HIROSHIMA MATH. J.
29 (1999), 529–577

Nonlocal nonlinear systems of transport equations in weighted L^1 spaces: An operator theoretic approach

Toshiyuki YAMAGUCHI

(Received January 14, 1999)

ABSTRACT. Mathematical models of a general class for muscle contraction are studied in terms of linear semigroup theory. Two-state and four-state cross-bridge dynamics are described as nonlocal nonlinear transport systems. The initial-value problem for the nonlinear transport equation is reformulated as an abstract evolution equation in certain weighted L^1 spaces and a natural notion of mild solution to the evolution problem is introduced. The existence, blowing-up at a finite time, and uniqueness of the mild solutions are discussed under natural assumptions.

1. Introduction

This paper is concerned with nonlocal nonlinear transport systems of the form

$$\text{(NNS)} \qquad \begin{cases} \partial_t \boldsymbol{u} + z'(t)\partial_x \boldsymbol{u} = \boldsymbol{\varphi}(t, x, \boldsymbol{u}, z(t)), & (t, x) \in (0, T) \times \mathbf{R}, \\ z(t) = L\left(\displaystyle\int_{-\infty}^{+\infty} \boldsymbol{w}(y) \cdot \boldsymbol{u}(t, y) dy\right), & t \in [0, T]. \end{cases}$$

Here $\boldsymbol{u} : [0, T] \times \mathbf{R} \to \mathbf{R}^N$ is an unknown function, $[0, T]$ is a given time interval, N is a given positive integer, ∂_t and ∂_x stand for the partial differential operators with respect to the time and space variables, respectively, z' means the time derivative of z, and $\boldsymbol{w}(y) \cdot \boldsymbol{u}(t, y)$ means the inner product of $\boldsymbol{w}$ and $\boldsymbol{u}$ in $\mathbf{R}^N$. Moreover, the function $\boldsymbol{\varphi} : [0, T] \times \mathbf{R} \times \mathbf{E} \times \mathbf{R} \to \mathbf{R}^N$ is supposed to be continuous in $(t, \boldsymbol{u}, z)$, where $\mathbf{E} = \{(u^1, \dots, u^N) \in \mathbf{R}^N \mid u^1, \dots, u^N \geq 0, u^1 + \cdots + u^N \leq 1\}$, $\boldsymbol{\varphi}$ need not be continuous in x, $L : (a, b) \to \mathbf{R}$ is a continuous, decreasing function, and $\boldsymbol{w} : \mathbf{R} \to \mathbf{R}^N$ is a continuous weight function whose components are all nondecreasing. The precise assumptions for the system are made later.

The coefficient $z'(t)$ of $\partial_x \boldsymbol{u}$ in (NNS) may vanish and need not have a constant sign. Hence the system (NNS) of partial differential equations may degenerate to a system of ordinary differential equations. Further, the

1991 *Mathematics Subject Classification.* 35Q80, 92C30, 47H20.

Key words and phrases. Muscle contraction, nonlocal nonlinearity, transport equation, abstract evolution equation, nonlinear constraint, quasi-dissipativity, invertible evolution operator.

transport term $z'(t)\partial_x \boldsymbol{u}$ is a product of the space derivative of the unknown function $\boldsymbol{u}$ and the function $z'(t)$ which contains $\boldsymbol{u}$ in a nonlocal way. The nonlocal nonlinearity of this type is not straightforward to treat.

In case $N = 1$, many authors have treated (NNS) under various assumptions and discussed the existence and uniqueness of solutions. In early works [8, 26], the inhomogeneous term $\varphi(t, x, u, z)$ is of a linear form $\gamma(t)f(x)(1-u) - g(x)u$ and is assumed to be smooth, $w(x) = x$, $L(\tau)$ is a specific function and the initial data is identically equal to zero. Under these conditions, classical solutions or Lipschitz continuous strong solutions with compact support were studied. In [4, 13], Lipschitz continuous strong solutions with compact supports are investigated in the case where φ is of the form $F(t, x, z) - G(t, x, z)u$ and is locally Lipschitz continuous in x, $L(\tau)$ is a general function, and the initial function has a compact support. Using the vanishing viscosity method, Colli and Grasselli [7] treated the equation in $L^2(\mathbf{R})$ for $\varphi(t, x, u, z) = F(t, x, z) - G(t, x, z)u$. For the function φ of the general form, Kato and Yamaguchi [17] chose the space of bounded, uniformly continuous functions on $\mathbf{R}$ as the base space. In addition, in [15] the equation in $L^1(\mathbf{R})$ is treated in the cases where $\varphi(t, x, u, z) = \gamma(t)f(x, z)(1-u)^p - g(x, z)u^{\bar{p}}$ and $\gamma(t)f(x, z)(1-u^p) - g(x, z)u^{\bar{p}}$ with $p, \bar{p} \geq 1$. In the recent paper [21], Matsumoto, Oharu and Yamaguchi have considered weak solutions with compact supports and showed the well-posedness in $L^1(\mathbf{R})$ in the case where $\varphi(t, x, u, z) = F(t, x, z, u) - G(t, x, z, u)u$, $w(x)$ is of class C^1 and bi-Lipschitz continuous, $L(\tau)$ is a general function, and the initial data have compact supports.

In case $N = 4$, Comincioli *et al.* [11] studied the case where $\boldsymbol{\varphi} = (\varphi^1, \varphi^2, \varphi^3, \varphi^4)$,

$$\varphi^i(t, x, u^1, u^2, u^3, u^4, z) = \sum_{j=i\pm 1} [a_{ij}(t, x)u^j - a_{ji}(t, x)u^i], \qquad i = 1, 2, 3, 4,$$

$a_{i,i\pm 1}(t, x)$, $i = 1, 2, 3, 4$, are bounded and of class C^1 on $[0, T] \times \mathbf{R}$, $\boldsymbol{w}(x) = (0, 0, x-\delta, x)$ (δ a constant), and

$$L(\tau) = \log\left(1 + \int_{-\infty}^{+\infty} [(x-\delta)u_0^3(x) + xu_0^4(x)]dx\right) - \log(1+\tau),$$

$a = -1$ and $b = +\infty$. Here $(u_0^1, u_0^2, u_0^3, u_0^4)$ stands for an initial datum. They established existence and uniqueness theorems for global classical solutions. See [1, 3, 5, 6, 7, 9, 10, 15, 16, 20, 27] for the mathematical researchs in the other model equations.

The unknown function $\boldsymbol{u}(t, x) = (u^1(t, x), \dots, u^N(t, x))$ represents an N-vector of densities or that of populations. Therefore each component is not always continuous in x, and it is preferable that those should be found in

$L^1(\mathbf{R})^N$. Furthermore, the function $\boldsymbol{\varphi}(t, x, \boldsymbol{u}, z)$ proposed originally by A. F. Huxley is discontinuous in x, and a solution is required to have the property that $x \mapsto \boldsymbol{w}(x) \cdot \boldsymbol{u}(t, x)$ is integrable on $\mathbf{R}$ because the integral $\int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}(t, x) dx$ has to make sense in (NNS). For these reasons, it is most suitable that the system is treated in the product space $X \equiv L^1(\mathbf{R}; (1 + |w^1(x)|)dx) \times \cdots \times L^1(\mathbf{R}; (1 + |w^N(x)|)dx)$ of weighted L^1 spaces by $1 + |w^i(x)|$, $i = 1, \ldots, N$. Here $w^i(x)$, $i = 1, \ldots, N$, are components of $\boldsymbol{w}(x)$. The precise definition of the weighted L^1 spaces is given in § 5. On the other hand, as seen later, it is possible to assume that the function $\boldsymbol{\varphi}$ can be eventually smooth in t.

The purpose of the present paper is to study the local existence together with blowing-up phenomena, global existence, and the global uniqueness of weak solutions to the Cauchy problem for (NNS) formulated in the weighted L^1 spaces. The main theorems may be stated as follows (the precise statements are given in § 5):

THEOREM A (THEOREM 5.4; LOCAL EXISTENCE). *Suppose that* $\boldsymbol{w}$ *is smooth enough, and that* L *is strictly decreasing and locally Lipschitz continuous. Assume that* $\boldsymbol{\varphi}$ *is Lipschitz continuous in* $(\boldsymbol{u}, z)$ *and grows at most linearly in* $\boldsymbol{u}$, *and furthermore that* $\boldsymbol{\varphi}$ *enjoys a subtangential condition. Let an initial value* $\boldsymbol{u}_0 \in X$ *be such that* $\boldsymbol{u}_0(x) \in \mathbf{E}$ *a.e. and* $a < \int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}_0(x) dx < b$. *Then the Cauchy problem for* (NNS) *has a local weak solution. Moreover, if* $[0, T_{\max})$, $T_{\max} < T$, *is the maximal interval of existence of weak solutions, and if* $\boldsymbol{u}$ *is a weak solution on* $[0, T_{\max})$, *then*

$$\limsup_{t \uparrow T_{\max}} \left| L\left(\int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}(t, x) dx \right) \right| = \infty.$$

THEOREM B (THEOREM 5.5; GLOBAL EXISTENCE). *Assume that* $\boldsymbol{w}(\cdot) \equiv (0, \ldots, 0, w^k(\cdot), \ldots, w^N(\cdot))$ *and* $w^k(\cdot), \ldots, w^N(\cdot)$ *are bi-Lipschitz continuous for some* $1 \leq k \leq N$. *Suppose also that the functions* $\boldsymbol{\varphi}$ *and* L *satisfy the same conditions as in the previous theorem. Let* $\boldsymbol{u}_0 \in X$ *be such that* $\boldsymbol{u}_0(x) \in \mathbf{E}$ *a.e. and* $a < \int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}_0(x) dx < b$. *Then the Cauchy problem for* (NNS) *has a weak solution on the whole interval* $[0, T]$.

THEOREM C (THEOREM 5.7; UNIQUENESS). *Suppose that the functions* $\boldsymbol{w}$ *and* L *satisfy the same conditions as in the first theorem. Assume that* $\boldsymbol{\varphi}$ *is Lipschitz continuous in* $(\boldsymbol{u}, z)$, *sufficiently regular in* x, *and grows at most linearly in* $\boldsymbol{u}$. *Then weak solutions to* (NNS) *are uniquely determined by the initial data.*

As for the nonlinearity of $\boldsymbol{\varphi}(t, x, \boldsymbol{u}, z)$, we consider not only the Lipschitz continuity with respect to $\boldsymbol{u}$ but also the quasi-dissipativity with respect to $\boldsymbol{u}$. This kind of generalization and the introduction of weighted L^1 spaces have not been made so far in the study of mathematical models of muscle contraction. Treating the system in the product space X, we regard the mapping $\boldsymbol{u}(t,\cdot) \mapsto \int_{-\infty}^{+\infty} \boldsymbol{w}(x)\cdot\boldsymbol{u}(t,x)dx$ as a continuous linear functional on X.

One may discuss strong solutions of (NNS), but it is necessary to assume that $x \mapsto \boldsymbol{\varphi}(t,x,\boldsymbol{u},z)$ is absolutely continuous, since the first equation is hyperbolic and any smoothing effect can not be expected. Accordingly, we do not treat strong solutions here because we are interested in the nonlinear term $\boldsymbol{\varphi}$ which is discontinuous in x.

In a way similar to the past researches, we first reduce (NNS) to an equivalent equation for $z(t)$ rather than $\boldsymbol{u}(t,x)$. The main reason is that if $\boldsymbol{u}(t,x)$ is first regarded as the unknown function then (NNS) becomes a fully nonlinear system, and so that the approach from this point of view is not straightforward. Therefore we make an attempt to formulate an appropriate equation for $z(\cdot)$ and find the nonlocal term $z(\cdot)$ by applying Schauder's fixed point theorem. Such $z(\cdot)$ is obtained on a "small" subinterval of $[0,T]$. We then prolong $z(t)$ onto $[0,T]$ step by step. This approach is essentially made in [20] for the parabolic regularizations and does not require *a priori* estimates. In the previous papers, various a priori estimates for $z(t)$ were given to guarantee the global existence of $z(t)$.

The notion of "weak solution" does not mean a solution in the sense of distributions which does not make sense in (NNS) if $z(t)$ is not differentiable. Therefore it is natural to employ mild solutions in the theory of abstract evolution equations. (See Definitions 2.2, 3.1 and 5.2 below.) For this reason we convert the evolution problem for (NNS) to an abstract nonlinear evolution system (AES) in a real ordered Banach space X. The system may be formulated as a semilinear evolution equation coupled with a nonlinear constraint:

$$\text{(AES)} \qquad \begin{cases} u'(t) + z'(t)\Lambda u(t) = F(t,u(t),z(t)), & t \in (0,T), \\ \boldsymbol{f}(u(t)) \in \Gamma(z(t)), & t \in [0,T]. \end{cases}$$

Here $u:[0,T]\to X$ and $z:[0,T]\to\mathbf{R}$ are unknown functions, $\Lambda:\mathscr{D}(\Lambda)\subset X \to X$ is the generator of a linear C_0-group on X, D stands for a natural class of elements of X in which solutions u take their values, and $F:[0,T]\times D\times\mathbf{R}\to X$ is a continuous nonlinear mapping. In the nonlinear constraint, $\boldsymbol{f}:X\to\mathbf{R}$ is a continuous linear functional on X, a multi-valued function $\Gamma:\mathscr{D}(\Gamma)\subset\mathbf{R}\to 2^{\mathbf{R}}$ is a nonlinear m-dissipative operator in $\mathbf{R}$, and $'$ stands for the differentiation with respect to t.

A one-parameter family $\{S(t)\}_{t\in\mathbf{R}}$ of continuous linear operators from a Banach space $(X,\|\cdot\|)$ into itself is said to be a linear C_0-group on X, if $S(s+t)=S(s)S(t)$ for all $s,t\in\mathbf{R}$, $S(0)=I_X$, the identity operator in X, and $\|S(t)v-v\|\to 0$ as $t\to 0$ for $v\in X$. The generator Λ of $\{S(t)\}_{t\in\mathbf{R}}$ is defined by $\mathscr{D}(A) = \{v \in X \mid \lim_{h \to 0} h^{-1}(S(h)v - v)$ exists in $X\}$ and $Av = \lim_{h\to 0} h^{-1} \cdot (S(h)v - v)$ for $v \in \mathscr{D}(A)$. The C_0-group $\{S(t)\}_{t \in \mathbf{R}}$ is of type $\omega \in \mathbf{R}$, if $\|S(t)\| \le e^{\omega|t|}$ for all $t \in \mathbf{R}$. Let A be a nonlinear, possibly multi-valued, operator in X. Its range $\mathscr{R}(A)$ is defined by $\mathscr{R}(A) = \bigcup_{u \in \mathscr{D}(A)} Au$. We often write $(u, v) \in A$ if $u \in \mathscr{D}(A)$ and $v \in Au$. In case $X = \mathbf{R}$, A is m-dissipative if and only if $(v_1 - v_2)(u_1 - u_2) \le 0$ for $(u_i, v_i) \in A$, $i = 1, 2$, and $\mathscr{R}(I - \lambda A) = X$ for $\lambda > 0$; in other words, $-A$ has a maximal monotone graph. In what follows, we deal with a real Banach space $(X, \|\cdot\|)$ equipped with a partial ordering $\leqslant$. Namely, $u_1 \leqslant u_2$ in X implies $u_1 + v \leqslant u_2 + v$ and $\alpha u_1 \leqslant \alpha u_2$ in X for $v \in X$ and $\alpha \ge 0$. We denote by X_+ the positive cone in X, i.e., $X_+ = \{u \in X \mid u \geqslant 0 \text{ in } X\}$. The system $(X, \|\cdot\|, \leqslant)$ is called an ordered Banach space, whenever X_+ is norm closed. We refer to [2, 12, 25] for linear (semi-)groups, [2, 22] for nonlinear dissipative operators, and [2, 12] for ordered Banach spaces.

This evolution system (AES) is of a specific form, although it extracts the characteristic features of the nonlocal nonlinear transport system (NNS) in such a way that (NNS) is reduced to a nonlinear evolution equation in X coupled with a nonlinear constraint in terms of a continuous linear functional f on X.

This paper is organized as follows: §2 is devoted to the analysis of semilinear evolution equations (SE; z) formulated for a given function $z(\cdot)$:

$$(\mathrm{SE}; z) \qquad u' + z'(t)Au = F(t, u, z(t)), \qquad t \in (0, T).$$

This analysis is required to reduce the abstract evolution system (AES). In §3, the notion of mild solution to (AES) is introduced and our main results concerning the local and global existence and uniqueness of the mild solutions are stated. The uniqueness theorem for (AES) is proved in §3. In addition, we describe the reduction from (AES) to equivalent equations for $z(\cdot)$. In §4 we discuss the existence theorems for (AES) via a fixed point argument. In §5 our main results for (NNS) are stated. We make basic assumptions for w, φ and L in (NNS) here and introduce weighted L^1 spaces and then the notion of weak solution to (NNS). Moreover, we give a local existence theorem together with a result concerning blowing up solutions and then a global existence theorem as well as a uniqueness theorem for weak solutions. These results are proved in §6 by applying the abstract results given in §3. Finally, in §7, we prove some technical estimates for mild solutions to (SE; z).

Features of the model. The evolution system (NNS) is interpreted as a mathematical model describing the *cross-bridge* dynamics in muscle contraction phenomena. The constitutive unit of a muscle is called a *sarcomere* which consists of a thick filament (*myosin*) and thin filaments (*actins*). Pioneering researches in muscle contraction were made by H. E. Huxley and A. F. Huxley.

Nowadays there is a general agreement that muscle contraction can be explained in terms of sliding filament theory proposed by A. F. Huxley. According to his theory, the generation of muscular force is due to interactions between myosins and actins. Under the influence of the intercellular calcium ions which are emitted through nerve impulses, the so-called cross-bridges connect the thick filaments and the thin filaments, and then act like Hookean springs. As a result, the muscular force is generated and works on relative slides between those filaments. The cross-bridges are composed of myosin molecules standing out from the myosin filaments, and it is inferred that they are approximately governed by the linear elasticity.

The x-axis is placed on a myosin (thick) filament and the origin $(x = 0)$ is taken at the root of each cross-bridge. Then the position x represents the orthogonal projection to the x-axis of each *subfragment-1* (myosin head). See, for instance, [8, 11, 26] for more explanations. Two states of the cross-bridges can be considered: the state in which a cross-bridge attaches to the actin and the state in which a cross-bridge does not attach the actin. We then denote by $u^1(t,x)$ and $u^2(t,x)$, respectively, the detached and attached cross-bridge densities in the half-sarcomere under observation at time t and position x. Then the functions u^1 and u^2 are governed by

$$\partial_t u^i + v(t)\partial_x u^i = \varphi^i(t,x,u^1,u^2,z(t)), \qquad (t,x) \in (0,T) \times \mathbf{R}, \ \ i = 1,2,$$

where $\partial_t u^i + v(t)\partial_x u^i$ indicates the material derivative of $u^i(t,x)$ and $z(t)$ stands for the length (of shortening) of the half-sarcomere at time t. Furthermore, $v(t)$ stands for the velocity of contraction, and hence $v(t) = z'(t)$. The functions φ^1 and φ^2 take the forms of $\varphi^1(t,x,u^1,u^2,z) = g_1(x)u^2 - \gamma(t)f_1(x)u^1$ and $\varphi^2(t,x,u^1,u^2,z) = \gamma(t)f_2(x)u^1 - g_2(x)u^2$. Here $f_i(x)$ and $g_i(x)$ are the attachment and detachment rate functions, respectively; $\gamma(t)$ stands for the change in time of the concentration of calcium ions and is a nonnegative smooth function. If the contraction is *twitch*, $\gamma(t)$ rises from zero to a single peak soon and then decays back to zero; if it is *tetanus*, $\gamma(t)$ rises from zero to a peak soon and keeps up the maximum till the arrival of the last impulse, and then decays back to zero. For $N = 2$, the interpretation of the first equation of (NNS) may be made in this way. The force generated by the attached cross-bridges at x is given by $\kappa x u^2(t,x)$ (κ being a positive constant). The support of function $\boldsymbol{u} = (u^1,u^2)$ is contained in a sufficiently large bounded interval of the space variable, because the lengths of cross-bridges are bounded above. The density is normalized in the sense that the total density of cross-bridges of the position x in the half-sarcomere under observation is the unity, namely, $u^1 + u^2 = 1$ on such interval.

On the other hand, the generation of muscular force or tension of a whole muscle is generally explained in terms of rheology by using three-component model of Hill. In this model a muscle is composed of a contractile component CC, a series elastic component SEC and a parallel elastic component PEC. The active contraction of muscles is due to CC which represents the half-sarcomere. The components of SEC and PEC are passive in the sense that they generate the force only when the whole muscle is activated by the outer and inner force. Both elastic components represent muscular tissues, tendons, blood vessels, and so on. They are not Hookean springs but assumed to be nonlinear elastic systems. This assumption implies an exponential relation between the contractile force $\tau := \int_{-\infty}^{+\infty} \kappa x u^2(t,x)dx$ generated by CC and the length $\zeta + \text{const.}$ of the (half-)sarcomere ($\zeta := z(t)$). Roughly speaking, considering the inverse of the exponential relation, we obtain the second equation of (NNS). In this case the weight function $\boldsymbol{w}(x)$ is given by $\boldsymbol{w}(x) = (0, \kappa x)$. If the contraction is *isometric*, the function $L(\tau)$ is given by $L(\tau) = -\log(1+\tau)$, $a = -1$, $b = +\infty$; if it is *isotonic*, $L(\tau) = \log[(Q-\tau)/q(1+\tau)]$, $a = -1$, $b = Q$ for some $0 < q < Q < +\infty$; if it is *isometric-isotonic*, $L(\tau) = \log[(q - (\tau - Q + q)^+)/q(1+\tau)]$, $a = -1$, $b = Q$ for some $0 < q < Q < +\infty$, where $c^+ = \max\{c, 0\}$. We refer to [26] for the derivations of these functions $L(\tau)$. In addition, we refer to [1] for different models of the isometric or isotonic contraction phenomena; and we refer to [9] for a model which is considered the nonlinear viscoelasticity in place of the nonlinear elasticity.

In the above models we have considered two states in which cross-bridges connect or do not connect with actin filaments. In this sense the model is called a two-state cross-bridge model. In the past researches, (NNS) with $N = 1$ had been considered as the standard model equation for a two-state cross-bridge model: The attached cross-bridge density $u(t,x)$ is governed by

$$\begin{cases} \partial_t u + z'(t)\partial_x u = \varphi(t,x,u,z(t)), & (t,x) \in (0,T) \times \mathbf{R}, \\ z(t) = L\left(\int_{-\infty}^{+\infty} w(y)u(t,y)dy\right), & t \in [0,T]. \end{cases}$$

In particular, the function φ proposed by A. F. Huxley is of the form $\varphi(t,x,u,z) = \gamma(t)f(x)(1-u) - g(x)u$. Here $f(x)$ and $g(x)$ are the attachment and detachment rate functions which are, respectively, given by

$$f(x) = \begin{cases} k_1 x/h, & \text{if } 0 \le x \le h, \\ 0, & \text{otherwise}, \end{cases} \qquad g(x) = \begin{cases} k_2, & \text{if } x < 0, \\ k_3 x/h, & \text{otherwise}, \end{cases}$$

where k_1, k_2, k_3 and h are positive constants. Notice that the detached cross-bridge density is represented by $1 - u(t,x)$. Put $u^1 = 1 - u$ and $u^2 = u$. Then $\boldsymbol{u} = (u^1, u^2)$ is (formally) a solution of (NNS) with $N = 2$. In this case, $\boldsymbol{\varphi} = (\varphi^1, \varphi^2)$ is given by $\varphi^2(t,x,u^1,u^2,z) = -\varphi^1(t,x,u^1,u^2,z) = \gamma(t)f(x)u^1 - g(x)u^2$, and $\boldsymbol{w} = (w^1, w^2)$ is given by $w^1(x) \equiv 0$ and $w^2(x) = w(x)$. Thus, the case $N = 1$ can be considered as a special case of the system with $N = 2$. Accordingly, we should adopt the system with $N = 2$ as a two-state cross-bridge model rather than the case $N = 1$.

In the two-state cross-bridge model we consider only two states in which cross-bridges connect with actin filaments or not. On the other hand, there is a model such that a cross-bridge connects with an actin filament in two states and not in the other two states. This is called a four-state cross-bridge model. States 1 and 2 are states in which cross-bridges do not connect with actins. The other two states are specified in such a way that if the attached cross-bridges do not generate forces when the attachment angle between the sub-fragment-1 and actin filament is 90° (resp., 45°) then the state is called State 3 (resp., State 4). Introducing the space variable x in a way similar to the case $N = 2$, we put the origin at which the cross-bridge in State 4 does not generate force. Thus, the force generated by the cross-bridges of the position x in State 4 is expressed by $\kappa_4 x u^4(t,x)$ (κ_4 being a positive constant, $u^4(t,x)$ the density of the cross-bridges in State 4 at t and x). If the cross-bridge in State 3 does not generate force at $x = \delta(>0)$, then the force generated by the cross-bridges in this state at x is given by $\kappa_3(x-\delta)u^3(t,x)$ ($\kappa_3 > 0$, $u^3(t,x)$ denotes the density of cross-bridges in State 3 at t and x).

The density function (u^1, u^2, u^3, u^4) describing the respective states in which cross-bridges are is governed by the equations

$$\partial_t u^i + v(t)\partial_x u^i = \sum_{j=i\pm 1} [a_{ij}(t,x)u^j - a_{ji}(t,x)u^i], \qquad i = 1,2,3,4.$$

Here $v(t)$ represents the contracting velocity of a half-sarcomere. Hence $v(t) = z'(t)$ and $z(t) + \text{const.}$ means the contracting length of a half-sarcomere. The coefficients $a_{ij}(t,x)$ are the rate functions of the transition from State j to State i. In a way similar to the case $N = 2$, a rheological model implies a relation

$$z(t) = \log\left(1 + \int_{-\infty}^{+\infty} \kappa_3(x-\delta)u_0^3(x)dx + \int_{-\infty}^{+\infty} \kappa_4 x u_0^4(x)dx\right) - \log(1+\tau)$$

between the length of a half-sarcomere and the contractile force, where $(u_0^1, u_0^2, u_0^3, u_0^4)$ is a vector of initial densities of respective states of cross-bridges and the contractile force τ generated by the cross-bridges in a half-sarcomere is expressed by

$$\tau = \int_{-\infty}^{+\infty} \kappa_3(x-\delta)u^3(t,x)dx + \int_{-\infty}^{+\infty} \kappa_4 x u^4(t,x)dx.$$

In this case, $\boldsymbol{w}(x)$ is chosen as $\boldsymbol{w}(x) = (0, 0, \kappa_3(x-\delta), \kappa_4 x)$. We may interprete (NNS) for $N = 4$ in this manner. We refer to [11] and the references therein for detailed explanations for the four-state cross-bridge models and refer to [18] for related topics.

Therefore, it is natural to employ (NNS) as a mathematical model for the two-state and four-state cross-bridge models of muscle contraction phenomena. It is interesting to compare the two models from a mathematical point of view.

Acknowledgment

The author wishes to thank Professor S. Oharu for his valuable advice and kind encouragement and Professor N. Kato for his serious guidance for a long time. His special thanks are also extended to Dr. T. Matsumoto for his thoughtful suggestions and helpful discussions.

2. Semilinear evolution equations associated with (AES)

In this section we study the semilinear evolution equation $(\mathrm{SE}; z)$ for a given function $z(\cdot)$.

We first state our basic hypotheses. Condition (BS) below is a hypothesis on the Banach space X and (GR) is a hypothesis on the linear operator Λ:

(BS) $(X, \|\cdot\|, \leqslant)$ is a real, ordered Banach space. D is a nonempty, closed subset of X which is contained in the positive cone X_+. Moreover, $u, v, v - u \in X_+$ (i.e., $0 \leqslant u \leqslant v$ in X) imply $\|u\| \leq \|v\|$;

(GR) The linear operator $-\Lambda : \mathscr{D}(\Lambda) \subset X \to X$ generates a C_0-group $\{S(\sigma)\}_{\sigma \in \mathbf{R}}$ of type $\omega \geq 0$ on X such that $S(\sigma)X_+ \subset X_+$ and $S(\sigma)D \subset D$ for $\sigma \in \mathbf{R}$.

Let T be an arbitrary but fixed positive number. We hereafter assume that the nonlinear mapping $F : [0, T] \times D \times \mathbf{R} \to X$ is continuous. In addition, we put conditions (F1) and (F2). (F1) implies the Lipschitz continuity in u and (F2) is the so-called subtangential condition. For the subtangential conditions, we refer to [19, 23, 24].

(F1) For each $r > 0$ there exists a constant $\bar{K}(r) > 0$ such that

$$\|F(t, u_1, z) - F(t, u_2, z)\| \leq \bar{K}(r)\|u_1 - u_2\|$$

for $t \in [0, T]$, $u_1, u_2 \in D$ and $z \in [-r, r]$;

(F2) For each $(t, u, z) \in [0, T] \times D \times \mathbf{R}$, $\liminf_{h \downarrow 0} h^{-1} d(u + hF(t, u, z), D) = 0$ holds, where $d(v, D)$ stands for the distance from v to D, that is, $d(v, D) = \inf_{u \in D}\|v - u\|$.

To define a mild solution to $(\mathrm{SE}; z)$, we need an evolution operator $U_z(t, s)$ determined by Λ and $z(\cdot)$. For $z \in W^{1,1}(0, T)$ and for almost every $t \in (0, T)$, we define a linear operator $A_z(t)$ in X by

$$\mathscr{D}(A_z(t)) := \begin{cases} \mathscr{D}(\Lambda), & \text{if } z'(t) \neq 0, \\ X, & \text{if } z'(t) = 0, \end{cases} \qquad A_z(t) := -z'(t)\Lambda.$$

Moreover, for each $z \in C([0,T])$, we put $U_z(t,s) = S(z(t) - z(s))$, $s, t \in [0,T]$, where $\{S(\sigma)\}_{\sigma \in \mathbf{R}}$ is the C_0-group generated by $-\Lambda$. Then we obtain the following proposition, cf. [14].

PROPOSITION 2.1. *Assume* (BS) *and* (GR). *Let* $z \in C([0,T])$. *Then the two-parameter family* $\{U_z(t,s)\}_{t,s \in [0,T]}$ *has the following properties*:
(i) $U_z(t,s) : X \to X$ *is a continuous linear operator for* $(t,s) \in [0,T] \times [0,T]$.
(ii) $(t,s) \mapsto U_z(t,s)$ *is X-strongly continuous on* $[0,T] \times [0,T]$.
(iii) $U_z(t,s)U_z(s,r) = U_z(t,r)$, $U_z(s,s) = I$ *for* $r,s,t \in [0,T]$.
(iv) $U_z(t,s)Y \subset Y$, *and* $(t,s) \mapsto U_z(t,s)$ *is Y-strongly continuous on* $[0,T] \times [0,T]$, *where* $Y := \mathscr{D}(\Lambda)$ *is endowed with the graph norm of* Λ.
(v) *If* $z \in W^{1,1}(0,T)$ *and* $u \in Y$, *then*

$$U_z(t,s)u - u = \int_s^t A_z(\tau)U_z(\tau,s)u\,d\tau = \int_s^t U_z(t,\tau)A_z(\tau)u\,d\tau,$$

$$(t,s) \in [0,T] \times [0,T].$$

(vi) *The operator* $U_z(t,s)$ *is invertible and* $U_z(t,s)^{-1} = U_z(s,t)$ *for* $s,t \in [0,T]$. *Thus,* $\{U_z(t,s)\}_{t,s \in [0,T]}$ *is a unique (linear) evolution operator on X generated by* $\{A_z(t)\}_t$.

Let $0 \leq s < \varsigma \leq T$. Given $z \in C([s,\varsigma])$, we define mild solutions to

$$(\mathrm{SE}; z) \qquad u' + z'(t)\Lambda u = F(t,u,z(t)), \qquad t \in (s,\varsigma).$$

DEFINITION 2.2. A function $u : [s,\varsigma] \to X$ is called a *mild solution* to (SE; z) on $[s,\varsigma]$, if $u \in C([s,\varsigma]; D)$ and satisfies the integral equation

$$u(t) = S(z(t) - z(s))u(s) + \int_s^t S(z(t) - z(\tau))F(\tau, u(\tau), z(\tau))\,d\tau, \qquad t \in [s,\varsigma].$$

Here the integral is taken in X in the sense of Bochner.

Our first goal is to prove the following theorem.

THEOREM 2.3. *Assume* (BS), (GR), (F1) *and* (F2). *Let* $0 \leq s < \varsigma \leq T$, $z \in C([s,\varsigma])$ *and* $u_s \in D$. *Then the Cauchy problem for* (SE; z) *on* $[s,\varsigma]$ *with initial condition* $u(s) = u_s$ *possesses a unique mild solution* u_z *in* $C([s,\varsigma]; D)$.

To prove this theorem we adopt the "method of characteristics" which has been employed in most of the papers concerning mathematical models for muscle contraction phenomena.

Let $0 \le s < \varsigma \le T$. Given $z \in C([s,\varsigma])$ and $u_s \in D$, we formulate an initial-value problem for the ordinary differential equation

$$\text{(ODE}; z) \qquad v'(t) = S(-z(t))F(t, S(z(t))v(t), z(t)), \qquad t \in [s,\varsigma],$$

in X with initial condition $v(s) = S(-z(s))u_s$, or equivalently, the integral equation

$$(2.1) \quad v(t) = S(-z(s))u_s + \int_s^t S(-z(\tau))F(\tau, S(z(\tau))v(\tau), z(\tau))d\tau, \qquad t \in [s,\varsigma],$$

where $\{S(\sigma)\}_{\sigma \in \mathbf{R}}$ is the C_0-group generated by $-A$.

PROPOSITION 2.4. *Assume* (BS) *and* (GR). *Let* $0 \le s < \varsigma \le T$, $z \in C([s,\varsigma])$ *and* $u_s \in D$. *Then the initial-value problem for* (SE; z) *with* $u(s) = u_s$ *and the initial-value problem for* (ODE; z) *with* $v(s) = S(-z(s))u_s$ *are equivalent in the following sense*:

(i) *Let* $u \in C([s,\varsigma]; D)$ *be a mild solution of* (SE; z) *with* $u(s) = u_s$ *and put* $v(t) = S(-z(t))u(t)$. *Then* $v \in C([s,\varsigma]; D) \cap C^1([s,\varsigma]; X)$ *gives a classical solution of* (ODE; z) *satisfying* $v(s) = S(-z(s))u_s$;

(ii) *Let* $v \in C([s,\varsigma]; D) \cap C^1([s,\varsigma]; X)$ *be a classical solution of* (ODE; z) *with* $v(s) = S(-z(s))u_s$ *and put* $u(t) = S(z(t))v(t)$. *Then* $u \in C([s,\varsigma]; D)$ *becomes a mild solution of* (SE; z) *with* $u(s) = u_s$.

PROOF. (i) Let $u \in C([s,\varsigma]; D)$ be a mild solution to the initial-value problem for (SE; z) on $[s,\varsigma]$ with $u(s) = u_s$. Set $v(t) = S(-z(t))u(t)$. Then it is clear that $v \in C([s,\varsigma]; D)$ and it satisfies (2.1) by (GR), and so $v \in C^1([s,\varsigma]; X)$. Notice here that the operator $S(\sigma)$ is invertible and $S(\sigma)^{-1} = S(-\sigma)$ for any $\sigma \in \mathbf{R}$. The implication from (ii) to (i) is verified. □

PROPOSITION 2.5. *Under the same assumptions as in Theorem* 2.3, *the initial-value problem for* (ODE; z) *on* $[s,\varsigma]$ *with initial condition* $v(s) = S(-z(s))u_s$ *has a unique classical solution* v_z *in* $C([s,\varsigma]; D) \cap C^1([s,\varsigma]; X)$.

PROOF. We begin by extending $F(t,u,z)$ to $\mathbf{R} \times D \times \mathbf{R}$ by

$$F(t,u,z) = \begin{cases} F(0,u,z), & \text{if } t < 0, \\ F(T,u,z), & \text{if } t > T. \end{cases}$$

Hence we may assume that $F \in C(\mathbf{R} \times D \times \mathbf{R}; X)$ and satisfies (F1) and (F2) for all $t \in \mathbf{R}$.

Fix $z \in C([s,\varsigma])$ and set

$$z(t) = \begin{cases} z(s), & \text{if } t < s, \\ z(\varsigma), & \text{if } t > \varsigma. \end{cases}$$

Then $z \in C(\mathbf{R})$. Put $G(t,v) := G_z(t,v) := S(-z(t))F(t, S(z(t))v, z(t))$ for $(t,v) \in$

$\mathbf{R} \times D$. Then $G \in C(\mathbf{R} \times D; X)$ since the C_0-group $\{S(\sigma)\}_{\sigma \in \mathbf{R}}$ is of type $\omega \geq 0$. By (F1), we see that $G(t, v)$ is Lipschitz continuous in v uniformly for $t \in \mathbf{R}$:

$$\|G(t, v_1) - G(t, v_2)\| \leq e^{2\omega r}\bar{K}(r)\|v_1 - v_2\|, \qquad t \in \mathbf{R},\ v_1, v_2 \in D,$$

where $r = \sup_{\tau \in \mathbf{R}}|z(\tau)| = \sup_{s \leq \tau \leq \varsigma}|z(\tau)| < \infty$ and $\bar{K}(r)$ is the constant employed in (F1). As a result, G is quasi-dissipative in the following sense:

$$(2.2) \qquad (1 - \lambda C_r)\|v_1 - v_2\| \leq \|v_1 - v_2 - \lambda[G(t, v_1) - G(t, v_2)]\|,$$

$$\lambda > 0,\ t \in \mathbf{R},\ v_1, v_2 \in D,$$

where $C_r = e^{2\omega r}\bar{K}(r)$. Moreover, G satisfies the subtangential condition

$$(2.3) \qquad \liminf_{h \downarrow 0} h^{-1} d(v + hG(t, v), D) = 0, \qquad t \in \mathbf{R},\ v \in D,$$

where $d(u, D) = \inf_{v \in D}\|u - v\|$ for $u \in X$. In fact, $S(-\sigma)S(\sigma) = I$ and $S(\sigma)D = D$ for $\sigma \in \mathbf{R}$ by (GR). Therefore we know that

$$d(v + hS(-\sigma)F(t, S(\sigma)v, \sigma), D) \leq e^{\omega|\sigma|} d(S(\sigma)v + hF(t, S(\sigma)v, \sigma), D)$$

for $h > 0$ and $(t, v, \sigma) \in \mathbf{R} \times D \times \mathbf{R}$. Thus, by (F2), it follows that

$$\liminf_{h \downarrow 0} h^{-1} d(v + hS(-\sigma)F(t, S(\sigma)v, \sigma), D) = 0, \qquad (t, v, \sigma) \in \mathbf{R} \times D \times \mathbf{R},$$

which implies (2.3).

Given $v_s \in D$, we consider the integral equation

$$(2.4) \qquad v(t) = v_s + \int_s^t G(\tau, v(\tau))d\tau, \qquad t \in [s, +\infty).$$

Since G belongs to $C(\mathbf{R} \times D; X)$ and satisfies (2.2), (2.3), we can apply Pavel [23, Corollary 1.1] to obtain a unique solution $v \in C([s, \infty); D)$ of (2.4). Consequently, for $0 \leq s < \varsigma \leq T$, $z \in C([s, \varsigma])$ and $u_s \in D$, the integral equation (2.1) has one and only one solution $v_z \in C([s, \varsigma]; D)$. Since (2.1) is equivalent to (ODE; z) with $v(s) = S(-z(s))u_s$, the proof is complete. □

PROOF OF THEOREM 2.3. The proof follows directly from Propositions 2.4 and 2.5. □

We next investigate the continuous dependence of solutions of (ODE; z) and (SE; z) on the function z. To do this, we need the Lipschitz continuity of F in z:

(F3) For every $r > 0$ there exist a nonnegative function $\nu_r \in L^1(0, T)$ and a nondecreasing function $\bar{\rho}_r : [0, \infty) \to [0, \infty)$ with $\bar{\rho}_r(+0) = 0$ such that

$$\|F(t, u, z_1) - F(t, u, z_2)\| \leq \nu_r(t)\bar{\rho}_r(|z_1 - z_2|)$$

for almost every $t \in (0, T)$, $u \in D$ with $\|u\| \leq r$, and $z_1, z_2 \in [-r, r]$.

LEMMA 2.6. *Assume* (BS), (GR) *and* (F1) *through* (F3). *Let* $0 \le s < \varsigma \le T$ *and* $u_s \in D$. *Let* $v_z \in C([s,\varsigma]; D)$ *be a solution of the initial-value problem for* (ODE; z) *on* $[s,\varsigma]$ *with initial condition* $v_z(s) = S(-z(s))u_s$. *Then* $z \mapsto v_z$ *is a continuous mapping from* $C([s,\varsigma])$ *into* $C([s,\varsigma]; X)$. *Here the spaces* $C([s,\varsigma])$ *and* $C([s,\varsigma]; X)$ *are equipped with the supremum-norm* $|\cdot|_\infty$ *and the usual norm* $\|u\|_\infty = \sup_{s \le t \le \varsigma}\|u(t)\|$, *respectively.*

PROOF. Suppose that $z_n \to z$ in $C([s,\varsigma])$ and that $v_n, v \in C([s,\varsigma]; D)$ are solutions of (ODE; z_n) and (ODE; z) on $[s,\varsigma]$ such that $v_n(s) = S(-z_n(s))u_s$ and $v(s) = S(-z(s))u_s$, respectively. Then we have

$$\begin{aligned}
&\|v_n(t) - v(t)\| \\
&\le \|S(-z_n(s))u_s - S(-z(s))u_s\| \\
&\quad + \int_s^t \|S(-z_n(\tau))[F(\tau, S(z_n(\tau))v_n(\tau), z_n(\tau)) - F(\tau, S(z(\tau))v(\tau), z_n(\tau))]\|d\tau \\
&\quad + \int_s^t \|S(-z_n(\tau))[F(\tau, S(z(\tau))v(\tau), z_n(\tau)) - F(\tau, S(z(\tau))v(\tau), z(\tau))]\|d\tau \\
&\quad + \int_s^t \|[S(-z_n(\tau)) - S(-z(\tau))]F(\tau, S(z(\tau))v(\tau), z(\tau))\|d\tau, \qquad t \in [s,\varsigma].
\end{aligned}$$

We here denote by J_1 and J_2, respectively, the second and third terms of the right-hand side of the above inequality. Put $\hat{r} = \sup_m |z_m|_\infty$. By (F1), we obtain

$$J_1 \le e^{\omega\hat{r}}\bar{K}(\hat{r})\left\{e^{\omega\hat{r}}\int_s^t \|v_n(\tau) - v(\tau)\|d\tau + \int_s^t \|[S(z_n(\tau)) - S(z(\tau))]v(\tau)\|d\tau\right\},$$

since $\{S(\sigma)\}_{\sigma\in\mathbf{R}}$ is of type ω. Let $r = \max\{\hat{r}, e^{\omega\hat{r}}\sup_{s\le\tau\le\varsigma}\|v(\tau)\|\}$. We have

$$J_2 \le e^{\omega\hat{r}}\int_0^T \nu_r(\tau)d\tau\bar{\rho}_r(|z_n - z|_\infty)$$

by (F3). Here $\nu_r \in L^1(0,T)$ and the local modulas $\bar{\rho}_r$ of continuity are the functions employed in (F3). Thus, it follows that

$$\|v_n(t) - v(t)\| \le \Upsilon_n + e^{2\omega\hat{r}}\bar{K}(\hat{r})\int_s^t \|v_n(\tau) - v(\tau)\|d\tau, \qquad t \in [s,\varsigma],$$

where

$$\begin{aligned}
\Upsilon_n &= \|[S(-z_n(s)) - S(-z(s))]u_s\| \\
&\quad + e^{\omega\hat{r}}\bar{K}(\hat{r})\int_s^\varsigma \|[S(z_n(\tau)) - S(z(\tau))]v(\tau)\|d\tau + e^{\omega\hat{r}}\int_0^T \nu_r(\tau)d\tau\bar{\rho}_r(|z_n - z|_\infty) \\
&\quad + \int_s^\varsigma \|[S(-z_n(\tau)) - S(-z(\tau))]F(\tau, S(z(\tau))v(\tau), z(\tau))\|d\tau.
\end{aligned}$$

Gronwall's Lemma then gives

$$\sup_{s \le t \le \varsigma} \|v_n(t) - v(t)\| \le \Upsilon_n \exp(T\bar{K}(\hat{r}) \exp(2\omega\hat{r})). \tag{2.5}$$

Since $\Upsilon_n \to 0$ by the Bounded Convergence Theorem, letting $n \to \infty$ in (2.5) implies that $v_n \to v$ in $C([s, \varsigma]; X)$. Thus, the desired result follows. □

We put the following condition on the linear functional f.

(LF) The continuous linear functional $f : X \to \mathbf{R}$ is not identically equal to zero and the composition $fA : \mathscr{D}(A) \subset X \to \mathbf{R}$ is continuous on the linear subspace $(\mathscr{D}(A), \|\cdot\|)$ of $(X, \|\cdot\|)$. Furthermore, the unique extension g of fA to X satisfies $g(u) \le 0$ for all $u \in D$.

REMARK 2.7. The domain $\mathscr{D}(A)$ of A is a dense linear subspace of X since $-A$ is the generator of a C_0-group on X. Hence a continuous linear functional fA on $\mathscr{D}(A)$ is uniquely extended to all of X as a continuous linear functional.

LEMMA 2.8. *Assume* (BS), (GR) *and* (LF). *Then* (i) *for each* $v \in X$ *we have*

$$f(S(\sigma)v) = f(v) - \int_0^\sigma g(S(\tau)v)d\tau, \qquad \sigma \in \mathbf{R},$$

(ii) *for each* $r > 0$ *we have*

$$\|fS(\sigma_1) - fS(\sigma_2)\|_* \le \|g\|_* e^{\omega r}|\sigma_1 - \sigma_2|, \qquad \sigma_1, \sigma_2 \in [-r, r],$$

where $\|\cdot\|_*$ *denotes the norm of continuous linear functionals on* X.

PROOF. (i) Since $\mathscr{D}(A)$ is dense in X, it suffices to show that for $v \in \mathscr{D}(A)$ the result holds. Let $v \in \mathscr{D}(A)$. Since $-A$ is the generator of $\{S(\sigma)\}_{\sigma \in \mathbf{R}}$, it follows that $S(\sigma)v = v - \int_0^\sigma AS(\tau)v d\tau$ for $\sigma \in \mathbf{R}$. Noting that $fA = g$ on $\mathscr{D}(A)$ by (LF), we see that

$$f(S(\sigma)v) = f(v) - \int_0^\sigma f(AS(\tau)v)d\tau = f(v) - \int_0^\sigma g(S(\tau)v)d\tau, \qquad \sigma \in \mathbf{R}.$$

Assertion (ii) follows directly from (i). □

The next lemma asserts the continuous dependence of mild solutions of (SE; z) on z.

LEMMA 2.9. *Assume* (BS), (GR), (F1) *through* (F3) *and* (LF). *Let* $0 \le s < \varsigma \le T$, $u_s \in D$, *and let* $u_z \in C([s, \varsigma]; D)$ *be a mild solution of the initial-value problem for* (SE; z) *on* $[s, \varsigma]$ *with* $u_z(s) = u_s$. *Then* $z \mapsto fu_z$ *is a continuous mapping from* $C([s, \varsigma])$ *into itself, where* $C([s, \varsigma])$ *is the Banach space endowed with the supremum-norm* $|\cdot|_\infty$.

PROOF. Let $z_n \to z$ in $C([s,\varsigma])$ and let $u_n, u \in C([s,\varsigma];D)$ be the associated mild solutions to $(\mathrm{SE}; z_n)$ and $(\mathrm{SE}; z)$ on $[s,\varsigma]$ satisfying $u_n(s) = u(s) = u_s$, respectively. Set $v_n(t) = S(-z_n(t))u_n(t)$ and $v(t) = S(-z(t))u(t)$. Then v_n and v are the solutions of $(\mathrm{ODE}; z_n)$ and $(\mathrm{ODE}; z)$ satisfying, respectively, $v_n(s) = S(-z_n(s))u_s$ and $v(s) = S(-z(s))u_s$ by Proposition 2.4. Therefore, Lemma 2.8 (ii) implies that

$$
\begin{aligned}
|\boldsymbol{f}(u_n(t)) - \boldsymbol{f}(u(t))| &= |\boldsymbol{f}(S(z_n(t))v_n(t)) - \boldsymbol{f}(S(z(t))v(t))| \\
&\le \|\boldsymbol{f}\|_* e^{\omega\hat{r}} \|v_n(t) - v(t)\| + \|\boldsymbol{g}\|_* e^{\omega\hat{r}} |z_n(t) - z(t)|\, \|v(t)\|, \quad t \in [s,\varsigma],
\end{aligned}
$$

where $\hat{r} = \sup_n |z_n|_\infty$. Taking the supremum over $t \in [s,\varsigma]$ and then letting $n \to \infty$, we conclude that $\boldsymbol{f}u_n \to \boldsymbol{f}u$ in $C([s,\varsigma])$ by Lemma 2.6. □

REMARK 2.10. Under the assumptions in the above lemma, one can not expect the Lipschitz or Hölder continuity of $z \mapsto \boldsymbol{f}u_z : (C([s,\varsigma]), |\cdot|_\infty) \to (C([s,\varsigma]), |\cdot|_\infty)$.

The following lemma concerning the regularity of the function $\boldsymbol{f}(u_z(t))$ is applied to the proofs of the existence theorems for (AES).

LEMMA 2.11. *Assume* (BS), (GR) *and* (LF). *Let* $0 \le s < \varsigma \le T$, $1 \le p \le \infty$ *and* u_z *a mild solution of* $(\mathrm{SE}; z)$ *on* $[s,\varsigma]$. *If* $z \in W^{1,p}(s,\varsigma)$, *then* $\boldsymbol{f}(u_z(\cdot)) \in W^{1,p}(s,\varsigma)$ *and*

$$
(\boldsymbol{f}u_z)'(t) = -z'(t)\boldsymbol{g}(u_z(t)) + \boldsymbol{f}F(t, u_z(t), z(t)) \qquad a.e.\ (s,\varsigma).
$$

PROOF. For $v \in X$ it is seen that $(d/dt)\boldsymbol{f}(S(z(t))v) = -z'(t)\boldsymbol{g}(S(z(t))v)$ a.e. (s,ς) by Lemma 2.8 (i). Put $v_z(t) = S(-z(t))u_z(t)$. Then $v_z \in C([s,\varsigma];D) \cap C^1([s,\varsigma];X)$ and is a classical solution to $(\mathrm{ODE}; z)$ on $[s,\varsigma]$ with $v_z(s) = S(-z(s))u_z(s)$ by Proposition 2.4. Hence $\boldsymbol{f}u_z(t) = \boldsymbol{f}S(z(t))v_z(t)$. From this we see that

$$
\begin{aligned}
(\boldsymbol{f}u_z)'(t) &= (\boldsymbol{f}S(z(t)))'v_z(t) + \boldsymbol{f}S(z(t))v_z'(t) \\
&= -z'(t)\boldsymbol{g}(S(z(t))v_z(t)) + \boldsymbol{f}S(z(t))S(-z(t))F(t, S(z(t))v_z(t), z(t)) \\
&= -z'(t)\boldsymbol{g}(u_z(t)) + \boldsymbol{f}F(t, u_z(t), z(t)) \qquad \text{a.e. } (s,\varsigma).
\end{aligned}
$$

This shows that $(\boldsymbol{f}u_z)'(\cdot) \in L^p(s,\varsigma)$. The proof is now complete. □

3. Semilinear evolution equations coupled with nonlinear constraints

In this section we discuss the existence and uniqueness of mild solutions for the nonlinear evolution system (AES) which is an abstract form of (NNS). The existence theorems are proved in the next section. As in the previous section, we assume that the nonlinear mapping F is continuous on $[0, T] \times D \times \mathbf{R}$ throughout this section.

To establish our results for (AES), we need the following growth condition on F.

(F4) There exist an X-valued function $\mathscr{F} \in L^1(0, T; X_+)$ and a constant $\overline{M} > 0$ such that $F(t, u, z) \leqslant \mathscr{F}(t) + \overline{M}u$ in X for almost every $t \in (0, T)$, $u \in D$ and $z \in \mathbf{R}$.

On the function Γ, we impose a dissipativity condition:

(G) The multi-valued function $\Gamma : \mathscr{D}(\Gamma) \subset \mathbf{R} \to 2^{\mathbf{R}}$ is a nonlinear m-dissipative operator in $\mathbf{R}$, and locally quasi-dissipative in the following sense: For each $r > 0$ there exists a constant $\bar{\beta}_r > 0$ such that

$$(1 + \lambda\bar{\beta}_r)|\zeta_1 - \zeta_2| \leq |\zeta_1 - \zeta_2 - \lambda(\tau_1 - \tau_2)|$$

for $\lambda > 0$ and $(\zeta_i, \tau_i) \in \Gamma$ with $|\zeta_i| \leq r$, $i = 1, 2$.

We consider the Cauchy problem for the semilinear evolution equation coupled with the nonlinear constraint:

$$\text{(AES)} \qquad \begin{cases} u'(t) + z'(t)Au(t) = F(t, u(t), z(t)), & t \in (0, T), \\ \boldsymbol{f}(u(t)) \in \Gamma(z(t)), & t \in [0, T], \end{cases}$$

under the initial condition (IC)

$$\text{(IC)} \qquad (z(0), u(0)) = (z_0, u_0).$$

Definition 3.1. A pair of functions $(z, u) \in C([0, T]) \times C([0, T]; D)$ is said to be a *mild solution* of (AES) on $[0, T]$, if the function u is a mild solution of (SE; z) on $[0, T]$, and $(z(t), \boldsymbol{f}(u(t))) \in \Gamma$ for $t \in [0, T]$ in the sense that $z(t) \in \mathscr{D}(\Gamma)$ and $\boldsymbol{f}(u(t)) \in \Gamma(z(t))$.

We are now in a position to state the local existence theorem for the mild solutions to the Cauchy problem (AES)–(IC).

Theorem 3.2. *Assume* (BS), (GR), (F1) *through* (F4), (LF) *and* (G). *Let* $(z_0, u_0) \in \mathscr{D}(\Gamma) \times D$ *and* $\boldsymbol{f}(u_0) \in \Gamma(z_0)$. *Then there exist* $\hat{T} \in (0, T]$ *and a mild solution* $(z, u) \in C([0, \hat{T}]) \times C([0, \hat{T}]; D)$ *to the initial-value problem* (AES)–(IC) *on* $[0, \hat{T}]$ *such that* z, $\boldsymbol{f}u \in W^{1,\infty}(0, \hat{T})$. *Furthermore, let* $[0, T_{\max})$ *be the maximal interval of existence of mild solutions,* $0 < T_{\max} \leq T$, *and let* (z, u) *be a mild solution on* $[0, T_{\max})$. *If* $T_{\max} < T$, *then* $\limsup_{t \uparrow T_{\max}} |z(t)| = \infty$.

To obtain the global existence result we need the following additional hypotheses which are naturally satisfied for the system (NNS).

(AdLF) For the continuous linear functionals $\bar{\boldsymbol{f}}$, $\boldsymbol{h} : X \to \mathbf{R}$, $\sigma \in \mathbf{R}$ and $u \in X_+$, $\bar{\boldsymbol{f}}(u) \geq 0$ and $\boldsymbol{h}(S(\sigma)u) = \boldsymbol{h}(u)$. There are constants C_1, $C_2 > 0$ such that $0 < C_1\boldsymbol{h}(u) \le -\boldsymbol{g}(u) \le C_2\boldsymbol{h}(u)$ for nonzero $u \in X_+$. Furthermore, the composition $\bar{\boldsymbol{f}}A$ is continuous on the linear subspace $(\mathscr{D}(A), \|\cdot\|)$ of $(X, \|\cdot\|)$, and its unique extension $\bar{\boldsymbol{g}}$ on X satisfies $|\bar{\boldsymbol{g}}(u)| \le -\boldsymbol{g}(u)$ for $u \in X_+$;

(AdF) $F(t,u,z) \geqslant -\overline{M}u$ holds in X for $(t,u,z) \in [0,T] \times D \times \mathbf{R}$. Moreover, there exists a nonnegative function $\xi \in L^\infty(0,T)$ such that $|\boldsymbol{f}F(t,u,z)| \le \xi(t) + \overline{M}\bar{\boldsymbol{f}}(u)$ for almost every $t \in (0,T)$ and $(u,z) \in D \times \mathbf{R}$. Here $\overline{M}$ is the same constant as employed in (F4);

(AdG) The multi-valued function Γ satisfies either $0 \notin \mathscr{R}(\Gamma)$ or $(0,0) \in \Gamma$.

Under these conditions we have the following

Theorem 3.3. *Assume* (BS), (GR), (F1) *through* (F4), (LF), (G), (AdLF), (AdF) *and* (AdG). *Let* $(z_0, u_0) \in \mathscr{D}(\Gamma) \times D$ *and* $\boldsymbol{f}(u_0) \in \Gamma(z_0)$. *Then there exists a mild solution* (z,u) *to the initial-value problem* (AES)–(IC) *on* $[0,T]$ *such that* z, $\boldsymbol{f}u \in W^{1,\infty}(0,T)$.

To guarantee the uniqueness, we need an additional hypothesis which impose the Lipschitz continuity of the function $\sigma \mapsto S(-\sigma)F(t,S(\sigma)u,z)$.

(F5) For each $r > 0$ there exists a nonnegative function $\bar{\theta}_r \in L^1(0,T)$ such that

$$\|S(-\sigma_1)F(t,S(\sigma_1)u,z) - S(-\sigma_2)F(t,S(\sigma_2)u,z)\| \le \bar{\theta}_r(t)|\sigma_1 - \sigma_2|$$

for almost every $t \in (0,T)$, $u \in D$ with $\|u\| \le r$, and σ_1, σ_2, $z \in [-r,r]$.

Then we obtain the following uniqueness result.

Theorem 3.4. *Assume* (F5) *in addition to* (BS), (GR), (F1), (F3) *with* $\bar{\rho}_r(s) = C_r s$ *(*C_r *being some constant),* (F4), (LF) *and* (G). *Then for any pair of mild solutions* (z_i, u_i), $i = 1,2$, *to* (AES) *on* $[0,T]$ *we have*

$$|z_1 - z_2|_\infty \le C\|S(-z_1(0))u_1(0) - S(-z_2(0))u_2(0)\|. \tag{3.1}$$

Here $|\cdot|_\infty$ *denotes the supremum-norm over* $[0,T]$. *The positive constant* C *may depend upon a fixed number* $R \ge \max\{r_1, r_2\}$, *where* $r_1 = \max\{|z_1|_\infty, |z_2|_\infty\}$ *and* $r_2 = e^{\omega r_1 + \overline{M}T}(\max\{\|u_1(0)\|, \|u_2(0)\|\} + \int_0^T \|\mathscr{F}(\tau)\|d\tau)$. *In particular, a mild solution to* (AES) *is unique, if it exists.*

Proof. Conditions (F5) and (F1) together imply that for each $r > 0$ there is a nonnegative function $\bar{\theta}_r \in L^1(0,T)$ such that

$$\begin{aligned}(1 - \lambda\bar{\theta}_r(t))\|u_1 - u_2\| - \lambda\bar{\theta}_r(t)|\sigma_1 - \sigma_2| \\ \le \|u_1 - u_2 - \lambda[S(-\sigma_1)F(t,S(\sigma_1)u_1,z) - S(-\sigma_2)F(t,S(\sigma_2)u_2,z)]\|\end{aligned} \tag{3.2}$$

for $\lambda > 0$, almost every $t \in (0,T)$, u_1, $u_2 \in D$ with $\|u_i\| \le r$, and σ_1, σ_2, $z \in$

$[-r,r]$. Furthermore, it follows from (3.2) and (F3) with $\bar{\rho}_r(s) = C_r s$ that for each $r > 0$ there is a nonnegative function $\theta_r \in L^1(0,T)$ such that

$$(3.3)\quad (1-\lambda\theta_r(t))\|u_1-u_2\| - \lambda\theta_r(t)|\sigma_1-\sigma_2| \\ \leq \|u_1-u_2-\lambda[S(-\sigma_1)F(t,S(\sigma_1)u_1,\sigma_1) - S(-\sigma_2)F(t,S(\sigma_2)u_2,\sigma_2)]\|$$

for $\lambda > 0$, almost every $t \in (0,T)$, u_1, $u_2 \in D$ with $\|u_i\| \leq r$, and σ_1, $\sigma_2 \in [-r,r]$.

Let (z_i,u_i), $i = 1,2$, be any pair of mild solutions to (AES) on $[0,T]$. Put $v_i(t) = S(-z_i(t))u_i(t)$. Then v_i is a solution to (ODE; z_i) on $[0,T]$ by Proposition 2.4. In order to establish (3.1), we first show that

$$(3.4)\qquad \bar{\beta}_r|z_1(t)-z_2(t)| \leq \|\boldsymbol{f}\|_* e^{\omega r}\|v_1(t)-v_2(t)\|, \qquad t \in [0,T],$$

where $\bar{\beta}_r$ is the constant employed in (G) and $r \geq r_1 = \max\{|z_1|_\infty, |z_2|_\infty\}$. If $z_1(t) = z_2(t)$ at t, then (3.4) is trivial. We then suppose that $z_1(t) < z_2(t)$ at some t. Since Γ is a dissipative operator in $\mathbf{R}$ and $(z_i(t), \boldsymbol{f}(u_i(t))) \in \Gamma$, we see that $(z_1(t)-z_2(t))(\boldsymbol{f}(u_1(t)) - \boldsymbol{f}(u_2(t))) \leq 0$, and hence that $\boldsymbol{f}(u_1(t)) \geq \boldsymbol{f}(u_2(t))$. Thus, we obtain the estimate

$$\bar{\beta}_r|z_1(t)-z_2(t)| \leq |\boldsymbol{f}(u_1(t)) - \boldsymbol{f}(u_2(t))| = \boldsymbol{f}(u_1(t)) - \boldsymbol{f}(u_2(t))$$

by the local quasi-dissipativity of Γ. Moreover, noting that $\sigma \mapsto \boldsymbol{f}(S(\sigma)v_1(t))$ is nondecreasing by Lemma 2.8 (i) and using the fact that $\boldsymbol{f}(u_i(t)) = \boldsymbol{f}(S(z_i(t))v_i(t))$, we have

$$\bar{\beta}_r|z_1(t)-z_2(t)| \leq \boldsymbol{f}(S(z_2(t))v_1(t)) - \boldsymbol{f}(S(z_2(t))v_2(t)) \leq \|\boldsymbol{f}\|_* e^{\omega|z_2(t)|}\|v_1(t)-v_2(t)\|.$$

This implies (3.4). We next demonstrate that

$$(3.5)\quad \|v_1(t)-v_2(t)\| \\ \leq \exp\left(\int_0^T \theta_R(\tau)d\tau\right)\left(\|v_1(0)-v_2(0)\| + \int_0^t \theta_R(\tau)|z_1(\tau)-z_2(\tau)|d\tau\right) \\ \text{for } t \in [0,T],$$

where $\theta_R(\cdot) \in L^1(0,T)_+$, $R \geq \max\{r_1, r_2\}$ and

$$r_2 = e^{\omega r_1 + \bar{M}T}\left(\max\{\|u_1(0)\|, \|u_2(0)\|\} + \int_0^T \|\mathscr{F}(\tau)\|d\tau\right).$$

Let $0 < h \leq t \leq T$. Then v_i satisfies

$$v_i(t) = v_i(t-h) + \int_{t-h}^t S(-z_i(\tau))F(\tau, S(z_i(\tau))v_i(\tau), z_i(\tau))d\tau \\ = v_i(t-h) + hS(-z_i(t))F(t, S(z_i(t))v_i(t), z_i(t)) + o_i(h),$$

and hence

(3.6) $$v_i(t) - hS(-z_i(t))F(t, S(z_i(t))v_i(t), z_i(t)) = v_i(t-h) + o_i(h),$$

where $o_i(h)$ depends on t but $h^{-1}\|o_i(h)\| \to 0$ as $h \downarrow 0$. We then use the next estimate which is proved in §7.

LEMMA 3.5. *Let $z \in C([0,T])$ and $u_0 \in D$. If $v_z \in C([0,T]; D)$ is a solution of* (ODE; z) *on* $[0,T]$ *with initial condition* $v_z(0) = S(-z(0))u_0$, *then*

$$\|v_z(t)\| \le e^{\omega|z|_\infty + \bar{M}T}\left(\|u_0\| + \int_0^T \|\mathscr{F}(\tau)\| d\tau\right) \qquad \text{for } 0 \le t \le T.$$

In view of the above estimates, we have $\|v_i(t)\| \le r_2$ for $t \in [0,T]$. Therefore, by (3.3) with $r = R$ and (3.6), it follows that

$$\begin{aligned}
(1 - h\theta_R(t))&\|v_1(t) - v_2(t)\| - h\theta_R(t)|z_1(t) - z_2(t)| \\
&\le \|v_1(t) - v_2(t) - h[S(-z_1(t))F(t, S(z_1(t))v_1(t), z_1(t)) \\
&\qquad\qquad - S(-z_2(t))F(t, S(z_2(t))v_2(t), z_2(t))]\| \\
&\le \|v_1(t-h) - v_2(t-h)\| + \|o_1(h)\| + \|o_2(h)\|.
\end{aligned}$$

This leads us to the estimate

$$\begin{aligned}
(\|v_1(t-h) &- v_2(t-h)\| - \|v_1(t) - v_2(t)\|)/(-h) \\
&\le \theta_R(t)(\|v_1(t) - v_2(t)\| + |z_1(t) - z_2(t)|) + o(h)/h,
\end{aligned}$$

where $o(h)/h \to 0$ as $h \downarrow 0$. Taking the limit suprema of both sides as $h \downarrow 0$, we have

$$D^-\|v_1(t) - v_2(t)\| \le \theta_R(t)(\|v_1(t) - v_2(t)\| + |z_1(t) - z_2(t)|) \qquad \text{a.e. } (0,T),$$

where $D^- f(t) = \limsup_{k \uparrow 0} k^{-1}(f(t+k) - f(t))$. Solving this differential inequality, we get (3.5). Notice that we do not use the relation that $f(u_i(t)) \in \Gamma(z_i(t))$ to show (3.5).

It is clear from (3.4) with $r = R$ and (3.5) that

$$|z_1(t) - z_2(t)| \le C\left(\|v_1(0) - v_2(0)\| + \int_0^t \theta_R(\tau)|z_1(\tau) - z_2(\tau)| d\tau\right), \qquad t \in [0,T],$$

for some positive constant C depending on R. By Gronwall's Lemma, we get

$$|z_1(t) - z_2(t)| \le C\|v_1(0) - v_2(0)\| \exp\left(C\int_0^t \theta_R(\tau) d\tau\right) \qquad \text{for } t \in [0,T],$$

which implies (3.1).

It remains to show that (3.1) implies the uniqueness. Assume that $(z_1(0), u_1(0)) = (z_2(0), u_2(0))$. Then it is obvious that $z_1 \equiv z_2$ by (3.1). If a mild solution u_z to the Cauchy problem for (SE; z) on $[0, T]$ is at most one for $z \in C([0, T])$, then we deduce that $u_1 \equiv u_2$, and hence that $(z_1, u_1) \equiv (z_2, u_2)$. Namely, a mild solution to (AES) is at most one.

Now, (3.5) implies the uniqueness of solutions to (ODE; z) on $[0, T]$ for $z \in C([0, T])$. Consequently, a mild solution of (SE; z) on $[0, T]$, if it exists, is uniquely determined for $z \in C([0, T])$. See Proposition 2.4. The proof of Theorem 3.4 is now complete. □

REMARK 3.6. In the proof of the uniqueness, $z_1(0) = z_2(0)$ follows from $u_1(0) = u_2(0)$ and the local quasi-dissipativity of Γ plays an important role.

The following condition, (F6), is more general than the combination of (F1) and (F3), but Theorems 3.2 through 3.4 are still valid under (F6).

(F6) For each $r > 0$ there is a positive constant C_r such that

$$(1 - \lambda C_r)\|v_1 - v_2\| - \lambda C_r|z_1 - z_2|$$
$$\leq \|v_1 - v_2 - \lambda[S(-z_1)F(t, S(z_1)v_1, z_1) - S(-z_2)F(t, S(z_2)v_2, z_2)]\|$$

for $\lambda > 0$, $t \in [0, T]$ and $(v_i, z_i) \in D \times [-r, r]$, $i = 1, 2$.

THEOREM 3.7. *In Theorems* 3.2 *and* 3.3 *(resp. in Theorem* 3.4*), assume* (F6) *instead of* (F1) *and* (F3) *(resp. instead of* (F1), (F3) *and* (F5)*). Then the same assertions are valid.*

The remaining part of this section is devoted to the reduction of the Cauchy problem for (AES) to equivalent problems. Given $(z_s, u_s) \in \mathbf{R} \times X$, consider the following problems which are equivalent to (AES) on $[s, \varsigma]$ with initial condition $(z(s), u(s)) = (z_s, u_s)$:

Find $z \in C([s, \varsigma])$ satisfying the nonlinear constraint

(NC) $$(z(t), \boldsymbol{f}(u_z(t))) \in \Gamma, \quad t \in [s, \varsigma], \qquad \text{or equivalently,}$$
$$z(t) \in D(\Gamma) \quad \text{and} \quad \boldsymbol{f}(u_z(t)) \in \Gamma(z(t))$$

and the initial condition $(z(s), u_z(s)) = (z_s, u_s)$, where u_z is a mild solution of the initial-value problem for (SE; z) on $[s, \varsigma]$ with initial condition $u_z(s) = u_s$;

Find $z \in C([s, \varsigma])$ satisfying the equation

(FE) $$z(t) = (I - \lambda\Gamma)^{-1}(z(t) - \lambda\boldsymbol{f}(u_z(t))), \quad t \in [s, \varsigma]$$

for some positive constant λ which is independent of t, as well as the initial condition $(z(s), u_z(s)) = (z_s, u_s)$, where u_z is a mild solution of the initial-value problem for (SE; z) on $[s, \varsigma]$ with $u_z(s) = u_s$ and I the identity operator in $\mathbf{R}$. Notice that an inverse function $(I - \lambda\Gamma)^{-1}(\cdot)$ of $I - \lambda\Gamma$ is defined on all of $\mathbf{R}$ as a single-valued function, since Γ is an m-dissipative operator in $\mathbf{R}$.

THEOREM 3.8. *Assume* (BS), (GR), (F1), (F2), (LF) *and* (G). *Let* $0 \leq s < \varsigma \leq T$. *Under the initial condition* $(z(s), u(s)) = (z_s, u_s)$, *the initial-value problems for* (AES), (NC) *and* (FE) *on* $[s, \varsigma]$ *are equivalent to each other in the following sense*:

(i) *If* (z, u) *is a mild solution to* (AES), *then* z *is a solution to* (NC) *and* $u \equiv u_z$;

(ii) *If* z *is a solution to* (NC), *then* (z, u_z) *is a mild solution to* (AES);

(iii) z *is a solution to* (NC) *if and only if this function is a solution to* (FE).

Here u_z *is a unique mild solution to the initial-value problem for* (SE; z) *on* $[s, \varsigma]$ *with initial condition* $u_z(s) = u_s$.

PROOF. We see from the definitions of solutions and Theorem 2.3 that (i) and (ii) are satisfied. We then verify (iii). If $z \in C([s, \varsigma])$ satisfies $z(t) \in \mathscr{D}(\Gamma)$ and $\boldsymbol{f}(u_z(t)) \in \Gamma(z(t))$ on $[s, \varsigma]$, then $z(t) \in \mathscr{D}(\Gamma)$ and $z(t) - \lambda \boldsymbol{f}(u_z(t)) \in (I - \lambda\Gamma)(z(t))$ on $[s, \varsigma]$ for all $\lambda > 0$. Therefore, it follows that $(I - \lambda\Gamma)^{-1}(z(t) - \lambda \boldsymbol{f}(u_z(t))) = z(t)$ on $[s, \varsigma]$ for all $\lambda > 0$. Conversely, if $z \in C([s, \varsigma])$ and satisfies $(I - \lambda_0\Gamma)^{-1}(z(t) - \lambda_0 \boldsymbol{f}(u_z(t))) = z(t)$ on $[s, \varsigma]$ for some $\lambda_0 > 0$, then $z(t) \in \mathscr{D}(\Gamma)$ and $\boldsymbol{f}(u_z(t)) \in \Gamma(z(t))$ on $[s, \varsigma]$. It should be noted at this point that if $z \in C([s, \varsigma])$ satisfies $z(t) = (I - \lambda_0\Gamma)^{-1}(z(t) - \lambda_0 \boldsymbol{f}(u_z(t)))$ for some $\lambda_0 > 0$, then for any $\lambda > 0$ the function z satisfies $z(t) = (I - \lambda\Gamma)^{-1}(z(t) - \lambda \boldsymbol{f}(u_z(t)))$. □

REMARK 3.9. Theorem 3.8 states that if (z, u_z) is a mild solution of (AES), then z is a fixed point of the mapping $z \mapsto (I - \lambda\Gamma)^{-1}(z(\cdot) - \lambda \boldsymbol{f}(u_z(\cdot)))$, and *vice versa*.

4. Fixed point argument

In this section we give the proofs of Theorems 3.2, 3.3 and 3.7 stated in the previous section by applying Schauder's Fixed Point Theorem.

PROOF OF THEOREM 3.2. In view of Theorem 3.8, it suffices to show the existence of a solution z to equation (FE).

Let $(z_0, u_0) \in \mathscr{D}(\Gamma) \times D$ and $\boldsymbol{f}(u_0) \in \Gamma(z_0)$. Suppose that $0 \leq s < T$, and that $\hat{z} \in W^{1,\infty}(0, s)$ is a solution to (FE)–(IC) on $[0, s]$. We put

$$\alpha = e^{\overline{M}(T-s)+\omega}\left(\|\hat{u}(s)\| + \int_s^T \max\{\|\mathscr{F}(\tau)\|, 1\} d\tau\right), \qquad \lambda = (\alpha \max\{\|\boldsymbol{f}\|_*, \|\boldsymbol{g}\|_*\})^{-1},$$

$$r = 1 + |\hat{z}(s)| + |(I - \lambda\Gamma)^{-1}(0)|, \qquad \kappa = \|\boldsymbol{f}\|_*(2\overline{K}(r)\alpha + \sup_{\substack{0 \leq \tau \leq T \\ |\vartheta| \leq r}} \|F(\tau, \hat{u}(s), \vartheta)\|),$$

$$\delta = \kappa \bar{\beta}_{1+r}^{-1}, \qquad \varepsilon = (\bar{\beta}_{1+r}^{-1} + \lambda)^{-1}\kappa^{-1}, \qquad \text{and} \qquad \varsigma = \min\{s + \varepsilon, T\}.$$

Here $\overline{M}$, $\overline{K}(r)$, and $\bar{\beta}_{1+r}$ denote the constants employed, respectively, in (F4), (F1), and (G) with $1+r$; $\mathscr{F}(\cdot)$ is the X-valued function in (F4); $\hat{u}(\cdot)$ is a mild solution to (SE; $\hat{z}$) on $[0,s]$ with $\hat{u}(0)=u_0$, which is obtained by Theorem 2.3. The symbol $\|\cdot\|_*$ denotes the operator-norm of continuous linear functionals on X. Note that the continuous linear functional $\boldsymbol{f}$ is not identically zero. Hence α, λ, r, κ, δ and ε are positive and finite. We also note that $0 \le s < \varsigma \le T$ and $\varsigma \le s+\varepsilon$. We then define an operator $\Psi : \mathscr{K}_s \to C([s,\varsigma])$ by

$$\mathscr{K}_s = \{\zeta \in W^{1,\infty}(s,\varsigma) \mid \zeta(s) = \hat{z}(s), \quad |\zeta'|_\infty \le \delta\}, \tag{4.1}$$

$$(\Psi\zeta)(t) = (I-\lambda\Gamma)^{-1}(\zeta(t) - \lambda \boldsymbol{f}(u_\zeta(t))), \qquad t \in [s,\varsigma], \qquad \text{for } \zeta \in \mathscr{K}_s, \tag{4.2}$$

where u_ζ is a unique mild solution to the initial-value problem for (SE; ζ) on $[s,\varsigma]$ with $u_\zeta(s)=\hat{u}(s)$, which is obtained by Theorem 2.3, and $|\cdot|_\infty$ denotes the supremum-norm over $[s,\varsigma]$. Then $\mathscr{K}_s$ is a compact convex subset of $C([s,\varsigma])$ endowed with $|\cdot|_\infty$. We here apply Ascoli-Arzelà's Theorem to discuss the compactness.

Furthermore, we have

LEMMA 4.1. *The operator Ψ is well-defined as a continuous mapping from $(\mathscr{K}_s, |\cdot|_\infty)$ into $(C([s,\varsigma]), |\cdot|_\infty)$.*

PROOF. The multi-valued function Γ is m-dissipative in $\mathbf{R}$ by (G). Thus, the resolvent $(I-\lambda\Gamma)^{-1}$ of Γ is defined on $\mathbf{R}$ as a contraction operator on $\mathbf{R}$:

$$|(I-\lambda\Gamma)^{-1}(\zeta_1) - (I-\lambda\Gamma)^{-1}(\zeta_2)| \le |\zeta_1 - \zeta_2|, \qquad \zeta_1, \zeta_2 \in \mathbf{R}. \tag{4.3}$$

Let $\zeta \in \mathscr{K}_s$. Since $\zeta(\cdot)$ is Lipschitz continuous, $\boldsymbol{f}(u_\zeta(\cdot))$ is also Lipschitz continuous by Lemma 2.11. Therefore, it follows from the definition of Ψ that $(\Psi\zeta)(\cdot)$ is Lipschitz continuous, that is, $\Psi\zeta \in W^{1,\infty}(s,\varsigma)$, and so $\Psi : \mathscr{K}_s \to C([s,\varsigma])$ is well-defined.

We next prove the continuity of Ψ. Let $z_n, z \in \mathscr{K}_s$ and $|z_n - z|_\infty \to 0$. Then, by (4.3) we see that

$$|(\Psi z_n)(t) - (\Psi z)(t)| \le |z_n(t) - z(t)| + \lambda|\boldsymbol{f}(u_{z_n}(t)) - \boldsymbol{f}(u_z(t))|, \qquad t \in [s,\varsigma],$$

and hence $|\Psi z_n - \Psi z|_\infty \le |z_n - z|_\infty + \lambda|\boldsymbol{f}u_{z_n} - \boldsymbol{f}u_z|_\infty \to 0$ by Lemma 2.9. This shows that Ψ is continuous. $\square$

LEMMA 4.2. *The mapping Ψ has its values in $\mathscr{K}_s$, that is, $\Psi\mathscr{K}_s \subset \mathscr{K}_s$.*

PROOF. Let $z \in \mathscr{K}_s$. We have already shown that $\Psi z \in W^{1,\infty}(s,\varsigma)$ in the proof of Lemma 4.1. Since $\boldsymbol{f}(\hat{u}(s)) \in \Gamma(\hat{z}(s))$, we have

$$(I-\lambda\Gamma)^{-1}(z(s) - \lambda\boldsymbol{f}(u_z(s))) = (I-\lambda\Gamma)^{-1}(\hat{z}(s) - \lambda\boldsymbol{f}(\hat{u}(s))) = \hat{z}(s).$$

This means that $(\Psi z)(s) = \hat{z}(s)$. If $|(\Psi z)'|_\infty \leq \delta$, then the desired result follows. Let $t_1, t_2 \in [s, \varsigma]$. By the local quasi-dissipativity of Γ in (G), Γ satisfies

$$(4.4) \quad (1 + \lambda\bar{\beta}_{1+r})|\zeta_1 - \zeta_2| \leq |(\zeta_1 - \lambda\tau_1) - (\zeta_2 - \lambda\tau_2)| \quad \text{for } (\zeta_i, \tau_i) \in \Gamma, \quad |\zeta_i| \leq 1 + r, \quad i = 1, 2.$$

We first choose

$$\zeta_i := (\Psi z)(t_i) = (I - \lambda\Gamma)^{-1}(z(t_i) - \lambda \boldsymbol{f}(u_z(t_i))),$$

$$\tau_i := \lambda^{-1}[(I - \lambda\Gamma)^{-1} - I](z(t_i) - \lambda \boldsymbol{f}(u_z(t_i)))$$

for $i = 1, 2$. Then it follows that $(\zeta_i, \tau_i) \in \Gamma$, $i = 1, 2$. Indeed, $\mathscr{R}((I - \lambda\Gamma)^{-1}) = \mathscr{D}(\Gamma)$, and hence $\zeta_i \in \mathscr{D}(\Gamma)$. By the property of nonlinear dissipative operators, we see that $\tau_i \in \Gamma(I - \lambda\Gamma)^{-1}(z(t_i) - \lambda\boldsymbol{f}(u_z(t_i))) = \Gamma(\zeta_i)$, and hence $(\zeta_i, \tau_i) \in \Gamma$.

We next claim that $|\zeta_i| \leq 1 + r$ for $i = 1, 2$. In view of the choice of ζ_i and (4.3), we see that

$$(4.5) \quad |\zeta_i| \leq |z(t_i) - \lambda\boldsymbol{f}(u_z(t_i))| + |(I - \lambda\Gamma)^{-1}(0)|$$
$$\leq |\hat{z}(s) - \lambda\boldsymbol{f}(\hat{u}(s))| + \int_s^{t_i} |z'(\tau) - \lambda(\boldsymbol{f}u_z)'(\tau)|d\tau + |(I - \lambda\Gamma)^{-1}(0)|.$$

To estimate the last expression, we need the following

Lemma 4.3. (i) *For $z \in W^{1,\infty}(s, \varsigma)$ with $|z'|_\infty \leq \delta$, we have*

$$\|u_z(t)\| \leq e^{(\bar{M}+\omega\delta)(t-s)}\left(\|u_z(s)\| + \int_s^t \|\mathscr{F}(\tau)\|d\tau\right), \qquad t \in [s, \varsigma].$$

(ii) *For $z \in C([s, \varsigma)) \cap L^\infty(s, \varsigma)$, we have*

$$\|u_z(t)\| \leq e^{2\omega|z|_\infty}e^{\bar{M}(t-s)}\left(\|u_z(s)\| + \int_s^\varsigma \|\mathscr{F}(\tau)\|d\tau\right), \qquad t \in [s, \varsigma).$$

This lemma is proved in §7. Using the first estimate (i) in this lemma, we can check that $\|u_z(t)\| \leq \alpha$ for $t \in [s, \varsigma]$ since $\varsigma \leq T$ and $\delta(\varsigma - s) \leq \delta\varepsilon < 1$. Hence, it follows that $|\hat{z}(s) - \lambda\boldsymbol{f}(\hat{u}(s))| \leq |\hat{z}(s)| + 1$ and $-\lambda\boldsymbol{g}(u_z(t)) \leq 1$ for $t \in [s, \varsigma]$ by the definition of λ. Since $\boldsymbol{g}$ is nonpositive on D, and so $0 \leq 1 + \lambda\boldsymbol{g}(u_z(t)) \leq 1$ for $t \in [s, \varsigma]$. Noting that $|z(t)| \leq |\hat{z}(s)| + \int_s^t |z'(\tau)|d\tau \leq |\hat{z}(s)| + 1 \leq r$ for $t \in [s, \varsigma]$, and using (F1), we get

$$|\boldsymbol{f}F(t, u_z(t), z(t))| \leq \|\boldsymbol{f}\|_*(\|F(t, u_z(t), z(t)) - F(t, \hat{u}(s), z(t))\| + \|F(t, \hat{u}(s), z(t))\|) \leq \kappa \qquad \text{for } t \in [s, \varsigma].$$

Therefore, it follows from Lemma 2.4 that

$$(4.6)\quad \int_s^{t_i} |z'(\tau)-\lambda(\boldsymbol{f}u_z)'(\tau)|d\tau \le \int_s^{t_i} [|z'(\tau)|\,|1+\lambda \boldsymbol{g}(u_z(\tau))|+\lambda|\boldsymbol{f}F(\tau,u_z(\tau),z(\tau))|]d\tau$$

$$\le (\delta+\lambda\kappa)(t_i-s) \le (\kappa\bar{\beta}_{1+r}^{-1}+\lambda\kappa)(\varsigma-s) \le 1.$$

By (4.5), (4.6) and the fact that $|\hat{z}(s)-\lambda\boldsymbol{f}(\hat{u}(s))| \le |\hat{z}(s)|+1$, we have $|\zeta_i| \le 1+r$ as claimed.

Thus, we have $|(\Psi z)(t_1)-(\Psi z)(t_2)| = |\zeta_1-\zeta_2| \le (1+\lambda\bar{\beta}_{1+r})^{-1}|(\zeta_1-\lambda\tau_1)-(\zeta_2-\lambda\tau_2)|$ by (4.4). Moreover, it is easy to check that

$$\begin{aligned}|(\zeta_1-\lambda\tau_1)-(\zeta_2-\lambda\tau_2)| &= |(z(t_1)-\lambda\boldsymbol{f}(u_z(t_1)))-(z(t_2)-\lambda\boldsymbol{f}(u_z(t_2)))| \\ &\le \left|\int_{t_1}^{t_2} |z'(\tau)-\lambda(\boldsymbol{f}u_z)'(\tau)|d\tau\right| \\ &\le (\delta+\lambda\kappa)|t_1-t_2| \le (1+\lambda\bar{\beta}_{1+r})\delta|t_1-t_2|,\end{aligned}$$

and so $|(\Psi z)(t_1)-(\Psi z)(t_2)| \le \delta|t_1-t_2|$ for $t_1,t_2 \in [s,\varsigma]$. This completes the proof of Lemma 4.2. □

We then complete the proof of the local existence result. Lemmas 4.1 and 4.2 together allow us to apply Schauder's Fixed Point Theorem to get a fixed point $\bar{z} \in \mathscr{K}_s$ of Ψ. It is clear that $\bar{z}$ is a solution of (FE) on $[s,\varsigma]$ with $(\bar{z}(s),\bar{u}(s)) = (\hat{z}(s),\hat{u}(s))$. Setting

$$(4.7)\qquad z(t) = \begin{cases} \hat{z}(t) & \text{for } t \in [0,s], \\ \bar{z}(t) & \text{for } t \in (s,\varsigma], \end{cases}$$

we see that

$$u_z(t) = \begin{cases} \hat{u}(t) & \text{for } t \in [0,s], \\ \bar{u}(t) & \text{for } t \in (s,\varsigma], \end{cases}$$

and $z \in W^{1,\infty}(0,\varsigma)$ is a solution of (FE)–(IC) on $[0,\varsigma]$. Note that $\boldsymbol{f}(u_z(\cdot)) \in W^{1,\infty}(0,\varsigma)$ by Lemma 2.11.

Choosing $s=0$ in the above argument, we obtain a solution of (FE)–(IC) on $[0,\varsigma_1]$ for some $\varsigma_1 \in (0,T]$. Next, choosing $s=\varsigma_1$, we obtain a solution on $[0,\varsigma_2]$ for some $\varsigma_2 \in [\varsigma_1,T]$. Repeating this argument, we can extend a solution to some (maximal) subinterval $[0,\hat{T})$ of $[0,T]$.

It remains to show that the solution blows up at $\hat{T}$. Let $[0,T_{\max})$ be the maximal interval of existence of solutions to (FE)–(IC) (or, equivalently, (AES)–(IC)) and z a solution on $[0,T_{\max})$. Let us show by contradiction that $T_{\max} < T$ implies that $\limsup_{t\uparrow T_{\max}}|z(t)| = \infty$. Suppose that $\limsup_{t\uparrow T_{\max}}|z(t)|$

$< \infty$. Then we conclude that $\varrho := \sup_{0 \le \tau < T_{\max}} |z(\tau)| < \infty$. If not, then there is a sequence $\{\tau_n\} \subset [0, T_{\max})$ such that $\tau_n \uparrow T_{\max}$ and $|z(\tau_n)| > n$, which contradicts the assumption that $\limsup_{t \uparrow T_{\max}} |z(t)| < \infty$. Thus, it follows from Lemma 4.3 (ii) that

$$\|u_z(t)\| \le e^{2\omega\varrho} e^{\bar{M}T}\left(\|u_0\| + \int_0^T \|\mathscr{F}(\tau)\| d\tau\right), \qquad t \in [0, T_{\max}),$$

and so that $\sup_{0 \le \tau < T_{\max}} \|u_z(\tau)\| < \infty$.

Choose a sequence $\{t_n\}$ in $(0, T_{\max})$ such that $t_n \uparrow T_{\max}$. Put

$$\alpha = e^{\bar{M}T + \omega}\left(\sup_{0 \le \tau < T_{\max}} \|u_z(\tau)\| + \int_0^T \max\{\|\mathscr{F}(\tau)\|, 1\} d\tau\right),$$

$$\lambda = (\alpha \max\{\|\boldsymbol{f}\|_*, \|\boldsymbol{g}\|_*\})^{-1}, \qquad r = 1 + \varrho + |(I - \lambda\Gamma)^{-1}(0)|,$$

$$\kappa = \|\boldsymbol{f}\|_*(\bar{K}(r)(\alpha + \|u_0\|) + \sup_{\substack{0 \le \tau \le T \\ |\vartheta| \le r}} \|F(\tau, u_0, \vartheta)\|), \qquad \delta = \kappa \bar{\beta}_{1+r}^{-1},$$

$$\varepsilon = \min\{T - T_{\max}, (\bar{\beta}_{1+r}^{-1} + \lambda)^{-1}\kappa^{-1}\}, \qquad \text{and} \qquad \varsigma_n = t_n + \varepsilon.$$

Moreover, we define a mapping $\Psi : \mathscr{K}_{t_n} \to C([t_n, \varsigma_n])$ by (4.1) and (4.2) with $s = t_n$, $\varsigma = \varsigma_n$ and $\hat{z} = z$. Then, in the same way as above, we are able to extend the solution z on $[0, t_n]$ to $[0, \varsigma_n]$ for every n. In view of the choice of $\{t_n\}$ and the fact that ε is independent of n, it is clear that $\varsigma_n = t_n + \varepsilon \uparrow T_{\max} + \varepsilon$ as $n \to \infty$. Thus, the solution on $[0, T_{\max})$ can be extended beyond $T_{\max}$ since $\varepsilon > 0$. This contradicts the definition of $T_{\max}$. The proof of Theorem 3.2 is now complete. □

We next prove the global existence theorem.

Proof of Theorem 3.3. In a way similar to the proof of Theorem 3.2, it suffices to show the existence of a solution z to (FE). We split the proof into four steps.

Let $(z_0, u_0) \in \mathscr{D}(\Gamma) \times D$ and $\boldsymbol{f}(u_0) \in \Gamma(z_0)$. Suppose that $0 \le s < T$, and that $\hat{z} \in W^{1,\infty}(0, s)$ is a solution to (FE)–(IC) on $[0, s]$.

Step 1. We assume that $\hat{u}(s) \neq 0$. Put

$$\lambda = \left[C_2 e^{\bar{M}(T-s)}\left(\boldsymbol{h}(\hat{u}(s)) + \int_s^T \boldsymbol{h}(\mathscr{F}(\tau)) d\tau\right)\right]^{-1}, \qquad \varrho = C_1 e^{-\bar{M}(T-s)} \boldsymbol{h}(\hat{u}(s)),$$

$$\kappa = |\xi|_{L^\infty(0,T)} + \bar{M} e^{\bar{M}(T-s)}\left(\bar{\boldsymbol{f}}(\hat{u}(s)) + \int_s^T \bar{\boldsymbol{f}}(\mathscr{F}(\tau)) d\tau\right) + \bar{M} e^{\bar{M}T} \lambda^{-1},$$

$$\delta = \varrho^{-1}\kappa, \qquad \text{and} \qquad \varsigma = \min\{s + \delta^{-1}, T\}.$$

Here $\overline{M}$ is the constant appeared in (F4); C_1 and C_2 the constants employed in (AdLF); $\xi(\cdot)$ the function stated in (AdF); and $\hat{u}(\cdot)$ a mild solution to (SE; $\hat{z}$) on $[0,s]$ with $\hat{u}(0)=u_0$ obtained in Theorem 2.3. Note that λ, ϱ, κ and δ are positive and finite. In addition, $0 \le s < \varsigma \le T$ and $\varsigma \le s+\delta^{-1}$.

We then define an operator $\Psi : \mathscr{K}_s \to C([s,\varsigma])$ by (4.1) and (4.2). Then we see that $\mathscr{K}_s$ is a compact convex subset of $(C([s,\varsigma]), |\cdot|_\infty)$, and that Ψ is well-defined and continuous as seen in the proof of Theorem 3.2.

Let $z \in \mathscr{K}_s$. Then we see in a way similar to the proof of Lemma 4.2 that $\Psi z \in W^{1,\infty}(s,\varsigma)$ and $(\Psi z)(s) = \hat{z}(s)$. In order to show that $|(\Psi z)'|_\infty \le \delta$, we choose $s \le t_1 < t_2 \le \varsigma$. Then it follows from (4.3) and Lemma 2.11 that

$$(4.8)\quad |(\Psi z)(t_1)-(\Psi z)(t_2)| \le \int_{t_1}^{t_2} [|z'(\tau)|\,|1+\lambda \boldsymbol{g}(u_z(\tau))|+\lambda|\boldsymbol{f}F(\tau,u_z(\tau),z(\tau))|]d\tau.$$

To estimate further, we require the following lemma whose proof is given in §7.

LEMMA 4.4. *For* $z \in C([s,\varsigma])$, *we have*:

(i) $e^{-\overline{M}(t-s)}\boldsymbol{h}(u_z(s)) \le \boldsymbol{h}(u_z(t)) \le e^{\overline{M}(t-s)}(\boldsymbol{h}(u_z(s)) + \int_s^t \boldsymbol{h}(\mathscr{F}(\tau))d\tau)$ *for* $t \in [s,\varsigma]$.

(ii) $\boldsymbol{g}(u_z(t)) \le -C_1 e^{-\overline{M}(t-s)}\boldsymbol{h}(u_z(s))$ *for* $t \in [s,\varsigma]$.

(iii) *If* $z \in W^{1,\infty}(s,\varsigma)$ *and* $|z'|_\infty \le \delta$, *then*

$$\bar{\boldsymbol{f}}(u_z(t)) \le e^{\overline{M}(t-s)}\Bigg[\bar{\boldsymbol{f}}(u_z(s)) + \int_s^t \bar{\boldsymbol{f}}(\mathscr{F}(\tau))d\tau + C_2\delta(t-s)e^{\overline{M}(t-s)}\left(\boldsymbol{h}(u_z(s)) + \int_s^t \boldsymbol{h}(\mathscr{F}(\tau))d\tau\right)\Bigg] \quad \textit{for } t \in [s,\varsigma].$$

Using (AdLF) and Lemma 4.4 (i), we see that $-\boldsymbol{g}(u_z(t)) \le \lambda^{-1}$ for $t \in [s,\varsigma]$. Hence

$$(4.9)\qquad 0 \le 1+\lambda\boldsymbol{g}(u_z(t)) \le 1-\lambda\varrho \qquad \text{for } t \in [s,\varsigma]$$

by Lemma 4.4 (ii). Furthermore, we have

$$(4.10)\qquad |\boldsymbol{f}F(t,u_z(t),z(t))| \le \kappa \qquad \text{for } t \in [s,\varsigma]$$

by (AdF) and Lemma 4.4 (iii). Thus, it follows from (4.8) through (4.10) that

$$|(\Psi z)(t_1)-(\Psi z)(t_2)| \le [\delta(1-\lambda\varrho)+\lambda\kappa](t_2-t_1) = \delta(t_2-t_1).$$

This implies that $|(\Psi z)'|_\infty \le \delta$ as desired. Since $\Psi\mathscr{K}_s \subset \mathscr{K}_s$, we can apply Schauder's Fixed Point Theorem to find a fixed point $\bar{z} \in \mathscr{K}_s$ of Ψ. We see in the same way as in the proof of Theorem 3.2 that the function z defined by (4.7) is a solution of (FE)–(IC) on $[0,\varsigma]$, and $z \in W^{1,\infty}(0,\varsigma)$.

Finally, it should be noted that $\boldsymbol{h}(u_z(t)) \ge e^{-\overline{M}(t-s)}\boldsymbol{h}(\hat{u}(s)) > 0$ by Lemma 4.4 (i) and (AdLF), and so that $u_z(t) \ne 0$ on $[s,\varsigma]$.

Step 2. Assume that $u_0 \neq 0$. Letting $s = 0$ in Step 1, we obtain a solution $z \in W^{1,\infty}(0, \varsigma_1)$ of (FE)–(IC) on $[0, \varsigma_1]$, where

$$\lambda_1 = \left[C_2 e^{\bar{M}T}\left(\boldsymbol{h}(u_0) + \int_0^T \boldsymbol{h}(\mathscr{F}(\tau))d\tau\right)\right]^{-1}, \qquad \varrho_1 = C_1 e^{-\bar{M}T}\boldsymbol{h}(u_0),$$

$$\kappa_1 = |\xi|_{L^\infty(0,T)} + \bar{M}e^{\bar{M}T}\left(\bar{\boldsymbol{f}}(u_0) + \int_0^T \bar{\boldsymbol{f}}(\mathscr{F}(\tau))d\tau\right) + \bar{M}e^{\bar{M}T}\lambda_1^{-1},$$

$$\delta_1 = \varrho_1^{-1}\kappa_1, \qquad \text{and} \qquad \varsigma_1 = \min\{\delta_1^{-1}, T\}.$$

If $\varsigma_1 = T$, then this z is the desired global solution.

Let $\varsigma_1 < T$, that is, $\varsigma_1 = \delta_1^{-1}$. Since $u_z(\varsigma_1) \neq 0$, letting $s = \varsigma_1$ in Step 1 gives a solution $z \in W^{1,\infty}(0, \varsigma_2)$ on $[0, \varsigma_2]$. Here

$$\lambda_2 = \left[C_2 e^{\bar{M}(T-\varsigma_1)}\left(\boldsymbol{h}(u_z(\varsigma_1)) + \int_{\varsigma_1}^T \boldsymbol{h}(\mathscr{F}(\tau))d\tau\right)\right]^{-1}, \qquad \varrho_2 = C_1 e^{-\bar{M}(T-\varsigma_1)}\boldsymbol{h}(u_z(\varsigma_1)),$$

$$\kappa_2 = |\xi|_{L^\infty(0,T)} + \bar{M}e^{\bar{M}(T-\varsigma_1)}\left(\bar{\boldsymbol{f}}(u_z(\varsigma_1)) + \int_{\varsigma_1}^T \bar{\boldsymbol{f}}(\mathscr{F}(\tau))d\tau\right) + \bar{M}e^{\bar{M}T}\lambda_2^{-1},$$

$$\delta_2 = \varrho_2^{-1}\kappa_2, \qquad \text{and} \qquad \varsigma_2 = \min\{\varsigma_1 + \delta_2^{-1}, T\}.$$

By Lemma 4.4 (i), we see that $\varrho_2 \geq \varrho_1$ and $\lambda_2^{-1} \leq \lambda_1^{-1}$. Using Lemma 4.4 (iii), we have

$$\bar{\boldsymbol{f}}(u_z(\varsigma_1)) \leq e^{\bar{M}\varsigma_1}\left[\bar{\boldsymbol{f}}(u_0) + \int_0^{\varsigma_1} \bar{\boldsymbol{f}}(\mathscr{F}(\tau))d\tau + C_2 e^{\bar{M}\varsigma_1}\left(\boldsymbol{h}(u_0) + \int_0^T \boldsymbol{h}(\mathscr{F}(\tau))d\tau\right)\right].$$

Since $\lambda_2^{-1} \leq \lambda_1^{-1}$, we have

$$\begin{aligned}\kappa_2 &\leq |\xi|_{L^\infty(0,T)} + \bar{M}e^{\bar{M}T}\left(\bar{\boldsymbol{f}}(u_0) + \int_0^T \bar{\boldsymbol{f}}(\mathscr{F}(\tau))d\tau + \lambda_1^{-1}\right) + \bar{M}e^{\bar{M}T}\lambda_1^{-1} \\ &= \kappa_1 + \bar{M}e^{\bar{M}T}\lambda_1^{-1} \leq 2\kappa_1.\end{aligned}$$

Therefore, by this relation we deduce that

$$\delta_1^{-1} + \delta_2^{-1} = \delta_1^{-1} + \varrho_2\kappa_2^{-1} \geq \delta_1^{-1} + \varrho_1(2\kappa_1)^{-1} = (1 + 2^{-1})\delta_1^{-1},$$

since $\varrho_2 \geq \varrho_1$. This shows that $\varsigma_2 = \min\{\delta_1^{-1} + \delta_2^{-1}, T\} \geq \min\{(1 + 2^{-1})\delta_1^{-1}, T\}$. If $\varsigma_2 = T$, then the z above is the desired global solution.

Repeating the above argument, we find ς_n such that $\varsigma_n \geq \min\{(1 + 2^{-1} + \cdots + n^{-1})\delta_1^{-1}, T\}$ at the nth step. Since $\sum_{k=1}^n k^{-1} \nearrow +\infty$ as $n \to +\infty$, we see that $\varsigma_m = T$ for some m.

In this way, for $u_0 \neq 0$, we obtain a solution z on the whole interval $[0, T]$. Notice that z is in $W^{1,\infty}(0, T)$, and so that $\boldsymbol{f} u_z$ is also in $W^{1,\infty}(0, T)$ by Lemma 2.11. Therefore, in the case that $0 \notin \mathscr{R}(\Gamma)$, the proof of Theorem 3.3 is complete. In the case that $0 \in \mathscr{R}(\Gamma)$, we further proceed to Steps 3 and 4.

Step 3. We assume that $\hat{u}(s)$ vanishes. Since $\hat{z}$ is a solution to (FE) on $[0, s]$ by assumption, it follows that $0 = \boldsymbol{f}(\hat{u}(s)) \in \Gamma(\hat{z}(s))$. In addition, $0 \in \Gamma(0)$ by (AdG). Using the local quasi-dissipativity of Γ, we infer that $\hat{z}(s) = 0$. Put

$$\alpha = e^{\overline{M}T+\omega} \int_0^T \max\{\|\mathscr{F}(\tau)\|, 1\} d\tau, \qquad \lambda = (\alpha \max\{\|\boldsymbol{f}\|_*, \|\boldsymbol{g}\|_*\})^{-1},$$

$$\kappa = \|\boldsymbol{f}\|_* (\overline{K}(1)\alpha + \sup_{\substack{0 \le \tau \le T \\ |\vartheta| \le 1}} \|F(\tau, 0, \vartheta)\|), \qquad \delta = \kappa \bar{\beta}_1^{-1},$$

$$\varepsilon = (\bar{\beta}_1^{-1} + \lambda)^{-1} \kappa^{-1}, \qquad \text{and} \qquad \varsigma = \min\{s + \varepsilon, T\},$$

where $\overline{M}$, $\overline{K}(1)$ and $\bar{\beta}_1$ are the constants employed, respectively, in (F4), (F1) and (G) for $r = 1$. Note that $(I - \lambda\Gamma)^{-1}(0) = 0$, $\hat{z}(s) = 0$ and $\hat{u}(s) = 0$. Then, in a way similar to the proof of Theorem 3.2, we obtain a solution $\bar{z}$ in $W^{1,\infty}(s, \varsigma)$ such that $\bar{z}(s) = \hat{z}(s) (= 0)$. It is evident that the function z defined by (4.7) is a solution of (FE) on $[0, \varsigma]$ and $z \in W^{1,\infty}(0, \varsigma)$. Since ε is independent of s, we can extend $z(t)$ to any subinterval of $[0, T]$, whenever $u_z(t)$ vanishes.

Step 4. Assume that $u_0 = 0$. Choose $s = 0$ in Step 3. Then we get a solution $z \in W^{1,\infty}(0, \varsigma)$ for $\varsigma = \min\{\varepsilon, T\}$. If $\varepsilon \ge T$, then this z is the desired global solution. Let $\varepsilon < T$. If $u_z(\varepsilon) \neq 0$, then we repeat the same arguments as in Steps 1 and 2, and can extend $z(t)$ to $[0, T]$. On the other hand, if $u_z(\varepsilon) = 0$, then taking $s = \varepsilon$ in Step 3, we have a solution $z \in W^{1,\infty}(0, \varsigma)$ for $\varsigma = \min\{2\varepsilon, T\}$. If $2\varepsilon \ge T$, then this z is the desired solution on $[0, T]$. Let $2\varepsilon < T$. If $u_z(2\varepsilon) \neq 0$, then one employs the same arguments in Steps 1 and 2. If $u_z(2\varepsilon) = 0$, then choose $s = 2\varepsilon$ in Step 3. Repeating these arguments finite times, we gain the desired solution z on the whole interval $[0, T]$. Since $z, \boldsymbol{f} u_z \in W^{1,\infty}(0, T)$, the proof of Theorem 3.3 is now complete. □

We conclude this section with the proof of Theorem 3.7.

PROOF OF THEOREM 3.7. It is similar to the proofs of Theorems 3.2–3.4. In the proofs of Theorem 2.3 and Proposition 2.5, we apply (F6) instead of (F1). In the proof of Lemma 2.6, we apply (F6) in place of both (F1) and (F3) to obtain (3.5). This leads to an inequality analogous to (2.5). Thus, Lemma 2.9 is valid under (F6) instead of (F1) and (F3). Moreover, we employ (F6) in place of (3.3) in the proof of Theorem 3.4. We can prove the remaining part in the same way as in the proofs of Theorems 3.2–3.4. □

5. Main results for the nonlocal nonlinear system

In this section we state the existence and uniqueness results for weak solutions to the Cauchy problem for (NNS). These results are all proved in the next section.

We need some preparations to state our results. We begin by defining a weighted L^1 space to be our base Banach space. First, we formulate the following class of functions w from $\mathbf{R}$ into itself:

(W) $w : \mathbf{R} \to \mathbf{R}$ is absolutely continuous on bounded intervals, nondecreasing and satisfies

$$\operatorname*{ess.\,sup}_{x \in \mathbf{R}} \frac{|w'(x)|}{1 + |w(x)|} < \infty.$$

For a function $w(x)$ satisfying (W), we define an L^1 space with weight $1 + |w(x)|$ by

$$\begin{aligned} L^1(w) &:= L^1(\mathbf{R}; (1 + |w(x)|)dx) \\ &:= \left\{ v : \mathbf{R} \to \mathbf{R} \text{ measurable} \,\middle|\, \int_{-\infty}^{+\infty} |v(x)|(1 + |w(x)|)dx < \infty \right\}, \end{aligned}$$

and its norm by

$$|v|_w := \int_{-\infty}^{+\infty} |v(x)|(1 + |w(x)|)dx \qquad \text{for } v \in L^1(w).$$

It is clear that $(L^1(w), |\cdot|_w)$ is a real Banach space. Note that if $w(x) \equiv 0$ then $L^1(w)$ is the usual $L^1(\mathbf{R})$, and that a measurable function $v : \mathbf{R} \to \mathbf{R}$ belongs to $L^1(w)$ if and only if both v and wv belong to $L^1(\mathbf{R})$.

Let $N \geq 2$ be an integer. (For the case $N = 1$, see Remark 5.12 below.) We need at least a condition $(\mathrm{W})_N$ on the weight function:

$(\mathrm{W})_N$ $\boldsymbol{w} = (w^1, \ldots, w^N) : \mathbf{R} \to \mathbf{R}^N$ is not identically equal to zero and each component satisfies condition (W) and

$$\sup_{x \in \mathbf{R}} \frac{1 + |w^i(x)|}{1 + |w^{i+1}(x)|} < \infty, \qquad i = 1, \ldots, N - 1. \tag{5.1}$$

For such weight function $\boldsymbol{w}$, we can define a product space $\mathbf{L}^1(\boldsymbol{w}) := L^1(w^1) \times \cdots \times L^1(w^N)$ equipped with the norm $|\boldsymbol{v}|_{\boldsymbol{w}} = |v^1|_{w^1} + \cdots + |v^N|_{w^N}$ for $\boldsymbol{v} = (v^1, \ldots, v^N)$. Set

$$C = \max_{1 \leq i \leq N-1} \sup_{x \in \mathbf{R}} \frac{1 + |w^i(x)|}{1 + |w^{i+1}(x)|}.$$

Since $1 + |w^i(x)| \leq C(1 + |w^{i+1}(x)|)$ on $\mathbf{R}$ for $i = 1, \ldots, N - 1$ by (5.1), $L^1(w^1) \supset L^1(w^2) \supset \cdots \supset L^1(w^N)$.

Furthermore, for a function w satisfying (W), we define

$$L^\infty(w) := \left\{ v : \mathbf{R} \to \mathbf{R} \text{ measurable} \,\middle|\, |v(x)| \le \frac{C}{1+|w(x)|} \text{ a.e. for some } C > 0 \right\}$$

and its norm $\|v\|_w := \operatorname{ess.sup}_{x\in\mathbf{R}} |v(x)|(1+|w(x)|)$. If $w(x) \equiv 0$, then $L^\infty(w)$ is the usual $L^\infty(\mathbf{R})$.

For convenience, we introduce the cyclic rule on the indices: $i \equiv j \pmod N$, that is, for instance, $0 \equiv N$ and $N+1 \equiv 1$.

We denote by $\mathbf{R}^N_+$ the positive cone in $\mathbf{R}^N : \mathbf{R}^N_+ = \{(v^1, \dots, v^N) \in \mathbf{R}^N \mid v^1, \dots, v^N \ge 0\}$. Put $\mathbf{E} = \{(v^1, \dots, v^N) \in \mathbf{R}^N_+ \mid v^1 + \cdots + v^N \le 1\}$.

Let $0 < T < +\infty$ be an arbitrary but fixed number. On the function $\boldsymbol{\varphi} = (\varphi^1, \dots, \varphi^N) : [0,T] \times \mathbf{R} \times \mathbf{E} \times \mathbf{R} \to \mathbf{R}^N$, we impose the five conditions (P1)–(P5) below for the local existence.

(P1) A function $x \mapsto \boldsymbol{\varphi}(t,x,\boldsymbol{u},z)$ is measurable on $\mathbf{R}$ for every $(t,\boldsymbol{u},z) \in [0,T] \times \mathbf{E} \times \mathbf{R}$, and $(t,\boldsymbol{u},z) \mapsto \boldsymbol{\varphi}(t,x,\boldsymbol{u},z)$ is continuous on $[0,T] \times \mathbf{E} \times \mathbf{R}$ for almost every $x \in \mathbf{R}$;

(P2) For every $r > 0$ there exist nonnegative functions $\mathfrak{f}^{i,i+1}_r \in L^\infty(\mathbf{R})$, $i = 1, \dots, N-1$, $\mathfrak{f}^{N,1}_r \in L^\infty(w^N)$, $\mathfrak{f}^{1,N}_r \in L^\infty(\mathbf{R})$, $\mathfrak{f}^{i,i-1}_r \in L^\infty(w^i)$, $i = 2, \dots, N$, and a constant $K(r) > 0$ such that every component φ^i satisfies

$$|\varphi^i(t,x,\boldsymbol{u},z) - \varphi^i(t,x,\boldsymbol{v},z)| \le \sum_{j=i\pm1} \mathfrak{f}^{i,j}_r(x)|u^j - v^j| + K(r)\sum_{j=i}^{N} |u^j - v^j|$$

for $t \in [0,T]$, almost every $x \in \mathbf{R}$, $\boldsymbol{u} = (u^1, \dots, u^N)$, $\boldsymbol{v} = (v^1, \dots, v^N) \in \mathbf{E}$, and $z \in [-r,r]$;

(P3) For every $r > 0$ there exist nonnegative functions $(\mathfrak{g}^1_r, \dots, \mathfrak{g}^N_r) \in L^1(0,T;\mathbf{L}^1(\boldsymbol{w}))$ and a nondecreasing function $\rho_r : [0,\infty) \to [0,\infty)$ with $\rho_r(+0) = 0$ such that every component φ^i satisfies

$$|\varphi^i(t,x,\boldsymbol{u},z_1) - \varphi^i(t,x,\boldsymbol{u},z_2)| \le \left(\mathfrak{g}^i_r(t,x) + \sum\nolimits_{j=i}^{N} u^j\right)\rho_r(|z_1 - z_2|)$$

for almost every $(t,x) \in (0,T) \times \mathbf{R}$, $\boldsymbol{u} = (u^1, \dots, u^N) \in \mathbf{E}$, and z_1, $z_2 \in [-r,r]$;

(P4) There exist nonnegative functions $(\Phi^1, \dots, \Phi^N) \in C([0,T];\mathbf{L}^1(\boldsymbol{w}))$ and a constant $M > 0$ such that every φ^i satisfies

$$-Mu^i \le \varphi^i(t,x,\boldsymbol{u},z) \le \Phi^i(t,x) + Mu^i$$

for $t \in [0,T]$, almost every $x \in \mathbf{R}$, $\boldsymbol{u} = (u^1, \dots, u^N) \in \mathbf{E}$, and $z \in \mathbf{R}$;

(P5) (1) If $\boldsymbol{u} = (u^1, \dots, u^N) \in \mathbf{E}$ satisfies $\sum_{i=1}^N u^i = 1$, then $\sum_{i=1}^N \varphi^i(t,x,\boldsymbol{u},z) \le 0$ holds for $t \in [0,T]$, almost every $x \in \mathbf{R}$, and $z \in \mathbf{R}$;

(2) For every $r > 0$ there is a constant $\lambda_r > 0$ satisfying that $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbf{E}$ and $\boldsymbol{u} \leqslant \boldsymbol{v}$ in $\mathbf{R}^N$ together imply

$$\lambda_r \boldsymbol{u} + \boldsymbol{\varphi}(t,x,\boldsymbol{u},z) \leqslant \lambda_r \boldsymbol{v} + \boldsymbol{\varphi}(t,x,\boldsymbol{v},z) \qquad \text{in } \mathbf{R}^N$$

for $t \in [0, T]$, almost every $x \in \mathbf{R}$, and $z \in [-r, r]$. Here $\leqslant$ denotes the standard partial order relation in $\mathbf{R}^N$ such that $\boldsymbol{u} = (u^1, \dots, u^N) \leqslant \boldsymbol{v} = (v^1, \dots, v^N)$ in $\mathbf{R}^N$ if and only if $u^i \leq v^i$ for $i = 1, \dots, N$.

We then put a condition on the function $L : (a, b) \to \mathbf{R}$.

(L) $-\infty \leq a < b \leq +\infty$, $L \in C(a, b)$ is strictly decreasing, $L(a+0) = +\infty$ and $L(b-0) = -\infty$. Moreover, for every $r > 0$, there exists a constant $\beta_r > 0$ such that

$$(1 + \lambda\beta_r)|L(\tau_1) - L(\tau_2)| \leq |L(\tau_1) - L(\tau_2) - \lambda(\tau_1 - \tau_2)| \tag{5.2}$$

for $\lambda > 0$ and $\tau_1,\ \tau_2 \in [L^{-1}(r), L^{-1}(-r)]$.

Remark 5.1. In condition (L), $L^{-1}(r)$ and $L^{-1}(-r)$ make sense, since L is a bijection. Similarly, $L(\tau_1)$ and $L(\tau_2)$ make sense, since $a < L^{-1}(r) < L^{-1}(-r) < b$. Notice that (5.2) implies the local Lipschitz continuity of L with the Lipschitz constant β_r^{-1}. Theorems 5.4, 5.5, 5.7 and 5.10 below are valid, even if the possibly multi-valued inverse L^{-1} satisfies only condition (G) stated in §3.

We next define weak solutions to the nonlocal nonlinear transport system

$$\text{(NNS)} \qquad \begin{cases} \partial_t \boldsymbol{u} + z'(t)\partial_x \boldsymbol{u} = \boldsymbol{\varphi}(t, x, \boldsymbol{u}, z(t)), & (t, x) \in (0, T) \times \mathbf{R}, \\ z(t) = L\left(\displaystyle\int_{-\infty}^{+\infty} \boldsymbol{w}(y) \cdot \boldsymbol{u}(t, y)dy\right), & t \in [0, T]. \end{cases}$$

Definition 5.2. A function $\boldsymbol{u} : [0, T] \times \mathbf{R} \to \mathbf{R}^N$ is called a *weak solution* to (NNS) on $[0, T]$, if $\boldsymbol{u} \in C([0, T]; \mathbf{L}^1(\boldsymbol{w}))$, $\boldsymbol{u}(t, x) \in \mathbf{E}$ for $t \in [0, T]$, almost every $x \in \mathbf{R}$, $a < \int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}(t, x)dx < b$ for $t \in [0, T]$, the function $z(t) := L(\int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}(t, x)dx)$ is continuous on $[0, T]$, and the functions $\boldsymbol{u}(t, x)$ and $z(t)$ satisfy the integral equation

$$\boldsymbol{u}(t, x) = \boldsymbol{u}(0, x - z(t) + z(0)) + \int_0^t \boldsymbol{\varphi}(\tau, x - z(t) + z(\tau), \boldsymbol{u}(\tau, x - z(t) + z(\tau)), z(\tau))d\tau$$
$$\text{for } (t, x) \in [0, T] \times \mathbf{R}.$$

Remark 5.3. The above notion of weak solution is same as the notion of mild solution employed in the theory of abstract evolution equations rather than the notion of weak solution in the sense of distributions, cf. Definitions 2.2 and 3.1. Since (NNS) has the strong nonlinearity $z'(t)\partial_x \boldsymbol{u}$, weak solutions in the sense of distributions cannot be defined for (NNS) if $z(t)$ is not differentiable.

We now state a result of local existence for weak solutions to the Cauchy problem for (NNS).

THEOREM 5.4. *Assume* $(\mathrm{W})_N$, (P1)–(P5) *and* (L). *Let an initial data* $\boldsymbol{u}_0 \in \mathbf{L}^1(\boldsymbol{w})$ *satisfy* $\boldsymbol{u}_0(x) \in \mathbf{E}$ *a.e. and* $a < \int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}_0(x)dx < b$. *Then for some* $\hat{T} \in (0, T]$, *there exists a weak solution* $\boldsymbol{u} \in C([0, \hat{T}]; \mathbf{L}^1(\boldsymbol{w}))$ *of the initial-value problem for* (NNS) *on* $[0, \hat{T}]$ *such that the functions*

$$t \mapsto \int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}(t,x)dx \qquad \text{and} \qquad L\left(\int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}(t,x)dx\right)$$

belong to $W^{1,\infty}(0, \hat{T})$. *Moreover, let* $0 < T_{\max} \leq T$ *and* $[0, T_{\max})$ *the maximal interval of existence of the weak solution* $\boldsymbol{u}$. *If* $T_{\max} < T$, *then*

$$\limsup_{t \uparrow T_{\max}} \left| L\left(\int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}(t,x)dx\right)\right| = \infty.$$

To establish the global existence of weak solutions, we need a stronger condition than $(\mathrm{W})_N$.

(Ws) $w \in C^{0,1}(\mathbf{R})$ is strictly increasing and satisfies $\operatorname{ess.inf}_{x \in \mathbf{R}} w'(x) > 0$.

In other words, w is strictly increasing and bi-Lipschitz. For the weight function $\boldsymbol{w}$ we impose the following conditions:

$(\mathrm{Ws})_N$ For $\boldsymbol{w} = (w^1, \dots, w^N) : \mathbf{R} \to \mathbf{R}^N$, $w^1(x), \dots, w^{k-1}(x)$ are identically zero, and $w^k(x), \dots, w^N(x)$ satisfy condition (Ws) for some $1 \leq k \leq N$.

If $k = 1$, then $(\mathrm{Ws})_N$ is understood in such a way that all components of $\boldsymbol{w}$ satisfy (Ws). We now obtain the following result on the global existence.

THEOREM 5.5. *Assume* $(\mathrm{Ws})_N$, (P1)–(P5) *and* (L). *Let* $\boldsymbol{u}_0 \in \mathbf{L}^1(\boldsymbol{w})$ *be such that* $\boldsymbol{u}_0(x) \in \mathbf{E}$ *a.e. and* $a < \int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}_0(x)dx < b$. *Then there exists a weak solution* $\boldsymbol{u}$ *to the initial-value problem for* (NNS) *on* $[0, T]$ *such that the functions*

$$t \mapsto \int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}(t,x)dx \qquad \text{and} \qquad L\left(\int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}(t,x)dx\right)$$

belong to $W^{1,\infty}(0, T)$.

REMARK 5.6. If $w : \mathbf{R} \to \mathbf{R}$ satisfies (Ws), then we have

$$c|x - x_0| \leq |w(x)| \leq C|x - x_0| \qquad \text{on } \mathbf{R},$$

where $c = \operatorname{ess.inf}_{x \in \mathbf{R}} w'(x)$, $C = \operatorname{ess.sup}_{x \in \mathbf{R}} w'(x)$ and x_0 is a unique zero point of $w(\cdot)$. Therefore, if $\boldsymbol{w} : \mathbf{R} \to \mathbf{R}^N$ satisfies $(\mathrm{Ws})_N$, then $L^1(w^i) = L^1(\mathbf{R})$, $i = 1, \dots, k-1$, and $L^1(w^i) = L^1(\mathbf{R}; (1 + |x|)dx)$, $i = k, \dots, N$.

We now introduce two classes of functions in order to state our uniqueness result. First, for w satisfying condition (W), we employ the set (denoted by $\mathfrak{F}(w)$) of all measurable functions $\eta : \mathbf{R}^2 \to [0, \infty)$ such that for each $r > 0$ there exists a constant $C_r > 0$ and

$$\int_{-\infty}^{+\infty} \eta(x+\sigma_1, x+\sigma_2)(1+|w(x)|)dx \le C_r|\sigma_1-\sigma_2| \qquad \text{for } \sigma_1, \sigma_2 \in [-r, r].$$

We next employ the set (denoted by $\mathfrak{F}(w,T)$) of all measurable functions $\eta : (0,T)\times \mathbf{R}^2 \to [0,\infty)$ such that for each $r>0$ there exists a nonnegative function $\theta_r \in L^1(0,T)$ and

$$\int_{-\infty}^{+\infty} \eta(t, x+\sigma_1, x+\sigma_2)(1+|w(x)|)dx \le \theta_r(t)|\sigma_1-\sigma_2|$$

$$\text{for } t\in(0,T),\ \ \sigma_1,\sigma_2 \in [-r,r].$$

Let $w(x) = x$ and $f : \mathbf{R} \to \mathbf{R}$ be a Lipschitz continuous function with compact support. Then it is obvious that a function η defined by $\eta(x,y) = |f(x)-f(y)|$ belongs to the class $\mathfrak{F}(w)$, but that $\eta(x,y) = |f(x)-f(y)|$ defined for a Lipschitz continuous function f does not always belong to $\mathfrak{F}(w)$ if the support of f is not compact; for instance, $f(x) = x$. The continuity of a function f or the boundedness of its support does not necessarily imply that the function $\eta(x,y) = |f(x)-f(y)|$ belongs to $\mathfrak{F}(w)$. Such functions are found in Example 5.8. On the other hand, for nonnegative functions $\theta \in L^1(0,T)$ and $\eta \in \mathfrak{F}(w)$, a function $\bar{\eta}(t,x,y) = \theta(t)\eta(x,y)$ belongs to $\mathfrak{F}(w,T)$.

In order to obtain a global uniqueness result, we assume the following additional condition on the function $\boldsymbol{\varphi}$ on the right-hand side of (NNS):

(P6) For each $r>0$ there exist functions $\eta_r^i \in \mathfrak{F}(w^i,T)$, $i=1,\dots,N$, such that the components φ^i, $i=1,\dots,N$, satisfy

$$|\varphi^i(t,x_1,\boldsymbol{u},z) - \varphi^i(t,x_2,\boldsymbol{u},z)| \le \eta_r^i(t,x_1,x_2)$$

for almost all $t\in(0,T)$, x_1, $x_2 \in \mathbf{R}$, any $\boldsymbol{u}\in\mathbf{E}$ and $z\in[-r,r]$.

We now state our uniqueness theorem.

THEOREM 5.7. *Assume* (W)$_N$, (P1), (P2), (P3) *with* $\rho_r(s) = C_r s$ *(*C_r *being some constant),* (P4) *and* (L). *Assume further that* (P6) *holds. Let* $\boldsymbol{u}_1$ *and* $\boldsymbol{u}_2$ *be weak solutions to* (NNS) *on* $[0,T]$, *and set*

$$z_i(t) = L\left(\int_{-\infty}^{+\infty} \boldsymbol{w}(x)\cdot \boldsymbol{u}_i(t,x)dx\right), \qquad i=1,2.$$

Then we have

$$|z_1-z_2|_\infty \le C|\boldsymbol{u}_1(0,\cdot+z_1(0)) - \boldsymbol{u}_2(0,\cdot+z_2(0))|_{\boldsymbol{w}}. \tag{5.3}$$

Here $|\cdot|_\infty$ *is the supremum-norm over* $[0,T]$, $|\cdot|_{\boldsymbol{w}}$ *the norm of the product space* $\mathbf{L}^1(\boldsymbol{w})$, *and* C *some positive constant determined for any fixed number*

$R \geq \max\{r_1, r_2\}$, where $r_1 = \max\{|z_1|_\infty, |z_2|_\infty\}$, $r_2 = e^{\omega r_1 + MT}(\max\{|\boldsymbol{u}_1(0,\cdot)|_{\boldsymbol{w}}, |\boldsymbol{u}_2(0,\cdot)|_{\boldsymbol{w}}\} + \int_0^T |\boldsymbol{\Phi}(\tau,\cdot)|_{\boldsymbol{w}} d\tau)$, $\omega = \max_{1 \leq i \leq N} \operatorname{ess.sup}_{x \in \mathbf{R}} |(w^i)'(x)|/(1 + |w^i(x)|)$ and $\boldsymbol{\Phi} = (\Phi^1, \dots, \Phi^N)$. In particular, a weak solution to (NNS) *is unique if it exists.*

EXAMPLE 5.8. In case $N = 2$, one can formulate the following model: We first define $\boldsymbol{w}(x) \equiv (w^1(x), w^2(x)) = (0, w(x))$ for $w(\cdot)$ satisfying (Ws) and $\boldsymbol{\varphi} = (\varphi^1, \varphi^2)$ by

$$\varphi^1(t, x, u^1, u^2, z) = g_1(x)(u^2)^{p_{11}} - \gamma_1(t) f_1(x)(u^1)^{p_{12}},$$

$$\varphi^2(t, x, u^1, u^2, z) = \gamma_2(t) f_2(x)(u^1)^{p_{21}} - g_2(x)(u^2)^{p_{22}},$$

with powers $p_{11} \geq p_{22} \geq 1$ and $p_{21} \geq p_{12} \geq 1$. Here $\gamma_i(t)$, $f_i(x)$, $g_i(x)$, $i = 1, 2$, are nonnegative functions which satisfy that $\gamma_i \in C([0, T])$, $\gamma_1(t) \geq \gamma_2(t)$ on $[0, T]$, f_1, $g_2 \in L^\infty(\mathbf{R})$, $g_1 \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, $f_2 \in L^1(w) \cap L^\infty(w)$, $f_1(x) \geq f_2(x)$ and $g_1(x) \leq g_2(x)$ a.e. Moreover, functions $(x, y) \mapsto |f_i(x) - f_i(y)|$, $|g_i(x) - g_i(y)|$ belong to $\mathfrak{F}(w^i)$, $i = 1, 2$. Then the functions $\mathfrak{f}_r^{1,2}(\cdot)$, $\mathfrak{f}_r^{2,1}(\cdot)$ and the constant $K(r)$ stated in (P2) are chosen as $\mathfrak{f}_r^{1,2}(x) = p_{11} g_1(x)$, $\mathfrak{f}_r^{2,1}(x) = p_{21} |\gamma_2|_\infty f_2(x)$ and $K(r) = \max\{p_{12}|\gamma_1|_\infty |f_1|_\infty, p_{22}|g_2|_\infty\}$, respectively. The functions (Φ^1, Φ^2) and the constant M in (P4) can be chosen as $\Phi^1(t, x) = g_1(x)$, $\Phi^2(t, x) = \gamma_2(t) f_2(x)$ and $M = \max\{|\gamma_1|_\infty |f_1|_\infty, |g_2|_\infty\}$, respectively. The constant λ_r in (P5.2) and functions η_r^1 and η_r^2 in (P6) are defined to be $\lambda_r = \max\{p_{12}|\gamma_1|_\infty |f_1|_\infty, p_{22}|g_2|_\infty\}$ and $\eta_r^i(t, x, y) = \gamma_i(t)|f_i(x) - f_i(y)| + |g_i(x) - g_i(y)|$, $i = 1, 2$, respectively. In view of the specific properties of this function $\boldsymbol{\varphi}$, we can eliminate the condition that $g_1 \in L^1(\mathbf{R})$. More realistic forms of w, f_i and g_i, $i = 1, 2$, satisfying these conditions are given as follows: $w(x) = x$,

$$f_1(x) = f_2(x) = \begin{cases} k_1 x/h, & \text{if } 0 \leq x \leq h, \\ 0, & \text{otherwise,} \end{cases}$$

$$g_1(x) = g_2(x) = \begin{cases} k_2, & \text{if } x < 0, \\ k_3 x/h, & \text{if } 0 \leq x \leq ch/k_3, \\ c, & \text{otherwise,} \end{cases}$$

where k_1, k_2, k_3, and h are positive constants and c is any fixed constant with $c > k_2$.

For the function $L : (a, b) \to \mathbf{R}$, we consider the following forms: If muscle contraction is isometric, we take

$$L(\tau) = -\log(1 + \tau), \qquad a = -1, \ b = +\infty;$$

if muscle contraction is isotonic,

$$L(\tau) = \log \frac{Q - \tau}{q(1+\tau)}, \qquad a = -1, \ b = Q,$$

where $0 < q < Q < +\infty$; if it is isometric-isotonic,

$$L(\tau) = \log \frac{q - (\tau - Q + q)^+}{q(1+\tau)}, \qquad a = -1, \ b = Q,$$

where $0 < q < Q < +\infty$ and $c^+ = \max\{c, 0\}$. In these three cases we choose

$$\beta_r = e^{-r}, \qquad \frac{q(1+Q)e^{-r}}{(1+qe^r)^2}, \qquad \min\left\{ e^{-r}, \frac{q(1+Q)e^{-r}}{(1+qe^r)^2} \right\},$$

as the constant β_r in (L), respectively. We can then apply Theorems 5.5 and 5.7 to each case.

EXAMPLE 5.9. In case $N = 4$, one can formulate the following model: First, we define $\boldsymbol{w}(x) \equiv (w^1(x), w^2(x), w^3(x), w^4(x))$ by $(0, 0, w^3(x), w^4(x))$ for w^3 and w^4 satisfying (Ws) and $\boldsymbol{\varphi} = (\varphi^1, \varphi^2, \varphi^3, \varphi^4)$ by

$$\varphi^i(t, x, u^1, u^2, u^3, u^4, z) = \sum_{j=i\pm 1} [a_{ij}(t,x)(u^j)^{p_{ij}} - \hat{a}_{ji}(t,x)(u^i)^{\hat{p}_{ji}}], \qquad i = 1,2,3,4,$$

with powers $p_{i,i\pm 1} \geq \hat{p}_{i,i\pm 1} \geq 1$, $i = 1,2,3,4$, respectively. In accordance with the general cyclic rule for N we introduced before (P1)–(P5), we here adopt the cyclic rule for $N = 4$: $i \equiv j \pmod 4$. Here the functions $a_{i,i\pm 1}(t,x)$ and $\hat{a}_{i,i\pm 1}(t,x)$, $i = 1,2,3,4$, have the forms

$$a_{i,i\pm 1}(t,x) = \begin{cases} f_{i,i\pm 1}(x), & i = 1,2, \\ \gamma_{i,i\pm 1}(t) f_{i,i\pm 1}(x), & i = 3,4, \end{cases}$$

$$\hat{a}_{i,i\pm 1}(t,x) = \begin{cases} \hat{f}_{i,i\pm 1}(x), & i = 1,2, \\ \hat{\gamma}_{i,i\pm 1}(t) \hat{f}_{i,i\pm 1}(x), & i = 3,4. \end{cases}$$

The functions $\gamma_{i,i\pm 1}(t)$, $\hat{\gamma}_{i,i\pm 1}(t)$, $f_{i,i\pm 1}(x)$, $\hat{f}_{i,i\pm 1}(x)$ are assumed to be non-negative and such that $\gamma_{i,i\pm 1}, \hat{\gamma}_{i,i\pm 1} \in C([0,T])$, $\gamma_{i,i\pm 1}(t) \leq \hat{\gamma}_{i,i\pm 1}(t)$ on $[0,T]$, $i = 3,4$, $f_{i,i\pm 1} \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, $i = 1,2$, $f_{34} \in L^1(w^3) \cap L^\infty(\mathbf{R})$, $f_{32} \in L^1(w^3) \cap L^\infty(w^3)$, $f_{41}, f_{43} \in L^1(w^4) \cap L^\infty(w^4)$, $\hat{f}_{i,i\pm 1} \in L^\infty(\mathbf{R})$, $i = 1,2,3,4$, and $f_{i,i\pm 1}(x) \leq \hat{f}_{i,i\pm 1}(x)$ a.e., $i = 1,2,3,4$. Moreover, we assume that the functions $(x,y) \mapsto |f_{i,i\pm 1}(x) - f_{i,i\pm 1}(y)|$, $|\hat{f}_{i\pm 1,i}(x) - \hat{f}_{i\pm 1,i}(y)|$ belong to $\mathfrak{F}(w^i)$ for $i = 1,2,3,4$.

As $L : (a,b) \to \mathbf{R}$, one can choose the logarithmic function

$$L(\tau) = -\log(1+\tau) + \text{const.}, \qquad a = -1, \ b = +\infty.$$

$L(\cdot)$ may also be chosen in the same way as in Example 5.8.

We can apply Theorems 5.5 and 5.7 to this model. In view of this choice of $\boldsymbol{\varphi}$, we may replace the condition $f_{43} \in L^{\infty}(w^4)$ by a weaker condition $f_{43} \in L^{\infty}(\mathbf{R})$.

We may formulate the following type of condition which corresponds to (F6) introduced in §3.

(P7) For each $r > 0$ there exist nonnegative functions $\mu_r^{i,i+1} \in L^{\infty}(\mathbf{R})$, $i = 1, \dots, N-1$, $\mu_r^{N,1} \in L^{\infty}(w^N)$, $\mu_r^{1,N} \in L^{\infty}(\mathbf{R})$, $\mu_r^{i,i-1} \in L^{\infty}(w^i)$, $i = 2, \dots, N$, $\eta_r^i \in \mathfrak{F}(w^i)$, $i = 1, \dots, N$, such that the components φ^i, $i = 1, \dots, N$, satisfy

$$|u^i - v^i| - \lambda \sum_{j=i\pm 1} \mu_r^{i,j}(x_1)|u^j - v^j| - \lambda \eta_r^i(x_1, x_2)$$
$$\leq \left| u^i - v^i - \lambda[\varphi^i(t, x_1, \boldsymbol{u}, z_1) - \varphi^i(t, x_2, \boldsymbol{v}, z_2)] \right|$$

for $\lambda > 0$, $t \in [0, T]$, almost all x_1, $x_2 \in \mathbf{R}$, $\boldsymbol{u} = (u^1, \dots, u^N)$, $\boldsymbol{v} = (v^1, \dots, v^N) \in \mathbf{E}$, and z_1, $z_2 \in [-r, r]$.

THEOREM 5.10. *In Theorems* 5.4 *and* 5.5 *(resp. Theorem* 5.7*), assume* (P7) *in place of* (P2) *and* (P3) *(resp. in place of* (P2), (P3) *and* (P6)*). Then the same results hold.*

We mention a result concerning the supports of weak solutions of (NNS) with $N \geq 2$. The proof is very easy, and omitted. The condition (5.4) below is stronger than (P5.1), but natural in the mathematical models for muscle contraction phenomena.

PROPOSITION 5.11. *Assume that the function* $\boldsymbol{\varphi}$ *satisfies*

$$\sum_{i=1}^{N} \varphi^i(t, x, \boldsymbol{u}, z) = 0 \ (\textit{resp.}\ \leq 0) \qquad \textit{for}\ (t, x, \boldsymbol{u}, z) \in [0, T] \times \mathbf{R} \times \mathbf{E} \times \mathbf{R}. \tag{5.4}$$

Let $\boldsymbol{u}$ *be a possible weak solution to* (NNS) *with initial value* $\boldsymbol{u}_0$ *and set* $z(t) = L(\int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}(t, x) dx)$. *Then we have*

$$\operatorname{supp} \boldsymbol{u}(t, \cdot) = \operatorname{supp} \boldsymbol{u}_0(\cdot) + z(t) - z(0) \qquad \textit{for}\ t \in [0, T].$$
$$(\textit{resp.}\ \subset)$$

In particular, if the initial function $\boldsymbol{u}_0(\cdot)$ *is compactly supported, then the solution* $\boldsymbol{u}(t, \cdot)$ *is also compactly supported.*

REMARK 5.12. It should be mentioned that similar results are obtained for the case $N = 1$. For $N = 1$, conditions $(\mathrm{W})_N$ without (5.1), $(\mathrm{Ws})_N$, (P1), (P3) through (P6) make sense and $\mathbf{E} = [0, 1]$. In Theorems 5.4, 5.5 and 5.7, we may replace (P2) by:

$(\mathrm{P2})_{N=1}$ For every $r > 0$ there exists a constant $K(r) > 0$ such that

$$|\varphi(t,x,u,z) - \varphi(t,x,v,z)| \le K(r)|u - v|$$

for $t \in [0, T]$, almost every $x \in \mathbf{R}$, u, $v \in [0, 1]$ and $z \in [-r, r]$.

Also, in Theorem 5.10, (P7) can be replaced by:

$(\mathrm{P7})_{N=1}$ For each $r > 0$ there exist a constant $C_r > 0$ and a nonnegative function $\eta_r \in \mathfrak{F}(w)$ such that

$$(1 - \lambda C_r)|u - v| - \lambda\eta_r(x_1, x_2) \le \left|u - v - \lambda[\varphi(t, x_1, u, z_1) - \varphi(t, x_2, v, z_2)]\right|$$

for $\lambda > 0$, $t \in [0, T]$, almost every x_1, $x_2 \in \mathbf{R}$, u, $v \in [0, 1]$ and z_1, $z_2 \in [-r, r]$.

Under these conditions we obtain results similar to the case $N \ge 2$.

We conclude this section with an example for the case $N = 1$.

Example 5.13. In case $N = 1$, one can take $\varphi(t,x,u,z) = \gamma(t)f(x)(1-u)^p - g(x)u^{\hat{p}}$. Here $\gamma \in C([0,T])$, $f \in L^1(w) \cap L^\infty(\mathbf{R})$ and $g \in L^\infty(\mathbf{R})$. γ, f and g are nonnegative, $p, \hat{p} \ge 1$ and the function $w(\cdot)$ satisfies (Ws). Moreover, the functions $(x,y) \mapsto |f(x) - f(y)|$, $|g(x) - g(y)|$ belong to $\mathfrak{F}(w)$. The function $L : (a,b) \to \mathbf{R}$ is chosen in the same way as in Example 5.8. Then we can apply the results stated in Remark 5.12. Even if we assume $0 < p < 1$, in place of $p \ge 1$, φ satisfies $(\mathrm{P7})_{N=1}$ and hence the uniqueness result given in Remark 5.12 is applicable again.

6. Proofs of the results for (NNS)

In this section we apply the abstract results, and prove Theorems 5.4, 5.5, 5.7 and 5.10. Let $w : \mathbf{R} \to \mathbf{R}$ satisfy condition (W). We introduce the standard partial order relation $\leqslant$ in the weighted L^1 space $L^1(w)$, namely, $u \leqslant v$ in $L^1(w)$ if and only if $u(x) \le v(x)$ a.e. in $\mathbf{R}$. Then it is seen that $(L^1(w), |\cdot|_w, \leqslant)$ is an ordered Banach space and its positive cone is given by $L^1(w)_+ = \{v \in L^1(w) | v(x) \ge 0 \text{ a.e.}\}$. Moreover, we introduce the weighted Sobolev space

$$W^{1,1}(w) := W^{1,1}(\mathbf{R}; (1 + |w(x)|)dx) := \{v \in L^1(w) \,|\, v' \in L^1(w)\}$$

endowed with norm $|v|_w^{1,1} := |v|_w + |v'|_w$. It is easily seen that $(W^{1,1}(w), |\cdot|_w^{1,1})$ is a Banach space and that $C_0^\infty(\mathbf{R})$ is dense in $(L^1(w), |\cdot|_w)$ and $(W^{1,1}(w), |\cdot|_w^{1,1})$, respectively. Thus, $W^{1,1}(w)$ is dense in $(L^1(w), |\cdot|_w)$.

We then define three linear operators $S_\pm(\sigma) : L^1(w) \to L^1(w)$ for $\sigma \in \mathbf{R}$ and $A : \mathscr{D}(A) \subset L^1(w) \to L^1(w)$ as follows

$$(S_\pm(\sigma)u)(x) := u(x \pm \sigma) \qquad \text{for } x \in \mathbf{R},\ u \in L^1(w), \qquad \text{respectively,}$$

$$(Au)(x) := u'(x) \qquad \text{for } x \in \mathbf{R},\ u \in \mathscr{D}(A) := W^{1,1}(w).$$

Then we have

PROPOSITION 6.1. *One-parameter families* $\{S_\pm(\sigma)\}_{\sigma\in\mathbf{R}}$ *of the continuous linear operators in* $L^1(w)$ *are* C_0*-groups of type* $\bar{\omega}$ *and their generators are* $\pm A$, *respectively. Here*

$$\bar{\omega} = \operatorname*{ess.sup}_{x\in\mathbf{R}} \frac{|w'(x)|}{1+|w(x)|} (<\infty). \tag{6.1}$$

PROOF. Notice that $w(\cdot)$ satisfies that

$$\frac{1+|w(x)|}{1+|w(y)|} \le e^{\bar{\omega}|x-y|} \qquad \text{for } x,y\in\mathbf{R}. \tag{6.2}$$

Using (6.2), we obtain the results in a way similar to the proof of the fact that the differential operator $u\mapsto u'$ generates a contraction C_0-group in the usual $L^1(\mathbf{R})$. Note that it is not necessary to use the absolute continuity and the monotonicity of w. $\square$

LEMMA 6.2. *For every* $u\in W^{1,1}(w)$, *the function* wu *belongs to the usual Sobolev space* $W^{1,1}(\mathbf{R})$. *In particular,* $w(x)u(x)\to 0$ *as* $|x|\to\infty$.

PROOF. Let $u\in W^{1,1}(w)$. Then it is clear that u, wu, u' and wu' belong to $L^1(\mathbf{R})$ and $(wu)'(x) = w'(x)u(x) + w(x)u'(x)$ a.e. in $\mathbf{R}$. Hence

$$|(wu)'(x)| \le |w'(x)u(x)| + |w(x)u'(x)| \le \bar{\omega}(1+|w(x)|)|u(x)| + |w(x)u'(x)|,$$

where $\bar{\omega}$ is the constant defined by (6.1). Since the last expression belongs to $L^1(\mathbf{R})$, the derivative $(wu)'$ of wu also belongs to $L^1(\mathbf{R})$. Thus, we conclude that wu belongs to $W^{1,1}(\mathbf{R})$. Finally, it is well known that functions in $W^{1,1}(\mathbf{R})$ vanish at $\pm\infty$. $\square$

Assume that $(\mathrm{W})_N$ holds. Let $X := \mathbf{L}^1(\boldsymbol{w}) = L^1(w^1)\times\cdots\times L^1(w^N)$ and $\|\cdot\| := |\cdot|_{\boldsymbol{w}}$. The order relation $\leqslant$ is defined as follows: $(u^1,\dots,u^N)\leqslant(v^1,\dots,v^N)$ in X if and only if $u^i\leqslant v^i$ in $L^1(w^i)$ for all $i=1,\dots,N$. Then the positive cone X_+ is given by $X_+ = L^1(w^1)_+\times\cdots\times L^1(w^N)_+$ and it is clear that $\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}-\boldsymbol{u}\in X_+$ imply $\|\boldsymbol{u}\|\le\|\boldsymbol{v}\|$. Moreover, we define

$$\begin{aligned} D &:= \{(v^1,\dots,v^N)\in X_+ \mid v^1(x)+\cdots+v^N(x)\le 1 \text{ a.e.}\} \\ &= \{(v^1,\dots,v^N)\in X \mid (v^1(x),\dots,v^N(x))\in\mathbf{E} \text{ a.e.}\}, \end{aligned}$$

and linear operators $A:\mathscr{D}(A)\subset X\to X$ and $S(\sigma):X\to X$ by

$$\mathscr{D}(A) := W^{1,1}(w^1)\times\cdots\times W^{1,1}(w^N),$$

$$A\boldsymbol{u} := ((u^1)',\dots,(u^N)') \qquad \text{for } \boldsymbol{u} = (u^1,\dots,u^N)\in\mathscr{D}(A),$$

$$(S(\sigma)\boldsymbol{u})(x) := \boldsymbol{u}(x-\sigma) \qquad \text{for } x\in\mathbf{R},\ \boldsymbol{u}\in X,\ \sigma\in\mathbf{R}.$$

In addition, we set

$$\omega^i := \operatorname{ess.sup}_{x \in \mathbf{R}} \frac{|(w^i)'(x)|}{1 + |w^i(x)|}, \qquad \omega := \max_{1 \le i \le N} \omega^i.$$

Then, using Proposition 6.1, we have the following

PROPOSITION 6.3. *Assume* $(\mathrm{W})_N$. *Then the one-parameter family* $\{S(\sigma)\}_{\sigma \in \mathbf{R}}$ *of continuous linear operators in* X *gives a* C_0*-group of type* ω *and its generator is* $-\Lambda$. *In addition,* $S(\sigma)X_+ \subset X_+$ *and* $S(\sigma)D \subset D$ *for any* $\sigma \in \mathbf{R}$.

Next, we define two linear functionals on X by

$$(6.3) \quad \boldsymbol{f}(\boldsymbol{u}) := \int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}(x) dx, \quad \boldsymbol{g}(\boldsymbol{u}) := -\int_{-\infty}^{+\infty} \boldsymbol{w}'(x) \cdot \boldsymbol{u}(x) dx \quad \text{for } \boldsymbol{u} \in X,$$

where $\boldsymbol{w}'(x) = ((w^1)'(x), \dots, (w^N)'(x))$. Then we get

LEMMA 6.4. *Under condition* $(\mathrm{W})_N$, *the linear functionals* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *defined by* (6.3) *satisfy condition* (LF) *introduced in* §2.

PROOF. Clearly, $|\boldsymbol{f}(\boldsymbol{u})| \le \|\boldsymbol{u}\|$ for $\boldsymbol{u} \in X$, and so $\boldsymbol{f}$ is continuous. For $\boldsymbol{u} = (u^1, \dots, u^N) \in X$, we have

$$|\boldsymbol{g}(\boldsymbol{u})| \le \sum_{i=1}^N \int_{-\infty}^{+\infty} |(w^i)'(x) u^i(x)| dx \le \sum_{i=1}^N \omega^i \int_{-\infty}^{+\infty} (1 + |w^i(x)|) |u^i(x)| dx \le \omega \|\boldsymbol{u}\|.$$

Thus, $\boldsymbol{g}$ is continuous. Moreover, $w^i(x) \ge 0$ for all $i = 1, \dots, N$, and so $\boldsymbol{g}(\boldsymbol{u}) \le 0$ for $\boldsymbol{u} \in D$. By integration by parts and Lemma 6.2, we see that $\boldsymbol{f}(\Lambda \boldsymbol{u}) = \boldsymbol{g}(\boldsymbol{u})$ for $\boldsymbol{u} \in \mathscr{D}(\Lambda)$, and hence that $\boldsymbol{f}\Lambda$ is continuous on $(\mathscr{D}(\Lambda), \|\cdot\|)$ and $\boldsymbol{g}$ is a unique extension of $\boldsymbol{f}\Lambda$ to X. □

We then consider a function L satisfying condition (L). Since $L : (a, b) \to \mathbf{R}$ is continuous, $L(a+0) = +\infty$ and $L(b-0) = -\infty$, it follows that $\mathscr{R}(L) = \mathbf{R}$, i.e., L is onto. Since L is strictly decreasing, it has an inverse function. Set $\mathscr{D}(\Gamma) := \mathscr{R}(L) = \mathbf{R}$ and $\Gamma := L^{-1}$. Then it follows that Γ is dissipative in $\mathbf{R}$, because L^{-1} is decreasing. Since $L(a+0) = +\infty$ and $L(b-0) = -\infty$, we infer that $\Gamma(\zeta) \to a$ as $\zeta \to +\infty$ and $\Gamma(\zeta) \to b$ as $\zeta \to -\infty$, respectively. Thus, $(I - \Gamma)(\zeta) = \zeta - \Gamma(\zeta) \to \pm\infty$ as $\zeta \to \pm\infty$, respectively. This means that the range condition $\mathscr{R}(I - \Gamma) = \mathbf{R}$ holds, since $\Gamma = L^{-1}$ is continuous on $\mathbf{R}$. Consequently, Γ is m-dissipative. From (5.2), we get the local quasi-dissipativity of Γ. In view of the above-mentioned, we obtain

LEMMA 6.5. *Assume* (L). *Then* $\Gamma := L^{-1}$ *is single-valued and satisfies* (G) *introduced in* §3.

We now define a mapping $F : [0, T] \times D \times \mathbf{R} \to X$ by

(6.4) $(F(t, \boldsymbol{u}, z))(x) := \boldsymbol{\varphi}(t, x, \boldsymbol{u}(x), z)$ for $x \in \mathbf{R}$, $(t, \boldsymbol{u}, z) \in [0, T] \times D \times \mathbf{R}$.

LEMMA 6.6. *Assume that* $(\mathrm{W})_N$, (P1) *and* (P4) *are valid. Then* F *is well-defined as a continuous mapping.*

PROOF. By (P4), it is seen that F is well-defined. In order to show the continuity, let $(t_n, \boldsymbol{u}_n, z_n), (t, \boldsymbol{u}, z) \in [0, T] \times D \times \mathbf{R}$, $|t_n - t| + \|\boldsymbol{u}_n - \boldsymbol{u}\| + |z_n - z| \to 0$, and let $\boldsymbol{u}_n = (u_n^1, \dots, u_n^N), \boldsymbol{u} = (u^1, \dots, u^N)$. Using (P1) and (P4) and taking a subsequence if necessary, we see that for each $i = 1, \dots, N$,

$$\varphi^i(t_n, x, \boldsymbol{u}_n(x), z_n) \to \varphi^i(t, x, \boldsymbol{u}(x), z) \quad \text{as } n \to \infty, \quad \text{a.e. } x,$$

$$|\varphi^i(t_n, x, \boldsymbol{u}_n(x), z_n)| \le \Phi^i(t_n, x) + M u_n^i(x) \quad \text{a.e. } x,$$

$$\Phi^i(t_n, \cdot) + M u_n^i(\cdot), \quad \Phi^i(t, \cdot) + M u^i(\cdot) \in L^1(w^i),$$

$$\Phi^i(t_n, \cdot) + M u_n^i(\cdot) \to \Phi^i(t, \cdot) + M u^i(\cdot) \quad \text{in } L^1(w^i) \quad \text{as } n \to \infty.$$

Then the application of the Lebesgue Dominated Convergence Theorem implies that

$$\varphi^i(t_n, \cdot, \boldsymbol{u}_n(\cdot), z_n) \to \varphi^i(t, \cdot, \boldsymbol{u}(\cdot), z) \text{ in } L^1(w^i) \quad \text{as } n \to \infty, \quad i = 1, \dots, N,$$

and so that $\|F(t_n, \boldsymbol{u}_n, z_n) - F(t, \boldsymbol{u}, z)\| \to 0$ as $n \to \infty$. This shows that F is continuous. □

We are now in a position to prove Theorem 5.4.

PROOF OF THEOREM 5.4. We want to apply Theorem 3.2. By Proposition 6.3, Lemmas 6.4 and 6.5, we see that (BS), (GR), (LF) in §2 and (G) in §3 are satisfied. The mapping F is continuous by Lemma 6.6.

We next demonstrate that F satisfies (F1) introduced in §2. Let $t \in [0, T]$, $\boldsymbol{u} = (u^1, \dots, u^N), \boldsymbol{v} = (v^1, \dots, v^N) \in D$ and $|z| \le r$. Noting that

$$C := \max_{1 \le i \le N-1} \sup_{x \in \mathbf{R}} \frac{1 + |w^i(x)|}{1 + |w^{i+1}(x)|} \tag{6.5}$$

is finite, we see from (P2) that

$$\int_{-\infty}^{+\infty} |\varphi^1(t, x, \boldsymbol{u}(x), z) - \varphi^1(t, x, \boldsymbol{v}(x), z)|(1 + |w^1(x)|)dx$$
$$\le \int_{-\infty}^{+\infty} \mathfrak{f}_r^{1,2}(x)|u^2(x) - v^2(x)|(1 + |w^1(x)|)dx$$
$$+ \int_{-\infty}^{+\infty} \mathfrak{f}_r^{1,N}(x)|u^N(x) - v^N(x)|(1 + |w^1(x)|)dx$$

$$+K(r)\sum_{j=1}^{N}\int_{-\infty}^{+\infty}|u^j(x)-v^j(x)|(1+|w^1(x)|)dx$$

$$\leq C|\mathfrak{f}_r^{1,2}|_\infty|u^2-v^2|_{w^2}+C^{N-1}|\mathfrak{f}_r^{1,N}|_\infty|u^N-v^N|_{w^N}+K(r)\sum_{j=1}^{N}C^{j-1}|u^j-v^j|_{w^j}.$$

If $N\geq 3$, then, for $i=2,\dots,N-1$, we obtain in the same way as above

$$\int_{-\infty}^{+\infty}|\varphi^i(t,x,\boldsymbol{u}(x),z)-\varphi^i(t,x,\boldsymbol{v}(x),z)|(1+|w^i(x)|)dx$$

$$\leq C|\mathfrak{f}_r^{i,i+1}|_\infty|u^{i+1}-v^{i+1}|_{w^{i+1}}+\|\mathfrak{f}_r^{i,i-1}\|_{w^i}|u^{i-1}-v^{i-1}|_{w^{i-1}}$$

$$+K(r)\sum_{j=i}^{N}C^{j-i}|u^j-v^j|_{w^j}.$$

Recall that $\|\mathfrak{f}_r^{i,i-1}\|_{w^i}=\operatorname{ess.sup}_{x\in\mathbf{R}}|\mathfrak{f}_r^{i,i-1}(x)|(1+|w^i(x)|)$. Furthermore, in the same way,

$$\int_{-\infty}^{+\infty}|\varphi^N(t,x,\boldsymbol{u}(x),z)-\varphi^N(t,x,\boldsymbol{v}(x),z)|(1+|w^N(x)|)dx$$

$$\leq \|\mathfrak{f}_r^{N,1}\|_{w^N}|u^1-v^1|_{w^1}+\|\mathfrak{f}_r^{N,N-1}\|_{w^N}|u^{N-1}-v^{N-1}|_{w^{N-1}}+K(r)|u^N-v^N|_{w^N}.$$

Therefore, it follows that $\|F(t,\boldsymbol{u},z)-F(t,\boldsymbol{v},z)\|\leq C_r\|\boldsymbol{u}-\boldsymbol{v}\|$ for some positive constant C_r. This implies (F1) with $\overline{K}(r)=C_r$.

Thirdly, we show that F satisfies (F3) introduced in §2. Let $t\in(0,T)$, $\boldsymbol{u}\in D$, $\|\boldsymbol{u}\|\leq r$ and $z_1,z_2\in[-r,r]$. Then, using (P3), we have

$$\int_{-\infty}^{+\infty}|\varphi^i(t,x,\boldsymbol{u}(x),z_1)-\varphi^i(t,x,\boldsymbol{u}(x),z_2)|(1+|w^i(x)|)dx$$

$$\leq\left(|\mathfrak{g}_r^i(t,\cdot)|_{w^i}+r\sum_{j=i}^{N}C^{j-i}\right)\rho_r(|z_1-z_2|)$$

for $i=1,\dots,N$, where C is the constant defined by (6.5). Thus,

$$\|F(t,\boldsymbol{u},z_1)-F(t,\boldsymbol{u},z_2)\|\leq\sum_{i=1}^{N}\left(|\mathfrak{g}_r^i(t,\cdot)|_{w^i}+r\sum_{j=i}^{N}C^{j-i}\right)\rho_r(|z_1-z_2|).$$

This implies that (F3) holds for $\nu_r(t)=\sum_{i=1}^{N}(|\mathfrak{g}_r^i(t,\cdot)|_{w^i}+r\sum_{j=i}^{N}C^{j-i})$ and $\bar{\rho}_r(\cdot)=\rho_r(\cdot)$.

From (P4) it follows that F satisfies (F4) introduced in §3 for $\mathscr{F}(t)=(\Phi^1(t,\cdot),\dots,\ \Phi^N(t,\cdot))$ and $\overline{M}=M$.

We now show that (F2) introduced in §2 is satisfied. To this end, it suffices to show that for $r > 0$ there is a constant $\lambda_r > 0$ such that

(6.6) $\boldsymbol{u}+\lambda^{-1}\boldsymbol{\varphi}(t,x,\boldsymbol{u},z) \in \mathbf{E}$ for $\lambda \geq \lambda_r$, $(t,x,\boldsymbol{u},z) \in [0,T] \times \mathbf{R} \times \mathbf{E} \times [-r,r]$.

Indeed, if (6.6) holds, then, for $(t,\boldsymbol{u},z) \in [0,T] \times D \times \mathbf{R}$ and $h \in (0,\lambda_r^{-1}]$ with $r = |z|$, we have $\boldsymbol{u} + hF(t,\boldsymbol{u},z) = \boldsymbol{u}(\cdot) + h\boldsymbol{\varphi}(t,\cdot,\boldsymbol{u}(\cdot),z) \in D$. This shows that $d(\boldsymbol{u} + hF(t,\boldsymbol{u},z),D) = 0$, and so that (F2) is obtained.

We then check (6.6). Let $r > 0$. First, using (P5.2), we easily see that

(P5.2)′ if $\boldsymbol{u},\boldsymbol{v} \in \mathbf{E}$ satisfy $\boldsymbol{u} \leqslant \boldsymbol{v}$ in $\mathbf{R}^N$, then $\lambda\boldsymbol{u} + \boldsymbol{\varphi}(t,x,\boldsymbol{u},z) \leqslant \lambda\boldsymbol{v} + \boldsymbol{\varphi}(t,x,\boldsymbol{v},z)$ in $\mathbf{R}^N$ holds for $\lambda \geq \lambda_r$, $t \in [0,T]$, $x \in \mathbf{R}$, $z \in [-r,r]$.

Let $\lambda \geq \lambda_r$, $(t,x,\boldsymbol{u},z) \in [0,T] \times \mathbf{R} \times \mathbf{E} \times [-r,r]$ and $\boldsymbol{u} = (u^1,\dots,u^N)$. Then

(6.7) $$\mathbf{0} \leqslant \boldsymbol{\varphi}(t,x,\mathbf{0},z) \leqslant \lambda\boldsymbol{u} + \boldsymbol{\varphi}(t,x,\boldsymbol{u},z) \qquad \text{in } \mathbf{R}^N$$

by (P4) and (P5.2)′. On the other hand, there exists a vector $\hat{\boldsymbol{v}} = (v^1,\dots,v^N) \in \mathbf{E}$ such that $\sum_{i=1}^N v^i = 1$ and $\boldsymbol{u} \leqslant \hat{\boldsymbol{v}}$ in $\mathbf{R}^N$, and so (P5.2)′ and (P5.1) together imply

$$\sum_{i=1}^N(\lambda u^i + \varphi^i(t,x,\boldsymbol{u},z)) \leq \sum_{i=1}^N(\lambda v^i + \varphi^i(t,x,\hat{\boldsymbol{v}},z)) \leq \lambda.$$

This together with (6.7) yields (6.6). Consequently, F satisfies (F1) through (F4).

For an initial function $\boldsymbol{u}_0$ we assume that $\boldsymbol{u}_0 \in \mathbf{L}^1(\boldsymbol{w})$ and $\boldsymbol{u}_0(x) \in \mathbf{E}$ a.e. Hence $\boldsymbol{u}_0 \in D$. Put $z_0 := L(\int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}_0(x)dx)$, then $z_0 \in \mathscr{R}(L) = \mathscr{D}(\Gamma)$ and $\boldsymbol{f}(\boldsymbol{u}_0) = \int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}_0(x)dx = L^{-1}(z_0) = \Gamma(z_0)$.

Therefore, rewriting (NNS) in the form (AES), we can apply Theorem 3.2 to get the desired results. □

PROOF OF THEOREM 5.5. For this purpose, we want to apply Theorem 3.3. We first note that condition $(\mathrm{Ws})_N$ implies $(\mathrm{W})_N$. Conditions (BS), (GR), (F1)–(F4), (LF) and (G) are all satisfied in the same way as in the proof of Theorem 5.4.

We first show (AdG) introduced in §3. Since $\mathscr{R}(\Gamma) = \mathscr{D}(L) = (a,b)$, (AdG) holds if $0 \notin (a,b)$. Let $a < 0 < b$. If $L(0) = 0$, then $\Gamma(0) = L^{-1}(0) = 0$, and (AdG) is satisfied. In case $L(0) \neq 0$, we reduce the proof to the case $L(0) = 0$. Indeed, we define

$$\hat{\boldsymbol{\varphi}}(t,x,\boldsymbol{u},\hat{z}) := \boldsymbol{\varphi}(t,x,\boldsymbol{u},\hat{z} + L(0)), \qquad \hat{L}(\tau) := L(\tau) - L(0).$$

Then $\hat{\boldsymbol{\varphi}}$ satisfies (P1) through (P5) and $\hat{L}$ satisfies (L). Moreover, $\hat{L}(0) = 0$. Let $\boldsymbol{u}(t,x)$ be a weak solution of (NNS) and $z(t) := L(\int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}(t,x)dx)$. Set $\hat{z}(t) := z(t) - L(0)$. Then it is obvious that $\hat{z}(t) = \hat{L}(\int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}(t,x)dx)$

and

$$\boldsymbol{u}(t,x) = \boldsymbol{u}(0, x-\hat{z}(t)+\hat{z}(0)) + \int_0^t \hat{\boldsymbol{\varphi}}(\tau, x-\hat{z}(t)+\hat{z}(\tau), \boldsymbol{u}(\tau, x-\hat{z}(t)+\hat{z}(\tau)), \hat{z}(\tau))d\tau,$$

and hence $\boldsymbol{u}(t,x)$ is a weak solution to

$$\begin{cases} \partial_t \boldsymbol{u} + \hat{z}'(t)\partial_x \boldsymbol{u} = \hat{\boldsymbol{\varphi}}(t,x,\boldsymbol{u},\hat{z}(t)), & (t,x) \in (0,T) \times \mathbf{R}, \\ \hat{z}(t) = \hat{L}\left(\int_{-\infty}^{+\infty} \boldsymbol{w}(y) \cdot \boldsymbol{u}(t,y)dy\right), & t \in [0,T]. \end{cases}$$

This shows that the problem is reduced to the case $L(0) = 0$, if $L(0) \neq 0$.

To check (AdLF) and (AdF) introduced in §3, we consider condition $(\mathrm{Ws})_N$ in the following two cases:

Case 1. $k = 1$, that is, each component $w^i(x)$ of $\boldsymbol{w}(x)$ satisfies (Ws);

Case 2. there is a number $2 \leq k \leq N$ such that $w^1(x) = \cdots = w^{k-1}(x) \equiv 0$, and $w^k(x), \dots, w^N(x)$ satisfy (Ws).

Case 1. We define linear functionals $\bar{\boldsymbol{f}}$, $\bar{\boldsymbol{g}}$ and $\boldsymbol{h}$ by

$$\bar{\boldsymbol{f}}(\boldsymbol{u}) := \int_{-\infty}^{+\infty} |\boldsymbol{w}|(x) \cdot \boldsymbol{u}(x)dx, \qquad \bar{\boldsymbol{g}}(\boldsymbol{u}) := -\int_{-\infty}^{+\infty} |\boldsymbol{w}|'(x) \cdot \boldsymbol{u}(x)dx,$$

$$\boldsymbol{h}(\boldsymbol{u}) := \sum_{i=1}^{N} \int_{-\infty}^{+\infty} u^i(x)dx$$

for $\boldsymbol{u} = (u^1, \dots, u^N) \in X$, where $|\boldsymbol{w}|(x) = (|w^1(x)|, \dots, |w^N(x)|)$, $|\boldsymbol{w}|'(x) = (|w^1(x)|', \dots, |w^N(x)|')$ and $|w^i(x)|' = (d/dx)|w^i(x)|$. Then $|\bar{\boldsymbol{f}}(\boldsymbol{u})| \leq \|\boldsymbol{u}\|$, $|\boldsymbol{h}(\boldsymbol{u})| \leq \|\boldsymbol{u}\|$ and

$$|\bar{\boldsymbol{g}}(\boldsymbol{u})| \leq \sum_{i=1}^{N} \int_{-\infty}^{+\infty} |\,|w^i(x)|' u^i(x)|dx \leq \sum_{i=1}^{N} \omega^i \int_{-\infty}^{+\infty} (1 + |w^i(x)|)|u^i(x)|dx \leq \omega\|\boldsymbol{u}\|$$

for $\boldsymbol{u} = (u^1, \dots, u^N) \in X$, where $\omega^i = \operatorname{ess.sup}_{x\in\mathbf{R}} |(w^i)'(x)|/(1 + |w^i(x)|)$ and $\omega = \max_{1\leq i\leq N} \omega^i$. Thus, $\bar{\boldsymbol{f}}$, $\bar{\boldsymbol{g}}$ and $\boldsymbol{h}$ are continuous on X. In addition, for any nonzero $\boldsymbol{u} \in X_+$ it is clear that $\bar{\boldsymbol{f}}(\boldsymbol{u}) \geq 0$, $\boldsymbol{h}(\boldsymbol{u}) > 0$ and $\boldsymbol{h}(S(\sigma)\boldsymbol{u}) = \boldsymbol{h}(\boldsymbol{u})$ for $\sigma \in \mathbf{R}$. We define

$$C_1 := \min_{1\leq i\leq N} \operatorname*{ess.inf}_{x\in\mathbf{R}} (w^i)'(x), \qquad C_2 := \max_{1\leq i\leq N} \operatorname*{ess.sup}_{x\in\mathbf{R}} (w^i)'(x).$$

Then $0 < C_1 \leq C_2 < +\infty$, because each $w^i(\cdot)$ is bi-Lipschitz and increasing. Recall that $\boldsymbol{g}(\boldsymbol{u}) = -\int_{-\infty}^{+\infty} \boldsymbol{w}'(x) \cdot \boldsymbol{u}(x)dx$. Therefore, $C_1\boldsymbol{h}(\boldsymbol{u}) \leq -\boldsymbol{g}(\boldsymbol{u}) \leq C_2\boldsymbol{h}(\boldsymbol{u})$ and $|\bar{\boldsymbol{g}}(\boldsymbol{u})| \leq -\boldsymbol{g}(\boldsymbol{u})$ for any $\boldsymbol{u} \in X_+$. Moreover, by integration by parts and Lemma 6.2, we have $\bar{\boldsymbol{f}}A\boldsymbol{u} = \bar{\boldsymbol{g}}(\boldsymbol{u})$ for any $\boldsymbol{u} \in \mathscr{D}(A)$. Hence $\bar{\boldsymbol{g}}$ is a unique extension of $\bar{\boldsymbol{f}}A$, and (AdLF) holds.

By (P4) and the choice of $f, \bar{f}$ and F, it is easy to check that (AdF) is also satisfied for the function $\xi(t) := \sum_{i=1}^{N} \int_{-\infty}^{+\infty} |w^i(x)| \Phi^i(t,x) dx$. Therefore, we apply Theorem 3.3 to get the desired results for $k = 1$.

Case 2. Let f, g, $\bar{f}$ and $\bar{g}$ be the linear functionals treated above:

$$\boldsymbol{f}(\boldsymbol{u}) = \sum_{i=k}^{N} \int_{-\infty}^{+\infty} w^i(x) u^i(x) dx, \qquad \boldsymbol{g}(\boldsymbol{u}) = -\sum_{i=k}^{N} \int_{-\infty}^{+\infty} (w^i)'(x) u^i(x) dx,$$

$$\bar{\boldsymbol{f}}(\boldsymbol{u}) = \sum_{i=k}^{N} \int_{-\infty}^{+\infty} |w^i(x)| u^i(x) dx, \qquad \bar{\boldsymbol{g}}(\boldsymbol{u}) = -\sum_{i=k}^{N} \int_{-\infty}^{+\infty} |w^i(x)|' u^i(x) dx.$$

This time, the function $\xi(\cdot)$ is taken as $\xi(t) = \sum_{i=k}^{N} \int_{-\infty}^{+\infty} |w^i(x)| \Phi^i(t,x) dx$. We define $\boldsymbol{h}$ by $\boldsymbol{h}(\boldsymbol{u}) := \sum_{i=k}^{N} \int_{-\infty}^{+\infty} u^i(x) dx$. Furthermore, we define

$$C_1 := \min_{k \le i \le N} \operatorname*{ess.inf}_{x \in \mathbf{R}} (w^i)'(x), \qquad C_2 := \max_{k \le i \le N} \operatorname*{ess.sup}_{x \in \mathbf{R}} (w^i)'(x).$$

Then (AdLF) and (AdF) hold except for the case where $\boldsymbol{h}(\boldsymbol{u}) > 0$ for any nonzero $\boldsymbol{u} \in X_+$. It is not possible to apply Theorem 3.3 to the present case, because $\boldsymbol{h}$ fails to satisfy $\boldsymbol{h}(\boldsymbol{u}) > 0$ for any nonzero $\boldsymbol{u} \in X_+$. Therefore we prove the theorem in the following way: We first note that for $\boldsymbol{u} = (u^1, \dots, u^N) \in X_+$ the functional $\boldsymbol{h}$ satisfies $\boldsymbol{h}(\boldsymbol{u}) > 0$ for $(u^k, \dots, u^N) \not\equiv 0$, $\boldsymbol{h}(\boldsymbol{u}) = 0$ for $(u^k, \dots, u^N) \equiv 0$. In the proof of Theorem 3.3, we replace the condition $\hat{u}(s) \neq 0$ in Step 1 by $(\hat{u}^k(s), \dots, \hat{u}^N(s)) \neq 0$; $u_0 \neq 0$ in Step 2 by $(u_0^k, \dots, u_0^N) \neq 0$; $\hat{u}(s) = 0$ in Step 3 by $(\hat{u}^k(s), \dots, \hat{u}^N(s)) = 0$; $u_0 = 0$ in Step 4 by $(u_0^k, \dots, u_0^N) = 0$, respectively. Then we can employ the same arguments as in the proof of Theorem 3.3 and complete the proof of Theorem 5.5. □

PROOF OF THEOREM 5.7. We here use Theorem 3.4. In a way similar to the proof of Theorem 5.4, we may check the validity of (BS), (GR), (LF) and (G) by Proposition 6.3, Lemmas 6.4 and 6.5. In addition, using $(\mathrm{W})_N$, (P1), (P3), (P4) and Lemma 6.6, we see that F is continuous and satisfies (F3) and (F4). (F5) introduced in § 3 follows from (P6). Finally, (F1) is shown in the same way as in the proof of Theorem 5.4. The proof is complete. □

PROOF OF THEOREM 5.10. Employ Theorem 3.7. We can check the assumptions in Theorem 3.7, similarly to the proofs of Theorems 5.4, 5.5 and 5.7. □

7. Proofs of the technical lemmas

In this final section we give the proofs of the technical lemmas, Lemmas 3.5, 4.3 and 4.4, which have been deferred. For convenience we give the statements of the lemmas again.

LEMMA 3.5. *Assume that* (BS), (GR) *and* (F4) *hold. Let* $z \in C([0, T])$ *and* $u_0 \in D$. *If* v_z *is a solution in* $C([0, T]; D)$ *to the Cauchy problem for* (ODE; z) *on* $[0, T]$ *with* $v_z(0) = S(-z(0))u_0$, *then*

$$\|v_z(t)\| \le e^{\omega|z|_\infty + \overline{M}T}\left(\|u_0\| + \int_0^T \|\mathscr{F}(\tau)\|d\tau\right), \qquad t \in [0, T].$$

PROOF. Since $v_z(t) \in D \subset X_+$ and $S(\sigma)X_+ \subset X_+$, we see from (F4) that

$$0 \leqslant v_z(t) \leqslant S(-z(0))u_0 + \int_0^t S(-z(\tau))\mathscr{F}(\tau)d\tau + \overline{M}\int_0^t v_z(\tau)d\tau \qquad \text{in } X,$$

$$t \in [0, T].$$

Thus, (BS) and (GR) together imply that

$$\|v_z(t)\| \le e^{\omega|z|_\infty}\|u_0\| + e^{\omega|z|_\infty}\int_0^T \|\mathscr{F}(\tau)\|d\tau + \overline{M}\int_0^t \|v_z(\tau)\|d\tau, \qquad t \in [0, T].$$

The application of Gronwall's inequality implies the desired result. $\square$

LEMMA 4.3. *Assume that* (BS), (GR) *and* (F4) *are valid. Let* $0 \le s < \varsigma \le T$ *and* δ, $r > 0$. *Then*
(i) *for* $z \in W^{1,\infty}(s, \varsigma)$ *with* $|z'|_\infty \le \delta$, *we have*

$$\|u_z(t)\| \le e^{(\overline{M}+\omega\delta)(t-s)}\left(\|u_z(s)\| + \int_s^t \|\mathscr{F}(\tau)\|d\tau\right), \qquad t \in [s, \varsigma],$$

(ii) *for* $z \in C([s, \varsigma)) \cap L^\infty(s, \varsigma)$ *with* $|z|_\infty \le r$, *we have*

$$\|u_z(t)\| \le e^{2\omega r}e^{\overline{M}(t-s)}\left(\|u_z(s)\| + \int_s^\varsigma \|\mathscr{F}(\tau)\|d\tau\right), \qquad t \in [s, \varsigma),$$

where u_z *is a mild solution of* (SE; z).

PROOF. (i) Since $u_z(t) \in D \subset X_+$ and $S(\sigma)X_+ \subset X_+$, we see that

$$0 \leqslant u_z(t) \leqslant S(z(t) - z(s))u_z(s) + \int_s^t S(z(t) - z(\tau))[\mathscr{F}(\tau) + \overline{M}u_z(\tau)]d\tau$$

$$\text{for } t \in [s, \varsigma]$$

by (F4). Hence it follows from (BS) and (GR) that

$$\|u_z(t)\| \le e^{\omega|z(t)-z(s)|}\|u_z(s)\| + \int_s^t e^{\omega|z(t)-z(\tau)|}[\|\mathscr{F}(\tau)\| + \overline{M}\|u_z(\tau)\|]d\tau, \qquad t \in [s, \varsigma].$$

Noting that $|z(t) - z(\tau)| \le \int_\tau^t |z'(\xi)|d\xi \le \delta(t - \tau)$ for $s \le \tau \le t \le \varsigma$, we have

$$\|u_z(t)\| \le e^{\omega\delta(t-s)}\|u_z(s)\| + \int_s^t e^{\omega\delta(t-\tau)}[\|\mathscr{F}(\tau)\| + \overline{M}\|u_z(\tau)\|]d\tau,$$

and hence

$$e^{-\omega\delta t}\|u_z(t)\| \le e^{-\omega\delta s}\|u_z(s)\| + \int_s^t e^{-\omega\delta\tau}\|\mathscr{F}(\tau)\|d\tau + \overline{M}\int_s^t e^{-\omega\delta\tau}\|u_z(\tau)\|d\tau, \quad t \in [s,\varsigma].$$

By Gronwall's inequality, we get

$$e^{-\omega\delta t}\|u_z(t)\| \le e^{\overline{M}(t-s)}e^{-\omega\delta s}\left(\|u_z(s)\| + \int_s^t \|\mathscr{F}(\tau)\|d\tau\right), \qquad t \in [s,\varsigma],$$

and (i) holds. We next set $v_z(t) = S(-z(t))u_z(t)$. Then v_z is a solution to (ODE; z) on $[s,\varsigma)$. In a way similar to the proof of Lemma 3.5, we obtain

$$\|v_z(t)\| \le e^{\omega r + \overline{M}(t-s)}\left(\|u_z(s)\| + \int_s^{\varsigma} \|\mathscr{F}(\tau)\|d\tau\right), \qquad t \in [s,\varsigma).$$

Since $\|u_z(t)\| = \|S(z(t))v_z(t)\| \le e^{\omega r}\|v_z(t)\|$, we have (ii). $\square$

LEMMA 4.4. *We assume* (BS), (GR), (F4), (AdLF) *and* (AdF). *Let* $0 \le s < \varsigma \le T$ *and* u_z *a mild solution to* (SE; z) *on* $[s,\varsigma]$. *Then for* $z \in C([s,\varsigma])$ *we have*:
(i) $e^{-\overline{M}(t-s)}\boldsymbol{h}(u_z(s)) \le \boldsymbol{h}(u_z(t)) \le e^{\overline{M}(t-s)}(\boldsymbol{h}(u_z(s)) + \int_s^t \boldsymbol{h}(\mathscr{F}(\tau))d\tau)$ *for* $t \in [s,\varsigma]$.
(ii) $\boldsymbol{g}(u_z(t)) \le -C_1 e^{-\overline{M}(t-s)}\boldsymbol{h}(u_z(s))$ *for* $t \in [s,\varsigma]$.
(iii) *If* $z \in W^{1,\infty}(s,\varsigma)$ *and* $|z'|_\infty \le \delta$, *then*

$$\bar{\boldsymbol{f}}(u_z(t)) \le e^{\overline{M}(t-s)}\left[\bar{\boldsymbol{f}}(u_z(s)) + \int_s^t \bar{\boldsymbol{f}}(\mathscr{F}(\tau))d\tau + C_2\delta(t-s)e^{\overline{M}(t-s)}\left(\boldsymbol{h}(u_z(s)) + \int_s^t \boldsymbol{h}(\mathscr{F}(\tau))d\tau\right)\right] \qquad \textit{for } t \in [s,\varsigma].$$

PROOF. We first show that

$$(7.1) \qquad u_z(t) \geqslant e^{-\overline{M}(t-s)}S(z(t)-z(s))u_z(s) \text{ in } X, \qquad t \in [s,\varsigma].$$

Set $v_z(t) = S(-z(t))u_z(t)$ for $t \in [s,\varsigma]$. Then $v_z \in C([s,\varsigma];D) \cap C^1([s,\varsigma];X)$ is a classical solution of (ODE; z) on $[s,\varsigma]$ by Proposition 2.4. Hence,

$$\overline{M}v_z(t) + v_z'(t) = S(-z(t))[\overline{M}u_z(t) + F(t,u_z(t),z(t))] \geqslant 0, \qquad t \in [s,\varsigma]$$

by (GR) and (AdF). From this we see that $(e^{\overline{M}t}v_z(t))' \geqslant 0$ for $t \in [s,\varsigma]$. Integrating over $[s,\varsigma]$ and using the fact that X_+ is norm-closed, we obtain $v_z(t) \geqslant e^{-\overline{M}(t-s)}v_z(s)$ for $t \in [s,\varsigma]$. Consequently, it follows that

$$u_z(t) = S(z(t))v_z(t) \geqslant e^{-\overline{M}(t-s)}S(z(t))v_z(s) = e^{-\overline{M}(t-s)}S(z(t)-z(s))u_z(s), \quad t \in [s,\varsigma],$$

as desired. We next prove (i). By (7.1) and (AdLF), we have

$$e^{-\overline{M}(t-s)}\boldsymbol{h}(u_z(s)) = e^{-\overline{M}(t-s)}\boldsymbol{h}S(z(t)-z(s))u_z(s) \le \boldsymbol{h}(u_z(t)), \qquad t \in [s,\varsigma].$$

This implies the first inequality in (i). Since u_z is a mild solution of (SE; z) on $[s, \varsigma]$, the application of (F4), (GR) and (AdLF) implies

$$0 \leq \boldsymbol{h}(u_z(t)) \leq \boldsymbol{h}(u_z(s)) + \int_s^t \boldsymbol{h}(\mathscr{F}(\tau))d\tau + \overline{M} \int_s^t \boldsymbol{h}(u_z(\tau))d\tau, \qquad t \in [s, \varsigma].$$

Therefore, the second inequality in (i) follows from Gronwall's Lemma.

By (7.1) and (AdLF), we obtain (ii). To show (iii), we first note that

$$\bar{\boldsymbol{f}}(S(\sigma)v) = \bar{\boldsymbol{f}}(v) - \int_0^\sigma \bar{\boldsymbol{g}}(S(\tau)v)d\tau \tag{7.2}$$

holds for $\sigma \in \mathbf{R}$ and $v \in X$. This relation is obtained in a way similar to the proof of Lemma 2.8 (i), because $\bar{\boldsymbol{g}}$ is the extension of $\bar{\boldsymbol{f}}\Lambda$. It follows from (GR), (F4), (AdLF) and (7.2) that

$$\begin{aligned} 0 \leq \bar{\boldsymbol{f}}(u_z(t)) \leq \bar{\boldsymbol{f}}(u_z(s)) &+ \int_0^{|z(t)-z(s)|} |\bar{\boldsymbol{g}}(S(\sigma)u_z(s))|d\sigma \\ &+ \int_s^t \Bigg[\bar{\boldsymbol{f}}(\mathscr{F}(\tau) + \overline{M}u_z(\tau)) \\ &+ \int_0^{|z(t)-z(\tau)|} |\bar{\boldsymbol{g}}(S(\sigma)(\mathscr{F}(\tau) + \overline{M}u_z(\tau)))|d\sigma\Bigg]d\tau, \qquad t \in [s, \varsigma]. \end{aligned}$$

Using (AdLF), we know that

$$\begin{aligned} \int_0^{|z(t)-z(s)|} |\bar{\boldsymbol{g}}(S(\sigma)u_z(s))|d\sigma &\leq -\int_0^{|z(t)-z(s)|} \boldsymbol{g}(S(\sigma)u_z(s))d\sigma \\ &\leq C_2 \int_0^{|z(t)-z(s)|} \boldsymbol{h}(S(\sigma)u_z(s))d\sigma \\ &= C_2|z(t)-z(s)|\boldsymbol{h}(u_z(s)) \leq C_2\delta(t-s)\boldsymbol{h}(u_z(s)), \quad t \in [s, \varsigma]. \end{aligned}$$

By the same reason, we also get that

$$\int_s^t d\tau \int_0^{|z(t)-z(\tau)|} |\bar{\boldsymbol{g}}(S(\sigma)(\mathscr{F}(\tau)+\overline{M}u_z(\tau)))|d\sigma \leq C_2 \int_s^t \delta(t-\tau)\boldsymbol{h}(\mathscr{F}(\tau)+\overline{M}u_z(\tau))d\tau, \qquad t \in [s, \varsigma].$$

Thus, it follows that

$$\begin{aligned} 0 \leq \bar{\boldsymbol{f}}(u_z(t)) \leq \bar{\boldsymbol{f}}(u_z(s)) + C_2\delta(t-s)\left[\boldsymbol{h}(u_z(s)) + \int_s^t \boldsymbol{h}(\mathscr{F}(\tau) + \overline{M}u_z(\tau))d\tau\right] \\ + \int_s^t \bar{\boldsymbol{f}}(\mathscr{F}(\tau))d\tau + \overline{M}\int_s^t \bar{\boldsymbol{f}}(u_z(\tau))d\tau, \qquad t \in [s, \varsigma]. \end{aligned}$$

By Gronwall's Lemma and (i), we obtain the result. □

References

[1] G. Allain and P. Colli, A mathematical study of a muscle contraction model in which the fibre is a continuum of elements, Adv. Applied Math. **9** (1988), 104–126.

[2] Ph. Clément, H. J. A. M. Heijmans, S. Angenent, C. J. van Duijn and B. de Pagter, One-Parameter Semigroups, CWI Monograph 5, North-Holland, Amsterdam, 1987.

[3] P. Colli, A mathematical model of heterogeneous behavior of single muscle fibres, J. Math. Biol. **24** (1986), 103–118.

[4] P. Colli, On a nonlinear and nonlocal evolution equation related to muscle contraction, Nonlinear Anal. TMA **13** (1989), 1149–1162.

[5] P. Colli, V. Comincioli, G. Naldi and A. Torelli, A mathematical study of the plasticity effects in muscle contraction, Appl. Math. Optim. **22** (1990), 1–26.

[6] P. Colli and M. Grasselli, Mathematical study of a nonlinear transport-diffusion problem related to muscle contraction, Differential Integral Equations **3** (1990), 837–849.

[7] P. Colli and M. Grasselli, Parabolic perturbation of a nonlinear hyperbolic problem arising in physiology, J. Differential Equations **101** (1993), 178–212.

[8] V. Comincioli and A. Torelli, Mathematical aspects of the cross-bridge mechanism in muscle contraction, Nolinear Anal. TMA **7** (1983), 661–683.

[9] V. Comincioli and A. Torelli, A mathematical model of contracting muscle with viscoelastic elements, SIAM J. Math. Anal. **19** (1988), 593–612.

[10] V. Comincioli and A. Torelli, A mathematical study of a continuum-state cross-bridge model of muscle contraction, Math. Methods Appl. Sci. **10** (1988), 289–302.

[11] V. Comincioli, A. Torelli, C. Poggesi and C. Reggiani, A four-state cross bridge model for muscle contraction. Mathematical study and validation, J. Math. Biol. **20** (1984), 277–304.

[12] D. Daners and P. Koch Medina, Abstract Evolution Equations, Periodic Problems and Applications, Research Notes in Math. 279, Longman, New York, 1992.

[13] L. Gastaldi and F. Tomarelli, A nonlinear and nonlocal evolution equation describing the muscle contraction, Nonlinear Anal. TMA **11** (1987), 163–182.

[14] S. Ishii, Linear evolution equations $du/dt + A(t)u = 0$: a case where $A(t)$ is strongly uniform-measurable, J. Math. Soc. Japan **34** (1982), 413–424.

[15] N. Kato and T. Yamaguchi, Nonlinear nonlocal differential equations related to muscle contraction mechanism, Nonlinear Anal. TMA **30** (1997), 3909–3915.

[16] N. Kato and T. Yamaguchi, Nonlinear nonlocal transport-diffusion equations arising in physiology, Nihonkai Math. J. **10** (1999), 1–30.

[17] N. Kato and T. Yamaguchi, Nonlinear nonlocal hyperbolic equations arising in physiology, Adv. Math. Sci. Appl. **9** (1999), 993–1006.

[18] C. A. Lazalde and L. Barr, Four-state models and regulation of contraction of smooth muscle. I. Physical considerations, stability, and solutions, II. Properties of the solutions and identification, Math. Biosci. **112** (1992), 1–30, 31–54.

[19] R. H. Martin Jr., Nonlinear Operators and Differential Equations in Banach Spaces, Pure & Applied Math., John Wiley & Sons, New York, 1976.

[20] T. Matsumoto, Time-dependent nonlinear perturbations of analytic semigroups in Banach spaces, Adv. Math. Sci. Appl. **7** (1997), 119–163.

[21] T. Matsumoto, S. Oharu and T. Yamaguchi, A nonlinear nonlocal transport system related to the cross-bridge mechanism of muscle contraction, Nonlinear Anal. TMA (to appear).

[22] I. Miyadera, Nonlinear Semigroups, Transl. Math. Monogr. 109, AMS, Providence, Rhode Island, 1992.

[23] N. H. Pavel, Semilinear equations with dissipative time-dependent domain perturbations, Israel J. Math. **46** (1983), 103–122.

[24] N. H. Pavel, Differential Equations, Flow Invariance and Applications, Research Notes in Math. 113, Pitman, Boston, 1984.

[25] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Appl. Math. Sci. 44, Springer-Verlag, New York, 1983.

[26] A. Torelli, A non linear hyperbolic equation related to the dynamics of cardiac muscle, Portugal. Math. **41** (1982), 171–188.

[27] A. Torelli, Study of a mathematical model for muscle contraction with deformable elements, Rend. Sem. Mat. Univ. Pol. Torino **55** (1997), 241–271.

Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima 739-8526, Japan