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The homotopy groups of an L_2-localized type one finite spectrum at the prime 2

Dedicated to Professor Teiichi Kobayashi on his 60th birthday

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Abstract. In this paper we determine the homotopy groups as the title indicates. This is a grip to understand the homotopy groups of $\pi_*(L_2S^0)$, as well as the category of L_2-local CW-spectra at the prime 2. For example, the result indicates that an analogue of the Hopkins-Gross theorem on duality would require the condition $2 \cdot 1_X = 0$ if it holds at the prime 2.

1. Introduction

For each prime number p, let $K(n)_*$ denote the n-th Morava K-theory with coefficient ring $K(n)_* = F_p[v_n, v_n^{-1}]$ for $n > 0$ and $K(0)_* = \boldsymbol{Q}$. Here v_n has dimension $2p^n - 2$ and corresponds to the generators v_n of the coefficient ring $BP_* = \boldsymbol{Z}_{(p)}[v_1, v_2, \dots]$ of the Brown-Peterson spectrum BP at the prime p. A p-local finite spectrum F has type n if $K(i)_*(F) = 0$ for $i < n$ and $K(n)_*(F) \neq 0$. Let L_n denote the Bousfield localization functor with respect to the spectrum $K(0) \vee K(1) \vee \cdots \vee K(n)$ (or equivalently to $v_n^{-1}BP$) from the category of p-local CW-spectra to itself. In this paper we compute the homotopy groups of the L_2-localization of a type 1 finite spectrum W with $BP_*(W) = BP_*/(2) \otimes \Lambda(t_1, t_1^2, t_2)$ as a $BP_*(BP)$-comodule at the prime 2. Notice that S^0 is a type 0. Since W is a type 1 finite spectrum, it is closer to S^0 than a type 2 spectrum or an infinite spectrum. By virtue of Hopkins and Ravenel's chromatic convergence theorem, we can say that the homotopy groups $\pi_*(L_nS^0)$ will play a central role to understand the category of L_n-local spectra.

Besides, the Hopkins-Gross theorem says that the L_n-localization of the Spanier-Whitehead dual of a type n finite spectrum F is equivalent to the Brown-Comenetz dual up to some kind of suspension in the category of $K(n)_*$-local spectra if $p \cdot 1_F = 0$, and if the prime is large so that the Adams-Novikov

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spectral sequence for L_nF collapses. Note that $L_nF = L_{K(n)}F$ for a type n finite spectrum F. By the computations [9], [14] at the prime 3, the analogue of Hopkins-Gross theorem seems to hold even at a small prime number. Our theorem here shows that the analogue of Hopkins-Gross theorem should also require the condition $2 \cdot 1_X = 0$ at the prime 2 if it holds. Note that for a large prime, Devinatz and Hopkins [3] shows the necessity of the condition.

Throughout this paper, the prime is fixed to be 2 and every spectrum is 2-localized. In order to state our results, we prepare some notation:

$$k(n)_* = F_2[v_n],$$

$$K(n)_* = v_n^{-1}k(n)_* = F_2[v_n, v_n^{-1}],$$

$$C(k)\langle x_\alpha\rangle = \left\{\sum\nolimits_\alpha \lambda_\alpha x_\alpha/v_1^k \mid \lambda_\alpha \in k(1)_* \otimes K(2)_* \text{ with } v_1^k x_\alpha/v_1^k = 0\right\},$$

$$C(\infty)\langle y_\alpha\rangle = \varinjlim_k C(k)\langle y_\alpha\rangle;$$

$$W(2k) = \text{the cofiber of } v^k : \Sigma^{4k} W \longrightarrow W,$$

$$W(\infty) = \operatorname{ho}\varinjlim_k W(2k).$$

Here W is the cofiber of Hopkins-Mahowald's self map $\gamma : \Sigma^5 V \to V$ [4], where $V = M_2 \wedge M_\eta \wedge M_\nu$ for the cofiber M_ξ of the elements $\xi = 2, \eta \in \pi_1(S^0)$ and $\nu \in \pi_3(S^0)$, and $v : \Sigma^4 W \to W$ is the essential map given by $v \in [M_2 \wedge M_\eta, M_2 \wedge M_\eta]_4$ inducing $BP_*(v) = v_1^2$ (see Lemma 2.3).

THEOREM 1.1. *The Adams-Novikov E_∞-term for computing $\pi_*(L_2 W(\infty))$ is a $k(1)_*$-module*

$$(C(\infty)\langle 1, h_{21}, h_{30}, h_{21}h_{30}\rangle \oplus C(3)\langle h_{31}, h_{30}h_{31}\rangle) \otimes \Lambda(\rho).$$

This theorem implies the following:

COROLLARY 1.2. *The Adams-Novikov E_∞-term for computing $\pi_*(L_2 W(2k))$ for some $k \geq 1$ is a $k(1)_*$-module isomorphic to*

$$C_2 \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho),$$

if $k = 1$, and

$$(C_{2k}\langle 1, h_{21}, h_{30}, h_{21}h_{30}\rangle \oplus C_3\langle h_{31}, h_{30}h_{31}\rangle) \otimes \Lambda(\rho),$$

if $k > 1$. Here $C_k\langle x_\alpha\rangle$ denotes a $k(1)_*$-module isomorphic to the direct sum of $K(1)_*[v_1]/(v_1^k)$ generated by x_α's, which is also isomorphic to $C(k)\langle x_\alpha\rangle$.

Since $2 \cdot 1_{W(k)} \neq 0$ (see Corollary 7.2), the condition $2 \cdot 1_X = 0$ is necessary for the analogue of the Hopkins-Gross theorem at the prime 2. In fact, Corollary 1.2 shows that the homotopy groups of the finite spectrum $W(k)$ ($k > 1$) does not satisfy the duality, which is expected to hold if the analogue of Hopkins-Gross theorem without the condition is valid in this case. As is seen in Corollary 1.2 above, $W(2)$ satisfies the duality in the E_2-term (or E_∞-term) and $2 \cdot 1_{W(2)} \neq 0$. So this indicates that there would be some non-trivial extension in the spectral sequence, by which the duality fails to hold in the homotopy groups of $W(2)$.

As we have noticed above, W is a type one finite spectrum. The following would be a mile stone to understand the homotopy groups $\pi_*(L_2S^0)$:

COROLLARY 1.3. *The Adams-Novikov E_∞-term for computing $\pi_*(L_2W)$ is a $k(1)_*$-module isomorphic to*

$$k(1)_* \oplus K(1)_*b \oplus C'(\infty)\langle 1\rangle \oplus K(1)_*h_{20} \otimes \Lambda(b) \oplus C(\infty)\rho$$
$$\oplus\, (C(\infty)\langle h_{21}, h_{30}, h_{21}h_{30}\rangle \oplus C(3)\langle h_{31}, h_{30}h_{31}\rangle) \otimes \Lambda(\rho).$$

Here $C'(\infty)\langle 1\rangle = \{v_2^i/v_1^j \mid i, j \in \mathbf{Z}, i \neq 0, j > 0\}$ *and* $b \in \pi_4(L_2W)$.

2. Finite spectra

We denote BP the Brown-Peterson spectrum and $E(2)$ the Johnson-Wilson spectrum. The coefficient rings are $B = BP_* = \mathbf{Z}_{(2)}[v_1, v_2, \ldots]$ and $E = E(2)_* = \mathbf{Z}_{(2)}[v_1, v_2, v_2^{-1}]$. We also have $P = BP_*(BP) = BP_*[t_1, t_2, \ldots]$ and $L = E(2)_* \otimes_B P \otimes_B E(2)_*$, and (B, P) and (E, L) are the Hopf algebroids. Then the E_2-terms of the Adams-Novikov spectral sequences for computing the homotopy groups $\pi_*(X)$ and $\pi_*(L_2X)$ are given by $\mathrm{Ext}_P^*(B, BP_*(X))$ and $\mathrm{Ext}_L^*(E, E(2)_*(X))$, respectively. Here we denote $L_2 : \mathscr{S} \to \mathscr{S}$ the Bousfield localization functor with respect to $E(2)$, in which $\mathscr{S}$ denotes the homotopy category of 2-local CW-spectra. The Ext groups $\mathrm{Ext}_G^*(F, M)$ for a Hopf algebroid (F, G) and a G-comodule M are obtained as a cohomology of a cobar complex $(\Omega_G^s M, d_s : \Omega_G^s M \to \Omega_G^{s+1} M)_s$. Here

$$\Omega_G^s M = M \otimes_F G \otimes_F \cdots \otimes_F G \quad (s \text{ factors of } G),$$

and

$$d_s(m \otimes g) = \psi(m) \otimes g + \sum_{i=1}^{s} m \otimes \Delta_i(g) - (-1)^s m \otimes g \otimes 1,$$

for the comodule structure $\psi : M \to M \otimes_F G$ and $\Delta_i : G^{\otimes s} \to G^{\otimes(s+1)}$ defined by $\Delta_i(g_1 \otimes \cdots \otimes g_s) = g_1 \otimes \cdots \otimes \Delta(g_i) \otimes \cdots \otimes g_s$, where $\Delta : G \to G \otimes_F G$ is the diagonal of G.

Let M_α for an element $\alpha \in \pi_k(S^0)$ denote a cofiber of the map $a : S^k \to S^0$ representing the homotopy class α. Consider a spectrum $X = M_\eta \wedge M_\nu$ and the inclusion $i : S^0 \to X$ to the bottom cell.

LEMMA 2.1. [4] *There exits an essential map* $\gamma : \Sigma^5 X \to X$ *such that* $\gamma i \in \pi_*(X)$ *is detected by the class* $h_{20} = [t_2 + \cdots]$ *of the* E_2*-term* $\mathrm{Ext}^1_P(B, BP_*(X))$.

PROOF. The cofiber sequences

$$S^1 \xrightarrow{\eta} S^0 \xrightarrow{i_\eta} M_\eta \xrightarrow{\pi_\eta} S^2, \quad \text{and}$$

$$S^3 \xrightarrow{\nu} S^0 \xrightarrow{i_\nu} M_\nu \xrightarrow{\pi_\nu} S^4$$

induce the exact sequences

$$\cdots \longrightarrow \pi_{s-3}(M_\eta) \xrightarrow{\nu_*} \pi_s(M_\eta) \xrightarrow{i_{\nu *}} \pi_s(X) \xrightarrow{\pi_{\nu *}} \pi_{s-4}(M_\eta) \longrightarrow \cdots, \quad \text{and}$$

$$\cdots \longrightarrow \pi_{s-1}(S^0) \xrightarrow{\eta_*} \pi_s(S^0) \xrightarrow{i_{\eta *}} \pi_s(M_\eta) \xrightarrow{\pi_{\eta *}} \pi_{s-2}(S^0) \longrightarrow \cdots.$$

We further know the homotopy groups of spheres:

$$\begin{aligned} (2.2) \qquad & \pi_0(S^0) = \boldsymbol{Z}, \quad \pi_1(S^0) = \boldsymbol{Z}/2\langle\eta\rangle, \quad \pi_2(S^0) = \boldsymbol{Z}/2\langle\eta^2\rangle, \\ & \pi_3(S^0) = \boldsymbol{Z}/8\langle\nu\rangle \quad \pi_4(S^0) = 0 = \pi_5(S^0), \quad \pi_6(S^0) = \boldsymbol{Z}/2\langle\nu^2\rangle \end{aligned}$$

with a relation $4\nu = \eta^3$. By these, we obtain $\pi_5(X) = \boldsymbol{Z}/2\langle i_\nu \tilde{\nu}\rangle$ and $\pi_6(X) \cong \boldsymbol{Z}$, where $\tilde{\nu} \in \pi_5(M_\eta)$ and $\pi_\eta \tilde{\nu} = \nu$. We further see that $\nu^*(i_\nu \tilde{\nu}) = i_\nu \tilde{\nu}\nu = i_\nu \nu \tilde{\nu} = 0$. Chasing the commutative diagram

$$\begin{tikzcd}
{[X,X]_5} \arrow[r, "i_\nu^*"] & {[M_\eta, X]_5} \arrow[r, "\nu^*"] \arrow[d, "i_\eta^*"] & {[M_\eta, X]_8} \arrow[d, "i_\eta^*"] \\
 & \pi_5(X) \arrow[r, "\nu^*"] \arrow[d, "\eta^*"'] & \pi_8(X) \\
 & \pi_6(X) &
\end{tikzcd}$$

we have an element $\gamma \in [X, X]_5$ such that $i_\eta^* i_\nu^*(\gamma) = i_\nu \tilde{\nu}$.

Note that $\nu \in \pi_3(S^0)$ is detected by $h_{11} = [t_1^2 - v_1 t_1] \in \mathrm{Ext}_P^{1,4}(B, B)$ and $h_{20} = [t_2 + a(t_1^2 - v_1 t_1)] \in \mathrm{Ext}^{1,6}(B, BP_*(M_\eta))$ is sent to h_{11} by $\pi_{\eta *}$. Here $BP_*(M_\eta) = B \otimes \Lambda(a)$ with $\psi(a) = a + t_1$ and $|a| = 2$. Since $\pi_* \tilde{\nu} = \nu$, filt $\tilde{\nu} \leq 1$, and so $\tilde{\nu}$ is detected by h_{20}. q.e.d.

LEMMA 2.3. *There exists a map* $v : \Sigma^4 M_2 \wedge M_\eta \to M_2 \wedge M_\eta$ *such that* $BP_*(v) = v_1^2$.

PROOF. Consider the exact sequence

$$\pi_4(M_\eta) \xrightarrow{i_*} \pi_4(M_2 \wedge M_\eta) \xrightarrow{\pi_*} \pi_3(M_\eta) \xrightarrow{2} \pi_3(M_\eta)$$

associated to the cofiber sequence $S^0 \xrightarrow{2} S^0 \xrightarrow{i} M_2 \xrightarrow{\pi} S^1$. Then we obtain $\pi_4(M_2) = \boldsymbol{Z}/2\langle\tilde{\eta}\rangle$ by a computation with (2.2), where $\pi\tilde{\eta} = \eta^2$. Note that the element $\tilde{\eta}$ is detected by v_1^2 of the E_2-term of the Adams-Novikov spectral sequence for computing $\pi_*(M_2 \wedge M_\eta)$. In fact, the differential d_3 induces the connecting homomorphism on E_3-terms, which sends v_1^2 to h_{10}^2 since $d_3(v_1^2) = h_{10}^3$ in the E_2-term for $\pi_*(M_2)$. Since $2\tilde{\eta} = 0$ and $\eta\tilde{\eta} = 0$, $\tilde{\eta}$ is extended to $v \in [M_2 \wedge M_\eta, M_2 \wedge M_\eta]_4$ as desired. q.e.d.

COROLLARY 2.4. *There exists a spectrum* Y_2 *such that* $BP_*(Y_2) = BP_*/(2, v_1^2) \otimes \Lambda(a)$ *with* $|a| = 2$.

Consider the spectrum $W = M_2 \wedge D(A_1)$, where $D(A_1)$ denotes the cofiber of γ (*cf.* [4]). That is, W fits into the cofiber sequence

$$\Sigma^5 M_2 \wedge X \xrightarrow{1 \wedge \gamma} M_2 \wedge X \longrightarrow W \tag{2.5}$$

Then by Lemma 2.3, we obtain the self map $v : \Sigma^4 W \to W$ such that $BP_*(v) = v_1^2$. We write $v_1^{-1} W = \operatorname{holim}_{\overrightarrow{v}} W$ and define a spectrum $W(\infty)$ by the cofiber sequence

$$W \hookrightarrow v_1^{-1} W \longrightarrow W(\infty). \tag{2.6}$$

Note that $W(\infty)$ is given another way: Define a spectrum $W(2k)$ by the cofiber sequence

$$\Sigma^{4k} W \xrightarrow{v^k} W \longrightarrow W(2k),$$

and the map $w(k) : \Sigma^4 W(2k) \to W(2k+2)$ by the commutative diagram

$$\begin{tikzcd}
\Sigma^{4k+4} W \arrow[r, "v^k"] \arrow[d, equal] & \Sigma^4 W \arrow[r] \arrow[d, "v"] & \Sigma^4 W(2k) \arrow[d, "w(k)"] \\
\Sigma^{4k+4} W \arrow[r, "v^{k+1}"] & W \arrow[r] & W(2k+2).
\end{tikzcd} \tag{2.7}$$

Now $W(\infty)$ is given by

$$W(\infty) = \operatorname{holim}_{\overrightarrow{w(k)}} W(2k).$$

These show the following

PROPOSITION 2.8. *The $E(2)_*$-homology of these spectra are as follows*:

$$E(2)_*(X) = E(2)_* \otimes \Lambda(a,b),$$
$$E(2)_*(W) = E(2)_*/(2) \otimes \Lambda(a,b,c),$$
$$E(2)_*(v_1^{-1}W) = v_1^{-1}E(2)_*/(2) \otimes \Lambda(a,b,c),$$
$$E(2)_*(W(2k)) = E(2)_*/(2, v_1^{2k}) \otimes \Lambda(a,b,c),$$
$$E(2)_*(W(\infty)) = E(2)_*/(2, v_1^{\infty}) \otimes \Lambda(a,b,c).$$

Here $|a| = 2$, $|b| = 4$ *and* $|c| = 6$ *with coaction* $\psi(a) = a + t_1$, $\psi(b) = b + t_1^2$ *and* $\psi(c) = c + t_2 + at_1^2 + v_1at_1$.

3. $H^*K(2)_*$

In this section we will compute $H^*K(2)_* = \mathrm{Ext}_L^*(E, K(2)_* \otimes \Lambda(a,b,c))$. Here $H^*M = \mathrm{Ext}_L^*(E, M \otimes \Lambda(a,b,c))$ for an L-comodule M, and $K(2)_*$ is the L-comodule $K(2)_* = E(2)_*/(2, v_1) = F_2[v_2, v_2^{-1}]$.

To compute these modules, we introduce Hopf algebroids $(B, P_2) = (B, B[t_2, t_3, \dots])$ whose structure inherits from (B, P), and

$$(A, \Sigma) = (A, A \otimes_B P_2 \otimes_B A) = (F_2[v_1, v_2, v_2^{-1}], A[t_2, t_3, \dots]/(\eta_R(v_i) : i > 2)).$$

Since we see that

$$M \otimes \Lambda(a,b) = M \square_{\Sigma} A,$$

the change of rings theorem (*cf.* [12, Th. A1.3.12]) shows

$$H^*M = \mathrm{Ext}_{\Sigma}^*(A, M \otimes \Lambda(c)). \tag{3.1}$$

Take $M = K(2)_*$. Then we have a short exact sequence

$$0 \longrightarrow K(2)_* \longrightarrow K(2)_* \otimes \Lambda(c) \longrightarrow K(2)_* \longrightarrow 0 \tag{3.2}$$

of Σ-comodules.

THEOREM 3.3. $H^*K(2)_* = K(2)_* \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho)$, *where the generators are represented by the cocycles of the cobar complex as follows*: $h_{21} = [t_2^2]$, $h_{3i} = [t_3^{2^i}]$ $(i = 0, 1)$ *and* $\rho = [v_2^{-5}t_4 + v_2^{-10}t_4^2]$.

PROOF. Note first that $\mathrm{Ext}_{\Sigma}^*(A, K(2)_*) = \mathrm{Ext}_{\Sigma'}^*(K(2)_*, K(2)_*)$ for $\Sigma' = \Sigma/(v_1)$. Since $K(2)_*$ consists of primitive elements,

$$\mathrm{Ext}_{\Sigma}^*(A, K(2)_*) = K(2)_* \otimes \mathrm{Ext}_{S(2,2)}^*(F_2, F_2),$$

whose right hand factor is determined in [6, p. 239] to be $F_2[h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho)$. Apply the functor $\mathrm{Ext}^*_\Sigma(A, -)$ to the short exact sequence (3.2), and we have the long exact one

$$\cdots \longrightarrow \mathrm{Ext}^{s-1}_\Sigma(A, K(2)_*) \xrightarrow{\delta} \mathrm{Ext}^s_\Sigma(A, K(2)_*) \longrightarrow H^s K(2)_* \longrightarrow \cdots,$$

where $\delta(x) = h_{20}x$ since the comodule structure on c shows $[d_0(c)] = h_{20}$ by definition of d_0 of the cobar complex. This shows the theorem. q.e.d.

4. Bockstein spectral sequence

Consider the Σ-comodule $M^1_1 = E(2)_*/(2, v_1^\infty) = \varinjlim_k E(2)_*/(2, v_1^k)$. Then the colimit of short exact sequences

$$0 \longrightarrow K(2)_* \xrightarrow{v_1^k} E(2)_*/(2, v_1^{k+1}) \longrightarrow E(2)_*/(2, v_1^k) \longrightarrow 0$$

for $k > 0$ gives rise to another short exact one

$$0 \longrightarrow K(2)_* \xrightarrow{\varphi} M^1_1 \xrightarrow{v_1} M^1_1 \longrightarrow 0,$$

where $\varphi(x) = x/v_1$. Noticing that H^*- is a homology functor, we have the long exact sequence

$$0 \longrightarrow H^0K(2)_* \xrightarrow{\varphi_*} H^0M^1_1 \xrightarrow{v_1} H^0M^1_1$$
$$\xrightarrow{\delta} H^1K(2)_* \xrightarrow{\delta_*} H^1M^1_1 \xrightarrow{v_1} H^1M^1_1 \longrightarrow \cdots.$$

Then by [8, Remark 3.11], we can show

LEMMA 4.1. *If a submodule $B^s = \sum_\alpha C(\infty)\langle x_\alpha\rangle \oplus \sum_\beta C(n_\beta)\langle y_\beta\rangle$ of $H^sM^1_1$ satisfies the following two conditions, then $H^sM^1_1 = B^s$.*

1. $\mathrm{Im}\,\varphi_* \subset B^s$,
2. *The set $\{\delta(v_2^t y_\beta/v_1^{n_\beta})\}_{t,\beta}$ is linearly independent over F_2.*

In fact, we obtain the exact sequence $\longrightarrow H^sK(2)_* \xrightarrow{\varphi_*} B^s \xrightarrow{v_1} B^s \xrightarrow{\delta} H^{s+1}K(2)_* \longrightarrow$ if B^s satisfies the conditions of Lemma 4.1. Then just use [8, Remark 3.11] to certify the lemma.

LEMMA 4.2. *In the cobar complex $\Omega^2_\Sigma A \otimes \Lambda(c)$,*

$$d_1(t_{30}) = 0.$$

$$d_1(t_{31}) \equiv v_1^3 v_2^{-3} t_2^2 \otimes t_3^2.$$

Here $t_{30} = t_3 + v_1 c t_2$ *and* $t_{31} = t_3^2 + v_1 v_2^2 t_3 + v_1^2 v_2^{-1} t_4 + v_1^3(v_2^{-16} t_5^2 + v_2^{-2} t_2 t_3^2 + v_2^{-2} c t_3^2)$.

PROOF. By Hazewinkel's and Quillen's formulae, we obtain

$$\Delta(t_3) = \sum_{i=0}^{3} t_i \otimes t_{3-i}^{2^i} - v_1(t_1 \otimes t_1^2(t_2 \otimes 1 + 1 \otimes t_2) + t_2 \otimes t_2)$$
$$+ v_1^2(t_1 \otimes t_1)\Delta(t_2) - v_1^3(t_1 \otimes t_1)\Delta(t_1^2) - 2v_2(t_1 \otimes t_1)\Delta(t_1^2)$$

in $P \otimes_B P$. Now sending t_1 to 0 and the formula $\psi(c) = c + t_2$ show the first equation.

For the second, we compute:

$$d_1(t_3^2) = v_1^2 t_2^2 \otimes t_2^2,$$
$$d_1(v_1 v_2^2 t_3) = v_1^2 v_2^2 t_2 \otimes t_2,$$
$$d_1(v_1^2 v_2^{-1} t_4) \equiv v_1^2 v_2^{-1} t_2 \otimes t_2^4 + v_1^2 t_2^2 \otimes t_2^2 + v_1^3 v_2^{-1} t_3 \otimes t_3 \mod(v_1^4),$$
$$d_1(v_1^3 v_2^{-17} t_5^2) \equiv v_1^3 v_2^{-17}(t_2^2 \otimes t_3^8 + t_3^2 \otimes t_2^{16} + v_2^2 t_3^4 \otimes t_3^4) \mod(v_1^4),$$
$$\equiv v_1^3 v_2^{-3} t_2^2 \otimes t_3^2 + v_1^3 v_2^{-2} t_3^2 \otimes t_2 + v_1^3 v_2^{-1} t_3 \otimes t_3 \mod(v_1^4),$$
$$d_1(v_1^3 v_2^{-2} t_2 t_3^2) \equiv v_1^3 v_2^{-2}(t_2 \otimes t_3^2 + t_3^2 \otimes t_2) \mod(v_1^4),$$
$$d_1(v_1^s v_2^{-2} c t_3^2) \equiv v_1^3 v_2^{-2} t_2 \otimes t_3^2 \mod(v_1^4).$$

Now using the relations $v_i = 0 = \eta_R(v_i)$ in Σ for $i > 2$, we see the second equation. q.e.d.

LEMMA 4.3. *We have a cochain $R_k \in \Omega_\Sigma^1 A$ such that $d_1(R_k) \equiv 0 \mod(v_1^k)$ and $R_k \equiv v_2^{-5} t_4 + v_2^{-10} t_4^2 \mod(v_1)$.*

PROOF. Note that $t_4^4 \equiv v_2^{15} t_4 \mod(v_1)$ in Σ by the relation $\eta_R(v_6) = 0$, and $d_1(R) \equiv 0 \mod(v_1)$ for $R = v_2^{-5} t_4 + v_2^{-10} t_4^2$ since $\rho = [R]$. Now put $R_k = R^{2^k}$, and we see the lemma. q.e.d.

For the next theorem, we introduce the $k(1)_*$-modules $F(s)_*$:

$$F(s)_* = 0 \quad (s < 0, 2 < s),$$
$$F(0)_* = C(\infty)\langle 1 \rangle,$$
$$F(1)_* = C(\infty)\langle h_{21}, h_{30} \rangle \oplus C(3)\langle h_{31} \rangle,$$
$$F(2)_* = C(\infty)\langle h_{21} h_{30} \rangle \oplus C(3)\langle h_{30} h_{31} \rangle.$$

By definition, there exists an integer $k > 0$ for each element $x \in F(s)_* \subset$

$E_2^s(W(\infty))$ such that $v_1^k x = 0$. Then Lemma 4.3 shows that $xR_k \in E_2^*(W(\infty))$, and then we denote it by $x\rho$.

THEOREM 4.4. *The E_2-term $E_2^{s,*}(L_2 W(\infty))$ of the Adams-Novikov spectral sequence computing $\pi_*(W(\infty))$ is isomorphic to a direct sum of $k(1)_*$-modules* $F(s)_*$ and $F(s-1)_*\rho$.

PROOF. We proceed to prove the theorem by checking the conditions 1 and 2 of Lemma 4.1 for each s. Put $B^0 = C(\infty)$, and we see easily that the conditions 1 and 2 are satisfied.

For $s = 1$, we just check the condition 2, that is, if the set $\{\delta(h_{31}/v_1^3)\}$ is independent. By Lemma 4.2, we compute $\delta(v_2^s h_{31}/v_1^3) = v_2^{s-3} h_{21} h_{31}$, which is obviously non-zero.

This shows that $\operatorname{Im} \varphi_* = \{x/v_1 \mid x \in H^2 K(2)_*, x \notin K(2)_* \langle h_{21} h_{31} \rangle\}$. Thus $B^2 = F(2)_* \oplus F(1)_* \rho$ contains $\operatorname{Im} \varphi_*$. Lemma 4.2 also shows

$$\delta(v_2^s h_{30} h_{31}/v_1^3) = v_2^{s-3} h_{21} h_{30} h_{31} \quad \text{and}$$

$$\delta(v_2^s h_{31} \rho / v_1^3) = v_2^{s-3} h_{21} h_{31} \rho.$$

Thus the condition 2 for B^2 is satisfied and so $H^2 M_1^1 = B^2$. Besides, the formulae above show that the image of φ_* in $H^3 M_1^1$ is the $K(2)_*$-module over $\{h_{21} h_{30} \rho / v_1, h_{30} h_{31} \rho / v_1\}$. Furthermore, we see that

$$\delta(v_2^s h_{30} h_{31} \rho / v_1^3) = h_{21} h_{30} h_{31} \rho.$$

Therefore we obtain $H^3 M_1^1$ and $\operatorname{Im} \varphi_* = 0 \subset H^4 M_1^1$. For $n \geq 4$, since $\operatorname{Im} \varphi_* = 0$, we set $B^n = 0$ and get $H^n M_1^1 = 0$ by Lemma 4.1. q.e.d.

5. The Adams-Novikov differentials

In this section, we compute differentials of the Adams-Novikov spectral sequence. By Theorem 4.4, we see that $E_2^s(W(\infty)) = 0$ if $s > 3$, and so the all Adams-Novikov differentials d_r are zero except for $d_3 : E_2^0(W(\infty)) \to E_2^3(W(\infty))$. In order to study the exceptional case, recall [6], [5] the spectra D and Z (which is denoted by X in [5]). Let $X\langle 1 \rangle$ be the Mahowald ring spectrum with $BP_*(X\langle 1 \rangle) = B/(2)[t_1]$. Then $v_1 \in \pi_2(X\langle 1 \rangle)$ is extended to the self map $v_1 : \Sigma^2 X\langle 1 \rangle \to X\langle 1 \rangle$, whose cofiber is D. C is defined by the cofiber sequence $X\langle 1 \rangle \to v_1^{-1} X\langle 1 \rangle \to C$ and Z is the cofiber of $\gamma : \Sigma^5 C \to C$ defined by $h_{20} \in \pi_5(X\langle 1 \rangle)$. Note that $C = \underrightarrow{\operatorname{holim}}\, C(n)$ and $Z = \underrightarrow{\operatorname{holim}}\, Z_n$, where $C(n)$ and Z_n is defined by the following commutative diagram of cofiber sequences:

(5.1)
$$\begin{tikzcd}
\Sigma^{2n+5}X\langle 1\rangle \arrow[r, "\gamma"] \arrow[d, "v_1^n"'] & \Sigma^{2n}X\langle 1\rangle \arrow[r] \arrow[d, "v_1^n"] & \Sigma^{2n}C_\gamma \arrow[d] \\
\Sigma^5 X\langle 1\rangle \arrow[r, "\gamma"] \arrow[d] & X\langle 1\rangle \arrow[r] \arrow[d] & C_\gamma \arrow[d] \\
\Sigma^5 C(n) \arrow[r] & C(n) \arrow[r] & Z_n
\end{tikzcd}$$

Then

(5.2)
$$Z = \operatorname{holim}_{\longrightarrow} Z_n$$

and since $D = C(1)$,

(5.3) Z_1 is a cofiber of $\gamma : \Sigma^5 D \to D$, where γ is obtained from the element $h_{20} \in \pi_5(X\langle 1\rangle)$.

PROPOSITION 5.4. *The E_∞-term of the Adams-Novikov spectral sequence computing $\pi_*(L_2Z_2)$ is the tensor product of $\Lambda(h_{30}, h_{31}, \rho)$ and a direct sum of $k(1)_*$-modules $K(2)_*[v_3^2], v_1K(2)_*[v_3], h_{21}K(2)_*[v_3]$ and $v_1v_3h_{21}K(2)_*[v_3^2]$.*

PROOF. By (5.3), we have an exact sequence

$$E_2^{s-1}(L_2D) \xrightarrow{h_{20}} E_2^s(L_2D) \longrightarrow E_2^s(L_2Z_1) \longrightarrow E_2^s(L_2D)$$

and $E_2^*(L_2D) = K(2)_*[v_3, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho)$ by [6, Th. 2.1]. Therefore, we obtain

(5.5)
$$E_\infty(L_2Z_1) = K(2)_*[v_3] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho).$$

In fact, we can deduce that $d_3(v_3^s) = 0$ from [6, Th. 7.1], and so we see the special sequence collapses. By the definition (5.1) of Z_n, we have the cofiber sequence $\Sigma^2 Z_1 \xrightarrow{v_1} Z_2 \longrightarrow Z_1$. This gives rise to the long exact sequence

$$\longrightarrow E_2^{s-1}(L_2Z_1) \xrightarrow{\delta} E_2^s(L_2Z_1) \xrightarrow{v_1} E_2^s(Z_2) \longrightarrow E_2^s(L_2Z_1) \xrightarrow{\delta}$$

of E_2-terms. Since $\delta(v_3) = h_{21}$ as is seen in [5], the proposition follows from (5.5). q.e.d.

PROPOSITION 5.6. *In the Adams-Novikov spectral sequence computing $\pi_*(L_2W(\infty)), d_3(v_2^s/v_1^j) = 0$.*

Proof. By Theorem 4.4, we see that

$$d_3(v_2^s/v_1^j) = \lambda v_2^t h_{21} h_{30} \rho / v_1^\varepsilon \tag{5.7}$$

for $\varepsilon \in \{1,2\}$ and for some $\lambda \in F_2$ in the $E_3 = E_2$-term of the Adams-Novikov spectral sequence for $\pi_*(L_2 W(\infty))$. Here $6t = 6s - 2j - 22 + 2\varepsilon$. In fact, $d_3(v_2^s/v_1^j)$ should be infinitely v_1-divisible because of the naturality of differentials and existence of the map $v : \Sigma^4 W(\infty) \to W(\infty)$. Consider now the cofiber sequence

$$\longrightarrow \Sigma^4 W(2) \longrightarrow \Sigma^4 W(\infty) \xrightarrow{v} W(\infty) \xrightarrow{i} \Sigma^5 W(2) \longrightarrow$$

obtained from the homotopy colimit of cofiber sequences $\Sigma^{4k} W(2) \xrightarrow{w^k} W(2k+2) \longrightarrow W(2k) \longrightarrow \Sigma^{4k+1} W(2)$, where $w^k = w(k) \cdots w(2)$ for $w(k)$ in (2.7). Since $d_3(v_2^s/v_1^{j-2}) = v_* d_3(v_2^s/v_1^j) = 0$ in the E_2-term by (5.7), v_2^s/v_1^{j-2} is a permanent cycle of the spectral sequence for $\pi_*(L_2 W(\infty))$. Therefore, the equation (5.7) also produces the relation

$$i_*(v_2^s/v_1^{j-2}) = \lambda v_1^{2-\varepsilon} v_2^t h_{21} h_{30} \rho \tag{5.8}$$

in homotopy groups $\pi_*(W(2))$. Consider the commutative diagram

$$\begin{array}{ccccc} & & W(\infty) & \xrightarrow{i} & W(2) \\ & & \Big\downarrow \iota & & \Big\downarrow \iota \\ Z & \xrightarrow{v_1^2} & Z & \xrightarrow{i} & Z_2. \end{array}$$

Now send (5.8) by ι, we have

$$i_*(v_2^s/v_1^{j-2}) = \lambda \iota_* v_1^{2-\varepsilon} v_2^t h_{21} h_{30} \rho.$$

Since v_2^s/v_1^j is a permanent cycle in the spectral sequence for $\pi_*(L_2 Z)$ by the main theorem of [5], $i_*(v_2^s/v_1^{j-2}) = i_*(v_1^2(v_2^s/v_1^j)) = 0$. On the other hand, $\iota_* v_1^{2-\varepsilon} v_2^t h_{21} h_{30} \rho$ is not zero if $\varepsilon = 2$ by Proposition 5.4. Therefore we see that $\lambda = 0$ in this case.

Now suppose that $\varepsilon = 1$. Put $V = M_2 \wedge M_\eta \wedge M_\nu = M_2 \wedge X$. Then we have a cofiber sequence $\Sigma^5 V \xrightarrow{1 \wedge \gamma} V \to W$ by the definition (2.5) of W. The inclusion map $V \to W$ also yields the map $V_{2n} \to W(2n)$ for each n, where V_n is a cofiber of $v_1^n : \Sigma^{2n} V \to V$ in which the map v_1 is given in [2]. We also have a map $v_1 : V_n \to V_{n+1}$ fitting into the commutative diagram

$$\begin{array}{ccc} V_{2n} & \longrightarrow & W(2n) \\ {\scriptstyle v_1^2}\Big\downarrow & & \Big\downarrow{\scriptstyle w(n)} \\ V_{2n+2} & \longrightarrow & W(2n+2). \end{array}$$

Taking its homotopy colimit gives us a map $\kappa : V(\infty) \to W(\infty)$. The relation (5.7) is pulled first back to $d_3(v_2^s/v_1^{j+2}) = \lambda v_2^t h_{21} h_{30} \rho / v_1^3$ in $E_2^3(L_2 W(\infty))$ by v_* and then back it to the one in the spectral sequence for $\pi_*(L_2 V(\infty))$ by κ_*. Thus,

$$(5.9) \qquad d_3(v_2^s/v_1^{j+2}) = \lambda v_2^t h_{21} h_{30} \rho / v_1^3 + h_{20} x$$

for some $x \in E_2^2(L_2 V(\infty))$. This is sent to

$$d_3(v_2^s/v_1^{j+1}) = \lambda v_2^t h_{21} h_{30} \rho / v_1^2 + v_1 h_{20} x,$$

by the map $v_1 : \Sigma^2 V(\infty) \to V(\infty)$. Send this to $E_2^3(L_2 W(\infty))$ again, and we obtain $d_3(v_2^s/v_1^{j+1}) = \lambda v_2^t h_{21} h_{30} \rho / v_1^2$. This is the case where $\varepsilon = 2$, and so we obtain $\lambda = 0$ as we have studied above. q.e.d.

This proposition and Theorem 4.4 imply that $d_r = 0$ for all r in the Adama-Novikov spectral sequence for computing $\pi_*(W(\infty))$, and hence we obtain

THEOREM 5.10. *The Adams-Novikov spectral sequence for computing $\pi_*(W(\infty))$ collapses from E_2-term. That is, $E_\infty^* = E_2^*$.*

By this and Theorem 4.4, we see Theorem 1.1 in the introduction.

6. Homotopy groups

Recall [2] the self map $v_1 : \Sigma^2 Y \to Y$ for $Y = M_2 \wedge M_\eta$. Then Ravenel's computation [10] shows the following

LEMMA 6.1. *$\pi_*(v_1^{-1} Y) = K(1)_* \otimes \Lambda(\rho_1)$, where ρ_1 is represented by the cocycle $v_1^{-3}(t_2 - t_1^3) + v_1^{-4} v_2 t_1$ of the cobar complex.*

PROOF. Since $BP_*(Y) = BP_*/(2) \otimes \Lambda(a)$ with coaction $\psi(a) = a + t_1$, the E_2-term of the Adams-Novikov spectral sequence computing $\pi_*(v_1^{-1} Y)$ is given by

$$E_2^s(v_1^{-1} Y) = \mathrm{Ext}^s_{K(1)_* K(1)}(K(1)_*, K(1)_* \otimes \Lambda(a))$$

by the change of rings theorem [7]. We then have a long exact sequence

$$\cdots \xrightarrow{\delta} \mathrm{Ext}^s_{K(1)_* K(1)}(K(1)_*, K(1)_*) \longrightarrow \mathrm{Ext}^s_{K(1)_* K(1)}(K(1)_*, K(1)_* \otimes \Lambda(a))$$

$$\longrightarrow \mathrm{Ext}^s_{K(1)_* K(1)}(K(1)_*, K(1)_*) \xrightarrow{\delta} \cdots,$$

in which $\mathrm{Ext}^s_{K(1)_* K(1)}(K(1)_*, K(1)_*) = K(1)_*[h_{10}] \otimes \Lambda(\rho_1)$ shown in [10]. Furthermore, the structure on a yields $\delta(x) = x h_{10}$. Thus we see that $E_2^s(v_1^{-1} Y)$

$= \mathrm{Ext}^s_{K(1)_*K(1)}(K(1)_*, K(1)_* \otimes \Lambda(a)) = K(1)_* \otimes \Lambda(\rho_1)$. Since $E_2^s(v_1^{-1}Y) = 0$ if $s > 1$, $d_r = 0$ in the Adams-Novikov spectral sequence, and we see that $E_\infty^s(v_1^{-1}Y) = E_2^s(v_1^{-1}Y)$. The sparseness of the spectral sequence implies the triviality of the problem of extension and we obtain the homotopy groups.

q.e.d.

LEMMA 6.2. *$\pi_*(v_1^{-1}M_2 \wedge X) = K(1)_* \otimes \Lambda(\rho_1, b)$, where $|b| = 4$ and the Adams-Novikov filtration of b is* 0.

PROOF. Note that $M_2 \wedge X = Y \wedge M_\nu$. The generator $\nu \in \pi_3(S^0)$ induces the map $\nu : \Sigma^3 v_1^{-1}Y \to v_1^{-1}Y$. Then, $\nu_* : BP_*(v_1^{-1}Y) \to BP_*(v_1^{-1}Y)$ is trivial and so we have a long exact sequence

$$\cdots \longrightarrow E_2^{s-1}(v_1^{-1}Y) \xrightarrow{\delta} E_2^s(v_1^{-1}Y) \longrightarrow E_2^s(v_1^{-1}Y \wedge M_\nu) \longrightarrow \cdots$$

of E_2-terms. We compute $BP_*(Y \wedge M_\nu) = BP_*/(2) \otimes \Lambda(a, b)$ with $|b| = 4$ and $\psi(b) = b + t_1^2$, and so we compute

$$\begin{aligned}\delta(x) &= [i^{-1}d(bx)] = [t_1^2 \otimes x] \\ &= [v_1 t_1 \otimes x] = [d(v_1 ax)] \\ &= 0,\end{aligned}$$

in which we use the relations $\eta_R(v_2) = 0 = v_2$ in $K(1)_*K(1)$ and $\eta_R(v_2) = v_2 + v_1 t_1^2 - v_1^2 t_1$. Thus we have the desired homotopy groups. The filtration of b is read off from the short exact sequence turned from the above long exact sequence.

q.e.d.

LEMMA 6.3. *$\pi_*(v_1^{-1}W) = K(1)_* \otimes \Lambda(b, h_{20})$, where $|h_{20}| = 5$ and the Adams-Novikov filtration of h_{20} is* 1.

PROOF. We see that the map $1 \wedge \gamma : \Sigma^5 M_2 \wedge X \to M_2 \wedge X$ induces an isomorphism $E_2^0(v_1^{-1}M_2 \wedge X) \overset{\delta}{\cong} E_2^1(v_1^{-1}M_2 \wedge X)$ by Lemma 2.1, since $\rho_1 = h_{20}$ and $\delta(x) = xh_{20}$. Now consider the exact sequence associated to the cofiber sequence (2.5) that defines W, and we obtain the lemma in the same manner as the above one.

q.e.d.

These lemmas imply the following

COROLLARY 6.4. *The E_2-term $E_2^s(v_1^{-1}W)$ of the Adams-Novikov spectral sequence for $\pi_*(v_1^{-1}W)$ is isomorphic to $K(1)_* \otimes \Lambda(b)$ if $s = 0, 1$, and 0 if $s > 1$.*

7. Self homotopy sets

By (2.3), we obtain $BP_*(W(2k)) = BP_*/(2, v_1^{2k}) \otimes \Lambda(a, b, c)$. The E_2-terms for computing $\pi_*(L_2W(2k))$ are read off from Theorem 4.4, which are stated in Corollary 1.2. Furthermore, we see that

PROPOSITION 7.1. $[W(2k), W(2k)]_{-4k-7} = \boldsymbol{Z}/4$ *for* $k > 0$.

PROOF. Note first that $[M_2, W(2k)]_s = 0$ if $s < -1$. A filtration given by the skeleton of $W(2k)$ yields a spectral sequence

$$\bigvee_{j \in J_k} [M_2, W(2k)]_{s+j} \Longrightarrow [W(2k), W(2k)]_s.$$

Here $J_k = \{0, 2, 4, 6, 4k+1, 4k+3, 4k+5, 4k+7\}$. Therefore, we have

$$[M_2, W(2k)]_0 \cong [W(2k), W(2k)]_{-4k-7}.$$

Besides, $[M_2, W(2k)]_0 = [M_2, M_2]_0 = \boldsymbol{Z}/4$ and we have the proposition.

q.e.d.

COROLLARY 7.2. $2 \cdot 1_{W(2k)} \neq 0$ *for* $k > 0$.

PROOF. Take a generator $x \in [W(2k), W(2k)]_{-4k-7}$. Then x induces a map $x_* : [W(2k), W(2k)]_0 \to [W(2k), W(2k)]_{-4k-7}$ such that $x_*(2 \cdot 1_{W(2k)}) = 2x \neq 0$ by Proposition 7.1. q.e.d.

8. Homotopy groups $\pi_*(L_2 W)$

Applying the homotopy theory $E(2)_*(-)$ to the cofiber sequence (2.6) generates the short exact sequence $0 \to E(2)_*(W) \to v_1^{-1}E(2)_*(W) \to E(2)_*(W(\infty)) \to 0$, and hence the long exact sequence

$$E_2^s(L_2 W) \longrightarrow E_2^s(v_1^{-1} W) \longrightarrow E_2^s(L_2 W(\infty)) \xrightarrow{\delta} E_2^{s+1}(L_2 W)$$

of E_2-terms. The E_2-terms $E_2^*(v_1^{-1} W)$ and $E_2^*(L_2 W(\infty))$ are determined in Corollary 6.4 and Theorem 4.4. Therefore, the long exact sequence splits into the exact sequences

$$\begin{aligned} 0 \to E_2^0(L_2 W) \to K(1)_* \otimes \Lambda(b) \to C(\infty)\langle 1 \rangle \\ \to E_2^1(L_2 W) \to K(1)_* \otimes \Lambda(b) \to 0, \quad \text{and} \\ 0 \to E_2^s(L_2 W(\infty)) \to E_2^{s+1}(L_2 W) \to 0 \quad (s > 0). \end{aligned}$$

These show Corollary 1.3 in the introduction.

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