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Representation of α-harmonic functions in Lipschitz domains

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Abstract. We give the Martin representation for nonnegative functions which are harmonic on Lipschitz domains with respect to symmetric stable processes.

1. Introduction

The purpose of the present paper is to derive the Martin representation of nonnegative functions which are α-harmonic on Lipschitz domains. In general any nonnegative function α-harmonic in a bounded domain decomposes into regular part and singular part [9]. Each part admits its Martin representation. The regular part is given by integrating the function outside the domain against the α-harmonic measures. In [3] we have seen that for Lipschitz domains the α-harmonic measures have jointly continuous densities with respect to the Lebesgue measure. In the present paper, by using the boundary Harnack principle proved in [3], we shall show that the singular part is given by integrating a family of singular α-harmonic functions which are parameterized by the points at the boundary. Our restriction to Lipschitz domains simplifies the argument and yields an explicit definition of the family in terms of the Green function.

Our development is a standard one (cf. [1]). Nevertheless most of the arguments are substantially modified in comparison with its classical counterpart. We use this opportunity to illustrate the theory of Martin representation by a straightforward construction. Martin representation in the case of arbitrary bounded domains is the subject of [9]. Reader interested in the Martin representation in a general setting of Markov processes is referred to [7].

We first review the notation and a few results on α-harmonic functions following [3]. For the rest of the paper, let $\alpha \in (0,2)$ and $d \geq 2$. We denote by (X_t, P^x) the standard rotation invariant α-stable Lévy process (i.e. homogeneous and with independent increments) in $\mathbf{R}^d$ with the characteristic

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function

$$E^0 e^{i\xi X_t} = e^{-t|\xi|^\alpha}, \qquad \xi \in \mathbf{R}^d,\ t \geq 0.$$

As usual, E^x denotes the expectation with respect to the distribution P^x of the process starting from $x \in \mathbf{R}^d$. We assume, as we can, that sample paths of X_t are right-continuous and have left limits a.s. The process is Markov with transition probabilities given by $P_t(x, A) = P^x(X_t \in A) = \mu_t(A - x)$, where μ_t is the distribution of X_t with respect to P^0. It is well-known that (X_t, P^x) is strong Markov with respect to the so-called standard filtration, see e.g. [2] (also for another definition of X_t using subordination). The process has the generator $\Delta^{\alpha/2}$:

$$\Delta^{\alpha/2} u(x) = \mathscr{A}(d, -\alpha) \int_{\mathbf{R}^d} \frac{u(x+y) - u(x)}{|y|^{d+\alpha}}\, dy, \tag{1.1}$$

where $\mathscr{A}(d, \gamma) = \Gamma((d-\gamma)/2)/(2^\gamma \pi^{d/2} |\Gamma(\gamma/2)|)$ (cf. [8]). The limiting classical case $\alpha = 2$ corresponds to the Brownian motion with Laplacian $\Delta = \sum_{i=1}^d \partial_i^2$ as the generator. Needless to say, the integro-differential operator $\Delta^{\alpha/2}$ is not of local type. The Fourier transforms of the $\Delta^{\alpha/2}$ and Δ satisfy the equation

$$\mathscr{F}(-\Delta^{\alpha/2})(\xi) = |\xi|^\alpha = (\mathscr{F}(-\Delta)(\xi))^{\alpha/2}. \tag{1.2}$$

A proof of (1.2) can be found in [8]. The notation $\Delta^{\alpha/2}$ suggests the negative of the fractional power of $-\Delta$ ([10, IX.11]).

For a Borel set $A \subset \mathbf{R}^d$, we define $\tau_A = \inf\{t \geq 0 : X_t \in A^c\}$, the *first entrance time* of the complement A^c of A. It is well known that τ_A is a Markov time with respect to the standard filtration (cf. [2]). If A is bounded then $\tau_A < \infty$ *a.s.* Otherwise, in writing expressions like $E^x u(X_{\tau_A})$, we adopt the usual convention $E^x u(X_{\tau_A}) = E^x\{u(X_{\tau_A}); \tau_A < \infty\}$.

DEFINITION 1. *Let $u \geq 0$ be a Borel measurable function on $\mathbf{R}^d$. We say that u is* regular *α-harmonic in an open set $V \subset \mathbf{R}^d$ and write $u \in \mathscr{H}_R^\alpha(V)$ if*

$$u(x) = E^x u(X_{\tau_V}) < \infty, \qquad x \in V. \tag{1.3}$$

We say that u is α-harmonic in V and write $u \in \mathscr{H}^\alpha(V)$ if $u \in \mathscr{H}_R^\alpha(B)$ for every bounded open set B with the closure $\bar{B}$ contained in V i.e.

$$u(x) = E^x u(X_{\tau_B}) < \infty, \qquad x \in B. \tag{1.4}$$

We say that u is singular *α-harmonic in V and write $u \in \mathscr{H}_0^\alpha(V)$ if $u \in \mathscr{H}^\alpha(V)$ and $u(x) = 0$, $x \in V^c$.*

We have that $P^x(\tau_A = 0) = 1$ for every $x \in A^c$ with this definition of τ_A. Thus the equalities in (1.4) and (1.3) in fact hold on all of $\mathbf{R}^d$. Since $\tau_B \leq \tau_V$

for $B \subset V$, it follows by the strong Markov property that regular α-harmonic functions are α-harmonic. The converse is not generally true, as we shall shortly exhibit the existence of nontrivial singular α-harmonic functions. For $u \in \mathscr{H}_R^\alpha(V)$, the values $u(x), x \in V^c$, serve as the boundary condition. Obviously, the only regular α-harmonic function which is also singular is the zero constant. For $x \in \mathbf{R}^d$, the P^x distribution of X_{τ_V} is called *α-harmonic measure* and denoted by ω_V^x. Clearly, ω_V^x is concentrated on V^c and for a function $u \in \mathscr{H}_R^\alpha(V)$ we have

$$u(x) = \int_{V^c} u(y)\omega_V^x(dy), \qquad x \in V. \tag{1.5}$$

For $x \in V^c$, by $P^x(X_0 = x) = 1$, we have $\omega_V^x = \delta_x$, the Dirac measure in x. For $x \in \mathbf{R}^d$ and $r > 0$ we denote $B(x,r) = \{y \in \mathbf{R}^d : |x - y| < r\}$. In the case of the ball $B = B(0,r)$, $r > 0$, the α-harmonic measure $\omega_B^x(\cdot)$, has for $x \in B$ the density function $P_r(x,\cdot)$ explicitly given by the formula

$$P_r(x,y) = \begin{cases} c_\alpha^d (r^2 - |x|^2)^{\alpha/2}(|y|^2 - r^2)^{-\alpha/2}|x - y|^{-d}, & |y| > r, \\ 0, & |y| \le r, \end{cases} \tag{1.6}$$

where $c_\alpha^d = \Gamma(d/2)\pi^{-d/2-1}\sin(\pi\alpha/2)$ (see [8]). Thus, every $u \in \mathscr{H}_R^\alpha(B(\theta,r))$ satisfies the equality

$$u(x) = \int_{|y-\theta|>r} P_r(x - \theta, y - \theta)u(y)dy, \qquad x \in B(\theta,r). \tag{1.7}$$

The integral in (1.7) is an analogue of the classical Poisson integral for the ball. By (1.6) and (1.7), if $V \neq \varnothing$ is open then every function $u \in \mathscr{H}^\alpha(V)$ is smooth on V and satisfies the condition

$$\int_{\mathbf{R}^d} u(y)(1 + |y|)^{-d-\alpha}dy < \infty.$$

Unlike for the Brownian motion, the α-harmonic measure ω_V^x is typically supported on all of V^c for $x \in V$, which reflects the existence of jumps of sample paths. The fact follows from (1.6) and an obvious inequality $\omega_V^x \ge \omega_B^x$ on V^c, where B is a ball with $x \in B \subset V$. Another consequence of (1.7) and (1.6) is the usual Harnack inequality. We shall use the following version of it (see [3, Lemma 2]).

Lemma 1. *Let $x_1, x_2 \in \mathbf{R}^d, r > 0$ and $k \in \mathbf{N}$ with $|x_1 - x_2| < 2^k r$. If $u \in \mathscr{H}^\alpha(B(x_1,r) \cup B(x_2,r))$ then*

$$J^{-1}2^{-k(d+\alpha)}u(x_2) \le u(x_1) \le J2^{k(d+\alpha)}u(x_2), \tag{1.8}$$

with a constant $J = J(d,\alpha)$.

The notation $C = C(x, y, z)$ means that the *constant* C depends *only* on x, y, z. Constants are always numbers in $(0, \infty)$.

As in the classical case ($\alpha = 2$) the notion of α-harmonicity can be also formulated in terms of the generator. Namely, as stated in [3], a nonnegative Borel measurable function u defined on all of $\mathbf{R}^d$ and C^2 in an open set V is α-harmonic in V if and only if $\Delta^{\alpha/2}u(x) = 0$, $x \in V$. In particular, these conditions are satisfied if for every $\theta \in V$ there is $r > 0$ such that (1.7) holds.

For the rest of the paper let D be a Lipschitz domain with the *localization radius* R_0 and the *Lipschitz constant* λ. It means that D is a *bounded* open set and for each $Q \in \partial D$ there are a Lipschitz function $\Gamma_Q : \mathbf{R}^{d-1} \to \mathbf{R}$ with the Lipschitz constant not greater than λ and an orthonormal coordinate system $y = (y_1, y_2, \ldots, y_d)$ such that

$$D \cap B(Q, R_0) = \{y : y_d > \Gamma_Q(y_1, y_2, \ldots, y_{d-1})\} \cap B(Q, R_0).$$

Lipschitz domains have the following simple geometric property.

There exists a constant $\kappa = \kappa(\lambda) \in (0, 1)$ such that for every $r \in (0, R_0)$ and $Q \in \partial D$, there is at least one point $A = A_r(Q)$ such that $B(A, \kappa r) \subset D \cap B(Q, r)$.

For Lipschitz domains α-harmonic measures have nice density functions ([3, Lemma 6]):

LEMMA 2. *The α-harmonic measure ω_D^x is concentrated on* $\operatorname{int} D^c$ *and is absolutely continuous with respect to the Lebesgue measure on D^c. There exists a density function $P(x, y)$ (the Poisson kernel) which is continuous in $(x, y) \in D \times \operatorname{int} D^c$.*

As usual, $\operatorname{int} D^c$ above denotes the interior of D^c.

It will be convenient to fix an increasing sequence of open sets $D_n \subset D$ such that $\bigcup_{n=1}^{\infty} D_n = D$ and each $\overline{D_n}$ is a compact subset of D. Since $\tau_D < \infty$ *a.s.*, we have that

$$\tau_{D_n} \uparrow \tau_D \quad a.s. \qquad \text{as } n \to \infty. \tag{1.9}$$

Moreover, the fact that ω_D^x does not charge ∂D for $x \in D$ has the following consequence:

$$\lim_{n \to \infty} P^x\{X_{\tau_{D_n}} = X_{\tau_D}\} = 1, \qquad x \in D \tag{1.10}$$

(see [3, (5.40)]). In the later discussion we need a specific choice for the sequence $\{D_n\}$. We let

$$D_n = \{x \in D : \operatorname{dist}(x, D^c) > 1/n\}, \qquad n = 1, 2, \ldots, \tag{1.11}$$

where dist denotes the Euclidean distance in $\mathbf{R}^d$.

The potential operator U_α of the process X_t is given by the Riesz kernel of order α. Namely, for nonnegative Borel measurable functions f on $\mathbf{R}^d$

$$U_\alpha f(x) = E^x \int_0^\infty f(X_t)dt = \mathscr{A}(d,\alpha) \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy, \qquad x \in \mathbf{R}^d.$$

Given a measure μ on $\mathbf{R}^d$, we let U_α^μ be its Riesz potential,

$$U_\alpha^\mu(x) = \mathscr{A}(d,\alpha) \int_{\mathbf{R}^d} \frac{d\mu(y)}{|x-y|^{d-\alpha}}, \qquad x \in \mathbf{R}^d. \tag{1.12}$$

We recall the definition of the Green function of the domain D:

$$G(x,y) = U_\alpha^{\delta_x}(y) - U_\alpha^{\omega_D^x}(y), \qquad x, y \in \mathbf{R}^d \tag{1.13}$$

(we have $G(x,x) = \infty$ if $x \in D$ and we put $G(x,x) = 0$ if $x \in D^c$), see [8]. For nonnegative Borel measurable functions f

$$E^x \int_0^{\tau_D} f(X_t)\, dt = \int_D f(y) G(x,y) dy, \qquad x \in \mathbf{R}^d. \tag{1.14}$$

We clearly have

$$G(x,y) \le \mathscr{A}(d,\alpha)|x-y|^{\alpha-d}, \qquad x, y \in \mathbf{R}^d. \tag{1.15}$$

It is well-known that $G(x,y) > 0$ on D. Also, $G(x,y) = G(y,x)$, $x, y \in \mathbf{R}^d$, (symmetry) and $G(x,y) = 0$ provided $x \in D^c$ or $y \in D^c$, see [8], [6]. For each $y \in D$, $G(x,y)$ is α-harmonic in $x \in D \setminus \{y\}$ and regular α-harmonic in $D \setminus B(y,r)$ for every $r > 0$. The same is of course true when the roles of x and y are interchanged. The next formula recovers the α-harmonic measure density from the Green function:

$$\begin{aligned} P(x,y) &= [\Delta^{\alpha/2} G(x,\cdot)](y) \\ &= \mathscr{A}(d,-\alpha) \int_{\mathbf{R}^d} \frac{G(x,v)}{|y-v|^{d+\alpha}} dv, \qquad x \in D,\ y \in \operatorname{int} D^c. \end{aligned} \tag{1.16}$$

Formula (1.16) follows from the connection between the Lévy measure and the harmonic measure established in [5] (see also [3, (4.32)]) and will be frequently used in the sequel.

The following lemma is a local, scaling-invariant version of the boundary Harnack principle for α-harmonic functions (BHP) compiled from [3, Lemma 16 and the proof of Theorem 1].

LEMMA 3 (BHP). *For all* $Q \in \partial D$, $r \in (0, R_0/2)$ *and functions* $u, v \in \mathscr{H}_R^\alpha(D \cap B(Q,2r))$ *which vanish on* $D^c \cap B(Q,2r)$ *and satisfy* $u(A_r(Q)) =$

$v(A_r(Q))>0$, the ratio $h(x)=u(x)/v(x)$ is Hölder continuous in $D\cap B(Q,r)$. In fact there exist $C_1=C_1(d,\alpha,\lambda)$ and $v=v(d,\alpha,\lambda)$ such that

$$|h(x)-h(y)|\le C_1(|x-y|/r)^{v},\qquad x,y\in D\cap B(Q,r). \tag{1.17}$$

In particular, there is a constant $C_2=C_2(d,\alpha,\lambda)$ such that

$$C_2^{-1}\le h(x)\le C_2,\qquad x\in D\cap B(Q,r) \tag{1.18}$$

and $\lim_{D\ni x\to Q'} h(x)$ exists for every $Q'\in\partial D\cap B(Q,r)$.

As for the decay rate of α-harmonic functions near the boundary ∂D, the next result provides useful absolute estimates. For a proof the reader is referred to [3, Lemma 3, Lemma 4 and Lemma 5].

Lemma 4. *There exist constants $C_3=C_3(d,\alpha,\lambda)$ and $\varepsilon=\varepsilon(d,\alpha,\lambda)$ such that for all $Q\in\partial D$, $r\in(0,R_0/2)$ and functions $u\in\mathscr{H}_R^\alpha(D\cap B(Q,2r))$ which vanish on $D^c\cap B(Q,2r)$ we have*

$$C_3^{-1}u(A)\rho(x)^{\alpha-\varepsilon}\le u(x)\le C_3u(A)\rho(x)^{\varepsilon},\qquad x\in D\cap B(Q,r), \tag{1.19}$$

where $\rho(x)=dist(x,D^c)/r$, $A=A_r(Q)$.

2. Martin representation

We keep denoting by D the generic Lipschitz domain in $\mathbf{R}^d$. By (1.5) and Lemma 2, every function $u\in\mathscr{H}_R^\alpha(D)$ has the representation

$$u(x)=\int_{D^c}P(x,y)u(y)\,dy,\qquad x\in D. \tag{2.20}$$

Let χ_{D^c} be the indicator function of D^c. We regard $u(y)\chi_{D^c}(y)dy$ as a representing measure, the kernel $P(x,y)$ being independent of u. The values of u on D^c are incorporated in the notion of α-harmonicity thus the representing measure is unique in (2.20). In this section we will reveal a representation valid for all the nonnegative α-harmonic functions.

Lemma 5. *Every function $u\in\mathscr{H}^\alpha(D)$ decomposes into the unique sum $u=r+s$, where $r\in\mathscr{H}_R^\alpha(D)$ and $s\in\mathscr{H}_0^\alpha(D)$.*

Proof. By the mean value property, for $x\in\mathbf{R}^d$ and D_n as in (1.11)

$$\begin{aligned}u(x)&=E^xu(X_{\tau_{D_n}})\\&=E^x\{u(X_{\tau_{D_n}});X_{\tau_{D_n}}\in D^c\}+E^x\{u(X_{\tau_{D_n}});X_{\tau_{D_n}}\in D\backslash D_n\}\\&=E^x\{u(X_{\tau_D});X_{\tau_{D_n}}\in D^c\}+E^x\{u(X_{\tau_{D_n}});X_{\tau_{D_n}}\in D\backslash D_n\}.\end{aligned}$$

Clearly the first term coincides with $u(x)$ if $x \in D^c$ while $\{X_{\tau_{D_n}} \in D^c\} \nearrow \{X_{\tau_D} \in D^c\}$ up to a P^x-null set by (1.10) if $x \in D$. Consequently $u(x) \geq E^x\{u(X_{\tau_D}); X_{\tau_{D_n}} \in D^c\} \nearrow E^x[u(X_{\tau_D})] = r(x)$ and $E^x\{u(X_{\tau_{D_n}}); X_{\tau_{D_n}} \in D \backslash D_n\}$ decreases to a limit, say $s(x)$, as $n \to \infty$. By definition $u(x) = r(x) + s(x)$. Clearly $r(x) = u(x)$ and $s(x) = 0$ if $x \in D^c$. Being finite, r is regular α-harmonic in D. If B is open relatively compact in D and $x \in B$ then

$$\begin{aligned} r(x) + s(x) = u(x) &= E^x u(X_{\tau_B}) \\ &= E^x r(X_{\tau_B}) + E^x s(X_{\tau_B}) = r(x) + E^x s(X_{\tau_B}). \end{aligned}$$

Subtracting $r(x) < \infty$, we see that s is (singular) α-harmonic in D. □

It is easy to verify that the composition of the regular part r with the process *stopped* on leaving the domain is a closed martingale. Such a composition with s yields a supermartingale with the expectation tending to 0.

We now fix a reference point $x_0 \in D$.

LEMMA 6. *For every $Q \in \partial D$ and $x \in \mathbf{R}^d$*

$$K(x, Q) = \lim_{D \ni \xi \to Q} \frac{G(x, \xi)}{G(x_0, \xi)} \tag{2.21}$$

exists. The mapping $(x, Q) \mapsto K(x, Q)$ is continuous on $D \times \partial D$. For every $Q \in \partial D$ the function $K(\cdot, Q)$ is singular α-harmonic in D with $K(x_0, Q) = 1$. If $Q, S \in \partial D$ and $Q \neq S$ then $K(x, Q) \to 0$ as $x \to S$.

PROOF. For each fixed $x \in D$, the existence of the limit in (2.21) is an immediate consequence of Lemma 3 trivially extended by relaxing the assumption $u(A_r(Q)) = v(A_r(Q))$. We now prove that $K(x, Q)$ is α-harmonic in $x \in D$. To this end we fix $Q \in \partial D$. Throughout the proof, we let $r \in (0, R_0/2)$ be so small that $|x_0 - Q| \geq 2r$ and we write $\xi_r = A_r(Q)$. With the conditions on r, Lemma 3 (extended as above) yields

$$C_2^{-1} \frac{G(y, \xi_r)}{G(x_0, \xi)} \leq \frac{G(y, \xi)}{G(x_0, \xi)} \leq C_2 \frac{G(y, \xi_r)}{G(x_0, \xi_r)}, \qquad \xi \in D \cap B(Q, r), \; y \in D \backslash B(Q, 2r). \tag{2.22}$$

Letting $\xi \to Q$ gives

$$C_2^{-1} \frac{G(y, \xi_r)}{G(x_0, \xi_r)} \leq K(y, Q) \leq C_2 \frac{G(y, \xi_r)}{G(x_0, \xi_r)}, \qquad y \in D \backslash B(Q, 2r). \tag{2.23}$$

Let $\theta \in D$ and $\rho = \mathrm{dist}(\theta, D^c)/3$. We fix $x \in B(\theta, \rho)$ and define $\sigma^x(dy) =$

$P_\rho(x-\theta, y-\theta)\,dy$. To prove α-harmonicity, it is enough to verify that

$$K(x,Q) = \int K(y,Q)\sigma^x(dy)$$

(see Introduction). By (1.7) and Fatou's lemma we only have

$$\begin{aligned} K(x,Q) &= \lim_{D\ni\xi\to Q} \frac{G(x,\xi)}{G(x_0,\xi)} \\ &= \lim_{D\ni\xi\to Q} \int \frac{G(y,\xi)}{G(x_0,\xi)}\sigma^x(dy) \\ &\geq \int K(y,Q)\sigma^x(dy). \end{aligned}$$

In particular, $K(\cdot\,,Q)$ is σ^x-integrable. In addition to the previous conditions on r, let $r \leq \rho/2$. For $y \in B(Q,2r)$ we now have $|y-\theta| \geq 2\rho$ and $P_\rho(x-\theta, y-\theta) \leq c_1 = c_1(d,\alpha,\rho)$. Therefore, using (1.15) and polar coordinates, we get

$$\begin{aligned} \int_{D\cap B(Q,2r)} \frac{G(y,\xi_r)}{G(x_0,\xi_r)}\,\sigma^x(dy) &\leq \frac{c_1}{G(x_0,\xi_r)} \int_{B(0,4r)} \frac{\mathscr{A}(d,\alpha)}{|z|^{d-\alpha}}\,dz \\ &= \frac{c_2}{G(x_0,\xi_r)} r^\alpha, \end{aligned}$$

with a constant $c_2 = c_2(d,\alpha,\rho)$. By Lemma 4, there is a constant $c_3 = c_3(d,\alpha,D,x_0)$ such that $G(x_0,\xi_r) \geq c_3 r^{\alpha-\varepsilon}$, and so

$$\int_{D\cap B(Q,2r)} \frac{G(y,\xi_r)}{G(x_0,\xi_r)}\,\sigma^x(dy) \leq c_2 c_3^{-1} r^\varepsilon \tag{2.24}$$

under the above restrictions on r. We recall that $K(\cdot\,,Q)$ is σ^x-integrable. By the left-hand side of (2.23) and by (2.24) the functions $G(\cdot\,,\xi_r)/G(x_0,\xi_r)$ are uniformly σ^x-integrable for all $r > 0$ small enough. Therefore

$$\begin{aligned} K(x,Q) &= \lim_{r\to 0^+} \int \frac{G(y,\xi_r)}{G(x_0,\xi_r)}\sigma^x(dy) \\ &= \int \lim_{r\to 0^+} G(y,\xi_r)/G(x_0,\xi_r)\sigma^x(dy) \\ &= \int K(y,Q)\sigma^x(dy). \end{aligned}$$

We have thus proved that $K(\cdot\,,Q)$ is (singular) α-harmonic in D. In particular $K(x,Q)$ is smooth in x for each fixed $Q \in \partial D$. Let $F \ni x_0$ be a compact subset of D. Let $U = \{\xi \in D : \operatorname{dist}(\xi, D^c) < \varepsilon\}$ where $\varepsilon = \min(R_0/2, \operatorname{dist}(F, D^c)/2)$. As the quotients $G(x,\xi)/G(x_0,\xi)$ equal 1 for $x = x_0$, by their α-harmonicity in x and Harnack inequality (1.8) there is a constant $c = c(\alpha, F, \varepsilon)$ such that

$$\frac{G(x,\xi)}{G(x_0,\xi)} \le c, \qquad x \in F,\ \xi \in U.$$

By Lemma 3 (extended as above) there are constants $c' = c'(d,\alpha,\lambda,\varepsilon)$ and $\nu = \nu(d,\alpha,\lambda)$ such that

$$\left|\frac{G(x,\xi_1)}{G(x_0,\xi_1)} - \frac{G(x,\xi_2)}{G(x_0,\xi_2)}\right| \le cc'|\xi_1 - \xi_2|^{\nu}, \qquad x \in F,\ \xi_1,\xi_2 \in U.$$

It follows that

$$|K(x,Q_1) - K(x,Q_2)| \le cc'|Q_1 - Q_2|^{\nu}, \qquad x \in F,\ Q_1,Q_2 \in \partial D.$$

Let $F \ni x \to y \in F$, $\partial D \ni Q \to R \in \partial D$. We have

$$\begin{aligned}|K(x,Q) - K(y,R)| &\le |K(x,Q) - K(x,R)| + |K(x,R) - K(y,R)| \\ &\le cc'|Q - R|^{\nu} + |K(x,R) - K(y,R)| \to 0.\end{aligned}$$

It follows that $K(\cdot,\cdot)$ is jointly continuous on $D \times \partial D$. The continuous decay of $K(x,Q)$ as $x \to S \in \partial D$, $S \neq Q$ follows from the right-hand side of (2.23) (see also Lemma 4). □

The argument proving α-harmonicity of $K(\cdot, Q)$ above differs from the Harnack convergence theorem which is used in the classical case ($\alpha = 2$). We do not have a direct analogue of the classical Harnack convergence theorem for $\alpha < 2$, see [9]. Also, the analogue of the function $K(\cdot, Q)$ for domains less regular than Lipschitz is not (singular) α-harmonic in general ([9]).

We are now ready to state and prove the main result of the paper.

THEOREM 1. *Let D be a Lipschitz domain in $\mathbf{R}^d$, $d \ge 2$. For every finite nonnegative Borel measure μ on ∂D, the function*

$$u(x) = \int_{\partial D} K(x,Q)\mu(dQ), \qquad x \in \mathbf{R}^d \tag{2.25}$$

is in $\mathscr{H}_0^{\alpha}(D)$ and $u(x_0) = \mu(\mathbf{R}^d)$. Conversely, for every function $u \in \mathscr{H}_0^{\alpha}(D)$ there is a unique finite nonnegative Borel measure μ on ∂D such that (2.25) *holds.*

PROOF. Since $K(x,Q) = 0$ if $x \in D^c$, for each $x \in \mathbf{R}^d, K(x,Q)$ is continuous in $Q \in \partial D$ by Lemma 6. Therefore the integral in (2.25) is well-defined, finite and nonnegative. By the joint continuity of $K(\cdot,\cdot), u(\cdot)$ is continuous (hence Borel measurable) in D. If B is open, relatively compact in D, by

Fubini–Tonelli and α-harmonicity of $K(\cdot,Q)$ we have

$$\begin{aligned} E^x u(X_{\tau_B}) &= \int u(y)\omega_B^x(dy) \\ &= \int_{\partial D}\int K(y,Q)\omega_B^x(dy)\mu(dQ) \\ &= \int_{\partial D} K(x,Q)\mu(dQ) = u(x), \qquad x \in \mathbf{R}^d. \end{aligned}$$

It follows that $u \in \mathscr{H}_0^\alpha(D)$. We now prove the uniqueness of the representation (2.25). Let $Q \in \partial D$ and D_n be as given by (1.11). We define $\nu_n^Q(dy) = K(y,Q)\omega_{D_n}^{x_0}(dy)$, $n = 1,2,\dots$ We have $\nu_n^Q \Rightarrow \delta_Q$ in the sense of weak convergence as $n \to \infty$. Indeed, for every $n, \nu_n^Q(\mathbf{R}^d) = \int K(y,Q)\omega_{D_n}^{x_0}(dy) = K(x_0,Q) = 1$. Also, for each ball $B = B(Q,r)$ with $r > 0$ we get

$$\begin{aligned} \nu_n^Q(B^c) &\le \omega_{D_n}^{x_0}(\mathbf{R}^d) \sup_{y \in D_n^c \setminus B} K(y,Q) \\ &= \sup_{y \in D_n^c \setminus B} K(y,Q) \to 0 \qquad \text{as } n \to \infty, \end{aligned}$$

where we have used Lemma 6. More generally, for $\nu_n(dy) = u(y)\omega_{D_n}^{x_0}(dy)$ with u satisfying (2.25) we have

$$\nu_n \Rightarrow \mu \qquad \text{as } n \to \infty. \tag{2.26}$$

Indeed, let ϕ be a bounded continuous function on $\mathbf{R}^d$. We may assume that $\phi \ge 0$. By Fubini–Tonelli theorem and bounded convergence theorem we get

$$\begin{aligned} \int \phi\, d\nu_n &= \int_{\partial D}\int \phi(y)K(y,Q)\omega_{D_n}^{x_0}(dy)\, d\mu(Q) \\ &= \int_{\partial D}\int \phi(y)\nu_n^Q(dy)\, d\mu(Q) \to \int_{\partial D} \phi(Q)\, d\mu(Q) \qquad \text{as } n \to \infty. \end{aligned}$$

Thus the function u determines the measure μ.

We now prove the existence part of the theorem. We denote by $G_n, P_{(n)}$ the Green function and the Poisson kernel of D_n, respectively. The existence of $P_{(n)}$, that is the absolute continuity of $\omega_{D_n}^x(\cdot)$ with respect to the Lebesgue measure (for $x \in D_n$), is here, at least for large n, a consequence of the Lipschitz character of D_n. Namely, if $n \ge n_0$, n_0 being an integer depending on D, the localization radii of the sets D_n may be chosen independent of n and their Lipschitz constants are not greater than λ. The verification of this geometric result is left to the reader.

Let $x \in D$ and let $n_1 \geq n_0$ be such that x, $x_0 \in D_{n_1}$. We use (1.16) and Fubini–Tonelli theorem in the following basic calculation ($n \geq n_1$):

$$u(x) = \int_{D_n^c} u(y) P_{(n)}(x, y) dy$$

$$= \int_{D_n^c} \int_{D_n} u(y) \mathscr{A}(d, -\alpha) G_n(x, \xi) |y - \xi|^{-d-\alpha} d\xi dy$$

$$= \int_{D_n} \frac{G_n(x, \xi)}{G_n(x_0, \xi)} \mu_n(d\xi), \tag{2.27}$$

where $\mu_n(d\xi) = G_n(x_0, \xi) \int_{D_n^c} u(y) \mathscr{A}(d, -\alpha) |y - \xi|^{-d-\alpha} dy\, d\xi$. We see that $\mu_n(\mathbf{R}^d) = u(x_0) < \infty$. For each compact $F \subset D$ and every $\xi \in F$, we get

$$\int_{D_n^c} u(y) \mathscr{A}(d, -\alpha) |y - \xi|^{-d-\alpha} dy$$

$$\leq \frac{\mathscr{A}(d, -\alpha)}{\operatorname{dist}(F, D_n^c)^{d+\alpha}} \int_{D_n^c} u(y)\, dy \to 0 \qquad \text{as } n \to \infty, \tag{2.28}$$

because u is integrable.

By (1.15) and (2.28) we have

$$\mu_n(F) \to 0 \qquad \text{as } n \to \infty. \tag{2.29}$$

The sequence being tight, some subsequence $\{\mu_{n_k}\}$ weakly converges to a measure μ supported on ∂D. Similarly to (2.29),

$$\int_F \frac{G_n(x, \xi)}{G_n(x_0, \xi)} \mu_n(d\xi) \to 0 \qquad \text{as } n \to \infty. \tag{2.30}$$

We now observe that

$$G_n(x, \xi) \uparrow G(x, \xi) \qquad \text{as } n \to \infty,\ \xi \in \mathbf{R}^d. \tag{2.31}$$

This is a consequence of (1.9), (1.14) and the continuity of the Green functions. By (2.31) and its analogue for x_0 we have

$$\frac{G_n(x, \xi)}{G_n(x_0, \xi)} \to \frac{G(x, \xi)}{G(x_0, \xi)}, \qquad \xi \in D, \quad \text{as } n \to \infty$$

(as usual if $x = x_0 = \xi \in D_n$ we let the quotients equal 1). Outside of D_{n_1}, by BHP, the functions $G_n(x, \xi)/G_n(x_0, \xi)$ are uniformly equicontinuous in ξ on their respective domains of definition. Therefore the convergence in (2.31) is uniform outside of D_{n_1} in the sense that for each $\varepsilon > 0$

$$\left| \frac{G_n(x, \xi)}{G_n(x_0, \xi)} - \frac{G(x, \xi)}{G(x_0, \xi)} \right| \leq \varepsilon, \qquad \xi \in D_n \backslash D_{n_1}, \tag{2.32}$$

for all n sufficiently large. With this ε and such n we have

$$\int_{D_{n_1}^c} \left| \frac{G(x,\xi)}{G(x_0,\xi)} - \frac{G_n(x,\xi)}{G_n(x_0,\xi)} \right| \mu_n(d\xi) \le \varepsilon \mu_n(\mathbf{R}^d) = \varepsilon u(x_0). \tag{2.33}$$

Now, by (2.30) with $F = \overline{D_{n_1}}$ it follows that

$$\begin{aligned} u(x) &= \lim_{k\to\infty} \int \frac{G_{n_k}(x,\xi)}{G_{n_k}(x_0,\xi)} \mu_{n_k}(d\xi) \\ &= \lim_{k\to\infty} \int_{D_{n_1}^c} \frac{G(x,\xi)}{G(x_0,\xi)} \mu_{n_k}(d\xi). \end{aligned}$$

By Lemma 6 the function $\xi \mapsto G(x,\xi)/G(x_0,\xi)$ extends continuously to $\bar{D}\backslash\{x\}$. Since $\operatorname{supp}\mu \subset \partial D$, we get

$$u(x) = \int_{\partial D} K(x,Q)\mu(dQ).$$

The subsequence $\{\mu_{n_k}\}$ was chosen independent of $x \in D$ and the existence of a representing measure is established. Actually, due to the uniqueness of such representation, we even have $\mu_n \Rightarrow \mu$ as $n \to \infty$. □

We also get the characterization of minimal singular α-harmonic functions on D. We call $f \in \mathscr{H}_0^\alpha(D)$ minimal if every $h \in \mathscr{H}_0^\alpha(D)$ dominated by f is a constant multiple of f. Indeed, the correspondence between u and μ in (2.25) is obviously linear and preserves the natural order i.e. $u_1 \le u_2$ if and only if $\mu_1 \le \mu_2$ (the "only if" part is a consequence of (2.26)). It follows that minimal singular α-harmonic functions f on D with $f(x_0) = 1$ are precisely the functions $\{K(\cdot,Q),\ Q \in \partial D\}$ corresponding to the Dirac measures on ∂D.

3. Examples

As before, D is a Lipschitz domain in $\mathbf{R}^d$, $x_0 \in D$ is fixed and $P(x,y) = \omega_D^x(dy)/dy$ is the Poisson kernel for D. It is convenient to have a formula for the kernel $K(\cdot,\cdot)$ in terms of the Poisson kernel.

Lemma 7. *For every $x \in D$ and $Q \in \partial D$*

$$K(x,Q) = \lim_{\operatorname{int} D^c \ni y \to Q} \frac{P(x,y)}{P(x_0,y)}. \tag{3.34}$$

Proof. Let $x \in D$ and $Q \in \partial D$. For $y \in \operatorname{int} D^c$ we have by (1.16)

$$P(x,y)/P(x_0,y) = \frac{\int G(x,v)|y-v|^{-d-\alpha}dv}{\int G(x_0,v)|y-v|^{-d-\alpha}dv}.$$

According to Lemma 4 there exist constants $c = c(d, \alpha, D, x_0)$ and $\varepsilon = \varepsilon(d, \alpha, \lambda)$ such that

$$G(x, v) \geq c|v - Q|^{\alpha-\varepsilon} \tag{3.35}$$

for v in an *inner cone* $\mathscr{C} \subset D$ with vertex in Q. By (1.16), Fatou's lemma and (3.35) we see that

$$\begin{aligned}\liminf_{\operatorname{int} D^c \ni y \to Q} P(x, y) &\geq \int \mathscr{A}(d, -\alpha) G(x, v) |Q - v|^{-d-\alpha} dv \\ &\geq c \int_{\mathscr{C}} \mathscr{A}(d, -\alpha) |Q - v|^{-d-\varepsilon} dv = \infty,\end{aligned}$$

hence $P(x, y) \to \infty$ as $\operatorname{int} D^c \ni y \to Q$. Let V be an arbitrary neighborhood of Q. If $y \to Q$ then

$$\int_{V^c} G(x, v)|y - v|^{-d-\alpha} dv \to \int_{V^c} G(x, v)|Q - v|^{-d-\alpha} dv < \infty,$$

hence we have

$$\begin{aligned}\limsup_{\operatorname{int} D^c \ni y \to Q} \frac{P(x, y)}{P(x_0, y)} &= \limsup_{\operatorname{int} D^c \ni y \to Q} \frac{\int_V G(x, v)|y - v|^{-d-\alpha} dv}{\int_V G(x_0, v)|y - v|^{-d-\alpha} dv} \\ &\leq \sup_{v \in V \cap D} \frac{G(x, v)}{G(x_0, v)}.\end{aligned}$$

Analogously,

$$\liminf_{\operatorname{int} D^c \ni y \to Q} \frac{P(x, y)}{P(x_0, y)} \geq \inf_{v \in V \cap D} \frac{G(x, v)}{G(x_0, v)}.$$

Letting V shrink to Q and using Lemma 6 we get (3.34). □

EXAMPLE 1. Let D be the ball $B(0, r) \in \mathbf{R}^d, r > 0$ and $x_0 = 0$. An application of Lemma 7 and (1.6) yield

$$K(x, Q) = r^{d-\alpha} \frac{(r^2 - |x|^2)^{\alpha/2}}{|x - Q|^d}, \qquad |x| < r \tag{3.36}$$

for all $Q \in \partial B(0, r)$. Formula (3.36) is an exact analogue of the classical Poisson kernel for the ball. The resulting Martin representation of nonnegative functions singular α-harmonic in a ball is quite similar to that of classical harmonic functions. As a consequence it follows easily that multiplication by the factor $r^{2-\alpha}(r^2 - |x|^2)^{\alpha/2-1}$ is an isomorphism from the class of nonnegative functions harmonic in $B(0, r)$ onto $\mathscr{H}_0^\alpha(B(0, r))$. The description was given before in [4].

Martin representation of nonnegative functions singular α-harmonic in certain unbounded domains can be derived from our results by means of the Kelvin transform. The Kelvin transform (with center at the origin) is the mapping $T : \mathbf{R}^d\backslash\{0\} \mapsto \mathbf{R}^d\backslash\{0\}$ given by $Tx = x/|x|^2$. For $A \subset \mathbf{R}^d\backslash\{0\}$, TA is the image of A under T as well as the inverse image since $T^{-1} = T$. Let u be a function on $\mathbf{R}^d\backslash\{0\}$. The Kelvin transform of u will mean the function Tu on $\mathbf{R}^d\backslash\{0\}$ defined by

$$Tu(y) = |y|^{\alpha-d}u(Ty) \tag{3.37}$$

(we drop α from the notation). We have the following result

LEMMA 8. *Let D be an open set in $\mathbf{R}^d\backslash\{0\}$. If $u \in \mathscr{H}^\alpha(D)$ then $Tu \in \mathscr{H}^\alpha(TD)$.*

Lemma 8 implies in particular that the classes $\mathscr{H}_0^\alpha(D)$ and $\mathscr{H}_0^\alpha(TD)$ are isomorphic. In consequence it yields a (Martin) representation for $\mathscr{H}_0^\alpha(TD)$ if such a representation is given for $\mathscr{H}_0^\alpha(D)$. Proof of Lemma 8 is given in Appendix. A similar result with u being a Riesz potential (1.12) is implicit in [8].

EXAMPLE 2. Let $P = (0, \dots, 0, 1) \in \mathbf{R}^d$. The Kelvin transform of the ball $B(P, 1)$ is the half-space $\{y = (y_1, \dots, y_d) \in \mathbf{R}^d : y_d > 1/2\}$. Using (4.42) below one can easily calculate the Kelvin transforms of the Martin functions (3.36). These are clearly (minimal) singular α-harmonic and give rise to a representation of general nonnegative functions singular α-harmonic for the half-space which is completely analogous to that for the ball. The details are left to the reader. We give the resulting Martin functions in the half-space $\Pi = \{y = (y_1, \dots, y_d) \in \mathbf{R}^d;\ y_d > 0\}$. We have the following functions

$$K(y, Q) = \frac{y_d^{\alpha/2}}{|y - Q|^d}(1 + |Q|^2)^{d/2}, \qquad y_d > 0 \tag{3.38}$$

with $Q \in \partial\Pi$ and

$$K(y, \infty) = y_d^{\alpha/2}, \qquad y_d > 0, \tag{3.39}$$

corresponding to the point at infinity. The functions are normalized so to satisfy $K(P, \cdot) \equiv 1$. Let us also remark that (3.38) and (3.39) can be derived (as in Lemma 7 and Example 1) from the explicit form of the following Poisson kernel for the half-space Π:

$$P(x, y) = c_\alpha^d \frac{x_d^{\alpha/2}}{|y_d|^{\alpha/2}}|x - y|^{-d}, \qquad y \in \operatorname{int} \Pi^c, \tag{3.40}$$

where $x \in \Pi$. Verification of (3.40) is left as an exercise to the reader.

4. Appendix

LEMMA 9. *Let u be a nonnegative Borel measurable function on $\mathbf{R}^d\backslash\{0\}$ and let Tu be its Kelvin transform* (3.37). *For every ball $B \subset \mathbf{R}^d\backslash\{0\}$ we have*

$$E^{Tx}Tu(X_{\tau_{TB}}) = |x|^{d-\alpha}E^x u(X_{\tau_B}), \qquad x \in B. \tag{4.41}$$

PROOF. The following basic relation can be easily verified:

$$|Tx - Ty| = \frac{|x-y|}{|x||y|}, \qquad x, y \neq 0. \tag{4.42}$$

To be specific let $B = B(Q, r)$ where $Q \in \mathbf{R}^d\backslash\{0\}$ and $0 < r < |Q|$. It is well known that TB is also a ball. Let $TB = B(S, \rho)$, thus defining S and ρ. Using (4.42) for points $x, y \in \partial D$ on the radius from 0 to Q we get

$$\rho = \frac{r}{(|Q|-r)(|Q|+r)} = \frac{r}{|Q|^2 - r^2}, \tag{4.43}$$

$$S = \frac{Q}{|Q|}\frac{1}{2}\left(\frac{1}{|Q|-r} + \frac{1}{|Q|+r}\right) = \frac{Q}{|Q|^2 - r^2}, \tag{4.44}$$

By (1.7)

$$E^{Tx}Tu(X_{\tau_{TB}}) = c_\alpha^d \int_{TB^c} \left[\frac{\rho^2 - |Tx-S|^2}{|y-S|^2 - \rho^2}\right]^{\alpha/2} |Tx-y|^{-d}|y|^{\alpha-d}u(Ty)dy.$$

We change the variable $y = Tz$. For the Jacobian of T we have the formula $|JT(z)| = |z|^{-2d}$ ((4.43) establishes the rate of expansion of the element of volume under T). Using (4.42), (4.43), the fact that $T^2 = id$ and (4.44) we get

$$\begin{aligned}
E^{Tx}Tu(X_{\tau_{TB}}) &= c_\alpha^d \int_{B^c} \left[\frac{\rho^2 - |Tx-S|^2}{|Tz-S|^2 - \rho^2}\right]^{\alpha/2} |Tx - Tz|^{-d}|z|^{d-\alpha}u(z)|z|^{-2d}dz \\
&= c_\alpha^d \int_{B^c} \left[\frac{\rho^2 - |Tx-S|^2}{|Tz-S|^2 - \rho^2}\right]^{\alpha/2} |x-z|^{-d}|z|^{-\alpha}|x|^d u(z)\,dz \\
&= |x|^{d-\alpha}c_\alpha^d \int_{B^c} \left[\frac{r^2(|Q|^2-r^2)^{-2} - |x-TS|^2|x|^{-2}|TS|^{-2}}{|z-TS|^2|z|^{-2}|TS|^{-2} - r^2(|Q|^2-r^2)^{-2}}\right]^{\alpha/2} \frac{|x|^\alpha u(z)\,dz}{|z|^\alpha|x-z|^d} \\
&= |x|^{d-\alpha}c_\alpha^d \int_{B^c} \left[\frac{r^2|x|^2 - |x-TS|^2|Q|^2}{|z-TS|^2|Q|^2 - r^2|z|^2}\right]^{\alpha/2} \frac{u(z)}{|x-z|^d}dz
\end{aligned}$$

Since $TS = Q\dfrac{|Q|^2 - r^2}{|Q|^2}$, we have

$$|Q|^2|x - TS|^2 = |Q|^2|x|^2 + |Q|^2|TS|^2 - 2|Q|^2(x, TS)$$
$$= |Q|^2|x|^2 + (|Q|^2 - r^2)^2 - 2(|Q|^2 - r^2)(x, Q),$$

where $(\cdot, \cdot)$ denotes the usual inner product in $\mathbf{R}^d$. It follows that

$$r^2|x|^2 - |Q|^2|x - TS|^2 = (|Q|^2 - r^2)(r^2 - |x - Q|^2).$$

Analogously $|z - TS|^2|Q|^2 - r^2|z|^2 = (|Q|^2 - r^2)(|z - Q|^2 - r^2)$. Therefore

$$E^{Tx}Tu(X_{\tau_{TB}}) = |x|^{d-\alpha}c_\alpha^d \int_{B^c} \left[\frac{r^2 - |x - Q|^2}{|z - Q|^2 - r^2}\right]^{\alpha/2} |x - z|^{-d}u(z)dz$$
$$= |x|^{d-\alpha}E^x u(X_{\tau_B}).$$

The proof of (4.41) is complete. □

PROOF OF LEMMA 8. By a remark in Introduction it is enough to prove that

$$Tu(Tx) = E^{Tx}Tu(X_{\tau_{TB}}), \qquad x \in B$$

for every ball $B \subset \bar{B} \subset D, D$ being the domain of α-harmonicity of u. Since $Tu(Tx) = |x|^{d-\alpha}u(x)$, this follows from (4.41) and (1.4). □

ADDED IN PROOF. During the conference on Geometric Stochastic Analysis and Fine Properties of Stochastic Processes on March 23–27, 1998 in Berkeley I was informed that independent research into this subject was carried out by Zhen-Qing Chen and Renming Song. Their interesting results, which partially overlap with those above, are given in preprint "*Martin boundary and integral representation for harmonic functions of symmetric stable process*".

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