HIROSHIMA MATH. J.
29 (1999), 591–630

Convergence to a viscosity solution for an advection-reaction-diffusion equation arising from a chemotaxis-growth model

Marie HENRY*, Danielle HILHORST*, and Reiner SCHÄTZLE†

(Received December 7, 1998)

ABSTRACT. We study the limiting behavior as ε tends to zero of the solution of a Cauchy problem for an advection-reaction-diffusion equation; this equation arises in a model for a chemotaxis growth process in biology. We consider the case of an arbitrary time interval and prove the convergence of the solution of this problem to the unique viscosity solution of a limit free boundary problem.

1. Introduction

In this paper, we study the limiting behavior as ε tends to zero of the solution ϕ^ε of an advection-reaction-diffusion equation arising from a chemotaxis-growth model proposed by Mimura and Tsujikawa [10]. We suppose that the density of the chemotactic substance is a known function $v(x,t)$. More precisely, we consider two Cauchy problems. The first one is given by

$$(P_1^\varepsilon)\begin{cases} \phi_t^\varepsilon = \Delta\phi^\varepsilon - \nabla.(\phi^\varepsilon\nabla\chi(v)) + \dfrac{1}{\varepsilon^2}f(\phi^\varepsilon,\varepsilon\alpha) & \text{in } R^N\times(0,T] \\ \phi^\varepsilon(x,0) = \phi_0^\varepsilon(x) & x\in R^N, \end{cases}$$

where $f(s,\tilde{\alpha}) = s(1-s)(s-1/2+\tilde{\alpha})$, and where α is a fixed constant. The functions ϕ^ε and v are respectively the population density and the concentration of chemotactic substance. Here, χ and v are supposed to be smooth functions. The population is subjected to three competitive effects: diffusion, growth induced by the nonlinear term $\phi^\varepsilon(1-\phi^\varepsilon)(\phi^\varepsilon-1/2+\tilde{\alpha})$ and a tendency of migrating towards higher gradients of the chemotactic substance induced by the advection term.

The second problem, that we consider has a slighly different scaling, namely

1991 *Mathematics Subject Classification.* 35K57, 35B25, 53A10, 35A05.
Key words and phrases. Advection-reaction-diffusion equation; Singular limit; Modified motion by mean curvature; Viscosity solution.

$$(P_2^\varepsilon)\begin{cases} \phi_t^\varepsilon = \varepsilon\Delta\phi^\varepsilon - \nabla.(\phi^\varepsilon\nabla\chi(v)) + \dfrac{1}{\varepsilon}f(\phi^\varepsilon,\alpha) & \text{in } R^N \times (0,T] \\ \phi^\varepsilon(x,0) = \phi_0^\varepsilon(x) & x \in R^N. \end{cases}$$

We study Problem (P_2^ε) in the case that $\alpha \in [0, 0.4]$. We remark that if $0 \leq \phi^\varepsilon(.,0) \leq 1$, the standard maximum principle implies that the function ϕ^ε satisfies

$$0 \leq \phi^\varepsilon \leq 1 \qquad \text{in } R^N \times [0,T]. \tag{1.1}$$

In the case of a more realistic coupled system describing the chemotaxis phenomenon, Mimura and Tsujikawa formally derive [10] the free boundary problem corresponding to taking the limit $\varepsilon \to 0$ in Problem (P_2^ε). As for Problem (P_1^ε) rigorous results are proved by Bonami, Hilhorst, Logak, and Mimura [2] in the case of a corresponding Neumann problem on a bounded domain. In order to state their results in a clear way, we now give hypotheses which will not be needed further on in this paper. We suppose that Γ_0 is a closed and smooth hypersurface without boundary. As ε tends to zero, ϕ^ε converges to a limit function

$$\phi = \begin{cases} 1 & \Omega_t^+ \\ 0 & \Omega_t^0 \end{cases}$$

for $t \in [0,T]$ and the equation of motion for the boundary $\Gamma = \{\bigcup \Gamma_t, t \in [0,T]\}$ where Γ_t separates Ω_t^+, Ω_t^0 is given by

$$(L_1)\begin{cases} V_n = -K + \dfrac{\partial\chi(v)}{\partial n} - (\tilde{c})'(0)\alpha & \text{on } \Gamma_t,\ t \in [0,T] \\ \Gamma_t|_{t=0} = \Gamma_0, \end{cases}$$

in the case of Problem (P_1^ε) and by

$$(L_2)\begin{cases} V_n = \dfrac{\partial\chi(v)}{\partial n} - \tilde{c}(\alpha) & \text{on } \Gamma_t,\ t \in [0,T] \\ \Gamma_t|_{t=0} = \Gamma_0 \end{cases}$$

in the case of Problem (P_2^ε). The time T is the existence time of a smooth solution of either Problem (L_1) or Problem (L_2). Here n denotes the outward unit normal vector to Γ_t which points from Ω_t^+ to Ω_t^0, V_n is the normal velocity of Γ_t and K is the mean curvature of Γ_t. The function $\tilde{c}(\alpha)$ is the velocity of the one-dimensional travelling wave $w(x,t) = \tilde{q}(x - \tilde{c}t, \alpha)$ of a related equation; the pair $(\tilde{q}(r,\alpha), \tilde{c}(\alpha))$, where $\tilde{C}(\alpha) = -\sqrt{2}\alpha$, satisfies the problem

$$(TW)\begin{cases} \tilde{q}_{rr} + \tilde{c}(\alpha)\tilde{q}_r + \tilde{q}(1-\tilde{q})(\tilde{q} - 1/2 + \alpha) = 0 \\ \tilde{q}(-\infty,\alpha) = 0, \quad \tilde{q}(+\infty,\alpha) = 1. \end{cases}$$

In this article, we prove convergence properties of the function ϕ^ε on an arbitrary time interval $[0,T]$. In general a classical solution of the limiting free

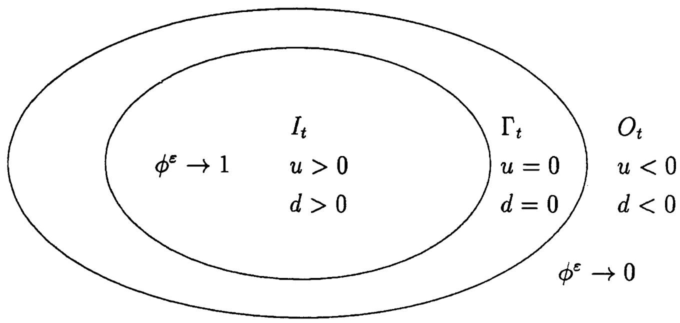

Fig. 1. A possible configuration in the limiting free boundary problem

boundary problems does not exist on such a time interval, which leads us to introduce viscosity solutions of these problems.

In what follows we transform the free boundary problems (L_1) and (L_2) by introducing a new unknown function u such that the free boundary Γ_t is a level set of u. We suppose that Γ_0 is a compact set in R^N, such that $R^N \backslash \Gamma_0 = O_0 \cup I_0$ where O_0 and I_0 are two open disjoint subsets of $R^N \backslash \Gamma_0$. We define for each $t > 0$

$$\Gamma_t = \{x \in R^N, u(.,t) = 0\}. \tag{1.2}$$

Furthermore we set

$$I_t = \{x \in R^N, u(.,t) > 0\} \tag{1.3}$$

and

$$O_t = \{x \in R^N, u(.,t) < 0\}, \tag{1.4}$$

and denote by $d(x,t)$ the signed distance function to Γ_t,

$$d(x,t) = \begin{cases} dist(x,\Gamma_t) & \text{for } x \in I_t \\ -dist(x,\Gamma_t) & \text{for } x \in O_t \end{cases}$$

for all (x,t) in $R^N \times [0,T]$.

A standard computation gives

$$V_n = -\frac{u_t}{|\nabla u|}, \qquad n = \frac{\nabla u}{|\nabla u|} \qquad \text{and} \qquad K = div(n) = div\left(\frac{\nabla u}{|\nabla u|}\right) \qquad \text{on } \Gamma_t.$$

This leads us to consider Problem (P_1^l) in the case of the first scaling,

$$(P_1^l)\begin{cases} u_t - \Delta u + \dfrac{(D^2u\nabla u.\nabla u)}{|\nabla u|^2} + (\nabla u.\nabla\chi(v)) - \sqrt{2}\alpha|\nabla u| = 0 & \text{in } R^N \times (0,T) \\ u(x,0) = U_0(x) := \max(-1, \min(1, d(x,0))), & x \in R^N, \end{cases}$$

and Problem (P_2^l) in the case of the second scaling,

$$(P_2^l)\begin{cases} u_t + (\nabla u.\nabla\chi(v)) - \sqrt{2}\alpha|\nabla u| = 0, & \text{in } R^N \times (0,T) \\ u(x,0) = U_0(x) & x \in R^N. \end{cases}$$

As it is done by Barles, Soner and Souganidis [1] and by Elliott and Schätzle [5] we consider viscosity solutions of both the problems (P_1^l) and (P_2^l). For studies about viscosity solutions of partial differential equations we refer to Giga, Goto, Ishii, and Sato [8] and to Crandall, Ishii and Lions [4]. We recall below the definition of a viscosity solution of a second order parabolic equation. In the sequel, we denote by LSC(.) and USC(.) the sets of lower semicontinuous and upper semicontinuous functions, and by K_* (K^* resp.) the lower (upper resp.) semicontinuous envelope of K. For instance we recall that $K_*(p,X) = \liminf_{\varepsilon\to 0}\{K(q,Y), |p-q| < \varepsilon, |X-Y| < \varepsilon\}$, and $K^* = -(-K)_*$.

DEFINITION 1.1. *Let $F \in C(R^N \times (0,T) \times R \times (R^N\backslash\{0\}) \times S(N))$, where $S(N)$ is the set of symmetric $N \times N$ matrices such that*

$$F \text{ is elliptic, i.e. } F(x,t,s,p,X) \le F(x,t,s,p,Y) \qquad \text{if } X \ge Y.$$

A function $u : R^N \times (0,T) \to R$ is called a viscosity subsolution (resp. supersolution) of

$$u_t + F(x,t,u,Du,D^2u) = 0, \qquad (x,t) \in R^N \times (0,T) \tag{1.5}$$

which we formally write as $u_t + F(x,t,u,Du,D^2u) \le 0, (x,t) \in R^N \times (0,T)$ (resp. ≥ 0) if u is $USC(R^N \times [0,T])$ (resp. $LSC(R^N \times [0,T])$) and if

$$\phi_t(x_0,t_0) + F_*(x_0,t_0,u(x_0,t_0)), D\phi(x_0,t_0), D^2\phi(x_0,t_0)) \le 0$$

$$(\textit{resp. } \phi_t(x_0,t_0) + F^*(x_0,t_0,u(x_0,t_0), D\phi(x_0,t_0), D^2\phi(x_0,t_0)) \ge 0)$$

for all ϕ in $C^{2,1}(R^N \times [0,T])$ and all local maxima (resp. local minima) (x_0,t_0) of the function $(u\text{-}\phi)$.

The function v is a viscosity solution of (1.5) *if it is both a sub- and a supersolution.*

The paper is organised as follows:

We prove in Section 2 the existence and uniqueness of the viscosity solution of both the limiting free boundary problems (P_1^l) and (P_2^l).

We show in Section 3 that as ε tends to zero, the solution ϕ^ε of Problem (P_1^ε) tends to the characteristic function of the moving domain $I_t \subset R^N$. More precisely we prove the following result.

THEOREM 1.2. *Let ϕ^ε be the solution of Problem (P_1^ε) with $\phi^\varepsilon(x,0) = \tilde{q}\left(\frac{U_0(x)}{\varepsilon}, 0\right)$ in R^N and let u be the viscosity solution of Problem (P_1^l). Then*

$$\begin{cases} \phi^\varepsilon(x,t) \to 0 & \text{if } (x,t) \in O \\ \phi^\varepsilon(x,t) \to 1 & \text{if } (x,t) \in I \end{cases}$$

as ε *tends to* 0, *where* $O := \{(x,t) \in R^N \times [0,T], u(x,t) < 0\}$ *and* $I := \{(x,t) \in R^N \times [0,T], u(x,t) > 0\}$. *Moreover this convergence is uniform on compact sets of* O *and* I.

The main steps of the proof are the following: one constructs approximate solutions $u^{\delta,a}$ for a class of problems related to Problem (P_1^l), one introduces an approximate distance to O which is the key ingredient for constructing viscosity supersolutions of Problem (P_1^ε); one finally proves the uniform convergence of the sequence $u^{\delta,a}$ to the solution u of Problem (P_1^l) on compact sets of $R^N \times [0,T]$ and concludes that ϕ^ε converges uniformly to zero on compact sets of O as $\varepsilon \downarrow 0$. The proof that ϕ^ε converges uniformly to one on compact sets of I as $\varepsilon \downarrow 0$ is very similar.

In Section 4 we show how one can adapt the proof given in Section 3 to prove the convergence of the solution of Problem (P_2^ε). More precisely we prove the following result.

THEOREM 1.3. *Let* ϕ^ε *be the solution of Problem* (P_2^ε) *with* $\phi^\varepsilon(x,0) = \tilde{q}\left(\frac{U_0(x)}{\varepsilon}, 0\right)$ *in* R^N *and let* u *be the viscosity solution of Problem* (P_2^l). *Then*

$$\begin{cases} \phi^\varepsilon(x,t) \to 0 & \text{if } (x,t) \in O \\ \phi^\varepsilon(x,t) \to 1 & \text{if } (x,t) \in I \end{cases}$$

as ε *tends to* 0. *Moreover this convergence is uniform on compact sets of* O *and* I.

Properties of travelling wave solutions $\tilde{q}$ are described in the Appendix. We remark that $\tilde{q}\left(\frac{U_0(x)}{\varepsilon}, \alpha\right) = \tilde{q}\left(\frac{U_0(x)}{\varepsilon}, 0\right)$.

Our methods of proof are closely related to those of Barles, Soner and Souganidis [1]. However the problems which we consider involve convection as well as reaction so that many proofs are much more technical. In particular [1] hardly consider the case of the scaling of Problem P_2^ε for which we use new perturbations of both the travelling wave equation and the limiting free boundary problem in order to be able to construct super- and subsolutions.

2. Existence and uniqueness of viscosity solutions of the Problems P_1^l and P_2^l

In this section we recall a result due to Giga, Goto, Ishii, and Sato [8] about the existence and the uniqueness of the viscosity solution of a general evolution problem. We apply their result to prove the existence and uniqueness of viscosity solutions of the problems (P_1^l) and (P_2^l). We consider the evolution problem

$$u_t + F(x,t,u,Du,D^2u) = 0 \qquad (x,t) \in R^N \times (0,T) \tag{2.1}$$

$$u(x,0) = u_0(x) \qquad x \in R^N, \tag{2.2}$$

where F satisfies for all $(x,t,s,p,X) \in R^N \times (0,T] \times R \times R^N \backslash \{0\} \times S(N)$ the hypotheses

(H1) $\quad F(x,t,s,\mu p, \mu X + \eta p p^t) = \mu F(x,t,s,p,X) \qquad$ for $\mu \in R^+$, $\eta \in R$,

(H2) $\quad F$ is elliptic, i.e. $F(x,t,s,p,X) \le F(x,t,s,p,Y) \qquad$ if $X \ge Y$,

as well as the following technical hypotheses

(H3) $\begin{cases} \text{The mapping } (x,t,s,p,X) \to F(x,t,s,p,X) \text{ is bounded for} \\ \text{bounded } (p,X), \text{ and continuous for } (x,t,s) \in R^N \times (0,T] \times R, \\ p \in B(0,r)\backslash\{0\}, \text{ and } \|X\| \le r \text{ for all } r > 0, \end{cases}$

(H4) $\begin{cases} \text{For every } R > \rho > 0 \text{ there exists a modulus } \sigma = \sigma_{R\rho} \text{ such that} \\ \text{for all } (x,t,s) \in R^N \times (0,T] \times R, \rho < |p|, |q| \le R, |X|, |Y| \le R, \\ |F(x,t,s,p,X) - F(x,t,s,q,Y)| \le \sigma_{R\rho}(|p-q| + |X-Y|), \end{cases}$

(H5) $\begin{cases} \text{There exists } \rho_0 > 0 \text{ and a modulus } \sigma_1 \text{ such that} \\ \text{for all } (x,t,s) \in R^N \times (0,T] \times R, \text{ and } |p|, |X| \le \rho_0 \\ F^*(x,t,s,p,X) - F^*(x,t,s,0,0) \le \sigma_1(|p| + |X|) \\ F_*(x,t,s,p,X) - F_*(x,t,s,0,0) \ge -\sigma_1(|p| + |X|), \end{cases}$

(H6) $\begin{cases} \text{There exists a modulus } \sigma_2 \text{ such that} \\ \text{for all } (x,t,s) \in R^N \times (0,T] \times R, p \in R^N\backslash\{0\}, X \in S(N) \\ |F(x,t,s,p,X) - F(y,t,s,p,X)| \le \sigma_2(|x-y|(|p|+1)), \end{cases}$

(H7) $\begin{cases} \text{There exists a constant } c_0 \text{ such that} \\ \text{for all } (x,t,s,p,X) \in R^N \times (0,T] \times R \times R^N \times S(N) \\ \text{the map } s \to F(x,t,s,p,X) + c_0 s \text{ is nondecreasing.} \end{cases}$

Finally we suppose that for all (x,t) in $R^N \times (0,T]$

(H8) $\quad F$ is elliptic, i.e. $F(x,t,s,p,X) \le F(x,t,s,p,Y) \qquad$ if $X \ge Y$,

and

(H9) $\begin{cases} \text{There exists a function } c(q) \in C^1([0,\infty)) \text{ such that } c(q) > c_0 > 0 \\ \text{and such that for all } (x,t,p) \in R^N \times (0,T] \times R^N, \\ F_*(x,t,p,-I) \le c(|p|), \quad F^*(x,t,p,I) \ge -c(|p|). \end{cases}$

We recall that F^* and F_* denote the upper- and lower-semicontinuous envelopes of F, respectively.

Giga, Goto, Ishii, and Sato [8] show the following result

THEOREM 2.1. *Assume that the function F is independent of s, and satisfies the hypotheses* (H1)–(H3), (H6), (H8), *and* (H9). *Let $u_0 \in C(R^N)$ be such that $u_0(x) = a$ for large values of $|x|$, where a is some fixed real constant. Then there exists a unique viscosity solution $u \in C(R^N \times [0, T])$ of Problem* (2.1)–(2.2) *such that u is equal to a for large values of $|x|$.*

We remark that the evolution of the front Γ_t only depends on F and on the sets $I_0^i = \{x \in R^N, u(x, 0) > 0\}$, $O_0^i = \{x \in R^N, u(x, 0) < 0\}$ and $\Gamma_0^i = \{x \in R^N, u(x, 0) = 0\}$.

In both the cases of the problems (P_1^l) and (P_2^l) the limiting equations are of the form (2.1) with respectively

$$F_1(x, t, p, X) = -tr(X) + \frac{(Xp.p)}{|p|^2} + (p.\nabla\chi(v)) - \sqrt{2}\alpha|p| \tag{2.3}$$

$$F_2(x, t, p) = (p.\nabla\chi(v)) - \sqrt{2}\alpha|p|. \tag{2.4}$$

Giga and Goto [7] check that the hypotheses of Theorem 2.1 are satisfied for a large class of geometric equations containing the equations in the problems (P_1^l) and (P_2^l). However for the sake of completeness, we check below that the function F_1 satisfies the asumptions (H1)–(H3), (H6), (H8), and (H9). As a trivial consequence F_2 will satisfy these hypotheses as well so that we will be able to apply Theorem 2.1 to both the problems (P_1^l) and (P_2^l).

We first check that F_1 satisfies (H1). We have that

$$F_1(x, t, \mu p, \mu X + \eta p \otimes p) = \mu F_2(x, t, p, X) + \eta\left[-tr(p \otimes p) + \frac{(p \otimes p\, p.p)}{|p|^2}\right].$$

If we denote by $(p_i)_{i=1,\dots,n}$ the coordinates of p and use that $p \otimes p = pp^t$, we deduce that $tr(p \otimes p) = \Sigma p_i^2$ and $((p \otimes p\, p).p) = (\Sigma p_i^2)^2$. Therefore F_1 satisfies (H1).

Let X be an arbitrary symmetric matrix. We denote by $(\lambda_i(X))_{i=1,\dots,N}$ its eigenvalues and suppose that $\lambda_1(X) \leq \cdots \leq \lambda_N(X)$. In what follows, we suppose that all the symmetric matrices which we consider are written in the basis of their eigenvectors. Next we check the hypotheses (H2), and (H3). Let $Z = X - Y$. We prove below that $Z \geq 0$ implies $-tr(Z) + \dfrac{(Zp.p)}{|p|^2} \leq 0$, for all $p \in R^N$. (We say that a symmetric matrix Z is positive if $(Zp.p) \geq 0$ for all $p \in R^N$). Since Z is symmetric, we present our computation in its basis of eigenvectors,

$$-tr(Z)+\frac{(Zp.p)}{|p|^2}=\frac{1}{\Sigma p_i^2}(-\Sigma p_i^2\Sigma\lambda_i(Z)+\Sigma\lambda_i(Z)p_i^2)$$
$$=\frac{1}{\Sigma p_i^2}(-\Sigma_{i\neq j}\lambda_i(Z)p_j^2).$$

Since $Z\geq 0$, its eigenvalues are nonnegative so that $-tr(Z)+\dfrac{(Zp.p)}{|p|^2}\leq 0$. This completes the proof of (H2). Furthermore we have for all $X\in S(N)$ $N\lambda_1(X)\leq trX\leq N\lambda_N(X)$, and $0\leq\dfrac{(Xp.p)}{|p|^2}=\dfrac{\Sigma\lambda_i p_i^2}{\Sigma p_i^2}\leq\lambda_N(X)$. Since moreover $p\in B(0,r)\backslash\{0\}$, the quantities $(p.\nabla\chi(v))$ and $\alpha|p|$ are also bounded, so that F_1 satisfies (H3).

Since $D^2(\chi(v))$ is bounded we deduce that

$$F_1(x,t,p,X)-F_1(y,t,p,X)=(p.(\nabla\chi(v)(x,t)-\nabla\chi(v)(y,t)))\leq K|x-y||p|$$

so that (H6) is satisfied.

In order to prove (H8) and (H9) we define $G(p,X)=-trX+\dfrac{(Xp.p)}{|p|^2}$. First we note that $G_*(p,X)=-\Sigma_{i=2}^{i=n}\lambda_i(X)$, and $G^*(p,X)=-\Sigma_{i=1}^{i=n-1}\lambda_i(X)$, so that in particular $(F_1)_*(x,t,0,0)=G_*(0,0)=0$ and also $(F_1)^*(x,t,0,0)=G^*(0,0)=0$. This implies (F8).

Finally we check that (H9) is satisfied. We have that

$$\begin{aligned}(F_1)_*(x,t,p,-I)&=G_*(p,-I)+(p.\nabla\chi(v))-\sqrt{2}\alpha|p|\\&=N-1+(p.\nabla\chi(v))-\sqrt{2}\alpha|p|\\&\leq N+|p|(\|\nabla\chi(v)\|_\infty+\sqrt{2}\alpha)\end{aligned}$$

and

$$\begin{aligned}(F_1)^*(x,t,p,I)&=G^*(p,I)+(p.\nabla\chi(v))-\sqrt{2}\alpha|p|\\&\geq-(N+|p|(\|\nabla\chi(v)\|_\infty+\sqrt{2}\alpha)).\end{aligned}$$

Therefore we have shown that (F_1) satisfies all the assumptions of Theorem 2.1, and trivially (F_2) satisfies the same assumptions. Thus we conclude that there exists a unique viscosity solution of the problems (P_1^l) and (P_2^l). Next we recall the comparison theorem given in [8], and we check that we can apply this result to the viscosity sub- and super-solutions of Problems (P_1^l) and (P_2^l), and to the solutions of Problems (P_1^ε) and (P_2^ε) involving variants of the Allen-Cahn equation.

THEOREM 2.2. *Suppose that F satisfies* (H2)–(H8). *Let u and v be respectively viscosity sub- and supersolutions of equation* (2.1) *in $R^N \times (0, T]$. Assume that u and v satisfy the following assumptions*

$$\begin{cases} \text{There exists a positive constant } K \text{ such that for all } (x,t) \in R^N \times (0,T] \\ u(x,t) \le K(|x|+1), v(x,t) \ge -K(|x|+1); \\ \text{there exists a modulus } m_T \text{ such that for all } (x,y) \in R^N \times R^N \\ u^*(x,0) - v_*(y,0) \le m_T(|x-y|); \\ \text{there exists a constant } K > 0 \text{ such that for all } (x,y) \in R^N \times R^N \\ u^*(x,0) - v_*(y,0) \le K(|x-y|+1). \end{cases}$$

Then there exists a modulus m such that

$$u^*(x,t) - v_*(y,t) \le m(|x-y|) \qquad \text{on } R^N \times (0,T].$$

We check below that we can apply this comparison principle to the viscosity sub- and supersolution of the problems (P_1^l) and (P_2^l). Since we have already checked the conditions (H2), (H3), (H6), (H8), we only have to prove that (H4) and (H5) are satisfied. Moreover we only write the proof for (P_1^l). By definition, we have that

$$F_1(x,t,p,X) - F_1(x,t,q,Y) = -tr(X) + tr(Y) + \frac{(Xp.p)}{|p|^2} - \frac{(Yq.q)}{|q|^2} + ((p-q).\nabla\chi(v)) - \sqrt{2}\alpha(|p| - |q|).$$

Since the map $X \to tr\,X$ is Lipschitz continuous, and since $(p,X) \to \dfrac{(Xp.p)}{|p|^2}$ is continuously differentiable on the compact set

$$\{p \in R^N, \rho < |p| \le R] \times [X \in S(N), |X| \le R\},$$

it follows that

$$|F_1(x,t,p,X) - F_1(x,t,q,Y)| \le K(|X-Y| + |p-q|).$$

Next we check (H5):

$$\begin{aligned} F_1^*(x,t,s,p,X) - F_1^*(x,t,s,0,0) &= G^*(p,X) + (p.\nabla\chi(v)) - \sqrt{2}\alpha|p| \\ &= -\Sigma_{i=1}^{i=N-1}\lambda_i(X) + (p.\nabla\chi(v)) - \sqrt{2}\alpha|p| \\ &\le |tr(X)| + |p|(|\nabla\chi(v)|_\infty - \sqrt{2}\alpha) \\ &\le K|X| + |p|(|\nabla\chi(v)|_\infty + \sqrt{2}\alpha). \end{aligned}$$

Similarly we have that

$$\begin{aligned}(F_1)_*(x,t,s,p,X)-(F_1)_*(x,t,s,0,0) &= G_*(p,X)+(p.\nabla\chi(v))-\sqrt{2}\alpha|p|\\ &\geq -|tr(X)|-|p|(|\nabla\chi(v)|_\infty+\sqrt{2}\alpha)\\ &\geq -K|X|-|p|(|\nabla\chi(v)|_\infty+\sqrt{2}\alpha).\end{aligned}$$

Next we check that we can apply this comparison principle to the problems (P_1^l) and (P_2^l). We define

$$\begin{aligned}\mathscr{H}_1(x,t,\phi^\varepsilon,\nabla\phi^\varepsilon,D^2\phi^\varepsilon) &= -tr(D^2\phi^\varepsilon)+(\nabla\phi^\varepsilon.\nabla\chi(v))+\phi^\varepsilon\nabla\chi(v)\\ &\quad -\frac{1}{\varepsilon^2}\phi^\varepsilon(1-\phi^\varepsilon)(\phi^\varepsilon-1/2+\varepsilon\alpha)\end{aligned}$$

and

$$\begin{aligned}\mathscr{H}_2(x,t,\phi^\varepsilon,\nabla\phi^\varepsilon,D^2\phi^\varepsilon) &= -\varepsilon\, tr(D^2\phi^\varepsilon)+(\nabla\phi^\varepsilon.\nabla\chi(v))+\phi^\varepsilon\nabla\chi(v)\\ &\quad -\frac{1}{\varepsilon}\phi^\varepsilon(1-\phi^\varepsilon)(\phi^\varepsilon-1/2+\alpha).\end{aligned}$$

We have to check that $\mathscr{H}_1$ and $\mathscr{H}_2$ satisfy (H2)–(H8). We only write the proof for $\mathscr{H}_1$. Since $\mathscr{H}_1(x,t,s,p,X)-\mathscr{H}_1(x,t,s,p,Y)=-tr(X-Y)$, (F2) is satisfied.

Moreover the mapping $(x,t,s,p,X)\to\mathscr{H}_2(x,t,s,p,X)$ is continuous on $R^N\times(0,T]\times R\times R^N\times S(N)$, so that the hypotheses (H3)–(H6), (H8) are trivially satisfied. Next we check (H7).

We have that

$$\begin{aligned}\mathscr{H}_1(x,t,s,p,X)-\mathscr{H}_1(x,t,r,p,X) &= (s-r)\Delta\chi(v)-\frac{1}{\varepsilon^2}(f(s,\varepsilon\alpha)-f(r,\varepsilon\alpha))\\ &\geq (s-r)[-\|\Delta\chi(v)\|_\infty-\frac{1}{\varepsilon^2}f'(\xi,\varepsilon\alpha)]\end{aligned}$$

where ξ is the maximum of f' on R. If we choose

$$c_0>\|\Delta\chi(v)\|_\infty+\frac{1}{\varepsilon^2}f'(\xi,\varepsilon\alpha),$$

then the function $s\to\mathscr{H}_1(x,t,s,p,X)+c_0 s$ is nondecreasing. Therefore (H7) is satisfied and we can apply the comparison principle to solutions of the problems (P_1^ε) and (P_2^ε); to that purpose we remark that the unique classical solutions of the problems (P_1^ε) and (P_2^ε) are also viscosity solutions in the sense of Definition 1.1.

3. Proof of the convergence Theorem 1.2

The convergence proof is organized as follows. In Section 3.1 we prove the convergence of the function ϕ^ε to zero in the subdomain O. The key idea of the proof is to construct sub- and super-viscosity solutions of Problem (P_1^l); to that purpose we make use of travelling wave solutions of a related one-dimensional parabolic problem and of a modified distance function. In Section 3.2 we prove the convergence of ϕ^ε to 1 in the subdomain I.

3.1. Convergence of the solution ϕ^ε of Problem (P_1^ε) in the set where $u < 0$

3.1.1. First definitions and preliminary lemmas

First we denote by $h_-(\varepsilon\alpha, \varepsilon a) < h_0(\varepsilon\alpha, \varepsilon a) < h_+(\varepsilon\alpha, \varepsilon a)$ the three solutions of the equation $f(s, \varepsilon\alpha) = s(1-s)(s - 1/2 + \varepsilon\alpha) = -\varepsilon a$. Note that $h_-(\varepsilon\alpha, 0) = 0$, $h_0(\varepsilon\alpha, 0) = 1/2 - \varepsilon\alpha$, and $h_+(\varepsilon\alpha, 0) = 1$.

We define by $(q(r, \varepsilon\alpha, \varepsilon a), c(\varepsilon\alpha, \varepsilon a))$ the travelling wave solution of the equation $u_t = u_{rr} + f(u, \varepsilon\alpha) + \varepsilon a$. Then (q, c) satisfies the problem

$$(TW)\begin{cases} q_{rr} + c(\varepsilon\alpha, \varepsilon a)q_r + q(1-q)(q - 1/2 + \varepsilon\alpha) = -\varepsilon a \\ q(-\infty, \varepsilon\alpha, \varepsilon a) = h_-(\varepsilon\alpha, \varepsilon a), \quad q(+\infty, \varepsilon\alpha, \varepsilon a) = h_+(\varepsilon\alpha, \varepsilon a), \end{cases}$$

and q is unique up to translation in r by constants. Finally, we set $C(\alpha, a) := \lim_{\varepsilon\to 0} \dfrac{c(\varepsilon\alpha, \varepsilon a)}{\varepsilon}$. For more precise information about the travelling wave q and the computation of $C(\alpha, a)$ we refer to the appendix. Next we introduce a sequence of problems related to Problem (P_1^l), namely

$$(P_1^{\delta,a})\begin{cases} u_t - \Delta u + \dfrac{(D^2u\nabla u.\nabla u)}{|\nabla u|^2} + (\nabla u.\nabla\chi(v)) + C(\alpha, a)|\nabla u| = 0 \\ u(x, 0) = U_0(x) + 2\delta, \end{cases}$$

where $U_0(x) = \max(-1, \min(1, d(x, 0)))$, and δ is a small enough positive constant. We set

$$F_1^a(x, t, p, X) = -tr(X) + \frac{(Xp.p)}{|p|^2} + (p.\nabla\chi(v)) + C(\alpha, a)|p|.$$

One can check just as in Section 2 that F_1^a satisfies the asumptions of both the theorems 2.2 and 2.1. Therefore Problem $(P_1^{\delta,a})$ has a unique viscosity solution $u^{\delta,a}$ wich satisfies a comparison principle.

We define the distance function

$$d^{\delta,a}(x, t) = \inf_{\{y, u^{\delta,a}(y,t)\le 0\}} |x - y| \tag{3.1}$$

and give some properties of $d^{\delta,a}$.

LEMMA 3.1. *$d^{\delta,a}(x,0) \geq u^{\delta,a}(x,0)$, for all $x \in R^N$.*

PROOF. If $u^{\delta,a}(x,0) \leq 0$ this inequality is obvious. Next we consider the case that $u^{\delta,a}(x,0) \geq 0$. There exists y such that $d^{\delta,a}(x,0) = |x-y|$ and $u^{\delta,a}(y,0) \leq 0$. Since U_0 has Lipschitz constant one, we have

$$|U_0(x) - U_0(y)| = |u^{\delta,a}(x,0) - u^{\delta,a}(y,0)| \leq |x-y| = d^{\delta,a}(x,0),$$

which in turn implies that $u^{\delta,a}(x,0) - u^{\delta,a}(y,0) \leq d^{\delta,a}(x,0)$. Since $u^{\delta,a}(y,0) \leq 0$ we deduce that $d^{\delta,a}(x,0) \geq u^{\delta,a}(x,0)$. This completes the proof of Lemma 3.1.

Next we state three lemmas, which are proved in [5].

LEMMA 3.2. *$d^{\delta,a}$ is lower-semicontinuous, that is if $(x_j,t_j) \to (x_0,t_0)$, then $d^{\delta,a}(x_0,t_0) \leq \liminf_{j\to+\infty} d^{\delta,a}(x_j,t_j)$.*

LEMMA 3.3. *$d^{\delta,a}$ is continuous in time from below, that is if $(x_j,t_j) \to (x_0,t_0)$ and $t_j \leq t_0$, then $d^{\delta,a}(x_0,t_0) = \lim_{j\to+\infty} d^{\delta,a}(x_j,t_j)$.*

LEMMA 3.4. *There exists a positive constant K such that $d^{\delta,a}$ satisfies the inequality*

$$d_t^{\delta,a} + (F_1^a)^*(x,t,\nabla d^{\delta,a}, D^2 d^{\delta,a}) \geq -K|\nabla d^{\delta,a}| d^{\delta,a} \tag{3.2}$$

in $R^N \times (0,T)$ in the sense of viscosity. Moreover $d^{\delta,a}$ satisfies

$$\begin{cases} |\nabla d^{\delta,a}| \geq 1 & (3.3) \\ -|\nabla d^{\delta,a}| \geq -1 & (3.4) \\ -(D^2 d^{\delta,a} \nabla d^{\delta,a} . \nabla d^{\delta,a}) \geq 0 \end{cases}$$

in $\{(x,t), d^{\delta,a}(x,t) > 0\}$ in the sense of viscosity.

We remark that in the case where $d^{\delta,a}$ is continuous and differentiable the inequalities (3.3) and (3.4) imply that $|\nabla d^{\delta,a}| = 1$. Next we prove the following result

LEMMA 3.5. *We have that*

$$|\nabla d^{\delta,a}| \geq 1,$$

$$-|\nabla d^{\delta,a}| \geq -1,$$

$$d_t^{\delta,a} - \Delta d^{\delta,a} + (\nabla d^{\delta,a} . \nabla \chi(v)) + C(\alpha,a)|\nabla d^{\delta,a}| \geq -K|\nabla d^{\delta,a}| d^{\delta,a},$$

in $\{(x,t), d^{\delta,a}(x,t) > 0\}$ in the sense of viscosity.

PROOF. Let $\varphi \in C^{2,1}(R^N \times (0,T))$, we assume that $d^{\delta,a} - \varphi$ has a strict minimum at point $(x_0, t_0) \in R^N \times (0,T)$. According to Lemma 3.4 we have that

$$\begin{cases} \varphi_t + (F_1^a)^*(x,t,\nabla\varphi, D^2\varphi) \geq -K|\nabla\varphi|d^{\delta,a} & (3.5) \\ |\nabla\varphi| = 1 & (3.6) \\ -(D^2\varphi\nabla\varphi.\nabla\varphi) \geq 0 & (3.7) \end{cases}$$

at the point (x_0, t_0). Since $\nabla\varphi(x_0, t_0) \neq 0$, we have that $(F_1^a)^*(x_0, t_0, \nabla\varphi(x_0, t_0), D^2\varphi(x_0, t_0)) = F_1^a(x_0, t_0, \nabla\varphi(x_0, t_0), D^2\varphi(x_0, t_0))$. In view of (3.7) we deduce from (3.5) that at the point (x_0, t_0)

$$\varphi_t - \Delta\varphi + (\nabla\varphi.\nabla\chi(v)) + C(\alpha, a)|\nabla\varphi| \geq -K|\nabla\varphi|d^{\delta,a}.$$

This complete the proof of lemma 3.5.

Following the proof of Theorem 9.1 in [1], we define

$$w^{\delta,a}(x,t) = \eta_\delta(d^{\delta,a}(x,t)), \tag{3.8}$$

where, as in [1], η_δ is a smooth function satisfying

$$(\mathrm{Def}_\eta)\begin{cases} \eta_\delta(z) = -\delta & \text{if } z \leq \delta/4 \\ \eta_\delta(z) = z - \delta & \text{if } z \geq \delta/2 \\ \eta_\delta(z) \leq -\delta/2 & \text{if } z \leq \delta/2 \\ 0 \leq \eta_\delta' \leq C & \text{and } |\eta_\delta''| \leq C\delta^{-1} \text{ on } R. \end{cases}$$

We remark that this definition implies that for all $z \in R$

$$\eta_\delta(z + 2\delta) \geq z \tag{3.9}$$

LEMMA 3.6. *There exists positive constants K and C such that for δ small enough we have that*

$$-|\nabla w^{\delta,a}| \geq -C, \tag{3.10}$$

and

$$\begin{aligned} w_t^{\delta,a} - \Delta w^{\delta,a} + (\nabla w^{\delta,a}.\nabla\chi(v)) + C(\alpha,a)|\nabla w^{\delta,a}| \\ \geq -K\delta^{-1} - K|\nabla w^{\delta,a}||w^{\delta,a}| \end{aligned} \tag{3.11}$$

in the sense of viscosity in $R^N \times (0,T)$.

Moreover we have that

$$|\nabla w^{\delta,a}| \geq 1, \tag{3.12}$$

$$-|\nabla w^{\delta,a}| \geq -1, \tag{3.13}$$

and

$$w_t^{\delta,a} - \Delta w^{\delta,a} + (\nabla w^{\delta,a}.\nabla\chi(v)) + C(\alpha,a)|\nabla w^{\delta,a}| + K|\nabla w^{\delta,a}|d^{\delta,a} \geq 0 \qquad (3.14)$$

in the sense of viscosity in $\{(x,t), d^{\delta,a}(x,t) > \delta/2\}$.

PROOF. Let $\varphi \in C^{2,1}(R^N \times (0,T))$; we suppose that $w^{\delta,a} - \varphi$ has a local minimum at the point $(x_0,t_0) \in R^N \times (0,T)$. Subtracting if necessary a constant from φ we may assume that $\varphi(x_0,t_0) = w^{\delta,a}(x_0,t_0)$, and moreover modifying φ we may also suppose that (x_0,t_0) is a strict minimum. (For instance we may replace φ by $\varphi + |x-x_0|^4 + |t-t_0|^4 - \phi(x_0,t_0)$, which does not modify the values of the first and second derivatives of φ at the point (x_0,t_0).) Using the notation $B_\rho(x_0,t_0) := \{(x,t), |x-x_0| + |t-t_0| < \rho\}$ we deduce that

$$w^{\delta,a}(x,t) \geq \varphi(x,t) \qquad \text{for all } (x,t) \text{ in } B_{\rho_0}(x_0,t_0)\backslash\{(x_0,t_0)\}, \qquad (3.15)$$

with equality at the point (x_0,t_0).

(i) We first consider the case where $d^{\delta,a}(x_0,t_0) > 0$.

Let $e > 0$; we set $\eta^e(z) = \eta_\delta(z) + ez$ for $z \in R$ and $\rho^e = (\eta^e)^{-1}$. Next we prove the following result, which will be useful to complete the proof of Lemma 3.6.

LEMMA 3.7. *The function* $\eta^e(d^{\delta,a}) - \varphi$ *attains its minimum in* $\bar{B}_{\rho_0}(x_0,t_0)$ *at a point* (x^e,t^e). *Moreover for e small enough we have that* (x^e,t^e) *is in* $B_{\rho_0}(x_0,t_0)$, *and that* $\lim_{e\to 0}(x^e,t^e) = (x_0,t_0)$. *Furthermore*

$$\lim_{e\to 0}\eta_\delta(d^{\delta,a}(x^e,t^e)) = \eta_\delta(d^{\delta,a}(x_0,t_0)). \qquad (3.16)$$

PROOF. Since $\eta^e(d^{\delta,a}) - \varphi$ is lower-semicontinuous on the compact set $\bar{B}_{\rho_0}(x_0,t_0)$, there exists (x^e,t^e) such that $\eta^e(d^{\delta,a}) - \varphi$ attains its minimum in (x^e,t^e) in $\bar{B}_{\rho_0}(x_0,t_0)$. We now prove that (x^e,t^e) is in $B_{\rho_0}(x_0,t_0)$, and moreover we show that for e small enough (x^e,t^e) tends to (x_0,t_0). For all $\rho < \rho_0$ we set

$$\lambda_\rho := \inf_{\bar{B}_{\rho_0}(x_0,t_0)\backslash\{B_\rho(x_0,t_0)\}} (\eta_\delta(d^{\delta,a}) - \varphi).$$

Since $\eta^e(d^{\delta,a}) - \varphi$ attains its minimum at (x^e,t^e) in $\bar{B}_{\rho_0}(x_0,t_0)$ we have

$$\begin{aligned}(\eta^e(d^{\delta,a}) - \varphi)(x^e,t^e) &\leq (\eta^e(d^{\delta,a}) - \varphi)(x_0,t_0)\\ &\leq (\eta_\delta(d^{\delta,a}) - \varphi)(x_0,t_0) + ed^{\delta,a}(x_0,t_0) = ed^{\delta,a}(x_0,t_0). \qquad (3.17)\end{aligned}$$

Moreover since (x_0,t_0) is a strict minimum of $\eta_\delta(d^{\delta,a}) - \varphi$ we have that $\lambda_\rho >$

$(\eta_\delta(d^{\delta,a}) - \varphi)(x_0, t_0) = 0$. Thus we deduce from (3.17) that if we define

$$e(\rho) := \frac{\lambda_\rho}{d^{\delta,a}(x_0, t_0)} > 0,$$

then

$$(\eta_\delta(d^{\delta,a}) - \varphi)(x^e, t^e) \le (\eta^e(d^{\delta,a}) - \varphi)(x_0, t_0) < \lambda_\rho,$$

for all $0 < e < e(\rho)$. This implies that (x^e, t^e) is not in $\bar{B}_{\rho_0}(x_0, t_0) \backslash B_\rho(x_0, t_0)$. Thus we conclude that (x^e, t^e) is in $B_\rho(x_0, t_0)$. More precisely we have shown that for all $\rho < \rho_0$, there exists $e(\rho)$ such that for all $e \in (0, e(\rho))$, $(x^e, t^e) \in B_\rho(x_0, t_0)$, which implies that $\lim_{e\to 0}(x^e, t^e) = (x_0, t_0)$.

Next we prove (3.16). It follows from the definition of (x_0, t_0) that

$$(\eta^e(d^{\delta,a}) - \varphi)(x^e, t^e) > (\eta_\delta(d^{\delta,a}) - \varphi)(x_0, t_0) + ed^{\delta,a}(x^e, t^e) \tag{3.18}$$

$$> (\eta_\delta(d^{\delta,a}) - \varphi)(x_0, t_0). \tag{3.19}$$

Letting e tend to zero in (3.17), and (3.19) we deduce that

$$\lim_{e\to 0}(\eta^e(d^{\delta,a}) - \varphi)(x^e, t^e) = (\eta_\delta(d^{\delta,a}) - \varphi)(x_0, t_0). \tag{3.20}$$

Using (3.20), and letting e tend to zero in (3.18) we deduce that $\lim_{e\to 0} ed^{\delta,a}(x^e, t^e) = 0$. In view of (3.20) and the fact that $\varphi(x^e, t^e)$ tends to $\varphi(x_0, t_0)$ it follows that $\lim_{e\to 0} \eta_\delta(d^{\delta,a}(x^e, t^e)) = \eta_\delta(d^{\delta,a}(x_0, t_0))$. This completes the proof of Lemma 3.7.

We now return to the proof of Lemma 3.6. From Lemma 3.7 we deduce that for all (x, t) in a neighborhood $B_{\rho_0}(x_0, t_0)$ of (x_0, t_0) we have that $(\eta^e(d^{\delta,a}) - \varphi)(x, t) \ge (\eta^e(d^{\delta,a}) - \varphi)(x^e, t^e)$, which in turn implies that

$$d^{\delta,a}(x, t) \ge \rho^e[(\eta^e(d^{\delta,a}) - \varphi)(x^e, t^e) + \varphi(x, t)] =: \psi^e(x, t), \tag{3.21}$$

where we have used the strict monotonicity of the function ρ^e. Note that the definition of ψ^e in (3.21) implies that $\psi^e(x^e, t^e) = d^{\delta,a}(x^e, t^e)$. Moreover since $d^{\delta,a}(x_0, t_0) > 0$ and since by Lemma 3.3 $d^{\delta,a}$ is lower semicontinuous we have that $d^{\delta,a}(x^e, t^e) > 0$ for e small enough. Using (3.21) and Lemma 3.5 we deduce that at the point (x^e, t^e)

$$|\nabla\psi^e(x^e, t^e)| = 1 \tag{3.22}$$

$$[(\psi^e)_t - \Delta\psi^e + (\nabla\psi^e.\nabla\chi(v)) + C(\alpha, a)|\nabla\psi^e| + K|\nabla\psi^e|d^{\delta,a}](x^e, t^e) \ge 0. \tag{3.23}$$

We now show (3.10). We deduce from (3.21) and (3.22) that

$$|\nabla\varphi(x^e, t^e)|\,|(\rho^e)'(\eta^e(d^{\delta,a}(x^e, t^e)))| = 1. \tag{3.24}$$

Moreover we note that $(\rho^e)'(\eta^e(d^{\delta,a}(x^e,t^e))).(\eta^e)'(d^{\delta,a}(x^e,t^e)) = 1$, which we substitute in (3.24) to obtain

$$|\nabla\varphi(x^e,t^e)| = |(\eta^e)'(d^{\delta,a}(x^e,t^e))| \tag{3.25}$$

Finally using that $e \le (\eta^e)' \le e + C$ and letting e tend to zero in (3.25), we obtain

$$-|\nabla\varphi(x_0,t_0)| \ge -C \tag{3.26}$$

which proves (3.10). Next we show (3.11). In view of (3.23) and the definition of ψ^e in (3.21) we obtain

$$\begin{aligned}(\rho^e)'(\eta^e(d^{\delta,a}(x^e,t^e)))[\varphi_t - \Delta\varphi + (\nabla\varphi.\nabla\chi(v)) + C(\alpha,a)|\nabla\varphi|\\ + K|\nabla\varphi|d^{\delta,a}](x^e,t^e) - |\nabla\varphi|^2(\rho^e)''(\eta^e(d^{\delta,a}(x^e,t^e))) \ge 0.\end{aligned} \tag{3.27}$$

Futhermore differentiating $\eta^e[\rho^e(s)] = s$ we obtain

$$\frac{(\rho^e)''(\eta^e(d^{\delta,a}(x^e,t^e)))}{(\rho^e)'(\eta^e(d^{\delta,a}(x^e,t^e)))} = -(\eta^e)''(\psi^e(x^e,t^e))[(\rho^e)'(\eta^e(d^{\delta,a}(x^e,t^e)))]^2, \tag{3.28}$$

which we substitute in (3.27) to deduce that

$$\begin{aligned}J(\varphi)(x^e,t^e) &:= (\varphi_t - \Delta\varphi + (\nabla\varphi.\nabla\chi(v)) + C(\alpha,a)|\nabla\varphi|)(x^e,t^e)\\ &\ge -K(|\nabla\varphi|d^{\delta,a})(x^e,t^e)\\ &\quad -|\nabla\varphi(x^e,t^e)|^2(\eta^e)''(\psi^e(x^e,t^e))[(\rho^e)'(\eta^e(x^e,t^e))]^2.\end{aligned} \tag{3.29}$$

Moreover since $|(\eta^e)''| = |\eta_\delta''| \le C\delta^{-1}$ and in view of (3.24) we obtain

$$J(\varphi)(x^e,t^e) \ge -K|\nabla\varphi(x^e,t^e)|d^{\delta,a}(x^e,t^e) - C\delta^{-1}. \tag{3.30}$$

Furthermore we deduce from (3.8), and (3.9) that $w^{\delta,a} \ge d^{\delta,a} - 2\delta$, which we substitute in (3.30) to obtain

$$J(\varphi)(x^e,t^e) \ge -K|\nabla\varphi(x^e,t^e)|\,|w^{\delta,a}(x^e,t^e)| - 2K\delta|\nabla\varphi(x^e,t^e)| - C\delta^{-1}.$$

Finally letting e tend to zero in the inequality above we conclude from the continuity of $w^{\delta,a}$ in (x_0,t_0) (see (3.16)), (3.26), and the definition of $J(\varphi)$ that for δ small enough

$$J(\varphi)(x_0,t_0) \ge -K|\nabla\varphi(x_0,t_0)|\,|w^{\delta,a}(x_0,t_0)| - 2\tilde{K}\delta^{-1}.$$

This completes the proof of (3.11). Next we show (3.12) and (3.13). We assume that $d^{\delta,a}(x_0,t_0) > \delta/2$, which implies, since $d^{\delta,a}$ is lower semicontinuous, that $d^{\delta,a}(x^e,t^e)) > \delta/2$ and thus that $(\eta^e)'(d^{\delta,a}(x^e,t^e)) = 1 + e$.

Substituting this in (3.25) and letting e tend to zero yields

$$|\nabla\varphi(x_0, t_0)| = 1, \tag{3.31}$$

which completes the proof of (3.12) and (3.13). We finally show (3.14). We first note that $d^{\delta,a}(x^e, t^e)) > \delta/2$ implies that $(\eta^e)''(d^{\delta,a}(x^e, t^e)) = 0$, which we substitute into (3.29) to obtain

$$J(\varphi)(x^e, t^e) \geq -K|\nabla\varphi(x^e, t^e)|d^{\delta,a}(x^e, t^e). \tag{3.32}$$

Using again that $d^{\delta,a} > \delta/2$ at the points (x_0, t_0) and (x^e, t^e) we have that at these two points $w^{\delta,a} = d^{\delta,a} - \delta$ and thus we deduce from the continuity of $w^{\delta,a}$ at the point (x_0, t_0) and from (3.16) that $\lim_{e\to 0} d^{\delta,a}(x^e, t^e) = d^{\delta,a}(x_0, t_0)$. Finally letting e tend to zero in (3.32) we conclude that

$$J(\varphi)(x_0, t_0) \geq -K|\nabla\varphi(x_0, t_0)|d^{\delta,a}(x_0, t_0),$$

which coincides with (3.14).
(ii) We now consider the case where $d^{\delta,a}(x_0, t_0) = 0$, which in view of (Def_η) implies that $w^{\delta,a}(x_0, t_0) = -\delta$. Set $B_h(x_0) := \{x \in R^N, |x - x_0| < h\}$. It follows from Lemma 3.3 that for h small enough,

$$d^{\delta,a}(x, t) < \delta/4 \qquad \text{for all } (x, t) \in B_h(x_0) \times [t_0 - h, t_0],$$

which in turn implies that $w^{\delta,a}(x, t) = -\delta$. Furthermore since the point (x_0, t_0) is a local minimum of the function $w^{\delta,a} - \varphi$, we conclude that for h small enough,

$$\varphi(x, t) \leq \varphi(x_0, t_0) \qquad \text{for all } (x, t) \in B_h(x_0) \times [t_0 - h, t_0].$$

This implies that

$$\begin{cases} \nabla\varphi(x_0, t_0) = 0, \\ (D^2\varphi(x_0, t_0)p.p) \leq 0 \qquad \text{for all } p \in R^N, \\ \text{and } \lim_{h\to 0} \dfrac{\varphi(x_0, t_0 - h) - \varphi(x_0, t_0)}{-h} = (\varphi(x_0, t_0))_t \geq 0. \end{cases}$$

We conclude that inequality (3.10) is satisfied and that

$$(\varphi_t - tr(D^2\varphi) + (\nabla\varphi.\nabla\chi(v)) + C(\alpha, a)|\nabla\varphi| + K|\nabla\varphi|\,|w^{\delta,a}|)(x_0, t_0) \geq 0,$$

which gives (3.11). This completes the proof of Lemma 3.6.

3.1.2. A supersolution for Problem (P_1^ε)

We suppose that $a \in [0, 1]$ and define

$$\bar{\phi}^\varepsilon(x, t) := q\left(\frac{w^{\delta,a}(x, t)}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right) \qquad \text{for all } (x, t) \in R^N \times [0, T]. \tag{3.33}$$

LEMMA 3.8. *For all $a > 0, \varepsilon < \delta^2$, and ε small enough $\bar{\phi}^{\varepsilon}$ is a viscosity supersolution of the parabolic equation in Problem (P_1^{ε}).*

PROOF. Let $\varphi \in C^{2,1}(R^N \times (0,T))$. We assume that $\bar{\phi}^{\varepsilon} - \varphi$ has a local minimum at the point $(x_0, t_0) \in R^N \times (0,T)$. Subtracting, if necessary, a constant from φ we may assume that there exists a neighborhood $N(x_0, t_0)$ of (x_0, t_0) such that

$$(\bar{\phi}^{\varepsilon} - \varphi)(x,t) \geq (\bar{\phi}^{\varepsilon} - \varphi)(x_0, t_0) = 0 \qquad \text{for all } (x,t) \in N(x_0, t_0).$$

Next we prove that

$$L_1^{\varepsilon}(\varphi(x_0,t_0)) = \left(\varphi_t - \Delta\varphi + (\nabla\varphi.\nabla\chi(v)) + \varphi\Delta\chi(v) - \frac{1}{\varepsilon^2} f(\varphi, \varepsilon\alpha)\right)(x_0, t_0) \geq 0, \tag{3.34}$$

which is the result of Lemma 3.8. In view of the strict monotonicity of the travelling wave q we have that $h_-(\varepsilon\alpha, \varepsilon a) < \bar{\phi}^{\varepsilon}(x_0, t_0) = \varphi(x_0, t_0) < h_+(\varepsilon\alpha, \varepsilon a)$. This implies that for all (x,t) in a neighborhood $N_1(x_0, t_0)$ of (x_0, t_0) we have that $h_-(\varepsilon\alpha, \varepsilon a) < \varphi(x,t) < h_+(\varepsilon\alpha, \varepsilon a)$. Using that q_r is strictly positive we deduce that there exists a function $y = y(x,t) \in C^{2,1}(N_1(x_0, t_0))$ such that for all (x,t) in $N_1(x_0, t_0)$ we have that

$$\varphi(x,t) = q\left(\frac{y(x,t)}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right) \tag{3.35}$$

$$\leq \bar{\phi}^{\varepsilon}(x,t) = q\left(\frac{w^{\delta,a}(x,t)}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right), \tag{3.36}$$

with equality at the point (x_0, t_0). Substituting (3.35) into (3.34) we deduce that

$$\begin{aligned} \varepsilon L_1^{\varepsilon}(\varphi(x_0,t_0)) = & \frac{1}{\varepsilon} q_{rr}\left(\frac{y}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right)(1 - |\nabla y|^2)(x_0, t_0) \\ & + q_r\left(\frac{y}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right)\left[y_t - \Delta y + (\nabla y.\nabla\chi(v)) + \frac{c(\varepsilon\alpha, \varepsilon a)}{\varepsilon}\right](x_0, t_0) \\ & + a + \varepsilon q\left(\frac{y}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right)\Delta\chi(v)(x_0, t_0). \end{aligned} \tag{3.37}$$

It follows from computations performed in the appendix (cf. (A.12) Lemma A.3) that

$$\lim_{\varepsilon\to 0} \frac{c(\varepsilon\alpha, \varepsilon a)}{\varepsilon} = C(\alpha, a) = -\sqrt{2}\alpha - 6\sqrt{2}a. \tag{3.38}$$

We also recall that by Lemma A.2

$$(|q_{rr}| + |q_r|)(r, \varepsilon\alpha, \varepsilon a) \leq K_1 e^{-K_2|r|} \tag{3.39}$$

for all r in R. Furthermore since q is strictly increasing we deduce from (3.36) that

$$y(x,t) \leq w^{\delta,a}(x,t) \qquad \text{for all } (x,t) \in N_1(x_0,t_0), \tag{3.40}$$

with equality at the point (x_0, t_0). This implies in view of Lemma 3.6 that

$$-|\nabla y(x_0,t_0)| \geq -C \tag{3.41}$$

$$\begin{aligned}(y_t - \Delta y + (\nabla y.\nabla\chi(v)) &+ C(\alpha,a)|\nabla y|)(x_0,t_0)\\ &\geq -K\delta^{-1} - K|\nabla y|\,|w^{\delta,a}|(x_0,t_0)\end{aligned} \tag{3.42}$$

and moreover when $d^{\delta,a}(x_0,t_0) > \delta/2$ the function y satisfies

$$|\nabla y(x_0,t_0)| = 1 \tag{3.43}$$

$$(y_t - \Delta y + (\nabla y.\nabla\chi(v)) + C(\alpha,a)|\nabla y|)(x_0,t_0) \geq -K|\nabla y| d^{\delta,a}(x_0,t_0). \tag{3.44}$$

(i) We first consider the case where $d^{\delta,a}(x_0,t_0) > \delta/2$. Substituting (3.43) and (3.44) in (3.37) we obtain

$$\begin{aligned}\varepsilon L_1^\varepsilon(\varphi(x_0,t_0)) \geq q_r\left(\frac{y(x_0,t_0)}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right)&\left[\frac{c(\varepsilon\alpha,\varepsilon a)}{\varepsilon} - C(\alpha,a) - Kd^{\delta,a}(x_0,t_0)\right]\\ &+ a + \varepsilon q\left(\frac{y}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right)\Delta\chi(v)(x_0,t_0).\end{aligned} \tag{3.45}$$

Moreover using that by $(\mathrm{Def}_\eta)\, y(x_0,t_0) = w^{\delta,a}(x_0,t_0) = d^{\delta,a}(x_0,t_0) - \delta$, and (3.39) we obtain,

$$\begin{aligned}\varepsilon L_1^\varepsilon(\varphi(x_0,t_0)) \geq &- KK_1\delta - K_1\left|\frac{c(\varepsilon\alpha,\varepsilon a)}{\varepsilon} - C(\alpha,a)\right|\\ &- Kq_r\left(\frac{y(x_0,t_0)}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right)|y(x_0,t_0)|\\ &+ a + \varepsilon q\left(\frac{y}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right)\Delta\chi(v)(x_0,t_0).\end{aligned} \tag{3.46}$$

Using again (3.39), and the inequality $se^{-s} < 1$, we deduce that

$$q_r\left(\frac{y(x_0,t_0)}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right)|y(x_0,t_0)| < K_1|y(x_0,t_0)|e^{-K_2(|y(x_0,t_0)|/\varepsilon)} < \frac{K_1}{K_2}\varepsilon.$$

Thus we deduce from (3.46) that

$$\varepsilon L_1^\varepsilon(\phi)(x_0,t_0) \geq a - KK_1\delta - K_1\left|\frac{c(\varepsilon\alpha,\varepsilon a)}{\varepsilon} - C(\alpha,a)\right| - K\frac{K_1}{K_2}\varepsilon - \tilde{C}\varepsilon,$$

where $\tilde{C}$ is an upper bound for the term $q(.,\varepsilon\alpha,\varepsilon a)\Delta\chi(v)$. Finally we deduce from (3.38) that for all $0<a<1$ there exist δ_0 and ε_0 such that for all $\delta\in(0,\delta_0)$ and $\varepsilon\in(0,\varepsilon_0)$ we have that $\varepsilon L_1^\varepsilon(\varphi(x_0,t_0)) \geq a/2 > 0$.

(ii) Next we consider the case $d^{\delta,a}(x_0,t_0)\leq\delta/2$. We substitute (3.42) into (3.37) to obtain

$$\begin{aligned}
&\varepsilon L_1^\varepsilon(\varphi(x_0,t_0))\\
&\quad\geq \frac{1}{\varepsilon}q_{rr}\left(\frac{y(x_0,t_0)}{\varepsilon},\varepsilon\alpha,\varepsilon a\right)(1-|\nabla y(x_0,t_0)|^2) + q_r\left(\frac{y(x_0,t_0)}{\varepsilon},\varepsilon\alpha,\varepsilon a\right)\\
&\quad\left[\frac{c(\varepsilon\alpha,\varepsilon a)}{\varepsilon} - C(\alpha,a)|\nabla y(x_0,t_0)| - K\delta^{-1} - K|\nabla y(x_0,t_0)|\,|y(x_0,t_0)|\right]\\
&\quad + a + \varepsilon q\left(\frac{y(x_0,t_0)}{\varepsilon},\varepsilon\alpha,\varepsilon a\right)\Delta\chi(v(x_0,t_0)).
\end{aligned} \tag{3.47}$$

Moreover by (Def$_\eta$) we have that $y(x_0,t_0) = w^{\delta,a}(x_0,t_0)\leq -\delta/2$, and it follows from (3.9) that $y(x_0,t_0)\geq d^{\delta,a}(x_0,t_0)-2\delta\geq -2\delta$; thus

$$\delta/2 \leq |y(x_0,t_0)| \leq 2\delta. \tag{3.48}$$

We deduce from (3.48) and (3.39) that

$$(|q_{rr}|+|q_r|)\left(\frac{y(x_0,t_0)}{\varepsilon},\varepsilon\alpha,\varepsilon a\right) \leq K_1 e^{-K_2(\delta/2\varepsilon)},$$

which we substitute in (3.47); also using (3.41) we obtain

$$\begin{aligned}
\varepsilon L_1^\varepsilon(\varphi)(x_0,t_0) \geq -K_1 e^{-K_2(\delta/2\varepsilon)}&\left[\frac{1}{\varepsilon}(1+C^2) + \left|\frac{c(\varepsilon\alpha,\varepsilon a)}{\varepsilon}\right| + C|C(\alpha,a)|\right.\\
&\left. + 2KC\delta + K\delta^{-1}\right] - \tilde{C}\varepsilon + a.
\end{aligned} \tag{3.49}$$

We choose $\delta > \sqrt{\varepsilon}$ in (3.49) to deduce that

$$\begin{aligned}
\varepsilon L_1^\varepsilon(\varphi)(x_0,t_0) \geq -K_1 e^{-(K_2/2\sqrt{\varepsilon})}&\left[\frac{1}{\varepsilon}(1+C^2) + \left|\frac{c(\varepsilon\alpha,\varepsilon a)}{\varepsilon}\right| + C|C(\alpha,a)| + K\frac{1}{\sqrt{\varepsilon}}\right]\\
&-4KC\frac{K_1}{K_2}\varepsilon - \tilde{C}\varepsilon + a.
\end{aligned} \tag{3.50}$$

Moreover since $C(\alpha, a)$ is bounded, we deduce from (3.38) that the quantity

$$\left[\left|\frac{c(\varepsilon\alpha, \varepsilon a)}{\varepsilon}\right| + C|C(\alpha, a)|\right]$$

is bounded. Finally we conclude that for $0 < a < 1$, and ε small enough we have $\varepsilon L_1^\varepsilon(\varphi)(x_0, t_0) \geq a/2 > 0$. This completes the proof of the Lemma 3.8.

3.1.3. Uniform convergence of $u^{\delta, a}$

In this section we prove the following result.

THEOREM 3.9. *$u^{\delta,a}$ (resp. $u^{-\delta,-a}$) tends to u uniformly on compact sets of $R^N \times [0, T]$ as (δ, a) tends to $(0,0)$.*

We first state three preliminary lemmas.

LEMMA 3.10. *Let (δ_0, a_0) be fixed positive real numbers. Then for all $0 \leq \delta \leq \delta_0$ and $0 \leq a \leq a_0$ we have that*

$$u^{-\delta_0, -a_0}(x, t) \leq u^{\delta, a}(x, t) \leq u^{\delta_0, a_0}(x, t), \tag{3.51}$$

for all $(x, t) \in R^N \times [0, T]$.

PROOF. We note that for all $0 \leq a \leq a_0$ (cf. Lemma A.3), we have that

$$C(\alpha, a) = -\sqrt{2}\alpha - 6\sqrt{2}a \geq C(\alpha, a_0),$$

which implies that $u^{\delta,a}$ is a subsolution of the equation

$$u_t + F_1^{a_0}(x, t, Du, D^2u) = 0.$$

Moreover for all $\delta \leq \delta_0$ we have $u^{\delta,a}(x, 0) = U_0(x) + 2\delta \leq u^{\delta_0, a_0}(x, 0)$, for all $x \in R^N$. Thus we deduce from the comparison principle that

$$u^{\delta,a}(x, t) \leq u^{\delta_0, a_0}(x, t), \qquad \text{for all } (x, t) \in R^N \times [0, T]. \tag{3.52}$$

In a similar way we can prove that $u^{-\delta_0, -a_0} \leq u^{\delta, a}$, which completes the proof of Lemma 3.10.

As it is done by Crandall, Ishii and Lions [4], we define

$$u^+(x, t) = \lim_{\{\nu \to 0\}} \sup \{u^{\delta,a}(z, \theta), \text{ for all } 0 \leq a \leq \nu, 0 \leq \delta \leq \nu, \text{ and for all } (z, \theta) \in R^N \times (0, T) \text{ such that } |x - z| \leq \nu, \text{ and } |t - \theta| \leq \nu\},$$

and

$$u^-(x, t) = \lim_{\{\nu \to 0\}} \inf \{u^{\delta,a}(z, \theta), \text{ for all } 0 \leq a \leq \nu, 0 \leq \delta \leq \nu, \text{ and for all } (z, \theta) \in R^N \times (0, T) \text{ such that } |x - z| \leq \nu, \text{ and } |t - \theta| \leq \nu\}.$$

Next we give some properties of u^+ and u^-.

Lemma 3.11. *$u^-(x,0) = u(x,0) = u^+(x,0)$, for all $x \in R^N$.*

Proof. Let (δ_0, a_0) be fixed positive real numbers, $\nu \in (0, \min(\delta_0, a_0))$, and let (δ, a) be such that $0 \le \delta \le \nu$ and $0 \le a \le \nu$; also let $x \in R^N$. We deduce from Lemma 3.10 that

$$u^{\delta,a}(z,\theta) \le u^{\delta_0,a_0}(z,\theta), \tag{3.53}$$

for all (z,θ) such that $|x-z| < \nu$ and $|\theta| < \nu$. Since u^{δ_0,a_0} is continuous, we deduce by letting ν tend to zero in (3.53) that $\lim_{\nu\to 0}\sup u^{\delta,a}(z,\theta) \le u^{\delta_0,a_0}(x,0)$. Thus

$$u^+(x,0) \le u^{\delta_0,a_0}(x,0) = U_0(x) + 2\delta_0, \qquad \text{for all } x \in R^N.$$

Similarly one can check that $u^-(x,0) \ge U_0(x) - 2\delta_0$. Thus we have shown that for all $\delta_0 \ge 0$ we have

$$U_0(x) - 2\delta_0 \le u^-(x,0) \le u^+(x,0) \le U_0(x) + 2\delta_0,$$

for all x in R^N. Letting δ_0 tend to zero we finally obtain that

$$u^-(x,0) = u^+(x,0) = U_0(x), \qquad \text{for all } x \in R^N.$$

This completes the proof of Lemma 3.11.

Lemma 3.12. *u^+ (resp. u^-) is a viscosity subsolution (resp. supersolution) of $u_t + F_1(x,t,\nabla u, D^2u) = 0$.*

Proof. First we note that u^+ is upper-semicontinuous. Indeed let μ positive be arbitrary and let (x_j, t_j) converge to a point (x,t) as j tends to $+\infty$. It follows from the definition of u^+ that there exists ν_0 positive such that for all $\nu \le \nu_0$

$$\left| u^+(x,t) - \sup_{a,\delta,|z-x|,|t-\theta|\le\nu} u^{\delta,a}(z,\theta) \right| \le \mu. \tag{3.54}$$

For j large enough we have that $|x_j - x| < \dfrac{\nu_0}{2}$ and $|t_j - t| < \dfrac{\nu_0}{2}$ so that

$$\sup_{a,\delta,|z-x_j|,|\theta-t_j|\le\nu/2} u^{\delta,a}(z,\theta) \le \sup_{a,\delta,|z-x|,|\theta-t|\le\nu_0} u^{\delta,a}(z,\theta) \le u^+(x,t) + \mu,$$

which together with the definition of u^+ implies that for j large enough

$$u^+(x_j,t_j) \le u^+(x,t) + \mu \qquad \text{for all } \mu > 0.$$

Letting j tend to $+\infty$ and μ tend to 0 we deduce that u^+ is upper-semicontinuous.

Let $\varphi \in C^{2,1}(R^N \times (0,T))$; we suppose that $u^+ - \varphi$ has a local maximum at the point $(x_0, t_0) \in R^N \times (0,T)$. Modifying φ if necessary, we may assume that (x_0, t_0) is a strict maximum of $u^+ - \varphi$ in a neighborhood $B_\rho(x_0, t_0)$. Next we prove the following result, which will be useful in the proof of Lemma 3.12.

LEMMA 3.13. *There exists a subsequence (δ_j, a_j) such that $(\delta_j, a_j) \to 0$ as j tends to $+\infty$ and such that the function $u^{\delta_j, a_j} - \varphi$ attains its maximum in $\bar{B}_{\rho_0}(x_0, t_0)$ at a point $(x_{\delta_j, a_j}, t_{\delta_j, a_j})$. Moreover for (δ_j, a_j) small enough we have that $(x_{\delta_j, a_j}, t_{\delta_j, a_j}) \in B_{\rho_0}(x_0, t_0)$ and $\lim_{(\delta_j, a_j)\to(0,0)}(x_{\delta_j, a_j}, t_{\delta_j, a_j}) = (x_0, t_0)$.*

PROOF. Since the function $(u^{\delta,a} - \varphi)$ is continuous, it admits a maximum at a point $(x_{\delta,a}, t_{\delta,a})$ in $\bar{B}_{\rho_0}(x_0, t_0)$. Moreover by the definition of u^+ there exists a subsequence $(\delta_j, a_j, x_j, t_j)$ such that $(\delta_j, a_j, x_j, t_j)$ tends to $(0, 0, x_0, t_0)$ as j tends to $+\infty$ and

$$u^+(x_0, t_0) = \lim_{j\to\infty} u^{\delta_j, a_j}(x_j, t_j). \tag{3.55}$$

Since the sequence $(x_{\delta_j, a_j}, t_{\delta_j, a_j})$ is in the compact set $\bar{B}_{\rho_0}(x_0, t_0)$ there exists a subsequence $(x_{\delta_{k(j)}, a_{k(j)}}, t_{\delta_{k(j)}, a_{k(j)}})$, which converges to some point $(\tilde{x}, \tilde{t})$ in $\bar{B}_{\rho_0}(x_0, t_0)$ as j tends to $+\infty$. Furthermore we note that for all $(x,t) \in \bar{B}_{\rho_0}(x_0, t_0)$ and all sequences $(\delta_n, a_n, x_n, t_n)$ converging to $(0, 0, x, t)$ we have

$$\limsup_{n\to\infty} u^{\delta_n, a_n}(x_n, t_n) \le u^+(x, t). \tag{3.56}$$

Applying (3.56) at the point $(\tilde{x}, \tilde{t})$ we deduce that

$$(u^+ - \varphi)(\tilde{x}, \tilde{t}) \ge \limsup_{j\to\infty}(u^{\delta_{k(j)}, a_{k(j)}} - \varphi)(x_{\delta_{k(j)}, a_{k(j)}}, t_{\delta_{k(j)}, a_{k(j)}}). \tag{3.57}$$

Using the fact that the function $u^{\delta_{k(j)}, a_{k(j)}} - \varphi$ attains its maximum at the point $(x_{\delta_{k(j)}, a_{k(j)}}, t_{\delta_{k(j)}, a_{k(j)}})$ we obtain

$$(u^{\delta_{k(j)}, a_{k(j)}} - \varphi)(x_{\delta_{k(j)}, a_{k(j)}}, t_{\delta_{k(j)}, a_{k(j)}}) \ge (u^{\delta_{k(j)}, a_{k(j)}} - \varphi)(x_{k(j)}, t_{k(j)}).$$

Substituting this into (3.57) and using (3.55) we deduce that

$$(u^+ - \varphi)(\tilde{x}, \tilde{t}) \ge \lim_{j\to\infty}(u^{\delta_{k(j)}, a_{k(j)}} - \varphi)(x_{k(j)}, t_{k(j)}) = (u^+ - \varphi)(x_0, t_0).$$

Since (x_0, t_0) is a strict maximum of $(u^+ - \varphi)$ we conclude that $(\tilde{x}, \tilde{t}) = (x_0, t_0)$. We have thus shown that the sequence $(x_{\delta_j, a_j}, t_{\delta_j, a_j})$ tends to (x_0, t_0) and consequently that for j large enough the point $(x_{\delta_j, a_j}, t_{\delta_j, a_j})$ is in $B_{\rho_0}(x_0, t_0)$. This completes the proof of Lemma 3.13.

We now return to the proof of Lemma 3.12. Using Lemma 3.13 and the fact that u^{δ_j, a_j} is a solution of the equation $u_t + F_1^{a_j}(x, t, \nabla u, D^2 u) = 0$, we deduce that

$$\varphi_t(x_{\delta_j,a_j}, t_{\delta_j,a_j}) + (F_1^{a_j})_*(x_{\delta_j,a_j}, t_{\delta_j,a_j}, \nabla\varphi(x_{\delta_j,a_j}, t_{\delta_j,a_j}), D^2\varphi(x_{\delta_j,a_j}, t_{\delta_j,a_j})) \leq 0.$$

By definition of $(F_1^{a_j})_*$ this can be rewritten as

$$\varphi_t(x_{\delta_j,a_j}, t_{\delta_j,a_j}) + (F_1)_*(x_{\delta_j,a_j}, t_{\delta_j,a_j}, \nabla\varphi(x_{\delta_j,a_j}, t_{\delta_j,a_j}), D^2\varphi(x_{\delta_j,a_j}, t_{\delta_j,a_j})) - 2\sqrt{2}a_j|\nabla\varphi(x_{\delta_j,a_j}, t_{\delta_j,a_j})| \leq 0. \tag{3.58}$$

Next we let a_j and δ_j tend to zero in (3.58) and we use the lower semicontinuity of $(F_1)_*$ and the fact that $\lim_{(\delta_j,a_j)\to(0,0)}(x_{\delta_j,a_j}, t_{\delta_j,a_j}) = (x_0, t_0)$ to deduce that

$$\varphi_t(x_0, t_0) + (F_1)_*(x_0, t_0, \nabla\varphi(x_0, t_0), D^2\varphi(x_0, t_0)) \leq 0.$$

Therefore u^+ is a viscosity subsolution of the equation $u_t + F_1(x, t, \nabla u, D^2u) = 0$. Similarly we can prove that u^- is a viscosity supersolution of this same equation, which completes the proof of Lemma 3.12.

LEMMA 3.14. $u^-(x,t) = u(x,t) = u^+(x,t)$, *for all* $(x,t) \in R^N \times (0,T]$.

PROOF. We deduce from the lemmas 3.11, 3.12 and from the comparison principle that

$$u^+(x,t) \leq u(x,t) \leq u^-(x,t), \qquad \text{for all } x \in R^N \times [0,T].$$

Moreover, since by definition $u^+ \geq u^-$, we conclude that $u^-(x,t) = u(x,t) = u^+(x,t)$ for all $x \in R^N \times [0,T]$. This completes the proof of Lemma 3.14.

We are now in a position to prove Theorem 3.10. We give a proof by contradiction. Suppose that $u^{\delta,a}$ does not tend to u uniformly on a compact set K of $R^N \times [0,T]$ as (δ,a) tends to $(0,0)$. This implies that there exists a real number $m > 0$, and a subsequence (d_j, a_j) such that $(d_j, a_j) \to 0$ as $j \to +\infty$, and a sequence $(x_j, t_j) \in K$ such that

$$u(x_j, t_j) - u^{\delta_j,a_j}(x_j, t_j) < -m \qquad \text{or} \qquad u(x_j, t_j) - u^{\delta_j,a_j}(x_j, t_j) > m. \tag{3.59}$$

Since K is compact we may suppose that the sequence (x_j, t_j) converges to a point $(x,t) \in K$. Moreover using (3.56) we have that

$$\limsup_{j\to\infty} u^{\delta_j,a_j}(x_j, t_j) \leq u^+(x,t). \tag{3.60}$$

Similarly we have that for all $(x,t) \in \bar{B}_{\rho_0}(x_0, t_0)$ and all sequences $(\delta_j, a_j, x_j, t_j)$ converging to $(0,0,x,t)$

$$\liminf_{j\to\infty} u^{\delta_j,a_j}(x_j, t_j) \geq u^-(x,t). \tag{3.61}$$

Letting j tend to $+\infty$ in (3.59) and using (3.60) and (3.61) yields

$$u(x,t) - u^+(x,t) \le -m \qquad \text{or} \qquad u(x,t) - u^-(x,t) \ge m. \tag{3.62}$$

Since $u^+ = u = u^-$ and m is strictly positive we deduce that (3.62) is impossible and we finally conclude that $u^{\delta,a}$ tends to u uniformly on compact sets of $R^N \times [0,T]$ as (δ, a) tends to $(0,0)$. This completes the proof of Theorem 3.9. One can prove in a similar way that $u^{-\delta,-a}$ tends to u uniformly on compact sets of $R^N \times [0,T]$ as (δ, a) tends to $(0,0)$.

We are now able to prove the first part of Theorem 1.2.

3.1.4. Proof of Theorem 1.2

We first prove the following Lemma.

LEMMA 3.15. *Let K be a compact set of O. Then*

$$\limsup_{\varepsilon \to 0} \sup_{(x,t)\in K} \bar{\phi}^\varepsilon(x,t) \le 0.$$

PROOF. Let $K \subset O$ be a compact set. We first note that the uniform convergence of $u^{\delta,a}$ to u on the compact set K implies that there exists (δ_0, a_0) such that for all $\delta \le \delta_0$ and $a \le a_0$ we have

$$u^{\delta,a}(x,t) < 0 \qquad \text{for all } (x,t) \in K. \tag{3.63}$$

Moreover in view of Lemma 3.1, and the fact that η_δ is nondecreasing, we deduce that $\eta_\delta(d^{\delta,a}(x,0)) \ge \eta_\delta(U_0(x) + 2\delta)$ for all $x \in R^N$. It follows from (3.9) that $w^{\delta,a}(x,0) \ge U_0(x)$. Since $q_r > 0$ and $q_a \ge 0$ (see Lemma A.2) we deduce that

$$q\left(\frac{w^{\delta,a}(x,0)}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right) \ge q\left(\frac{U_0(x)}{\varepsilon}, \varepsilon\alpha, 0\right) \qquad \text{for all } x \in R^N.$$

Since by (A.10) the function $s \to q(r,s,0)$ is constant we conclude that

$$\bar{\phi}^\varepsilon(x,0) = q\left(\frac{w^{\delta,a}(x,0)}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right) \ge q\left(\frac{U_0(x)}{\varepsilon}, 0, 0\right) = \tilde{q}\left(\frac{U_0(x)}{\varepsilon}, 0\right) = \phi^\varepsilon(x,0),$$

for all $x \in R^N$. Moreover since by Lemma 3.8 $\bar{\phi}^\varepsilon$ is a viscosity supersolution of the parabolic equation in Problem (P_1^ε) for all $a \in [0,1]$ and $\varepsilon < \delta^2$, we deduce by the comparison principle Theorem 2.2 that

$$q\left(\frac{w^{\delta,a}(x,t)}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right) \ge \phi^\varepsilon(x,t) \qquad \text{for all } (x,t) \in R^N \times (0,T). \tag{3.64}$$

Furthermore using the inequality (3.63) we deduce that for all $\delta \le \delta_0$ and

$a \leq a_0$ $d^{\delta,a}(x,t) = 0$ for all $(x,t) \in K$, which implies by the definition of $\eta_\delta(\mathrm{Def}_\eta)$ that $w^{\delta,a}(x,t) = -\delta$. Applying (3.64) for $\delta \leq \delta_0$ and $a \leq a_0$ we obtain

$$q\left(-\frac{\delta}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right) \geq \phi^\varepsilon(x,t), \qquad \text{for all } (x,t) \in K. \tag{3.65}$$

Integrating the inequality $q_r(r, \varepsilon\alpha, \varepsilon a) \leq K_1 e^{-K_2|r|}$ (cf (A.11)), on $\left(-\infty, -\frac{\delta}{\varepsilon}\right)$ we obtain

$$q\left(-\frac{\delta}{\varepsilon}, \varepsilon\alpha, \varepsilon a\right) \leq \frac{K_1}{K_2} e^{-K_2(\delta/\varepsilon)} + h_-(\varepsilon\alpha, \varepsilon a). \tag{3.66}$$

Substituting (3.66) into (3.65) and using that $\delta^2 > \varepsilon$ we obtain

$$\sup_{(x,t)\in K} \phi^\varepsilon(x,t) \leq \frac{K_1}{K_2} e^{-K_2(1/\sqrt{\varepsilon})} + h_-(\varepsilon\alpha, \varepsilon a),$$

where we let ε tend to zero to deduce that

$$\limsup_{\varepsilon\to 0} \sup_{(x,t)\in K} \phi^\varepsilon(x,t) \leq h_-(0,0) = 0. \tag{3.67}$$

This completes the proof of Lemma 3.15.

Inequality (3.67) together with the fact that $\phi^\varepsilon \geq 0$ implies the uniform convergence of ϕ^ε to zero as ε tends to zero in all compact sets of O. This concludes the first part of the Theorem 1.2.

3.2. Convergence of the solution ϕ^ε of Problem (P_1^ε) in the set where $u > 0$

In a similar way we prove that ϕ^ε tends to 1 uniformly on compact sets of I. Since the proof is based on the same method we only give the results. First we consider the sequence $(P_1^{-\delta,-a})$ of problems related to Problem (P_1^l), namely

$$(P_1^{-\delta,-a})\begin{cases} u_t - \Delta u + \dfrac{(D^2 u \nabla u.\nabla u)}{|\nabla u|^2} + (\nabla u.\nabla\chi(v)) + C(\alpha,-a)|\nabla u| = 0 \\ u(x,0) = U_0(x) - 2\delta. \end{cases}$$

We remark that Problem $(P_1^{-\delta,-a})$ has a unique viscosity solution, which we denote by $u^{-\delta,-a}$. We define the distance function

$$d^{-\delta,-a}(x,t) = -\inf_{\{y, u^{-\delta,-a}(y,t)\geq 0\}} |x-y|. \tag{3.68}$$

As in Section 3.1 we obtain the results

LEMMA 3.16. *$d^{-\delta,-a}(x,0) \leq u^{-\delta,-a}(x,0)$, for all $x \in R^N$.*

and

LEMMA 3.17. *We have that*

$$|\nabla d^{-\delta,-a}| \leq 1,$$

$$-|\nabla d^{-\delta,-a}| \leq -1,$$

$$d_t^{-\delta,-a} - \Delta d^{-\delta,-a} + (\nabla d^{-\delta,-a}.\nabla\chi(v)) + C(\alpha,a)|\nabla d^{-\delta,-a}| \leq -K|\nabla d^{-\delta,-a}|d^{-\delta,-a},$$

in $\{(x,t), d^{-\delta,-a}(x,t) < 0\}$ in the sense of viscosity.

Following the proof of Section 3.1 we define

$$w^{-\delta,-a}(x,t) = \tilde{\eta}_\delta(d^{-\delta,-a}(x,t)) \tag{3.69}$$

where $\tilde{\eta}_\delta : R \to R$ is a a smooth function such that

$$(\mathrm{Def}_{\tilde{\eta}})\begin{cases} \tilde{\eta}_\delta(z) = \delta & \text{if } z \geq -\delta/4 \\ \tilde{\eta}_\delta(z) = z + \delta & \text{if } z \leq -\delta/2 \\ \tilde{\eta}_\delta(z) \geq \delta/2 & \text{if } z \geq -\delta/2 \\ 0 \leq \tilde{\eta}'_\delta \leq C & \text{and } |\tilde{\eta}''_\delta| \leq C\delta^{-1} \text{ on } R. \end{cases}$$

One can show that

LEMMA 3.18. *There exist positive constants K and C such that for δ small enough*

$$-|\nabla w^{-\delta,-a}| \leq -C, \tag{3.70}$$

and

$$\begin{aligned} &w_t^{-\delta,-a} - \Delta w^{-\delta,-a} + (\nabla w^{-\delta,-a}.\nabla\chi(v)) + C(\alpha,a)|\nabla w^{-\delta,-a}| \\ &\quad \leq -K\delta^{-1} - K|\nabla w^{-\delta,-a}|\,|w^{-\delta,-a}| \end{aligned} \tag{3.71}$$

in the sense of viscosity in $R^N \times (0,T)$.

Moreover we have that

$$|\nabla w^{-\delta,-a}| \leq 1, \tag{3.72}$$

$$-|\nabla w^{-\delta,-a}| \leq -1, \tag{3.73}$$

and

$$\begin{aligned} &w_t^{-\delta,-a} - \Delta w^{-\delta,-a} + (\nabla w^{-\delta,-a}.\nabla\chi(v)) + C(\alpha,a)|\nabla w^{-\delta,-a}| \\ &\quad + K|\nabla w^{-\delta,-a}|d^{-\delta,-a} \leq 0 \end{aligned} \tag{3.74}$$

in the sense of viscosity in $\{(x,t), d^{-\delta,-a}(x,t) < -\delta/2\}$.

As in Section 3.1.2 we define

$$\underline{\phi}^{\varepsilon}(x,t) := q\left(\frac{w^{-\delta,-a}(x,t)}{\varepsilon}, \varepsilon\alpha, -\varepsilon a\right) \qquad \text{for all } (x,t) \in R^n \times [0,T]. \tag{3.75}$$

We deduce from Lemma 3.18 that

LEMMA 3.19. For all $a > 0$ *and* $\varepsilon < \delta^2$, $\underline{\phi}^{\varepsilon}$ *is a viscosity subsolution of the parabolic equation in Problem* (P_1^{ε}).

Using Lemma 3.19 and the fact that $u^{-\delta,-a}$ tends to u uniformly on compact sets of $R^N \times [0,T]$, we obtain a result analogous to that of Lemma 3.15 namely

LEMMA 3.20. *Let K be a compact subset of I. Then*

$$\liminf_{\varepsilon\to 0} \inf_{(x,t)\in K} \underline{\phi}^{\varepsilon}(x,t) \geq 1.$$

Lemma 3.20 together with the fact that $\phi^{\varepsilon} \leq 1$ implies the uniform convergence of ϕ^{ε} to 1 on all compact sets of I as ε tends to zero. This completes the proof of Theorem 1.2.

4. Convergence proof in the case of Problem (P_2^{ε})

In what follows we prove that the solution ϕ^{ε} of Problem (P_2^{ε}) converges to 0 in the set where $u < 0$. The proof that ϕ^{ε} converges to 1 in the set where $u > 0$ is similar.

4.1. First definitions and preliminary lemmas

As in Section 4, we denote by $h_+(\alpha,\varepsilon^{1/4}a) < h_0(\alpha,\varepsilon^{1/4}a) < h_-(\alpha,\varepsilon^{1/4}a)$ the three solutions of the equation $f(s,\alpha) := s(1-s)(s-1/2+\alpha) = -\varepsilon^{1/4}a$, and remark that $h_+(\alpha,0) = 0$, $h_0(\alpha,0) = 1/2-\alpha, h_-(\alpha,0) = 1$.

We define by $(q,c) = (q(\alpha,\varepsilon^{1/4}a), c(\alpha,\varepsilon^{1/4}a))$ the solution of the problem

$$(TW)\begin{cases} q_{rr} + c(\alpha,\varepsilon^{1/4}a)q_r + q(1-q)(q-1/2+\alpha) = -\varepsilon^{1/4}a \\ q(-\infty,\alpha,\varepsilon^{1/4}a) = h_+(\alpha,\varepsilon^{1/4}a), q(+\infty,\alpha,\varepsilon^{1/4}a) = h_-(\alpha,\varepsilon^{1/4}a). \end{cases}$$

Finally, as previously we introduce a sequence of approximating problems of (P_2^l), namely

$$(P_2^{\delta,b})\begin{cases} u_t + (\nabla u.\nabla\chi(v)) + c(\alpha+b,0)|\nabla u| = 0 \\ u(x,0) = U_0(x) + 2\delta \end{cases}$$

where $c(\alpha+b,0) = -\sqrt{2}(\alpha+b)$. We define

$$F_2^b(x,t,p) = (p.\nabla\chi(v)) + c(\alpha+b,0)|p|.$$

We have checked in Section 2 that F_2 satisfies the hypotheses of the theorems 2.1 and 2.2; this immediately implies that the function F_2^b satisfies them as well. Thus there exists a unique viscosity solution $u^{\delta,b}$ of Problem $(P_2^{\delta,b})$ and we can apply the comparison principle to the equation $u_t + F_2^b(x,t,Du) = 0$. Next we define a distance function, namely

$$d^{\delta,b}(x,t) = \inf_{\{y, u^{\delta,b}(y,t) \leq 0\}} |x-y|. \tag{4.1}$$

As in Section 3.1 we give some properties of $d^{\delta,b}$

Lemma 4.1. *$d^{\delta,b}(x,0) \geq u^{\delta,b}(x,0)$, for all $x \in R^N$.*

The proof of Lemma 4.1 is similar to that of Lemma 3.1. As in Section 3.1 we state for the function $d^{\delta,b}$ some results, which are proved in [5].

Lemma 4.2. (i) *$d^{\delta,b}$ is lower semicontinuous, that is if $(x_j,t_j) \to (x_0,t_0)$, then $d^{\delta,b}(x_0,t_0) \leq \liminf_{j\to+\infty} d^{\delta,b}(x_j,t_j)$.*

(ii) *$d^{\delta,b}$ is continuous in time from below, that is if $(x_j,t_j) \to (x_0,t_0)$ and $t_j \leq t_0$, then $d^{\delta,b}(x_0,t_0) = \lim_{j\to+\infty} d^{\delta,b}(x_j,t_j)$.*

Lemma 4.3. *There exists a positive constant K such that $d^{\delta,b}$ satisfies the inequality*

$$d_t^{\delta,b} + F_2^b(x,t,\nabla d^{\delta,b}) \geq -K|\nabla d^{\delta,b}| d^{\delta,b} \tag{4.2}$$

in $R^N \times (0,T)$ in the sense of viscosity. Moreover we have that

$$|\nabla d^{\delta,b}| \geq 1$$

$$-|\nabla d^{\delta,b}| \geq -1$$

in $R^N \times (0,T)$ in the sense of viscosity.

Next we prove a lower bound on $-\Delta d^{\delta,b}$ which is useful in this scaling.

Lemma 4.4. *We have that*

$$-\Delta d^{\delta,b} \geq -\frac{N-1}{d^{\delta,b}}$$

in $\{(x,t), d^{\delta,b}(x,t) > 0\}$ in the sense of viscosity.

Proof. Let $\varphi \in C^{2,1}(R^N \times (0,T))$; we assume that $d^{\delta,b} - \varphi$ has a strict minimum at the point $(x_0,t_0) \in R^N \times (0,T)$. Since $u^{\delta,b}$ is continuous, there exists $y \in R^N$ such that

$$d^{\delta,b}(x_0,t_0) = |x_0 - y|, \qquad \text{and} \qquad u^{\delta,b}(y,t_0) \leq 0. \tag{4.3}$$

Moreover in view of (4.1) and (4.3) we have that

$$|x-y| - \varphi(x,t_0) \geq d^{\delta,b}(x,t_0) - \varphi(x,t_0), \qquad \text{for all } x \in R^N.$$

Using the fact that (x_0, t_0) is a minimum of the function $d^{\delta,b} - \varphi$ we deduce that

$$|x-y| - \varphi(x,t_0) \geq d^{\delta,b}(x_0,t_0) - \varphi(x_0,t_0) = |x_0-y| - \varphi(x_0,t_0)$$

for all $x \in R^N$, which implies that x_0 is a minimum of the function $|x-y| - \varphi(x,t_0)$. Thus we deduce that

$$-\Delta\varphi(x_0,t_0) \geq -\Delta(|x_0-y|) = -\frac{N-1}{|x_0-y|}.$$

Finally using (4.3) we conclude that $-\Delta\varphi(x_0,t_0) \geq -\dfrac{N-1}{d^{\delta,b}(x_0,t_0)}$. This completes the proof of Lemma 4.4.

As in Section 3.1.1 we define

$$w^{\delta,b}(x,t) = \eta_\delta(d^{\delta,b}(x,t)) \tag{4.4}$$

where η_δ is the function defined by (Def_η) in Section 3.1.1.

LEMMA 4.5. *There exists positive constants K and C such that for δ small enough we have that*

$$-|\nabla w^{\delta,b}| \geq -C, \tag{4.5}$$

and

$$w_t^{\delta,b} + (\nabla w^{\delta,b}.\nabla\chi(v)) + c(\alpha+b,0)|\nabla w^{\delta,b}| \geq -K\delta^{-1} - K|\nabla w^{\delta,b}|\,|w^{\delta,b}| \tag{4.6}$$

in the sense of viscosity in $R^N \times (0,T)$.

Moreover we have that

$$|\nabla w^{\delta,b}| \geq 1, \tag{4.7}$$

$$-|\nabla w^{\delta,b}| \geq -1, \tag{4.8}$$

and

$$w_t^{\delta,b} + (\nabla w^{\delta,b}.\nabla\chi(v)) + c(\alpha+b,0)|\nabla w^{\delta,b}| + K|\nabla w^{\delta,b}|d^{\delta,b} \geq 0 \tag{4.9}$$

in the sense of viscosity in $\{(x,t), d^{\delta,b}(x,t) > \delta/2\}$.

PROOF. One can prove that Lemma 4.5 follows from Lemma 4.3 in the same way as we have deduced the result of Lemma 3.6 from that of Lemma 3.5. Next we deduce from Lemma 4.4 the following result.

LEMMA 4.6. *There exists a positive constant L such that*

$$-\Delta w^{\delta,b} \geq -\frac{L}{\delta} \tag{4.10}$$

in the sense of viscosity in $\{(x,t), d^{\delta,b}(x,t) \geq \delta/4\}$.

Moreover we have that

$$-\Delta w^{\delta,b} \geq 0 \tag{4.11}$$

in the sense of viscosity in $\{(x,t), d^{\delta,b}(x,t) < \delta/4\}$.

PROOF. Let $\varphi \in C^{2,1}(R^N \times (0,T))$; we suppose that $w^{\delta,b} - \varphi$ has a strict minimum at the point (x_0,t_0). This implies that there exists $\rho_0 > 0$ such that

$$(w^{\delta,b} - \varphi)(x,t) > (w^{\delta,b} - \varphi)(x_0,t_0) \qquad \text{for all } (x,t) \in B_{\rho_0}(x_0,t_0). \tag{4.12}$$

Next we prove (4.11); we suppose that $d^{\delta,b}(x_0,t_0) < \delta/4$. First we note that

$$d^{\delta,b}(x,t_0) - d^{\delta,b}(x_0,t_0) \leq |x - x_0| \qquad \text{for all } x \in R^N. \tag{4.13}$$

Let ρ be such that $0 < \rho < \min(\rho_0, \delta/4 - d^{\delta,b}(x_0,t_0))$. Using (4.13) we deduce that $d^{\delta,b}(x,t_0) < \delta/4$ for all $x \in B_\rho(x_0)$. This implies by (4.4) and (Def_η) that $w^{\delta,b}(x,t_0) = -\delta$. Substituting this in (4.12) gives

$$\varphi(x,t_0) \leq \varphi(x_0,t_0), \qquad \text{for all } x \in B_\rho(x_0).$$

Therefore x_0 is a maximum of the function $\varphi(.,t_0)$ in $B_\rho(x_0)$. Thus $-\Delta\varphi(x_0,t_0) \geq 0$, which implies (4.11).

Next we prove (4.10); we suppose that $d^{\delta,b}(x_0,t_0) \geq \delta/4$. As in the proof of Lemma 3.6 we introduce the functions $\eta^e(z) = \eta_\delta(z) + ez$ for $z \in R$ and $\rho^e = (\eta^e)^{-1}$. Applying Lemma 3.7 we have that $\eta^e(d^{\delta,b}) - \varphi$ attains its minimum in $B_{\rho_0}(x_0,t_0)$ at a point (x^e,t^e) and that moreover $\lim_{e\to 0}(x^e,t^e) = (x_0,t_0)$. By the definition of (x^e,t^e) we have that

$$(\eta^e(d^{\delta,b}) - \varphi)(x,t) \geq (\eta^e(d^{\delta,b}) - \varphi)(x^e,t^e), \quad \text{for all } (x,t) \in B_{\rho_0}(x_0,t_0),$$

which in turn implies that

$$d^{\delta,b}(x,t) \geq \rho^e[(\eta^e(d^{\delta,b}) - \varphi)(x^e,t^e) + \varphi(x,t)] =: \psi^e(x,t). \tag{4.14}$$

Note that the definition of ψ^e in (4.14) implies that $\psi^e(x^e,t^e) = d^{\delta,b}(x^e,t^e)$. Moreover since $d^{\delta,b}(x_0,t_0) > 0$ and since by Lemma 3.2 $d^{\delta,b}$ is lower-semicontinuous we have that $d^{\delta,b}(x^e,t^e) > 0$ for e small enough. Using (4.14) and Lemma 4.3 we deduce that

$$|\nabla\psi^e(x^e,t^e)| = 1. \tag{4.15}$$

In view of Lemma 4.4 ψ^e also satisfies the inequality

$$-\Delta\psi^e(x^e, t^e) \geq -\frac{N-1}{d^{\delta,b}(x^e, t^e)}. \tag{4.16}$$

Following the computation of the proof of Lemma 3.6 we have also using (3.28)

$$\begin{aligned}\Delta\psi^e(x^e, t^e) = (\rho^e)'(\eta^e(d^{\delta,b}(x^e, t^e)))\{\Delta\varphi(x^e, t^e) + |\nabla\varphi(x^e, t^e)|^2 \\ (-(\eta^e)''(\psi^e(x^e, t^e)))[(\rho^e)'(\eta^e(d^{\delta,b}(x^e, t^e)))]^2\}.\end{aligned} \tag{4.17}$$

Moreover we deduce from (4.14), and (4.15) that

$$|\nabla\varphi(x^e, t^e)|\,|(\rho^e)'(\eta^e(d^{\delta,b}(x^e, t^e)))| = 1. \tag{4.18}$$

Substituting (4.18) in (4.17) we obtain

$$\Delta\psi^e(x^e, t^e) = (\rho^e)'\{\eta^e(d^{\delta,b}(x^e, t^e))[\Delta\varphi(x^e, t^e) - (\eta^e)''(\psi^e(x^e, t^e)]\}.$$

Substituting this in (4.16) and using the fact that $(\rho^e)'(\eta^e(z)) = \dfrac{1}{(\eta^e)'(z)} > 0$ we obtain that

$$-\Delta\varphi(x^e, t^e) \geq -(\eta^e)'(\eta^e(d^{\delta,b}(x^e, t^e)))\frac{N-1}{d^{\delta,b}(x^e, t^e)} - (\eta^e)''(\psi^e(x^e, t^e)). \tag{4.19}$$

Moreover since $0 \leq (\eta^e)' \leq C + e$ and $|(\eta^e)''| \leq C\delta^{-1}$ we deduce that

$$-\Delta\varphi(x^e, t^e) \geq -(C+e)\frac{N-1}{d^{\delta,b}(x^e, t^e)} - C\delta^{-1}.$$

Letting e tend to zero in the inequality above we deduce from the lower semicontinuity of $d^{\delta,b}$ that

$$-\Delta\varphi(x_0, t_0) \geq -C\frac{N-1}{d^{\delta,b}(x_0, t_0)} - C\delta^{-1}.$$

Finally since in this case $d^{\delta,b}(x_0, t_0) \geq \delta/4$ we conclude that $-\Delta\varphi(x_0, t_0) \geq -\dfrac{L}{\delta}$. This completes the proof of Lemma 4.6.

4.2. A supersolution for Problem (P_2^ε)

We suppose that $a \in [0,1]$ and define

$$\bar{\phi}^\varepsilon(x, t) := q\left(\frac{w^{\delta,b}(x,t)}{\varepsilon}, \alpha, \varepsilon^{1/4}a\right) \qquad \text{for all } (x,t) \in R^N \times [0,T] \tag{4.20}$$

LEMMA 4.7. *For all $a > 0$, $\varepsilon^{1/3} > \delta > \varepsilon^{1/2}$, $b > \varepsilon^{1/8}$ and ε small enough, $\bar{\phi}^{\varepsilon}$ is a viscosity supersolution of the parabolic equation in Problem (P_2^{ε}).*

PROOF. Let $\varphi \in C^{2,1}(R^N \times (0,T))$. We assume that $\bar{\phi}^{\varepsilon} - \varphi$ has a local minimum at the point $(x_0, t_0) \in R^N \times (0,T)$ and that $(\bar{\phi}^{\varepsilon} - \varphi)(x_0, t_0) = 0$. We proceed in a similar way as in the proof of Lemma 3.8. There exist a neighborhood $N_1(x_0, t_0)$ and a function $y = y(x,t) \in C^{2,1}(N_1(x_0, t_0))$ such that

$$\varphi(x,t) = q\left(\frac{y(x,t)}{\varepsilon}, \alpha, \varepsilon^{1/4} a\right), \qquad \text{for all } (x,t) \in N_1(x_0, t_0). \tag{4.21}$$

Moreover the function y satisfies

$$y(x,t) \le w^{\delta,b}(x,t) \qquad \text{for all } (x,t) \in N_1(x_0,t_0), \tag{4.22}$$

with equality at the point (x_0, t_0). This implies in view of Lemma 4.5 that

$$-|\nabla y(x_0,t_0)| \ge -C \tag{4.23}$$

$$\begin{aligned}(y_t + (\nabla y . \nabla \chi(v)) + c(\alpha + b, 0)|\nabla y|)(x_0,t_0)&\\ \ge -K\delta^{-1} - K|\nabla y|\,|w^{\delta,b}|(x_0,t_0)&\end{aligned} \tag{4.24}$$

and in the case that $d^{\delta,b}(x_0,t_0) > \delta/2$ that the function y satisfies

$$|\nabla y(x_0,t_0)| = 1 \tag{4.25}$$

$$(y_t + (\nabla y . \nabla \chi(v)) + c(\alpha + b, 0)|\nabla y|)(x_0,t_0) \ge -K|\nabla y| d^{\delta,b}(x_0,t_0). \tag{4.26}$$

Moreover in view of Lemma 4.6 we also have

$$-\Delta y(x_0,t_0) \ge -\frac{L}{\delta} \tag{4.27}$$

in the case that $d^{\delta,b}(x_0,t_0) \ge \delta/4$, and

$$-\Delta y(x_0,t_0) \ge 0 \tag{4.28}$$

in the case that $d^{\delta,b}(x_0,t_0) < \delta/4$. Next we prove that

$$L_2^{\varepsilon}(\varphi(x_0,t_0)) = \left(\varphi_t - \varepsilon\Delta\varphi + (\nabla\varphi \cdot \nabla\chi(v)) + \varphi\Delta\chi(v) - \frac{1}{\varepsilon} f(\varphi,\alpha)\right)(x_0,t_0) \ge 0, \tag{4.29}$$

which is the result of Lemma 4.7. Substituting (4.21) into (4.29) we deduce that

$$\varepsilon L_2^\varepsilon(\varphi(x_0,t_0)) = q_{rr}\left(\frac{y}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)(1-|\nabla y|^2)(x_0,t_0)$$
$$+ q_r\left(\frac{y}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)[y_t - \varepsilon\Delta y + (\nabla y.\nabla\chi(v)) + c(\alpha,\varepsilon^{1/4}a)](x_0,t_0)$$
$$+ \varepsilon^{1/4}a + \varepsilon q\left(\frac{y}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)\Delta\chi(v)(x_0,t_0). \tag{4.30}$$

(i) We first consider the case that $d^{\delta,b}(x_0,t_0) > \delta/2$. Substituting (4.25) and (4.26) in (4.30) we obtain

$$\varepsilon L_2^\varepsilon(\varphi(x_0,t_0)) \geq q_r\left(\frac{y(x_0,t_0)}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)[-\varepsilon\Delta y + c(\alpha,\varepsilon^{1/4}a) - c(\alpha+b,0)$$
$$- Kd^{\delta,b}(x_0,t_0)] + \varepsilon^{1/4}a + \varepsilon q\left(\frac{y}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)\Delta\chi(v)(x_0,t_0). \tag{4.31}$$

Next we use that by $(\mathrm{Def}_\eta)\, y(x_0,t_0) = w^{\delta,b}(x_0,t_0) = d^{\delta,b}(x_0,t_0) - \delta$, inequality (3.39) and the fact that $se^{-s} \leq 1$ to obtain

$$\left|q_r\left(\frac{y}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)Kd^{\delta,b}\right|(x_0,t_0) \leq KK_1 e^{-K_2(|y(x_0,t_0)|/\varepsilon)}(|y(x_0,t_0)|+\delta)$$
$$\leq \frac{KK_1}{K_2}\varepsilon + KK_1\delta. \tag{4.32}$$

In view of (4.27) we deduce that

$$q_r\left(\frac{y}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)(-\varepsilon\Delta y(x_0,t_0)) \geq -K_1 L\frac{\varepsilon}{\delta}$$
$$\geq -K_1 L\sqrt{\varepsilon}, \tag{4.33}$$

for all $\delta^2 > \varepsilon$. Furthermore we have that

$$c(\alpha,\varepsilon^{1/4}a) - c(\alpha+b,0) = \sqrt{2}b + \varepsilon^{1/4}a\frac{\partial c}{\partial a}(\alpha,0) + O(\varepsilon^{1/2}).$$

Since $\dfrac{\partial c}{\partial a}(\alpha,0) = \dfrac{3}{\sqrt{2}}\dfrac{1}{\alpha^2-1/4}$ (cf. Lemma A.3) we have that for all $0<a<1$ and $b > \varepsilon^{1/8}$

$$c(\alpha,\varepsilon^{1/4}a) - c(\alpha+b,0) \geq \varepsilon^{1/8}\left(\frac{3}{\sqrt{2}}\frac{\varepsilon^{1/8}}{\alpha^2-1/4} + \sqrt{2} - C\varepsilon^{3/8}\right), \tag{4.34}$$

which is positive for ε small enough. Substituting (4.32), (4.33), (4.34) in (4.31) we obtain

$$\varepsilon L_2^\varepsilon(\varphi(x_0,t_0)) \geq -\frac{KK_1}{K_2}\varepsilon - KK_1\delta - K_1L\sqrt{\varepsilon} - \tilde{C}\varepsilon + \varepsilon^{1/4}a,$$

where $\tilde{C}$ is an upper bound for the term $q(.,\varepsilon\alpha,\varepsilon a)\Delta\chi(v)$. Finally we deduce that for all $0<a<1$, $\varepsilon^{1/3}>\delta>\varepsilon^{1/2}$ and $b>\varepsilon^{1/8}$ and ε small enough, the inequality $L_2^\varepsilon(\varphi(x_0,t_0))\geq 0$.

(ii) Next we consider the case that $d^{\delta,b}(x_0,t_0)\leq\delta/2$. We deduce from (4.24) and (4.30) that

$$\begin{aligned}
&\varepsilon L_2^\varepsilon(\varphi(x_0,t_0))\\
&\geq q_{rr}\left(\frac{y(x_0,t_0)}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)(1-|\nabla y(x_0,t_0)|^2)+q_r\left(\frac{y(x_0,t_0)}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)\\
&\quad[-\varepsilon\Delta y+c(\alpha,\varepsilon^{1/4}a)-c(\alpha+b,0)|\nabla y(x_0,t_0)|-K\delta^{-1}\\
&\quad-K|\nabla y(x_0,t_0)|\,|y(x_0,t_0)|\,]+\varepsilon^{1/4}a+\varepsilon q\left(\frac{y}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)\Delta\chi(v)(x_0,t_0). \qquad (4.35)
\end{aligned}$$

Moreover by (Def_η) we have that $y(x_0,t_0)=w^{\delta,b}(x_0,t_0)\leq-\delta/2$, and also use (3.9) to deduce that $y(x_0,t_0)\geq d^{\delta,b}(x_0,t_0)-2\delta\geq-2\delta$; thus

$$\delta/2\leq|y(x_0,t_0)|\leq 2\delta. \qquad (4.36)$$

We deduce from (4.36) and (3.39) that

$$(|q_{rr}|+|q_r|)\left(\frac{y(x_0,t_0)}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)\leq K_1e^{-K_2(\delta/2\varepsilon)},$$

which we substitute in (4.35); also using (4.23) and (4.27) or (4.28) we obtain

$$\begin{aligned}
\varepsilon L_2^\varepsilon(\varphi)(x_0,t_0)\geq{}&-K_1e^{-K_2(\delta/2\varepsilon)}\Big[1+C^2+L\frac{\varepsilon}{\delta}+K\delta^{-1}+KC2\delta\\
&+|c(\alpha,\varepsilon^{1/4}a)-c(\alpha+b,0)|\nabla y(x_0,t_0)||\Big]-\tilde{C}\varepsilon+\varepsilon^{1/4}a. \qquad (4.37)
\end{aligned}$$

Moreover using (4.23) and the fact that $c(\tilde{\alpha},\tilde{a})$ is bounded we obtain that $|c(\alpha,\varepsilon^{1/4}a)-c(\alpha+b,0)|\nabla y(x_0,t_0)||\leq C_2$. Substituting this in (4.37) and choosing $\delta>\sqrt{\varepsilon}$ we deduce that

$$\begin{aligned}
\varepsilon L_2^\varepsilon(\varphi)(x_0,t_0)\geq{}&-K_1e^{-(K_2/2\sqrt{\varepsilon})}\left[(1+C^2)+L\sqrt{\varepsilon}+C_2+2KC+K\frac{1}{\sqrt{\varepsilon}}\right]\\
&-\tilde{C}\varepsilon+\varepsilon^{1/4}a. \qquad (4.38)
\end{aligned}$$

Thus we have that $L_2^\varepsilon(\varphi)(x_0,t_0)\geq 0$. This completes the proof of Lemma 4.7.

4.3. Uniform convergence of $u^{\delta,b}$

In this section we prove the following result.

THEOREM 4.8. *$u^{\delta,b}$, (resp. $u^{-\delta,-b}$) tends to u uniformly on compact sets of $R^N \times [0,T]$ as (δ,b) tends to $(0,0)$.*

The proof of Theorem 4.8 is very similar to that of Theorem 3.9. As in Section 3.1.3 we first give three preliminary lemmas.

LEMMA 4.9. *Let (δ_0, b_0) be fixed. Then for all $0 \le \delta \le \delta_0$ and $0 \le b \le b_0$ we have that*

$$u^{-\delta_0,-b_0}(x,t) \le u^{\delta,b}(x,t) \le u^{\delta_0,b_0}(x,t), \tag{4.39}$$

for all $(x,t) \in R^N \times [0,T]$.

PROOF. For all $0 \le b \le b_0$ we deduce from Lemma A.1 and (A.4) that

$$c(\alpha + b, 0) = -\sqrt{2}(\alpha + b) \ge c(\alpha + b_0, 0),$$

which implies that $u^{\delta,b}$ is a subsolution of the equation

$$u_t + F_2^{b_0}(x,t,Du) = 0.$$

Moreover for all $\delta \le \delta_0$ we have $u^{\delta,b}(x,0) = U_0(x) + 2\delta \le u^{\delta_0,b_0}(x,0)$, for all $x \in R^N$. As previously we deduce from the comparison principle that

$$u^{\delta,b}(x,t) \le u^{\delta_0,b_0}(x,t), \qquad \text{for all } (x,t) \in R^N \times [0,T]. \tag{4.40}$$

In a similar way we can prove that $u^{-\delta_0,-b_0} \le u^{\delta,b}$, which together with (4.40) completes the proof of Lemma 4.9.

As we have done in Section 3.1.3 we define

$$u^+(x,t) = \lim_{\{\nu \to 0\}} \sup \{u^{\delta,b}(z,\theta), \text{ for all } 0 \le b \le \nu, 0 \le \delta \le \nu, \text{ and for all } (z,\theta) \in R^N \times (0,T) \text{ such that } |x-z| \le \nu, \text{ and } |t-\theta| \le \nu\},$$

and

$$u^-(x,t) = \lim_{\{\nu \to 0\}} \inf \{u^{\delta,b}(z,\theta), \text{ for all } 0 \le b \le \nu, 0 \le \delta \le \nu, \text{ and for all } (z,\theta) \in R^N \times (0,T) \text{ such that } |x-z| \le \nu, \text{ and } |t-\theta| \le \nu\}.$$

The proof of Theorem 4.8 then exactly follows as that of Theorem 3.9. We are now able to prove the first part of Theorem 1.3.

4.4. Proof of Theorem 1.3

We prove below the following result.

LEMMA 4.10. *For all K compact set of O, we have that*

$$\limsup_{\varepsilon\to 0}\sup_{(x,t)\in K}\phi^{\varepsilon}(x,t)\le 0.$$

PROOF. Let $K\subset O$ be a compact set. We first note that the uniform convergence of $u^{\delta,b}$ to u on the compact set K implies that there exists (δ_0,b_0) such that for all $\delta\le\delta_0$ and $b\le b_0$ we have

$$u^{\delta,b}(x,t)<0 \qquad \text{for all } (x,t)\in K. \tag{4.41}$$

Moreover in view of Lemma 3.1 and the fact that η_δ is nondecreasing, we deduce that $\eta_\delta(d^{\delta,b}(x,0))\ge\eta_\delta(U_0(x)+2\delta)$ for all $x\in R^N$, which together with (3.9) implies that $w^{\delta,b}(x,0)\ge U_0(x)$. Since by Lemma A.2 $q_r>0$ and $q_a\ge 0$ for $\alpha\in[0,0.4]$, we deduce that

$$q\left(\frac{w^{\delta,b}(x,0)}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)\ge q\left(\frac{U_0(x)}{\varepsilon},\alpha,0\right) \qquad \text{for all } x\in R^N.$$

Since by (A.10) the function $s\to q(r,s,0)$ is constant we conclude that

$$\bar{\phi}^{\varepsilon}(x,0)=q\left(\frac{w^{\delta,b}(x,0)}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)\ge q\left(\frac{U_0(x)}{\varepsilon},0,0\right)=\tilde{q}\left(\frac{U_0(x)}{\varepsilon},0\right)=\phi^{\varepsilon}(x,0),$$

for all $x\in R^N$. Moreover choosing δ such that $\varepsilon^{1/3}>\delta>\varepsilon^{1/2}$ we have by Lemma 4.7 that $\bar{\phi}^{\varepsilon}$ is a viscosity supersolution of the parabolic equation in Problem (P_2^{ε}) for all $a>0$, $b>\varepsilon^{1/8}$, and ε small enough. This implies by the comparison principle Theorem 2.2 that

$$q\left(\frac{w^{\delta,b}(x,t)}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)\ge\phi^{\varepsilon}(x,t), \qquad \text{for all } (x,t)\in R^N\times(0,T). \tag{4.42}$$

Furthermore using the inequality (4.41) we have for all $\delta\le\delta_0$ and $b\le b_0$ that $d^{\delta,b}(x,t)=0$ for all $(x,t)\in K$, which implies by the definition of $\eta_\delta(\mathrm{Def}_\eta)$ that $w^{\delta,b}(x,t)=-\delta$. Applying (4.42) for $\delta\le\delta_0$ and $b\le b_0$ we obtain

$$q\left(-\frac{\delta}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)\ge\phi^{\varepsilon}(x,t), \qquad \text{for all } (x,t)\in K. \tag{4.43}$$

Integrating the inequality $q_r(r,\varepsilon\alpha,\varepsilon a)\le K_1e^{-K_2|r|}$ (cf. (A.11)) on $\left(-\infty,-\frac{\delta}{\varepsilon}\right)$ we obtain

$$q\left(-\frac{\delta}{\varepsilon},\alpha,\varepsilon^{1/4}a\right)\le\frac{K_1}{K_2}e^{-K_2(\delta/\varepsilon)}+h_-(\alpha,\varepsilon^{1/4}a). \tag{4.44}$$

Substituting (4.44) into (4.43) and using that $\delta^2>\varepsilon$ we obtain

$$\sup_{(x,t)\in K} \phi^\varepsilon(x,t) \le \frac{K_1}{K_2} e^{-K_2(1/\sqrt{\varepsilon})} + h_-(\alpha, \varepsilon^{1/4} a),$$

where we let ε tend to zero to deduce that

$$\limsup_{\varepsilon\to 0} \sup_{(x,t)\in K} \phi^\varepsilon(x,t) \le h_-(\alpha, 0) = 0. \tag{4.45}$$

This completes the proof of Lemma 4.10.

Inequality (4.45) together with the fact that $\phi^\varepsilon \ge 0$ implies the uniform convergence of ϕ^ε to zero as ε tends to zero in all compact sets of O. This concludes the first part of the Theorem 1.3.

The proof that ϕ^ε converges uniformly to 1 as ε tends to zero on compact subsets of I is similar (see Section 3.2).

A Appendix. Travelling wave solutions

In this appendix we describe the main properties of travelling wave solutions of the equation

$$u_t = u_{rr} + u(1-u)(u - 1/2 + \tilde{\alpha}) + \tilde{a}. \tag{A.1}$$

We will use the results of this appendix with $\tilde{\alpha} = \varepsilon\alpha$, and $\tilde{a} = \pm\varepsilon a$ in the case of Problem (P_1^ε) and with $\tilde{\alpha} = \alpha$, and $\tilde{a} = \pm\varepsilon^{1/4} a$ in the case of Problem (P_2^ε). First we note that if $\tilde{\alpha} \in [0, 1/2)$ is a fixed constant and if $\tilde{a}$ is a small enough positive constant, then the equation

$$f(s,\alpha) := s(1-s)(s - 1/2 + \tilde{\alpha}) = -\tilde{a} \tag{A.2}$$

has three solutions

$$h_-(\tilde{\alpha}, \tilde{a}) < h_0(\tilde{\alpha}, \tilde{a}) < h_+(\tilde{\alpha}, \tilde{a}); \tag{A.3}$$

in the case that $\tilde{a} = 0$, they are explicitely given by

$$h_-(\tilde{\alpha}, 0) = 0, \qquad h_0(\tilde{\alpha}, 0) = 1/2 - \tilde{\alpha}, \qquad h_+(\tilde{\alpha}, 0) = 1. \tag{A.4}$$

Next we compute a travelling wave solution (q, c) of the equation

$$u_t = u_{rr} + f(u, \tilde{\alpha}) + \tilde{a}, \tag{A.5}$$

that is the solution of the problem

$$(TW)\begin{cases} q_{rr} + c(\tilde{\alpha}, \tilde{a})q_r + q(1-q)(q - 1/2 + \tilde{\alpha}) = -\tilde{a} \\ q(-\infty, \tilde{\alpha}, \tilde{a}) = h_-(\tilde{\alpha}, \tilde{a}), q(+\infty, \tilde{\alpha}, \tilde{a}) = h_+(\tilde{\alpha}, \tilde{a}). \end{cases}$$

One can show the following result.

LEMMA A.1.
The pair

$$(q,c)\begin{cases} q(r,\tilde{\alpha},\tilde{a}) = h_-(\tilde{\alpha},\tilde{a}) + \dfrac{\sqrt{2}\lambda(\tilde{\alpha},\tilde{a})}{1+e^{-\lambda(\tilde{\alpha},\tilde{a})r}} \\ c(\tilde{\alpha},\tilde{a}) = \dfrac{1}{\sqrt{2}}(2h_0 - h_- - h_+)(\tilde{\alpha},\tilde{a}) \end{cases}$$

where

$$\lambda(\tilde{\alpha},\tilde{a}) = \frac{1}{\sqrt{2}}(h_+ - h_-)(\tilde{\alpha},\tilde{a}) \tag{A.6}$$

is the unique solution of the system (TW) up to a translation constant.

Next we describe some qualitative properties of the travelling wave solution, which are proved for instance in [9].

LEMMA A.2. *There exist K_1, K_2 positive constants such that, for all $r \in R$, $\tilde{\alpha} \in [0, 0.4]$ and $\tilde{a}$ small enough, we have that*

$$h_-(\tilde{\alpha},\tilde{a}) = O(\tilde{a}), h_+(\tilde{\alpha},\tilde{a}) = 1 + O(\tilde{a}) \tag{A.7}$$

$$q_r(r,\tilde{\alpha},\tilde{a}) > 0, \tag{A.8}$$

$$q_a(r,\tilde{\alpha},\tilde{a}) \geq 0, \tag{A.9}$$

$$q_\alpha(r,\tilde{\alpha},0) = 0, \tag{A.10}$$

$$(|q_r| + |q_{rr}|)(r,\tilde{\alpha},\tilde{a}) \leq K_1 e^{-K_2|r|}. \tag{A.11}$$

In order to be able to evaluate the coefficient of α in the equations for the moving boundaries in the problems (P_1^ε) and (P_2^ε), one uses the following results.

LEMMA A.3.

$$C(\alpha,a) := \lim_{\varepsilon\to 0} \frac{c(\varepsilon\alpha,\varepsilon a)}{\varepsilon} = -\sqrt{2}\alpha - 6\sqrt{2}a \tag{A.12}$$

$$\frac{\partial c}{\partial a}(\alpha,0) = \frac{3}{\sqrt{2}}\frac{1}{\alpha^2 - 1/4}. \tag{A.13}$$

Acknowledgement

The authors are very grateful to Y. and M.-H. Giga for informing them of the reference [7].

References

[1] G. Barles, H. M. Soner and P. E. Souganidis, Front propagation and phase field theory, SIAM J. Control and Optimization **31** (1993), 439–469.

[2] A. Bonami, D. Hilhorst, E. Logak and M. Mimura, A free boundary problem arising in a chemotaxis model, in: Free Boundary Problems, Theory and Applications, M. Niezgódka and P. Strzelecki eds, Longman 1996.

[3] A. Bonami, D. Hilhorst, E. Logak and M. Mimura, Convergence of a chemotaxis-growth model to a free boundary problem, submitted.

[4] M. G. Crandall, H. Ishii and P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. **27** (1992), 1–67.

[5] C. M. Elliott and R. Schätzle, The limit of the fully anisotropic double-obstacle Allen-Cahn equation in the non-smooth case, SIAM J. Math. Analysis **28** (1997), 274–303.

[6] L. C. Evans, H. M. Soner and P. E. Souganidis, Phase transitions and generalized motion by mean curvature, Comm. Pure Appl. Math. **XLV** (1992), 1097–1123.

[7] Y. Giga, S. Goto, Motion of hypersurfaces and geometric equations, J. Math. Soc. Japan **44** (1992), 99–111.

[8] Y. Giga, S. Goto, H. Ishii, and M. H. Sato, Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains, Indiana Univ. Math. J. **40** (1991), 443–470.

[9] M. Henry, Doctoral Thesis, Systèmes de Réaction-Diffusion et Dynamique d'Interface en Chimie et en Biologie, Paris XI (1998).

[10] M. Mimura and T. Tsujikawa, Aggregating pattern dynamics in a chemotaxis model including growth, Physica A **230** (1996), 499–543.

**Analyse Numérique et EDP*
CNRS et Université de Paris-Sud
91405 Orsay Cedex
France

†Mathematisches Institut
Albert-Ludwigs Universität
D79104 Freiburg
Germany