D_k = Y_k$. $Y_{k-1} \cap D_k = \bigcup_{i=1}^{k-1} (D_i \cap D_k)$ (disjoint union). We have the Mayer-Vietoris exact sequence $\cdots \to \bigoplus_{i=1}^{k-1} H^{q-1}(D_i \cap D_k, \mathcal{S}) \to H^q(Y_k, \mathcal{S}) \to H^q(Y_{k-1}, \mathcal{S}) \oplus H^q(D_k, \mathcal{S}) \to \cdots$. By induction hypothesis $\dim H^q(Y_{k-1}, \mathcal{S}) < +\infty$. $D_k$ and $D_i \cap D_k$ are 1-convex. By Lemma 2 we have that $\dim H^q(D_k, \mathcal{S}) < +\infty$ and that $\dim H^{q-1}(D_i \cap D_k, \mathcal{S}) < +\infty$. Therefore it holds that $\dim H^q(Y_k, \mathcal{S}) < +\infty$. Since $X = Y_N$, the lemma is proved. □

LEMMA 6. *Let $X$ be a reduced complex space. Assume that there exist* 1*-convex analytic sets $X_1$ and $X_2$ of $X$ such that $X_1 \cup X_2 = X$ and that the intersection $X_1 \cap X_2$ is compact. Then $X$ is* 1*-convex.*

PROOF. Let $\mathscr{I}_\nu$ be the maximal defining ideal of $X_\nu (\nu = 1, 2)$. Let $\mathscr{J}$ be an arbitrary coherent ideal of $\mathscr{O}_X$. Consider the exact sequence of sheaves $0 \to \mathscr{J} \to (\mathscr{J}/\mathscr{I}_1\mathscr{J}) \oplus (\mathscr{J}/\mathscr{I}_2\mathscr{J}) \to \mathscr{J}/(\mathscr{I}_1 + \mathscr{I}_2)\mathscr{J} \to 0$. Then we have the exact sequence of cohomology groups $\cdots \to \Gamma(X_1 \cap X_2, (\mathscr{J}/(\mathscr{I}_1 + \mathscr{I}_2)\mathscr{J})|_{X_1 \cap X_2}) \to H^1(X, \mathscr{J}) \to H^1(X_1, (\mathscr{J}/\mathscr{I}_1\mathscr{J})|_{X_1}) \oplus H^1(X_2, (\mathscr{J}/\mathscr{I}_2\mathscr{J})|_{X_2}) \to \cdots$. Since $X_\nu$ is 1-convex, $H^1(X_\nu, (\mathscr{J}/\mathscr{I}_\nu\mathscr{J})|_{X_\nu})$ is finite dimensional by Lemma 2 $(\nu = 1, 2)$. Since the complex space $(X_1 \cap X_2, (\mathscr{O}_X/(\mathscr{I}_1 + \mathscr{I}_2))|_{X_1 \cap X_2})$ is compact, $\Gamma(X_1 \cap X_2, (\mathscr{J}/(\mathscr{I}_1 + \mathscr{I}_2)\mathscr{J})|_{X_1 \cap X_2})$ is finite dimensional by the finiteness theorem of Cartan-Serre [12, p. 186]. Therefore $H^1(X, \mathscr{J})$ is also finite dimensional. It follows that $X$ is 1-convex by Lemma 2. □

LEMMA 7. *Let $X$ be a complex space. Assume that the following three conditions are satisfied.*

i) $\dim H^2(X, \mathscr{S}) < +\infty$ *for every coherent analytic sheaf $\mathscr{S}$ on $X$.*

ii) $\dim H^1(X, \mathscr{O}_X) < +\infty$.

iii) *$X$ is K-separable outside a compact set $L$.*

*Then $X$ is 1-convex.*

PROOF. First we consider the case when $X$ is reduced and $\dim X < +\infty$. We proceed by induction on $\dim X$. The case $\dim X = 0$ is trivial. Assume that $\dim X \geq 1$. Take an arbitrary $f \in \mathscr{O}_X(X)$ which is not constant on any non-compact irreducible component of $X$. Let $A := \{x \in X | f(x) = 0\}$. Let $\{A_i\}_{i \in I}$ be the set of irreducible components of $A$. Let $I'$ be the set of $i \in I$ such that $A_i$ is a positive dimensional irreducible component of $X$. $I'' := I - I'$. $A' := \bigcup_{i \in I'} A_i$. $A'' := \bigcup_{i \in I''} A_i$. Since every $A_i, i \in I'$, is compact and contained in $L$, the analytic set $A'$ is compact. For every $i \in I''$ it holds that $\dim A_i = 0$ or that $\dim_x A_i < \dim_x X, x \in A_i$. Therefore it holds that $\dim A'' < \dim X$. The analytic set $A''$ is K-separable outside $L \cap A''$. We denote by $i(A'')$ the maximal defining ideal of $A''$. $\mathscr{O}_{A''} := (\mathscr{O}_X/i(A''))|_{A''}$ is the reduced complex structure sheaf of the analytic set $A''$. The exact sequence $0 \to i(A'') \to \mathscr{O}_X \to \mathscr{O}_X/i(A'') \to 0$ of sheaves induces the exact sequence of cohomology groups $\cdots \to H^1(X, \mathscr{O}_X) \to H^1(A'', \mathscr{O}_{A''}) \to H^2(X, i(A'')) \to \cdots$. Since $H^1(X, \mathscr{O}_X)$ and $H^2(X, i(A''))$ are finite dimensional, $H^1(A'', \mathscr{O}_{A''})$ is also finite dimensional. Let $\mathscr{S}$ be an arbitrary coherent analytic sheaf on $A''$. Let $\iota : A'' \to X$ be the inclusion. Then $\iota_*\mathscr{S}$ is a coherent analytic sheaf on $X$ by the extension principle. Therefore $H^2(A'', \mathscr{S}) = H^2(X, \iota_*\mathscr{S})$ is finite dimensional. By induction hypothesis the analytic set $A''$ is 1-convex. By Lemma 6 the analytic set $A = A' \cup A''$ is 1-convex. It follows that $X$ is 1-convex by Lemma 4.

Next we consider the case when $X$ is reduced and $\dim X = +\infty$. We have only to prove that $X$ is holomorphically convex. Let $\{X_i\}_{i \in I}$ be the set of irreducible components of $X$. Let $I' := \{i \in I | X_i \cap L \neq \emptyset\}$, $I'' := I - I'$,

$X' := \bigcup_{i \in I'} X_i$ and $X'' := \bigcup_{i \in I''} X_i$. Since $I'$ is finite, the analytic set $X'$ is finite dimensional. $X'$ is K-separable outside $L \cap X'$. By the same reasoning developed above we have that $\dim H^1(X, \mathcal{O}_{X'}) < +\infty$ and that $\dim H^2(X', \mathcal{S}) < +\infty$ for any coherent analytic sheaf $\mathcal{S}$ on $X'$. Therefore $X'$ is 1-convex by what was shown above. Take an arbitrary $i \in I''$. The analytic set $X_i$ is finite dimensional. By the same reasoning developed above we have that $\dim H^1(X_i, \mathcal{O}_{X_i}) < +\infty$ and that $\dim H^2(X_i, \mathcal{S}) < +\infty$ for any coherent analytic sheaf $\mathcal{S}$ on $X_i$. Therefore $X_i$ is 1-convex by what was shown above. Since $X_i \cap L = \emptyset$, the analytic set $X_i$ is K-complete. Therefore $X_i$ is Stein. It follows that $X''$ is Stein by Narasimhan [18]. Let $C$ be the maximal compact analytic set of $X'$ and $\varphi : X' \to Y'$ the Remmert reduction of $X'$. Then $P := \varphi(C)$ is a finite set of $Y'$. Since $\varphi^{-1}(P) = C$ and $C \subset L \subset X - X''$, there exists a neighborhood $U$ of $P$ such that $\varphi^{-1}(U) \subset\subset X - X''$. Let $Z$ be the direct sum of $X - C$ and $U$. Identifying such $z_1 \in \varphi^{-1}(U) - C$ and $z_2 \in U - P$ that $\varphi(z_1) = z_2$, we obtain the quotient space $Y$ of $Z$. Let $p : Z \to Y$ be the natural projection. Since we can verify that $Y$ is Hausdorff, $Y$ has a unique reduced complex structure such that both $p : X - C \to p(X - C)$ and $p : U \to p(U)$ are biholomorphic. Let $\psi : X \to Y$ be the map defined by $\psi(x) = p(x)$ for $x \in X - C$ and $\psi(x) = p(\varphi(x))$ for $x \in \varphi^{-1}(U)$. The map $\psi : X \to Y$ is a proper holomorphic surjection. We can also verify that the complex space $Y$ does not contain any positive dimensional compact analytic set. Let $\lambda : \hat{X} \to X$ be the normalization of $X$. Since every irreducible component of $X$ is holomorphically convex, every connected component of $\hat{X}$ is holomorphically convex. Therefore $\hat{X}$ is holomorphically convex. Since the composition $\psi \circ \lambda : \hat{X} \to Y$ is proper and surjective, $Y$ is Stein (cf. E.73b of [15, p. 314]). Since $\psi$ is proper, $X$ is holomorphically convex.

Finally we consider the case when $X$ is not reduced. $\tilde{\mathcal{O}} := \mathcal{O}_X / \mathcal{N}_X$ is the complex structure sheaf of $\operatorname{red} X$. We have the exact sequence $0 \to \mathcal{N}_X \to \mathcal{O}_X \to \tilde{\mathcal{O}} \to 0$. By the same reasoning developed above we have that $\dim H^1(X, \tilde{\mathcal{O}}) < +\infty$ and that $\dim H^2(X, \mathcal{S}) < +\infty$ for any coherent analytic sheaf $\mathcal{S}$ on $\operatorname{red} X$. Therefore $\operatorname{red} X$ is 1-convex by what was shown above. It follows that $X$ is 1-convex by Lemma 3. □

Theorem 8. *Let $X$ be a complex space. Assume that the following three conditions are satisfied.*

i) *There exists a locally finite* 1*-convex open covering of $X$ of order* $\leq 2$.

ii) *The dimension of $H^1(X, \mathcal{O}_X)$ is at most countably infinite.*

iii) *$X$ is K-separable outside a compact set.*

*Then $X$ is* 1*-convex.*

Proof. There exists a locally finite 1-convex open covering $\{D_i\}_{i \in I}$ of $X$ of order $\leq 2$. It holds that $\dim H^1(X, \mathcal{O}_X) < +\infty$ by Siu's theorem (Proposizione 7 of Ballico [3] or Théorème 2 of Jennane [14]). There exists a compact set $L$ of $X$ such that $X$ is K-separable outside $L$. Let $I' := \{i \in I | D_i \cap L \neq \emptyset\}$, $I'' := I - I'$, $Y := \bigcup_{i \in I'} D_i$ and $Z := \bigcup_{j \in I''} D_j$, then we have $Y \cup Z = X$ and $Y \cap Z = \bigcup_{(i,j) \in I' \times I''} (D_i \cap D_j)$ (disjoint union). Let $\mathscr{S}$ be an arbitrary coherent analytic sheaf on $X$. We have the Mayer-Vietoris exact sequence $\cdots \to H^1(Y \cap Z, \mathscr{S}) \to H^2(X, \mathscr{S}) \to H^2(Y, \mathscr{S}) \oplus H^2(Z, \mathscr{S}) \to H^2(Y \cap Z, \mathscr{S}) \to \cdots$. Since $Y \cap Z$ is Stein, $H^q(Y \cap Z, \mathscr{S}) = 0$ for every $q \geq 1$. Since $\{D_j\}_{j \in I''}$ is a Stein open covering of $Z$ of order $\leq 2$, $H^2(Z, \mathscr{S}) \cong H^2(\{D_j\}_{j \in I''}, \mathscr{S}) = 0$ (cf. [12, p. 35]). Therefore we have an isomorphism $H^2(X, \mathscr{S}) \cong H^2(Y, \mathscr{S})$. Since the set $I'$ is finite, it holds that $\dim H^2(X, \mathscr{S}) = \dim H^2(Y, \mathscr{S}) < +\infty$ by Lemma 5. It follows that $X$ is 1-convex by Lemma 7. □

Corollary. *Let $X$ be a K-complete complex space. Assume that the following two conditions are satisfied.*
i) *There exist two Stein open sets $D_1$ and $D_2$ of $X$ such that $D_1 \cup D_2 = X$.*
ii) *The dimension of $H^1(X, \mathcal{O}_X)$ is at most countably infinite.*
*Then $X$ is Stein.*

Theorem 8 is a generalization of Theorem 3 of Tovar [22]. It also gives a generalization of Proposition 3.4 of Cho-Shon [4] on the finite simple chain Stein open covering by the similar argument in the proof of Lemma 5.

In Theorem 8 we cannot replace the condition ii) by the weaker one that $H^1(X, \mathcal{O}_X)$ is Hausdorff with its canonical topology. For example let $X$ be the union of two Stein open sets $\{(z_1, z_2) \in \mathbf{C}^2 | |z_1| < 1, 0 < |z_2| < 1\}$ and $\{(z_1, z_2) \in \mathbf{C}^2 | 0 < |z_1| < 1, |z_2| < 1\}$ in $\mathbf{C}^2$. By Lemma 9 of Trapani [23] the topology of $H^1(X, \mathcal{O}_X)$ is Hausdorff. But $X$ is not Stein.

## 3. Increasing sequence of 1-convex open sets

If $X$ is a complex space which is the union of monotone increasing sequence of Stein open sets, then $X$ is not necessarily Stein as is shown by Fornæss [7, 8, 9] or Fornæss-Stout [10]. Markoe [16] and Silva [21] proved that a complex space $X$ is Stein if it satisfies the conditions that $X$ is the union of a monotone increasing sequence of Stein open sets and that the cohomology module $H^1(X, \mathcal{O}_X)$ is Hausdorff with its canonical topology. For the definition of the canonical topology of the cohomology modules $H^q(X, \mathcal{O}_X), q \geq 0$, we refer E.55h of [15, pp. 261–262]. We have the following 1-convex version of the theorem of Markoe-Silva [16, 21].

Theorem 9. *Let $X$ be a complex space which is the union of a monotone increasing sequence of 1-convex open sets. Assume that the following two conditions are satisfied.*

i) *$H^1(X, \mathcal{O}_X)$ is Hausdorff with its canonical topology.*
ii) *There exists a compact set L of X such that every nowhere discrete compact analytic set of X is contained in L.*

*Then X is* 1*-convex.*

PROOF. There exists a sequence $\{D_\nu\}_{\nu=1}^{\infty}$ of 1-convex open sets of $X$ such that $D_\nu \subset D_{\nu+1}$ for every $\nu \geq 1$ and that $X = \bigcup_{\nu=1}^{\infty} D_\nu$. We may assume that $L \subset D_1$. Let $C$ be the maximal compact analytic set of $D_1$. Then $C$ is also the maximal compact analytic set of $X$. By the same method as in the proof of Lemma 3 we construct a complex space $Y$ and a proper holomorphic surjection $\psi : X \to Y$. Then $Y$ contains no positive dimensional compact analytic set and $E_\nu := \psi(D_\nu)$ is an open set of $Y$ for every $\nu \geq 1$. Since the map $\psi_{D_\nu, E_\nu} : D_\nu \to E_\nu$ is proper and surjective, $E_\nu$ is Stein for every $\nu \geq 1$ (cf. E.73b of [15, p. 314]). Therefore $Y$ is the union of the monotone increasing sequence $\{E_\nu\}_{\nu=1}^{\infty}$ of Stein open sets. $\tilde{\psi}$ induces the isomorphisms $\tilde{\psi}^q : C^q(\{E_\nu\}_{\nu=1}^{\infty}, \mathcal{O}_Y) \to C^q(\{D_\nu\}_{\nu=1}^{\infty}, \mathcal{O}_X), \{g_{\nu_0 \cdots \nu_q}\} \mapsto \{\tilde{\psi}(g_{\nu_0 \cdots \nu_q})\}, q \geq 0$. By the definition of the product topology and by 55.6 ii) of [15, p. 258] $\tilde{\psi}^q$ is continuous for every $q \geq 0$. It holds that $\tilde{\psi}^{q+1} \circ \delta = \delta \circ \tilde{\psi}^q$ for every $q \geq 0$, where $\delta$ denotes the coboundary operator. Therefore $\tilde{\psi}^q$ induces a continuous isomorphism $H^q(\{E_\nu\}_{\nu=1}^{\infty}, \mathcal{O}_Y) \xrightarrow{\sim} H^q(\{D_\nu\}_{\nu=1}^{\infty}, \mathcal{O}_X)$ for every $q \geq 0$. Since $\{E_\nu\}_{\nu=1}^{\infty}$ is a Stein open covering of $Y$, the canonical homomorphism $H^1(\{E_\nu\}_{\nu=1}^{\infty}, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y)$ is isomorphic and homeomorphic. On the other hand the canonical homomorphism $H^1(\{D_\nu\}_{\nu=1}^{\infty}, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is injective and continuous. Therefore we have a continuous injection $H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)$. Since $H^1(X, \mathcal{O}_X)$ is Hausdorff, $H^1(Y, \mathcal{O}_Y)$ is also Hausdorff. It follows that $Y$ is Stein by the theorem of Markoe [16] or Silva [21]. Since the map $\psi : X \to Y$ is proper, $X$ is holomorphically convex. It follows that $X$ is 1-convex. □

We cannot drop the condition ii) in Theorem 9 above. For example let $X$ be the direct sum of countably infinite copies of the $n$-dimensional projective space $\mathbb{P}^n$. Then $X$ satisfies the condition i). But $X$ is not 1-convex.

## References


[1] M. Abe, On the limit of an increasing sequence of 1-convex open sets, Proceedings of the First Korean-Japanese Colloquium on Finite or Infinite Dimensional Complex Analysis, Pusan, Korea, 10–17 July 1993 (eds. J. Kajiwara, H. Kazama and K. H. Shon), Miura Bindery, Fukuoka, 1993, pp. 71–75.

[2] M. Abe, On the complex space which admits a 1-convex open covering, Proceedings of the 5th International Colloquium on Differential Equations, Plovdiv, Bulgaria, August 18–23, 1994, Vol. 1 (eds. D. Bainov and A. Dishliev), Science Culture Technology Publishing, Singapore, 1995, pp. 6–15.

[3] E. Ballico, Finitezza e annullamento di gruppi di coomologia su uno spazio complesso, Boll. Un. Mat. Ital., (6) 1-B (1982), 131–142.

[4] H. R. Cho and K. H. Shon, Stein-ness of open subsets in complex spaces of finite dimension, J. Korean Math. Soc., **31** (1994), 131–138.

[5] M. Coltoiu, Some open problems concerning Stein spaces, Rev. Roum. Math. Pures Appl., **36** (1991), 225–229.

[6] G. Fischer, Complex analytic geometry, Lecture Notes in Math. **538**, Springer, Berlin-Heidelberg-New York, 1976.

[7] J. E. Fornæss, An increasing sequence of Stein manifolds whose limit is not Stein, Math. Ann., **223** (1976), 275–277.

[8] J. E. Fornæss, 2 dimensional counterexamples to generalizations of the Levi problem, Math. Ann., **230** (1977), 169–173.

[9] J. E. Fornæss, A counterexample for the Levi problem for branched Riemann domains over $\mathbf{C}^n$, Math. Ann., **234** (1978), 275–277.

[10] J. E. Forneæss and E. L. Stout, *Polydiscs in complex manifolds*, Math. Ann., **227** (1977), 145–153.

[11] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann., **146** (1962), 331–368.

[12] H. Grauert and R. Remmert, Theory of Stein spaces, Springer, Berlin-Heidelberg-New York, 1979.

[13] H. Grauert and R. Remmert, Coherent analytic sheaves, Springer, Berlin-Heidelberg-New York-Tokyo, 1984.

[14] B. Jennane, Problème de Levi et espaces holomorphiquement séparés, Math. Ann., **268** (1984), 305–316.

[15] L. Kaup and B. Kaup, Holomorphic functions of several variables, Walter de Gruyter, Berlin-New York, 1983.

[16] A. Markoe, Runge families and inductive limits of Stein spaces, Ann. Inst. Fourier, **27** (1977), 117–127.

[17] K. Nagami, Dimension Theory, Academic Press, New York-London, 1970.

[18] R. Narasimhan, A note on Stein spaces and their normalizations, Ann. Scuola Norm. Sup. Pisa, **16** (1962), 327–333.

[19] R. Narasimhan, The Levi problem for complex spaces II, Math. Ann., **146** (1962), 195–216.

[20] V. K. Nguyen and D. S. Nguyen, On some characterizations of 1-convex spaces, Rev. Roum. Math. Pures Appl., **31** (1986), 237–245.

[21] A. Silva, Rungescher Satz and a condition for Steinness for the limit of an increasing sequence of Stein spaces, Ann. Inst. Fourier, **28** (1978), 187–200.

[22] L. M. Tovar, Open Stein subsets and domains of holomorphy in complex spaces, Topics in several complex variables (eds. E. Ramirez de Arellano and D. Sundararaman), Research Notes in Math. **112**, Pitman, Boston-London-Melbourne, 1985, pp. 183–189.

[23] S. Trapani, Inviluppi di olomorfia e gruppi di coomologia di Hausdorff, Rend. Sem. Mat. Univ. Padova, **75** (1986), 25–37.

[24] K.-W. Wiegmann, Über Quotienten holomorph-konvexer komplexer Räume, Math. Z., **97** (1967), 251–258.

*Oshima National College of Maritime Technology*
*Komatsu, Oshima-cho, Oshima-gun, Yamaguchi* 742-2193
*Japan*