HIROSHIMA MATH. J.
28 (1998), 419–435

Nonexistence of subsolutions of a nonlinear elliptic equation on bounded domains in a Riemannian manifold

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(Received March 14, 1997)
(Revised September 10, 1997)

ABSTRACT. We give some nonexistence results for positive subsolutions of a certain class of nonlinear elliptic equations including the scalar curvature equation on bounded domains in a Riemannian manifold.

1. Introduction

Let (M,g) be a Riemannian manifold ($n = \dim M \geq 3$), S_g its scalar curvature. For any smooth function f on M, f can be realized as the scalar curvature of some metric $\tilde{g}$ conformal to g, if and only if there exists a smooth solution u of the following equation:

$$(*)\qquad \begin{cases} -\frac{4(n-1)}{n-2}\Delta_g u + S_g u = f u^{(n+2)/(n-2)} \\ u > 0 \end{cases} \quad \text{on} \quad M,$$

where Δ_g is the Laplacian of g (i.e. $\Delta_g := g^{ij}\nabla_{ij}$). Indeed, the conformal metric $\tilde{g} = u^{4/(n-2)}g$ has the scalar curvature $S_{\tilde{g}} = f$.

In this paper, we consider a class of equations including $(*)$ on a certain type of open Riemannian manifolds, and give some nonexistence results in the case f is nonpositive. Before describing our results, we recall here some known facts for typical (M,g)'s. Throughout this paper, we use the notation "$f \sim \tilde{f}$" to mean that $f/\tilde{f}$ is bounded between two positive constants (i.e. $C\tilde{f} \leq f \leq C'\tilde{f}$ for some $C > 0$ and $C' > 0$).

FACT 1.1. *Let (M,g) be the Euclidean space $(\mathbf{R}^n, g_0)$. Denote the distance function to the origin by r. Then the following assertions hold*:

(1) *If $|f| \leq Cr^{-2-\varepsilon}$ near ∞ for some $C > 0$ and $\varepsilon > 0$, then $(*)$ has infinitely many solutions u satisfying $u \sim 1 = r^0$ near ∞* ([14]);

(2) *If $-Cr^{-2-\varepsilon} \leq f < 0$ near ∞ for some $C > 0$ and $\varepsilon > 0$, and $f \leq 0$ on $\mathbf{R}^n$, then $(*)$ has a solution u satisfying $u \geq C'r^{\varepsilon(n-2)/4}$ near ∞ for some $C' > 0$* ([4]);

1991 *Mathematics Subject Classification:* 35J60, 53C20.
Key words and phrases: Nonlinear elliptic equation, Scalar curvature equation.

(3) *If* $f \leq -Cr^{-2}$ *near* ∞ *for some* $C > 0$, *and* $f \leq 0$ *on* $\mathbf{R}^n$, *then* $(*)$ *has no solutions* ([12], [3]).

FACT 1.2. *Let* (M, g) *be the unit ball* $(B_1(0), g_0)$ *in* $\mathbf{R}^n$. *Then the following assertions hold*:

(1) *If* $|f| \leq C(1-r)^{-2+\varepsilon}$ *near* $\partial B_1(0)$ *for some* $C > 0$ *and* $\varepsilon > 0$, *then* $(*)$ *has infinitely many solutions* u *satisfying* $u \sim 1 = (1-r)^0$ *near* $\partial B_1(0)$ (*To see this, examine a supersolution and a subsolution* $u_{\pm} = \gamma\{2 \pm (1-r^2)^{\varepsilon}\}$.);

(2) *If* $-C(1-r)^{-2+\varepsilon} \leq f < 0$ *near* $\partial B_1(0)$ *for some* $C > 0$ *and* $\varepsilon > 0$, *and* $f \leq 0$ *on* $B_1(0)$, *then* $(*)$ *has a solution* u *satisfying* $u \geq C'(1-r)^{-\varepsilon(n-2)/4}$ *near* $\partial B_1(0)$ *for some* $C' > 0$ ([16]);

(3) *If* $f \leq -C(1-r)^{-2}$ *near* $\partial B_1(0)$ *for some* $C > 0$, *and* $f \leq 0$ *on* $B_1(0)$, *then* $(*)$ *has no solutions* ([16]).

Now, consider the case of open Riemannian manifolds of the form $M = \overline{M} \backslash \Sigma$, where $(\overline{M}, \bar{g})$ is a compact Riemannian manifold and Σ is its closed submanifold. The scalar curvature equation $(*)$ on such $M = \overline{M} \backslash \Sigma$ is studied as a problem of finding solutions with prescribed singularity Σ, or as an approach to an investigation on the space of positive solutions of the nonlinear equation $(*)$ with various Martin boundary.

It is well known that $\mathbf{R}^n$ is conformal to $\mathbf{S}^n \backslash \{p\}$, and that $B_1(0)$ is conformal to both a connected component of $\mathbf{S}^n \backslash \mathbf{S}^{n-1}$ and the hyperbolic space $\mathbf{H}^n$. Namely both $\mathbf{R}^n$ and $B_1(0)$ (or $\mathbf{H}^n$) belong to the class $\{M = \overline{M} \backslash \Sigma\}$. From this common viewpoint, Facts 1.1 and 1.2 seem to describe the same phenomenon on the different objects, and the difference of the orders in the assertions seems to be caused by the difference of the dimensions of the boundaries $\{p\}$ and $\mathbf{S}^{n-1}$. Indeed, there is a fact for the boundary of dimension $d \in [0, n-2)$ as below:

FACT 1.3. *Let* $(\overline{M}, \bar{g})$ *be a compact Riemannian manifold* ($n = \dim \overline{M} \geq 3$) *with positive scalar curvature, and* Σ *its closed* C^2-*submanifold* ($d = \dim \Sigma < n-2$). *Set* $(M, g) := (\overline{M} \backslash \Sigma, \bar{g}|_{\overline{M} \backslash \Sigma})$. *Denote the distance function to* Σ *by* r_{Σ}. *Then the following assertions hold*:

(1) *If* $|f| \leq C r_{\Sigma}^{2-4d/(n-2)+\varepsilon}$ *near* Σ *for some* $C > 0$ *and* $\varepsilon > 0$, *then* $(*)$ *has infinitely many solutions* u *satisfying* $u \sim r_{\Sigma}^{2-n+d}$ *near* Σ ([5], [10]);

(2) *If* $-C r_{\Sigma}^{2-4d/(n-2)+\varepsilon} \leq f < 0$ *near* Σ *for some* $C > 0$ *and* $\varepsilon > 0$, *and* $f \leq 0$ *on* M, *then* $(*)$ *has a solution* u *satisfying* $u \geq C' r_{\Sigma}^{2-n+d-\varepsilon(n-2)/4}$ *near* Σ *for some* $C' > 0$ ([7], [8]).

When we observe these results, it is natural to expect the following:

(3) *If* $f \leq -C r_{\Sigma}^{2-4d/(n-2)}$ *near* Σ *for some* $C > 0$, *and* $f \leq 0$ *on* M, *then* $(*)$ *has no solutions.*

The aim of this paper is to show more general nonexistence results which include this as a special case.

Let $(\check{M}, \check{g})$ be a Riemannian manifold ($n = \dim \check{M} \geq 3$), M a relatively compact domain in $\check{M}$, and $g := \check{g}|_M$. Let q be a number larger than 1, S (resp. f) a nonnegative (resp. nonpositive) locally Hölder continuous function on M. Under these assumptions, we consider the following equation:

$$(**) \qquad -\Delta_g u + Su = fu^q \quad \text{on } M.$$

Throughout this paper, we denote the distance function to ∂M (resp. Σ etc.) by $r_{\partial M}$ (resp. r_Σ etc.). We have the following

THEOREM I. *Let (M, g), q, S and f be as above, $\check{S}$ a nonnegative Hölder continuous function on $\check{M}$ such that $\check{S} \leq S$ on M, and φ a positive function on M satisfying $-\Delta_g \varphi + \check{S}\varphi = 0$. If f satisfies*

$$f \leq -Cr_{\partial M}^{-2} \varphi^{-(q-1)} \quad \textit{near } \partial M$$

for a positive constant C, then the equation $(**)$ does not possess a positive (sub-)solution.

In the proofs of Facts 1.1(3) and 1.2(3), analyses of ordinary differential inequalities played a cruicial role. However, it seems difficult to apply such methods to our case, and we employ a different method to show Theorem I.

In the assumption of Theorem I, if we choose $\check{S} \equiv 0$, then φ is a positive harmonic function. We can always choose $\varphi \equiv 1$, and, as a corollary, we get a generalization of Fact 1.2(3) to an arbitrary bounded domain in $\mathbf{R}^n$, or in a complete Riemannian manifold.

COROLLARY. *Let (M, g), q, S and f be as in Theorem I. If f satisfies $f \leq -Cr_{\partial M}^{-2}$ near ∂M for a positive constant C, then the equation $(**)$ does not possess a positive (sub-)solution.*

Note here that the set ∂M is free from any assumption on dimension and regularity. Moreover, by another special choice of φ, we get the following

THEOREM II. *Let (M, g), q, S and f be as in Theorem I. Assume one of the following conditions holds:*
(a) There exists a bounded domain $\Omega \subset\subset \check{M}$ which includes the closure $\overline{M}$ of M;
(b) $\check{M}$ is a compact Riemannian manifold without boundary, the closure $\overline{M}$ of M coincides with $\check{M}$, and S is bounded below by a positive constant.
Suppose $\partial M = \bigcup_{i=0}^{k} \Sigma_i$, and Σ_i is a compact C^2-submanifold of $\check{M}$ ($d_i = \dim \Sigma_i \leq n-2$) ($i = 1, \ldots, k$). If f satisfies

$$f \leq \begin{cases} -Cr_{\Sigma_0}^{-2} & \textit{when } i = 0 \\ -Cr_{\Sigma_i}^{-2}(\log(r_{\Sigma_i}^{-1}))^{-(q-1)} & \textit{when } d_i = n-2 \\ -Cr_{\Sigma_i}^{-2-(q-1)(2-n+d)} & \textit{when } d_i < n-2 \end{cases}$$

near Σ_i for a positive constant C, then the equation $(**)$ does not possess a positive (sub-)solution.

The expected Fact 1.3(3) for the scalar curvature equation $(*)$ is a special case of this theorem with the condition (b).

Recently, some nonexistence results were given for certain classes of complete Riemannian manifolds including $\mathbf{R}^n$ and $\mathbf{H}^n$, under curvature condition or volume growth condition ([17], [15], [19]). In particular, in [17] and [15], the orders between those of (1)–(2) and (3) in Facts 1.1 and 1.2 were considered. Also in our case, we have the following result. Here $\log^j$ means the j-times composition of the logarithmic function log.

THEOREM III. *Let (M,g), q, S and f be as in Theorem II with the condition (a) or (b). Suppose $\partial M = \bigsqcup_{i=1}^k \Sigma_i$ (disjoint union), and Σ_i is a compact C^2-submanifold of $\check{M}$ $(d_i = \dim \Sigma_i \leq n-1)$ $(i = 1, \dots, k)$. If f satisfies*

$$f \leq \begin{cases} -Cr_{\Sigma_i}^{-2}\prod_{j=1}^m(\log^j(r_{\Sigma_i}^{-1}))^{-1} & \textit{when } d_i = n-1 \\ -Cr_{\Sigma_i}^{-2}(\log(r_{\Sigma_i}^{-1}))^{-(q+1)}\prod_{j=2}^m(\log^j(r_{\Sigma_i}^{-1}))^{-1} & \textit{when } d_i = n-2 \\ -Cr_{\Sigma_i}^{-2-(q-1)(2-n+d)}\prod_{j=1}^m(\log^j(r_{\Sigma_i}^{-1}))^{-1} & \textit{when } d_i < n-2 \end{cases}$$

near Σ_i for a positive constant C, then the equation $(**)$ does not possess a positive (sub-)solution.

In the special case when $d_i \leq (n-2)/2$ for any i, there exists a complete scalar flat conformal metric on M. In this case, if $S \geq S_g(n-2)/4(n-1)$, then the assertion above can be shown also by using the transformation rule of the conformal Laplacian and combining [15, Theorem 4.1] and [10, Proposition 2, Theorem 2]. ([19, Theorem 3.1] also relates to this case.) On the other hand, [17, Theorem 4] treats the scalar curvature equation on complete Riemannian manifolds with negative scalar curvature which relates to the case when $d_i > (n-2)/2$ for any i. However, since it assumes the existence of a pole, our assertion cannot be derived from it. Moreover, our theorem considers mixed cases.

In Section 2, we recall and prepare some a priori upper estimates for subsolutions of the equation $(**)$, and prove Theorems I and II in Section 3. A proof of Theorem III is given in Section 4 with an improved estimate, and some applications to the scalar curvature equation $(*)$ is demonstrated in Section 5. In particular, we have the following uniqueness result for solutions of $(*)$.

THEOREM IV. *Let* (M, g), Σ *and* f *be as in Fact 1.3 with* $d = n - 2$. *Suppose* $f \sim -r_\Sigma^{-2}(\log(r_\Sigma^{-1}))^{-2n/(n-2)-\varepsilon}$ *near* Σ *for some* $\varepsilon > 0$, $f \leq 0$ *on* M, *and suppose* u *is a solution of* $(*)$. *If* $u/\log(r_\Sigma^{-1}) \to +\infty$ *as* $r_\Sigma \to 0$, *then* $u \sim (\log(r_\Sigma^{-1}))^{1+\varepsilon(n-2)/4}$ *near* Σ. *In particular,* u *coincides with the maximal solution of* $(*)$.

This corresponds to the results in the previous papers (see [8] for $d < n - 2$, [9] for $d = n - 1$).

2. A priori upper estimates for subsolutions

In this section, we observe a priori upper estimates for subsolutions.

By the same way as [16, Proposition 2.3], we can prove the following

LEMMA 2.1. *Let* (M, g), q, S *and* f *be as in Theorem I. Let* x *be a point in* M, *and* R *a positive number smaller than the injectivity radius of* $\overline{M}$. *Set* $R' := \min\{R, r_{\partial M}(x)\}$. *Then any nonnegative subsolution* u *of the equation* $(**)$ *satisfies the estimate*

$$u(x) \leq C_1 \inf_{0<r<R'} \left\{ \left(r^2 \min_{B_r(x)} |f| \right)^{-1/(q-1)} \right\},$$

where C_1 *is a positive constant which depends only on* n, q, R *and* $\max_{\overline{M}}|K_{\bar{g}}|$.

Here $B_r(x)$ denotes the r-neighborhood of x. We use notations of this type ($B_R(\partial M)$ etc.) throughout this paper.

We also use Harnack inequality of the following type (see e.g. [6, Theorem 8.20]):

LEMMA 2.2. *Let* (M, g) *be as in Theorem I,* $\check{S}$ *a nonnegative Hölder continuous function on* $\check{M}$, *and* φ *a positive solution of the equation* $-\Delta_g \varphi + \check{S}\varphi = 0$ *on* M. *Then there is a positive constant* C_2 *which satisfies* $\max_{\bar{B}_{r_{\partial M}(x)/5}(x)} \varphi \leq C_2 \varphi(x)$ *for any* $x \in M$ *close to* ∂M.

By using Lemmas 2.1 and 2.2, we have

LEMMA 2.3. *Let* (M, g), q, S *and* f *be as in Theorem I,* Σ *a connected component of* ∂M, R *as in Lemma 2.1,* $\check{S}$ *a nonnegative Hölder continuous function on* $\check{M}$ *such that* $\check{S} \leq S$ *on* M, *and* φ *a positive solution of the equation* $-\Delta_g \varphi + \check{S}\varphi = 0$ *on* M. *If* $f \leq -\psi(r_\Sigma)\varphi^{-(q-1)}$ *near* Σ *for a positive monotone function* ψ *on* $(0, 2R)$, *then, for* $C_3 := 4/5$ (*when* ψ *is nondecreasing*), $6/5$ (*when* ψ *is nonincreasing*), *any nonnegative subsolution* u *of the equation* $(**)$ *satisfies the estimate*

$$u \leq C_4 (r_\Sigma^2 \psi(C_3 r_\Sigma))^{-1/(q-1)} \varphi \quad \textit{near } \Sigma$$

for a positive constant C_4.

PROOF. Choose $r = r_\Sigma(x)/5$ in Lemma 2.1.

When ψ is nondecreasing, by using Lemma 2.2 for φ, we get the estimate

$$\begin{aligned} u(x) &\le C_1\left\{(r_\Sigma(x)/5)^2 \min_{\bar{B}_{r_\Sigma(x)/5}(x)} (\psi(r_\Sigma)\varphi^{-(q-1)})\right\}^{-1/(q-1)} \\ &\le C_1\{(r_\Sigma(x)/5)^2\psi(4r_\Sigma(x)/5)C_2^{-(q-1)}\varphi(x)^{-(q-1)}\}^{-1/(q-1)} \\ &\le 5^{2/(q-1)}C_1C_2\{r_\Sigma(x)^2\psi(4r_\Sigma(x)/5)\}^{-1/(q-1)}\varphi(x). \end{aligned}$$

When ψ is nonincreasing, we get the estimate

$$\begin{aligned} u(x) &\le C_1\left\{(r_\Sigma(x)/5)^2 \min_{\bar{B}_{r_\Sigma(x)/5}(x)} (\psi(r_\Sigma)\varphi^{-(q-1)})\right\}^{-1/(q-1)} \\ &\le C_1\{(r_\Sigma(x)/5)^2\psi(6r_\Sigma(x)/5)C_2^{-(q-1)}\varphi(x)^{-(q-1)}\}^{-1/(q-1)} \\ &\le 5^{2/(q-1)}C_1C_2\{r_\Sigma(x)^2\psi(6r_\Sigma(x)/5)\}^{-1/(q-1)}\varphi(x). \end{aligned}$$ q.e.d.

As a corollary to Lemma 2.3, we have the following a priori upper estimate which was partially obtained in [2], [7], [16] etc..

LEMMA 2.4. *Let (M,g), q, S, f, Σ, $\check{S}$ and φ be as in Lemma 2.3, If $f \le -C_5 r_\Sigma^\ell \varphi^{-(q-1)}$ near Σ for a positive constant C_5 and a real number ℓ, then any nonnegative subsolution u of the equation $(**)$ satisfies the estimate $u \le C_6 r_\Sigma^{-(\ell+2)/(q-1)}\varphi$ near Σ for a positive constant C_6.*

PROOF. By Lemma 2.3 with $\psi(t) = C_5 t^\ell$, we get the estimate

$$u \le C_4\{r_\Sigma^2 C_5(C_3 r_\Sigma)^\ell\}^{-1/(q-1)}\varphi \le C_4(C_5C_3^\ell)^{-1/(q-1)} r_\Sigma^{-(\ell+2)/(q-1)}\varphi.$$ q.e.d.

REMARK 2.5. Note here that $-(\ell+2)/(q-1) > 0$ for $\ell < -2$. Therefore, if $f \le -C_5 r_{\partial M}^\ell \varphi^{-(q-1)}$ for $\ell < -2$, then $u/\varphi \to 0$ as $r_{\partial M} \to 0$. Now, by the maximum principle, we have $u \le \varepsilon\varphi$ on M for any $\varepsilon > 0$, and consequently u must be equal to 0 on M. However, we want to show this for $\ell \le -2$, and hence we need to improve the estimate above.

3. Proofs of Theorems I and II

The following lemma is a key tool to prove our nonexistence results.

LEMMA 3.1. *Let (M,g), q, S and f be as in Theorem I, and R as in Lemma 2.1. Let φ be a positive C^2-function on $B_{2R}(\partial M) \cap M$ such that*

$$
\begin{cases} -\Delta_g \varphi + S\varphi \geq 0 \\ r_{\partial M}|\nabla_g \varphi| \leq C_7 \varphi \end{cases} \quad \text{on } B_{2R}(\partial M) \cap M \tag{3.1}
$$

for a positive constant C_7. Suppose f satisfies

$$
f \leq -C_8 r_{\partial M}^{-2} \varphi^{-(q-1)} \quad \text{on } B_{2R}(\partial M) \cap M
$$

for a positive constant C_8, Then, for any nonnegative subsolution u of the equation (**), if u satisfies the estimate $u \leq \gamma_0 \varphi$ on $B_R(\partial M) \cap M$ for a positive number γ_0, then it holds that $u/\varphi \to 0$ as $r_{\partial M} \to 0$.

Proof. First, note here that there exists a positive number β satisfying $|r_x \Delta_g r_x| \leq \beta$ on $B_R(x)$ for any $x \in \overline{M}$.

Now, we claim that for any $i \in \mathbf{N} \cup \{0\}$,

$$
u \leq \gamma_i \varphi \quad \text{on } B_{R/2^i}(\partial M) \cap M
$$

for a positive number γ_i given inductively by $\gamma_{i+1} := F^{-1}(\gamma_i)$, where

$$
F(\gamma) := \frac{C_8}{18(2C_7 + 1 + \beta)} \gamma^q + \gamma.
$$

By the assumption, this holds for $i = 0$. To see the claim, assume $u \leq \gamma_i \varphi$ on $B_{R/2^i}(\partial M) \cap M$ for some i. For any $x \in B_{R/2^{i+1}}(\partial M) \cap M$, set

$$
\begin{aligned}
u_x(y) := \{&4(\gamma_i - \gamma_{i+1}) r_{\partial M}(x)^{-2} r_x(y)^2 + \gamma_{i+1}\}\varphi(y) \\
&\text{for } y \in B_{r_{\partial M}(x)/2}(x) (\subset B_{3r_{\partial M}(x)/2}(\partial M) \cap M \subset B_{R/2^i}(\partial M) \cap M).
\end{aligned}
$$

By direct computation, we have

$$
\begin{aligned}
\Delta_g u_x(y) - S u_x(y) &\leq 8(\gamma_i - \gamma_{i+1}) r_{\partial M}(x)^{-2} \{2 r_x(y) \nabla_g r_x(y) \cdot \nabla_g \varphi(y) \\
&\qquad + (1 + r_x(y) \Delta_g r_x(y)) \varphi(y)\} \\
&\leq 8(\gamma_i - \gamma_{i+1}) r_{\partial M}(x)^{-2} \{2 r_{\partial M}(y) \cdot 1 \cdot |\nabla_g \varphi(y)| + (1 + \beta) \varphi(y)\} \\
&\leq 8(\gamma_i - \gamma_{i+1}) r_{\partial M}(x)^{-2} (2C_7 + 1 + \beta) \varphi(y) \\
&\leq 8(\gamma_i - \gamma_{i+1}) r_{\partial M}(x)^{-2} (2C_7 + 1 + \beta) \varphi(y) \{u_x(y)(\gamma_{i+1}\varphi(y))^{-1}\}^q \\
&= 8\{F(\gamma_{i+1}) - \gamma_{i+1}\} r_{\partial M}(x)^{-2} (2C_7 + 1 + \beta) \gamma_{i+1}^{-q} \varphi(y)^{-(q-1)} u_x(y)^q \\
&= C_8 (3 r_{\partial M}(x)/2)^{-2} \varphi(y)^{-(q-1)} u_x(y)^q \\
&\leq C_8 r_{\partial M}(y)^{-2} \varphi(y)^{-(q-1)} u_x(y)^q \\
&\leq |f(y)| u_x(y)^q = -f(y) u_x(y)^q \quad \text{for } y \in B_{r_{\partial M}(x)/2}(x).
\end{aligned}
$$

It is clear that

$$u_x(y) = \gamma_i \varphi(y) \geq u(y) \quad \text{for } y \in \partial B_{r_{\partial M}(x)/2}(x).$$

Therefore, by the maximum principle, $u_x(y) \geq u(y)$ for $y \in B_{r_{\partial M}(x)/2}(x)$. In particular, we have

$$u(x) \leq u_x(x) = \gamma_{i+1} \varphi(x).$$

Now, we get

$$u \leq \gamma_{i+1} \varphi \quad \text{on } B_{R/2^{i+1}}(\partial M) \cap M,$$

and the claim has been proved.

It is easy to see from the definition that the sequence $\{\gamma_i\}_{i \in \mathbf{N}}$ is strictly decleasing. Since F is continuous, we have $\lim_{i \to +\infty} \gamma_i = 0$. Therefore u/φ tends uniformly to 0 as $r_{\partial M} \to 0$. q.e.d.

To prove Theorem I, we also use the following interior estimates of derivatives (see e.g. [6, Theorem 6.2 (and Theorem 8.20)]).

LEMMA 3.2. *Let (M, g), $\check{S}$ and φ be as in Lemma 2.2. Then there is a positive constant C_7 which satisfies $r_{\partial M} |\nabla_g \varphi| \leq C_7 \varphi$ on M.*

PROOF OF THEOREM I. Clearly, it holds that

$$-\Delta_g \varphi + S\varphi \geq -\Delta_g \varphi + \check{S}\varphi = 0 \quad \text{on } M,$$

and, by Lemma 3.2, φ satisfies the condition (3.1). Furthermore, by Lemma 2.4, u satisfies $u \leq C_6 \varphi$ near ∂M. Hence, by Lemma 3.1, we get $u/\varphi \to 0$ as $r_{\partial M} \to 0$.

More precisely, for a positive number R small enough and any $i \in \mathbf{N}$, $u \leq \gamma_i \varphi$ on $B_{R/2^i}(\partial M) \cap M$, and

$$(\Delta_g - S)(\gamma_i \varphi - u) \leq 0 \quad \text{on } M.$$

Therefore, by the maximum principle, we have $u \leq \gamma_i \varphi$ on M. As $\lim_{i \to +\infty} \gamma_i = 0$, we get $u \equiv 0$ on M. This completes the proof. q.e.d.

Now, Theorem II follows from Theorem I and the following

LEMMA 3.3. *Let (M, g) and S be as in Theorem II with the condition (a) or (b). In the case of (a), set $\check{S} := 0$ (resp. In the case of (b), let $\check{S}$ be a positive constant such that $\check{S} \leq S$ on M). Suppose Σ is a compact C^2-submanifold of $\check{M}$ included in $\overline{M}$ $(d = \dim \Sigma \leq n - 2)$. Then there exists a positive solution G_Σ of the equation $-\Delta_g G_\Sigma + \check{S} G_\Sigma = 0$ on M which satisfies*

$$\left.\begin{matrix} C_9 \log (r_\Sigma^{-1}) \\ C_9 r_\Sigma^{2-n+d} \end{matrix}\right\} \le \quad G_\Sigma \quad \le \begin{cases} C_{10} \log(r_\Sigma^{-1}) & \textit{when } d = n-2 \\ C_{10} r_\Sigma^{2-n+d} & \textit{when } d < n-2 \end{cases}$$

$$C_{11} r_\Sigma^{1-n+d} \le |\nabla_g G_\Sigma| \le C_{12} r_\Sigma^{1-n+d}$$

near Σ for positive constants C_9, C_{10}, C_{11} and C_{12}.

Proof. In the case of (a), let $G(x,y)$ be the positive Dirichlet Green function of $-\Delta_g$ on a bounded domain Ω with smooth boundary such that $M \subset\subset \Omega \subset\subset \check{M}$ (resp. In the case of (b), let $G(x,y)$ be the Green function of $-\Delta_g + \check{S}$ on $\bar{M} = \check{M}$) (cf. [1, Chapter 4], [11, Theorem 2.8] etc.), and set $G_\Sigma(x) := \int_\Sigma G(x,y)\, d\sigma(y)$, where $d\sigma$ is the volume element of the induced metric on Σ. Then, by the same way as in the proof of [10, Proposition 2] (see also [18], [13], [5]), we get the estimates in the assertion above. q.e.d.

Proof of Theorem II. Let $\check{S}$ and G_{Σ_i} be as in Lemma 3.3, and set $\varphi := \sum_{i=1}^k G_{\Sigma_i}$. Then φ is also a positive solution of the equation $-\Delta_g \varphi + \check{S}\varphi = 0$ on M. By the definition, it is clear that $C_{13} := \min_M \varphi > 0$. For any $x \in M$ close to ∂M, choose an index i such that $r_{\partial M}(x) = r_{\Sigma_i}(x)$. By the assumption and Lemma 3.3, f satisfies

$$f(x) \le \begin{cases} -C r_{\Sigma_i}(x)^{-2} \le -C r_{\Sigma_i}(x)^{-2} (C_{13}^{-1} \varphi(x))^{-(q-1)} & \text{when } i = 0 \\ -C r_{\Sigma_i}(x)^{-2} (C_9^{-1} G_{\Sigma_i}(x))^{-(q-1)} & \text{when } i = 1, \dots, k \end{cases}$$

$$\le -C_{14} r_{\Sigma_i}(x)^{-2} \varphi(x)^{-(q-1)} = -C_{14} r_{\partial M}(x)^{-2} \varphi(x)^{-(q-1)},$$

where $C_{14} := C(\min\{C_9, C_{13}\})^{q-1}$. Now, by Theorem I, we get the assertion. q.e.d.

4. A proof of Theorem III

In this section, as we mentioned in Introduction, we treat orders expressed by the logarithmic function. From here, we denote the j times composition of log by $\log^j$. First, we see the following a priori estimate:

Lemma 4.1. *Let (M,g), q, S and f be as in Theorem I, and Σ a connected component of ∂M. If*

$$f \le -C_{15} r_\Sigma^{\ell_0} \prod_{j=1}^m (\log^j (r_\Sigma^{-1}))^{-\ell_j}$$

near Σ for a positive constant C_{15} and real numbers ℓ_j $(j = 0, 1, \dots, m)$, then any nonnegative subsolution u of the equation $(**)$ satisfies the estimate

$$u \leq C_{16} r_{\Sigma}^{-(\ell_0+2)/(q-1)} \prod_{j=1}^{m} (\log^j (r_{\Sigma}^{-1}))^{\ell_j/(q-1)}$$

near Σ for a positive constant C_{16}.

This lemma follows immediately from Lemma 2.3, since the function $t^{-\ell_0} \prod_{j=1}^{m} (\log^j t)^{-\ell_j}$ is monotone for t large enough. However, to improve Theorem II, we must show somewhat stronger estimate as follows:

LEMMA 4.2. *Let (M, g), q, S and f be as in Theorem I, and Σ a C^2-submanifold of $\check{M}$ ($d = \dim \Sigma$) which is a connected component of ∂M. If*

$$f \leq \begin{cases} -C_{17} r_{\Sigma}^{-2} \prod_{j=1}^{\mu-1} (\log^j (r_{\Sigma}^{-1}))^{-1} \prod_{j=\mu}^{m} (\log^j (r_{\Sigma}^{-1}))^{-\ell_j} & \text{when } d = n-1 \\ -C_{17} r_{\Sigma}^{-2} \prod_{j=1}^{m} (\log^j (r_{\Sigma}^{-1}))^{-\ell_j} & \text{when } d = n-2, \mu = 1 \\ -C_{17} r_{\Sigma}^{-2} (\log (r_{\Sigma}^{-1}))^{-(q+1)} \prod_{j=2}^{\mu-1} (\log^j (r_{\Sigma}^{-1}))^{-1} & \\ \quad \cdot \prod_{j=\mu}^{m} (\log^j (r_{\Sigma}^{-1}))^{-\ell_j} & \text{when } d = n-2, \mu \geq 2 \\ -C_{17} r_{\Sigma}^{-2-(q-1)(2-n+d)} \prod_{j=1}^{\mu-1} (\log^j (r_{\Sigma}^{-1}))^{-1} & \\ \quad \cdot \prod_{j=\mu}^{m} (\log^j (r_{\Sigma}^{-1}))^{-\ell_j} & \text{when } d < n-2 \end{cases}$$

near Σ for a positive constant C_{17}, a positive integer μ and real numbers ℓ_j ($j = \mu, \ldots, m$), then any nonnegative subsolution u of the equation $(**)$ satisfies the estimate

$$u \leq \begin{cases} C_{18} (\log^{\mu} (r_{\Sigma}^{-1}))^{(\ell_{\mu}-1)/(q-1)} \prod_{j=\mu+1}^{m} (\log^j (r_{\Sigma}^{-1}))^{\ell_j/(q-1)} & \text{when } d = n-1 \\ C_{18} (\log (r_{\Sigma}^{-1}))^{(\ell_1-2)/(q-1)} \prod_{j=2}^{m} (\log^j (r_{\Sigma}^{-1}))^{\ell_j/(q-1)} & \text{when } d = n-2, \mu = 1 \\ C_{18} \log (r_{\Sigma}^{-1}) (\log^{\mu} (r_{\Sigma}^{-1}))^{(\ell_{\mu}-1)/(q-1)} & \\ \quad \cdot \prod_{j=\mu+1}^{m} (\log^j (r_{\Sigma}^{-1}))^{\ell_j/(q-1)} & \text{when } d = n-2, \mu \geq 2 \\ C_{18} r_{\Sigma}^{2-n+d} (\log^{\mu} (r_{\Sigma}^{-1}))^{(\ell_{\mu}-1)/(q-1)} & \\ \quad \cdot \prod_{j=\mu+1}^{m} (\log^j (r_{\Sigma}^{-1}))^{\ell_j/(q-1)} & \text{when } d < n-2 \end{cases}$$

near Σ for a positive constant C_{18}.

To show this, we prepare the following

LEMMA 4.3. *Let (M, g), q, S and f be as in Theorem I, and Σ a connected component of ∂M. Let u be a nonnegative subsolution of the equation $(**)$. Suppose that there exists a positive supersolution ψ of the equation $(**)$ near Σ, and that there also exists a positive function v satisfying $-\Delta_g v + Sv \geq f \psi^{q-1} v$ near Σ, and $v/u \to +\infty$ as $r_{\Sigma} \to 0$. Then it holds that $u \leq C_{19} \psi$ near Σ for a positive constant C_{19}.*

Proof. Choose a positive number R such that all of our assumptions hold on $B_R(\Sigma) \cap M$. For any positive number ε, set $\psi_\varepsilon := \psi + \varepsilon v$. Then we have

$$\begin{aligned}
-\Delta_g \psi_\varepsilon + S\psi_\varepsilon &= -\Delta_g \psi + S\psi + \varepsilon(-\Delta_g v + Sv) \\
&\geq f\psi^q + \varepsilon f\psi^{q-1} v = f\psi^q(1 + \varepsilon v \psi^{-1}) \\
&\geq f\psi^q(1 + \varepsilon v\psi^{-1})^q = f\psi_\varepsilon^q,
\end{aligned}$$

namely, ψ_ε is a supersolution of the equation $(**)$ on $B_R(\Sigma) \cap M$. Set

$$C_{19} := \max\left\{1, \max_{\partial B_R(\Sigma)\cap M} (u\psi^{-1})\right\}.$$

Then we can easily show that $C_{19}\psi_\varepsilon$ is also a supersolution of the equation $(**)$ on $B_R(\Sigma) \cap M$. Now, by the definition of C_{19}, it is clear that $C_{19}\psi_\varepsilon \geq C_{19}\psi \geq u$ on $\partial B_R(\Sigma) \cap M$. On the other hand, since $C_{19}\psi_\varepsilon u^{-1} \geq C_{19}\varepsilon v u^{-1} \to \infty$ as $r_\Sigma \to 0$, it holds that $C_{19}\psi_\varepsilon \geq u$ on $B_{R'}(\Sigma) \cap M$ for a positive number $R' \leq R$. Hence, by the maximum principle, we have $C_{19}\psi_\varepsilon \geq u$ on $B_R(\Sigma) \cap M$, and we get $u \leq \lim_{\varepsilon \to +0}(C_{19}\psi_\varepsilon) = C_{19}\psi$ on $B_R(\Sigma) \cap M$. q.e.d.

For convenience, we denote

$$h(\alpha; t) = h(\alpha_0, \alpha_1, \dots, \alpha_m; t) := t^{\alpha_0} \prod_{j=1}^{m} (\log^j t)^{\alpha_j} \quad \text{on } \mathbf{R}^{m+1} \times (T, +\infty),$$

where m is a positive integer, and $T := 1$ $(m = 1)$, $\exp^{m-2} e$ $(m \geq 2)$. Clearly, it holds that

$$h(a\alpha + a'\alpha'; t) = h(\alpha; t)^a h(\alpha'; t)^{a'} \quad \text{for any } a, a' \in \mathbf{R}, \alpha, \alpha' \in \mathbf{R}^{m+1},$$

and that, if $\alpha_j = \alpha'_j$ $(j = 0, \dots, \mu - 1)$ and $\alpha_\mu > \alpha'_\mu$, then $h(\alpha; t) > h(\alpha'; t)$ for t large enough. Moreover, by direct computation, we have

$$h_1(\alpha; t) := \frac{\frac{dh}{dt}(\alpha; t)}{h(\alpha; t)} = \sum_{j=0}^{m} \alpha_j h(\underset{0}{\overset{\frown}{-1}}, \dots, \underset{j}{\overset{\frown}{-1}}, 0, \dots, \underset{m}{\overset{\frown}{0}}; t), \tag{4.1}$$

$$\begin{aligned}
h_2(\alpha; t) &:= \frac{\frac{d^2h}{dt^2}(\alpha; t)}{h(\alpha; t)} \\
&= \sum_{j,J=0; j<J}^{m} \alpha_J(2\alpha_j - 1) h(\underset{0}{\overset{\frown}{-2}}, \dots, \underset{j}{\overset{\frown}{-2}}, -1, \dots, \underset{J}{\overset{\frown}{-1}}, 0, \dots, \underset{m}{\overset{\frown}{0}}; t) \\
&\quad + \sum_{j=0}^{m} \alpha_j(\alpha_j - 1) h(\underset{0}{\overset{\frown}{-2}}, \dots, \underset{j}{\overset{\frown}{-2}}, 0, \dots, \underset{m}{\overset{\frown}{0}}; t).
\end{aligned} \tag{4.2}$$

PROOF OF LEMMA 4.2. Fix a positive integer m. We prove our assertion by the induction for μ. In this proof, $C_{20},\dots,C_{24}$ denote positive constants depending only on (M,g), q, S and f.

(THE CASE $d=n-1$) By the assumption, f satisfies

$$f \leq -C_{17}h(\underset{0}{\frown}{2},-1,\dots,-1,\underset{\mu}{\frown}{-\ell_\mu},-\ell_{\mu+1},\dots,\underset{m}{\frown}{-\ell_m};r_\Sigma^{-1}).$$

For any real number δ, define

$$w_\delta := h\left(\underset{0}{0},0,\dots,0,\underset{\mu}{\frac{\ell_\mu+\delta}{q-1}},\frac{\ell_{\mu+1}}{q-1},\dots,\underset{m}{\frac{\ell_m}{q-1}};r_\Sigma^{-1}\right).$$

When $\mu=1$, by Lemma 4.1 (resp. When $\mu\geq 2$, by the estimate for $\mu-1$), we have $u\leq C_{20}w_0$ near Σ. Set $\psi:=\gamma w_{-1}$ and $v:=w_1$. Then

$$|f|\psi^{q-1} \geq \gamma^{q-1}C_{17}h(\underset{0}{2},-1,\dots,-1,\underset{\mu}{-1},0,\dots,\underset{m}{0};r_\Sigma^{-1}).$$

On the other hand, by direct computation,

$$\psi^{-1}(\varDelta_g\psi-S\psi)=r_\Sigma^{-4}h_2(\dots;r_\Sigma^{-1})+r_\Sigma^{-3}(2-r_\Sigma\varDelta_g r_\Sigma)h_1(\dots;r_\Sigma^{-1})-S,$$

and, by (4.1) and (4.2), we have

$$\psi^{-1}(\varDelta_g\psi-S\psi)\leq C_{21}h(\underset{0}{2},-1,\dots,-1,\underset{\mu}{-1},0,\dots,\underset{m}{0};r_\Sigma^{-1})$$

near Σ. By the same way, we also have the same estimate for v. Now, if we choose $\gamma\geq(C_{21}/C_{17})^{1/(q-1)}$, then ψ and v satisfy $\psi^{-1}(\varDelta_g\psi-S\psi)\leq|f|\psi^{q-1}$ and $v^{-1}(\varDelta_g v-Sv)\leq|f|\psi^{q-1}$. Since $v/u\geq w_1/C_{20}w_0\to+\infty$ as $r_\Sigma\to 0$, by Lemma 4.3, we get $u\leq C_{19}\psi$ near Σ.

(THE CASE $d<n-2$) By Lemma 3.3, we have $G_\Sigma\sim r_\Sigma^{2-n+d}$. Hence our assumption can be rewritten as follows:

$$f\leq -C_{22}r_\Sigma^{-2}h(\underset{0}{-(q-1)},-1,\dots,-1,\underset{\mu}{-\ell_\mu},-\ell_{\mu+1},\dots,\underset{m}{-\ell_m};G_\Sigma).$$

For any δ, define

$$w_\delta := h\Big(\underset{0}{\overset{\frown}{1}}, 0, \dots, 0, \underset{\mu}{\overset{\frown}{\frac{\ell_\mu+\delta}{q-1}}}, \frac{\ell_{\mu+1}}{q-1}, \dots, \underset{m}{\overset{\frown}{\frac{\ell_m}{q-1}}}; G_\Sigma\Big).$$

When $\mu = 1$, by Lemma 4.1 (resp. When $\mu \geq 2$, by the estimate for $\mu - 1$), we have $u \leq C_{23} w_0$ near Σ. Set $\psi := \gamma w_{-1}$ and $v := w_1$. Then, by direct computation,

$$\psi^{-1}(\Delta_g \psi - S\psi) = |\nabla_g G_\Sigma|^2 h_2(\dots; G_\Sigma) + \Delta_g G_\Sigma h_1(\dots; G_\Sigma) - S.$$

If we choose γ large enough as before, then, by Lemma 3.3, (4.1) and (4.2), we can show

$$\begin{aligned}\psi^{-1}(\Delta_g \psi - S\psi) &\leq C_{24} r_\Sigma^{-2} h(\underset{0}{\overset{\frown}{0}}, -1, \dots, -1, \underset{\mu}{\overset{\frown}{-1}}, 0, \dots, \underset{m}{\overset{\frown}{0}}; G_\Sigma)\\ &\leq \gamma^{q-1} C_{22} r_\Sigma^{-2} h(\underset{0}{\overset{\frown}{0}}, -1, \dots, -1, \underset{\mu}{\overset{\frown}{-1}}, 0, \dots, \underset{m}{\overset{\frown}{0}}; G_\Sigma)\\ &\leq |f| \psi^{q-1}\end{aligned}$$

near Σ, and also $v^{-1}(\Delta_g v - Sv) \leq |f|\psi^{q-1}$. Hence, by Lemma 4.3, we get $u \leq C_{19}\psi$ near Σ.

(The case $d = n-2$ and $\mu = 1$) By Lemma 3.3, we have $G_\Sigma \sim \log(r_\Sigma^{-1})$. Hence our assumption can be rewritten as follows:

$$f \leq -C_{22} r_\Sigma^{-2} h(\underset{0}{\overset{\frown}{-\ell_1}}, -\ell_2, \dots, \underset{m-1}{\overset{\frown}{-\ell_m}}; G_\Sigma).$$

Define

$$w_\delta := h\Big(\underset{0}{\overset{\frown}{\frac{\ell_1+\delta}{q-1}}}, \frac{\ell_2}{q-1}, \dots, \underset{m-1}{\overset{\frown}{\frac{\ell_m}{q-1}}}; G_\Sigma\Big).$$

By Lemma 4.1, we have $u \leq C_{23} w_0$ near Σ. Set $\psi := \gamma w_{-2}$ and $v := w_1$. If we choose γ large as before, then, by the same way as for $d < n-2$, we have

$$\psi^{-1}(\Delta_g \psi - S\psi) \leq C_{24} r_\Sigma^{-2} h(\underset{0}{\overset{\frown}{-2}}, 0, \dots, \underset{m-1}{\overset{\frown}{0}}; G_\Sigma) \leq |f|\psi^{q-1}$$

near Σ, and also $v^{-1}(\Delta_g v - Sv) \leq |f|\psi^{q-1}$. Hence we get $u \leq C_{19}\psi$ near Σ.

(The case $d = n-2$ and $\mu \geq 2$) By the same reason as in the case $\mu = 1$, our assumption can be rewritten as follows:

$$f \le -C_{22} r_\Sigma^{-2} h(\underset{0}{\frown}{-(q+1)}, -1, \dots, -1, \underset{\mu-1}{-\ell_\mu}, -\ell_{\mu+1}, \dots, \underset{m-1}{-\ell_m}; G_\Sigma).$$

Define

$$w_\delta := h\left(\underset{0}{1}, 0, \dots, 0, \underset{\mu-1}{\frac{\ell_\mu + \delta}{q-1}}, \frac{\ell_{\mu+1}}{q-1}, \dots, \underset{m-1}{\frac{\ell_m}{q-1}}; G_\Sigma\right).$$

By the estimate for $\mu - 1$, we have $u \le C_{23} w_0$ near Σ. Set $\psi := \gamma w_{-1}$ and $v := w_1$. If we choose γ large, then we have

$$\psi^{-1}(\Delta_g \psi - S\psi) \le C_{24} r_\Sigma^{-2} h(\underset{0}{-2}, -1, \dots, -1, \underset{\mu-1}{-1}, 0, \dots, \underset{m-1}{0}; G_\Sigma) \le |f|\psi^{q-1}$$

near Σ, and also $v^{-1}(\Delta_g v - Sv) \le |f|\psi^{q-1}$. Hence we get $u \le C_{19}\psi$ near Σ. This completes the proof. q.e.d.

Now, by using Lemma 4.2, we can show the assertion of Theorem III.

PROOF OF THEOREM III. Let u be a nonnegative subsolution of the equation $(**)$. If $d_i = n-1$, then

$$f \le -C r_{\Sigma_i}^{-2} \prod_{j=1}^{m} (\log^j (r_{\Sigma_i}^{-1}))^{-1} = -C r_{\Sigma_i}^{-2} \prod_{j=1}^{m} (\log^j (r_{\Sigma_i}^{-1}))^{-1} \cdot (\log^{m+1}(r_{\Sigma_i}^{-1}))^{-0}.$$

By Lemma 4.2, we have $u \le C_{18}(\log^{m+1}(r_{\Sigma_i}^{-1}))^{-1/(q-1)}$ near Σ_i. Similarly, if $d_i \le n-2$, then we have $u \le C_{18} C_9^{-1} G_{\Sigma_i} (\log^{m+1}(r_{\Sigma_i}^{-1}))^{-1/(q-1)}$ near Σ_i, where G_{Σ_i} is as in Lemma 3.3. Set $\varphi := 1 + \Sigma_{i; d_i \le n-2} G_{\Sigma_i}$. Then φ is a positive function which satisfies $-\Delta_g \varphi + S\varphi \ge 0$ on M, and $u/\varphi \to 0$ as $r_{\partial M} \to 0$. Now, by using the maximum principle as in the proof of Theorem I, we have $u \equiv 0$. q.e.d.

5. Remarks on the scalar curvature equation

In the rest of this paper, we remark on some applications of our results to the scalar curvature equation $(*)$.

By Lemma 4.2 with $q = (n+2)/(n-2)$, $k = 1$ and $\mu = m = 1$, we immediately get the following

PROPOSITION 5.1. *Let (M, g), Σ and f be as in Fact 1.3 with $d \le n-1$. If*

$$f \le \begin{cases} -C_{25} r_\Sigma^{-2} (\log(r_\Sigma^{-1}))^{-\ell} & \text{when } d \ge n-2 \\ -C_{25} r_\Sigma^{2-4d/(n-2)} (\log(r_\Sigma^{-1}))^{-\ell} & \text{when } d < n-2 \end{cases}$$

near Σ for some $C_{25} > 0$ and ℓ, then any solution u of the scalar curvature equation $(*)$ *satisfies*

$$u \le \begin{cases} C_{26}(\log(r_\Sigma^{-1}))^{(\ell-1)(n-2)/4} & \text{when } d = n-1 \\ C_{26}(\log(r_\Sigma^{-1}))^{(\ell-2)(n-2)/4} & \text{when } d = n-2 \\ C_{26} r_\Sigma^{2-n+d}(\log(r_\Sigma^{-1}))^{(\ell-1)(n-2)/4} & \text{when } d < n-2 \end{cases}$$

near Σ for some $C_{26} > 0$.

We also have

Corollary 5.2. *Let (M, g), Σ and f be as in Proposition 5.1. If*

$$f \le \begin{cases} -C_{27} r_\Sigma^{-2}(\log(r_\Sigma^{-1}))^{-1} & \text{when } d = n-1 \\ -C_{27} r_\Sigma^{-2}(\log(r_\Sigma^{-1}))^{-2n/(n-2)} & \text{when } d = n-2 \\ -C_{27} r_\Sigma^{2-4d/(n-2)}(\log(r_\Sigma^{-1}))^{-1} & \text{when } d < n-2 \end{cases}$$

near Σ for some $C_{27} > 0$, then $(*)$ *has no solutions.*

When $d < n-2$, this improves Fact 1.3(3).

On the other hand, for instance, in the case when $d = n-2$, we can show the following fact by the same way as the proofs of [10, Theorems 2 and 3] and [7, Theorem IV].

Fact 5.3. *Let (M, g), Σ and f be as in Fact 1.3 with $d = n-2$. Then the following assertions hold:*

(1) *If $|f| \le C_{28} r_\Sigma^{-2}(\log(r_\Sigma^{-1}))^{-2n/(n-2)-\varepsilon}$ near Σ for some $C_{28} > 0$ and $\varepsilon > 0$, then* $(*)$ *has infinitely many solutions u satisfying $u \sim \log(r_\Sigma^{-1})$ near Σ;*

(2) *If $-C_{29} r_\Sigma^{-2}(\log(r_\Sigma^{-1}))^{-2n/(n-2)-\varepsilon} \le f < 0$ near Σ for some $C_{29} > 0$ and $\varepsilon > 0$, and $f \le 0$ on M, then* $(*)$ *has a solution u satisfying $u \ge C_{30}(\log(r_\Sigma^{-1}))^{1+\varepsilon(n-2)/4}$ near Σ for some $C_{30} > 0$.*

From this, the assumption of Corollary 5.2 is sharp in a sense. To compare the conditions in Theorem III (and Corollary 5.2) with some more general conditions for existence, see [10, Theorem 2′].

Furthermore, by applying the proof of [8, Theorem 2] to the case when $d = n-2$, we also get the uniqueness result described as Theorem IV in Introduction.

Proof of Theorem IV. By Proposition 5.1, we have $u \le C_{26}(\log(r_\Sigma^{-1}))^{1+\varepsilon(n-2)/4}$.

On the other hand, when $0 < \varepsilon < 1$, by the same way as the proof of [10, Theorem 2 (a)], the equation $(*)$ possesses a solution u_1 satisfying

$C_{31}G_\Sigma(1 - C_{32}G_\Sigma^{-\varepsilon}) \le u_1 \le C_{31}G_\Sigma$ for some $C_{31} > 0$ and $C_{32} > 0$. Since $1 - u_1/C_{31}G_\Sigma \le C_{32}G_\Sigma^{-\varepsilon} \le C_{32}C_9^{-\varepsilon}(\log(r_\Sigma^{-1}))^{-\varepsilon}$, by [8, Lemma 3.1], we have $u \ge C_{33}(\log(r_\Sigma^{-1}))^{1+\varepsilon(n-2)/4}$ for some $C_{33} > 0$.

When $\varepsilon \ge 1$, choose a positive number $\theta < \varepsilon^{-1}$. Then, by the same way as the proof of [8, Theorem 2], $u^\theta G_\Sigma^{1-\theta} \ge C_{33}(\log(r_\Sigma^{-1}))^{1+\theta\varepsilon(n-2)/4}$. Hence $u \ge C_{33}^{1/\theta}C_{10}^{1-1/\theta}(\log(r_\Sigma^{-1}))^{1+\varepsilon(n-2)/4}$.

Therefore, in both cases, we have $u \sim (\log(r_\Sigma^{-1}))^{1+\varepsilon(n-2)/4}$. Now, by [8, Theorem 1], u coincides with the maximal solution of $(*)$. q.e.d.

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