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On the Gysin isomorphism of rigid cohomology

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Abstract. We prove a comparison theorem of logarithmic Monsky-Washnitzer cohomology and rigid cohomology with overconvergent coefficients. Using this comparison theorem, we construct the Gysin isomorphism in rigid cohomology with overconvergent coefficients on small pairs of affine smooth varieties of positive characteristic. The Gysin isomorphism under the assumption "small" is sufficient to apply it to the finiteness problem of rigid cohomology with coefficients. We prove the finiteness theorem, Poincaré duality and Künneth formula of rigid cohomology for unit-root overconvergent F-isocrystals by our previous result of finite local monodromy theorem for them.

1. Introduction

The rigid cohomology with coefficient of overconvergent isocrystals, which was introduced by P. Berthelot, is a good candidate of the p-adic cohomology theory of varieties of positive characteristic p. If the rigid cohomology is a good cohomology, then it must have several expected properties, the finiteness, Poincaré duality, Künneth formula and so on. In [6] and [7] Berthelot proved the finiteness, Poincaré duality and Künneth formula of the rigid cohomology with the constant coefficient. In his proof the Gysin isomorphism played an important role.

In this article we construct the Gysin isomorphism of the rigid cohomology of overconvergent isocrystals on sufficiently small affine smooth varieties. For overconvergent F-isocrystals, this Gysin isomorphism commutes with Frobenius structures. We apply it to the finiteness, Poincaré duality and Künneth formula of the rigid cohomology of overconvergent unit-root F-isocrystals.

Let us explain the method of the construction of the Gysin isomorphism. First we introduce a logarithmic Monsky-Washnitzer cohomology and prove the comparison theorem with overconvergent coefficients between the logarithmic Monsky-Washnitzer cohomology and the rigid cohomology for an affine smooth variety with normal crossing divisor over a spectrum of field of positive characteristic. This comparison theorem is a p-adic analogue of A. Grothendieck and P. Deligne's comparison theorem of the logarithmic

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de Rham cohomology of complex analytic varieties. (See [11] and [12].) Applying the comparison theorem, we construct the Gysin isomorphism as in [12]. For the constant coefficient, our Gysin isomorphism coincides with the one in [6] and the commutativity of the Gysin isomorphism and Frobenius structures was proved in [8]. In the case of varieties over a finite field, the Gysin isomorphism was studied in [14] using p-adic functional analysis.

The key assertion of the comparison theorem is Lemma 3.7.5. The idea is essentially similar to that of P. Monsky, who studied the Gysin isomorphism of Monsky-Washnitzer cohomology for the pair of an affine smooth variety and its nonsingular hypersurface in [16].

We explain the contents of this paper. In §2 we review the theory of rigid cohomology. In §3 we define a logarithmic Monsky-Washnitzer cohomology with coefficients and prove the comparison theorem with overconvergent coefficients. In §4 we construct the Gysin morphism of rigid cohomology over a sufficiently small affine smooth variety. In §5 we give a comparison theorem between the crystalline cohomology and the rigid cohomology with coefficients. This comparison theorem is used in §6. In §6 we prove the finiteness theorem, Poincaré duality and Künneth formula of rigid cohomology for unit-root overconvergent F-isocrystals on a variety over a perfect field of characteristic p.

Throughout this paper, we fix the notation as follows;

p: a prime number;
k: a field of characteristic p;
V: a complete discrete valuation ring of mixed characteristics with residue field k;
$\boldsymbol{m}$: the maximal ideal of V;
K: the field of fraction of V;
$|\ |$: an absolute value of K;
σ: the Frobenius map on k.

We also denote by σ a lift of Frobenius endomorphism on V (resp. K) if it exists. If we mention F-isocrystals or Frobenius structures, we suppose the existence of a lift of Frobenius on K and we fix a Frobenius σ on K.

For a V-module M, we put $M_K = M \otimes_V K$.

Let (a_{ij}) be a matrix with entries in R. For a function f (resp. a norm $|\ |$) on R, we put $f((a_{ij})) = (f(a_{ij}))$ (resp. $|(a_{ij})| = \max\{|a_{ij}|\}$).

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2. Several properties of rigid cohomology

In this section we review several properties of rigid cohomology which are needed later. (See [4], [5], [6] and [9].) Throughout this section, we denote by X, $\overline{X}$, and $\hat{\mathscr{P}}$ a separated scheme of finite type over $\mathrm{Spec}\,k$, a proper scheme of finite type over $\mathrm{Spec}\,k$ with a k-open immersion $j : X \to \overline{X}$, and a formal scheme of finite type over $\mathrm{Spf}\,V$ with a closed immersion $\overline{X} \to \hat{\mathscr{P}}$ such that $\hat{\mathscr{P}}$ is smooth over $\mathrm{Spf}\,V$ around X, respectively. We denote by $\mathrm{Isoc}^{\dagger}(X/K)$ the category of overconvergent isocrystals on X/K and, for a positive integer a, by $F\text{-}\mathrm{Isoc}^{\dagger}(X/K, \sigma^a)$ the category of overconvergent F-isocrystals on X/K with respect to the Frobenius σ^a on K.

(2.1) For an object $(\mathscr{M}, \nabla)$ in $\mathrm{Isoc}^{\dagger}(X/K)$, we denote by $DR^{\bullet}(\mathscr{M})$ the de Rham complex

$$\cdots \to 0 \to \mathscr{M} \xrightarrow{\nabla} \mathscr{M} \otimes_{j^{\dagger}\mathcal{O}_{]\overline{X}[_{\hat{\mathscr{P}}}}} \Omega^1_{]\overline{X}[_{\hat{\mathscr{P}}}/K} \xrightarrow{\nabla} \mathscr{M} \otimes_{j^{\dagger}\mathcal{O}_{]\overline{X}[_{\hat{\mathscr{P}}}}} \Omega^2_{]\overline{X}[_{\hat{\mathscr{P}}}/K} \xrightarrow{\nabla} \cdots$$

of K-sheaves on $]\overline{X}[_{\hat{\mathscr{P}}}$ associated to $\mathscr{M}$, where we put $\mathscr{M}$ at the degree 0.

Let Z be a closed subscheme of X over $\mathrm{Spec}\,k$, and put $U = X - Z$ with the open immersion $j_U : U \to \overline{X}$. For a sheaf $\mathscr{E}$ of abelian groups on $]\overline{X}[_{\hat{\mathscr{P}}}$, we put

$$\underline{\Gamma}^{\dagger}_{]Z[_{\hat{\mathscr{P}}}}(\mathscr{E}) = \ker(\mathscr{E} \to j_U^{\dagger}\mathscr{E})$$

$$\Gamma_Z(\mathscr{E}) = \Gamma(]\overline{X}[_{\hat{\mathscr{P}}}, \underline{\Gamma}^{\dagger}_{]Z[_{\hat{\mathscr{P}}}}(\mathscr{E}))$$

to be the sheaf of overconvergent sections of $\mathscr{E}$ with supports in $]Z[_{\hat{\mathscr{P}}}$ and the group of global sections of $\mathscr{E}$ with supports in $]Z[_{\hat{\mathscr{P}}}$, respectively. For an object $(\mathscr{M}, \nabla)$ in $\mathrm{Isoc}^{\dagger}(X/K)$, the complex $\mathbf{R}\Gamma_Z(DR^{\bullet}(\mathscr{M}))$ is independent of the choices of $\overline{X}$ and $\hat{\mathscr{P}}$ in the derived category of complexes of K-vector spaces bounded below. We put

$$\mathbf{R}\Gamma_{Z,rig}(X/K, \mathscr{M}) = \mathbf{R}\Gamma_Z(DR^{\bullet}(\mathscr{M}))$$

and the rigid cohomology $H^l_{Z,rig}(X/K, \mathscr{M}) = \mathbf{R}^l\Gamma_Z(DR^{\bullet}(\mathscr{M}))$ with supports in Z. When $Z = X$, we simply denote $\mathbf{R}\Gamma_{rig}(X/K, \mathscr{M}) = \mathbf{R}\Gamma_X(DR^{\bullet}(\mathscr{M}))$ and $H^l_{rig}(X/K, \mathscr{M}) = H^l_{X,rig}(X/K, \mathscr{M})$. We define a distinguished triangle $\Delta_{rig}(X, Z, \mathscr{M})$ by

$$\mathbf{R}\Gamma_{Z,rig}(X/K, \mathscr{M}) \longrightarrow \mathbf{R}\Gamma_{rig}(X/K, \mathscr{M}) \longrightarrow \mathbf{R}\Gamma_{rig}(U/K, j_U^{\dagger}\mathscr{M}) \xrightarrow{+1}.$$

By the similar proof of [3, Proposition 2.4, 2.5] we have

Proposition 2.1.1. *With the notation as above, let* $(\mathscr{M}, \nabla)$ *be an object in* $\mathrm{Isoc}^{\dagger}(X/K)$.

(1) *If U is an open subscheme of X over* $\mathrm{Spec}\,k$ *such that $Z \subset U$, then there is a canonical isomorphism*

$$\mathbf{R}\Gamma_{Z,rig}(X/K, \mathscr{M}) \to \mathbf{R}\Gamma_{Z,rig}(U/K, j_U^{\dagger}\mathscr{M}).$$

(2) *If Z is a disjoint union of closed subschemes Z_1 and Z_2 of X over* $\operatorname{Spec} k$*, then there is a canonical isomorphism*

$$\mathbf{R}\Gamma_{Z_1,rig}(X/K,\mathscr{M}) \oplus \mathbf{R}\Gamma_{Z_2,rig}(X/K,\mathscr{M}) \to \mathbf{R}\Gamma_{Z,rig}(X/K,\mathscr{M}).$$

(3) *If T is a closed subscheme of Z over* $\operatorname{Spec} k$ *and if we put $Y = X - T$ and $Z_Y = Z - T$, then there exists a distinguished triangle*

$$\mathbf{R}\Gamma_{T,rig}(X/K,\mathscr{M}) \longrightarrow \mathbf{R}\Gamma_{Z,rig}(X/K,\mathscr{M}) \longrightarrow \mathbf{R}\Gamma_{Z_Y,rig}(Y/K, j_Y^\dagger\mathscr{M}) \xrightarrow{+1}$$

Here we denote by $j_Y : Y \to \overline{X}$ the open immersion.

Moreover, the induced K-homomorphisms on the rigid cohomology in (1), (2) *and* (3) *commute with Frobenius structures for an object in* $F\text{-Isoc}^\dagger(X/K,\sigma^a)$.

Let K' be an extension of K which is complete under the extension of valuation of K and denote by k' the residue field of K'. We put $X' = X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ (resp. $Z' = Z \times_{\operatorname{Spec} k} \operatorname{Spec} k'$, resp. $\overline{X}' = \overline{X} \times_{\operatorname{Spec} k} \operatorname{Spec} k'$, resp. $\hat{\mathscr{P}}' = \hat{\mathscr{P}} \times_{\operatorname{Spf} V} \operatorname{Spf} V'$) and denote by $j' : X' \to \overline{X}'$ (resp. $\tau_{K'/K} :]\overline{X}'[_{\hat{\mathscr{P}}'} \to]\overline{X}[_{\hat{\mathscr{P}}}$) the open immersion (resp. the natural morphism). Then $\tau_{K'/K}$ induces the inverse image functor

$$\tau_{K'/K}^* : \operatorname{Isoc}^\dagger(X/K) \to \operatorname{Isoc}^\dagger(X'/K').$$

If $\sigma'; K' \to K'$ is an extension of the Frobenius σ on K, then $\tau_{K'/K}$ induces the inverse image functor

$$\tau_{K'/K}^* : F\text{-Isoc}^\dagger(X/K,\sigma^a) \to F\text{-Isoc}^\dagger(X/K',(\sigma')^a)$$

for a positive integer a.

For an object $(\mathscr{M},\nabla)$ in $\operatorname{Isoc}^\dagger(X/K)$, if we put $(\mathscr{M}',\nabla') = \tau_{K'/K}^*(\mathscr{M},\nabla)$, then the natural homomorphism $\tau^{-1}\underline{\Gamma}_{]Z[}^\dagger(\mathscr{M}) \to \underline{\Gamma}_{]Z'[}^\dagger(\mathscr{M}')$ induces a canonical morphism

$$\tau_{K'/K}^* : \mathbf{R}\Gamma_{Z,rig}(X/K,\mathscr{M}) \otimes_K K' \to \mathbf{R}\Gamma_{Z',rig}(X'/K',\mathscr{M}')$$

in the derived category of complexes of K'-vector spaces. As a generalization of [6, Proposition 1.8] with coefficients we have

PROPOSITION 2.1.2. *With the notation as above, if K' is a finite extension of K, then the morphism*

$$\tau_{K'/K}^* : \mathbf{R}\Gamma_{Z,rig}(X/K,\mathscr{M}) \otimes_K K' \to \mathbf{R}\Gamma_{Z',rig}(X'/K',\mathscr{M}')$$

is an isomorphism. Moreover, if the Frobenius σ extends to the Frobenius on K', then the induced K'-homomorphism $\tau_{K'/K}^*$ on the rigid cohomology commutes with Frobenius structures for any object in $F\text{-Isoc}^\dagger(X/K,\sigma^a)$.

PROOF. By Proposition 2.1.1, one may assume that $Z = X$. Considering the Čech cohomology, the assertion follows easily from the fact that $\Gamma(\tau^{-1}(W), (j')^\dagger O_{]\bar{X}'[}) = \Gamma(W, j^\dagger O_{]\bar{X}[}) \otimes_K K'$ and that $H^l(W, \mathscr{M}) = H^l(\tau^{-1}(W), \mathscr{M}') = 0$ for $l \neq 0$ for any sufficiently small open affinoid W in $]\bar{X}[$. □

(2.2) We explain the relation between the rigid cohomology and the Monsky-Washnitzer cohomology. (See [5, 2.5].) We assume that there exists an affine smooth scheme $\mathscr{X} = \operatorname{Spec} A$ of finite type over $\operatorname{Spec} V$ with $X = \mathscr{X} \times_{\operatorname{Spec} V} \operatorname{Spec} k$. We fix a presentation

$$V[x_1, \dots, x_N]/I \cong A$$

over V. Put $\bar{\mathscr{X}}$ to be the Zariski closure of $\mathscr{X}$ in $\mathbf{P}^N$ ($\operatorname{Spec} V[\underline{x}]$ is the open subscheme defined by $x_0 \neq 0$), $\hat{\bar{\mathscr{X}}}$ to be the p-adic completion of $\bar{\mathscr{X}}$ and $\bar{X}$ to be the closure of X in $\hat{\bar{\mathscr{X}}}$. For $\lambda > 1$, we put a V-algebra

$$A_\lambda = V[\underline{x}]_\lambda / IV[\underline{x}]_\lambda,$$

where

$$V[\underline{x}]_\lambda = \left\{ \sum_{\underline{i} \geqq 0} a_{\underline{i}} \underline{x}^{\underline{i}} \in V[[x_1, \dots, x_N]] \,\middle|\, \begin{array}{c} a_{\underline{i}} \in V \\ |a_{\underline{i}}| \lambda^{|\underline{i}|} \to 0 (|\underline{i}| \to \infty) \end{array} \right\},$$

$\underline{i}$ is a multi index and $|\underline{i}| = i_1 + \cdots + i_N$. We define a Banach norm $\| \ \|_\lambda$ on $V[\underline{x}]_\lambda$ by

$$\left\| \sum a_{\underline{i}} \underline{x}^{\underline{i}} \right\|_\lambda = \sup\{ |a_{\underline{i}}| \lambda^{|\underline{i}|} \}$$

and define a Banach norm $\| \ \|_{\mathscr{X}_\lambda}$ on A_λ by the quotient norm of $\| \ \|_\lambda$ on $V[\underline{x}]_\lambda$. We define a V-algebra $A^\dagger$ and its norm $\| \ \|_{\mathscr{X}^\dagger}$ by

$$A^\dagger = \lim_{\lambda \to 1^+} A_\lambda$$

$$\| \ \|_{\mathscr{X}^\dagger} = \lim_{\lambda \to 1^+} \| \ \|_{\mathscr{X}_\lambda}.$$

$A^\dagger$ is the weak completion of A over V, independent of the choices of the presentation up to canonical isomorphism, and noetherian [17, Theorem 1.5, 2.1].

An algebra homomorphism $\varphi : A^\dagger \to A^\dagger$ is called Frobenius if and only if it is σ-linear and the induced map on $\Gamma(X, O_X) = A^\dagger / \boldsymbol{m} A^\dagger$ is the p-th power map.

Let $dt_1, \dots, dt_n$ be a local basis of the sheaf $\Omega^1_{\mathscr{X}/\operatorname{Spec} V}$ of the differential module of $\mathscr{X}$ over $\operatorname{Spec} V$ and let $\partial_1, \dots, \partial_n$ be a dual basis of $dt_1, \dots, dt_n$ in the sheaf $Der(\mathscr{X}/\mathrm{Spec}\, V)$ of derivation. Then ∂_i can extend on $A^\dagger$ and we use the same symbol ∂_i for this extension.

Let $\nabla : M \to M \otimes_A \Omega^1_{A/V}$ be a connection of a finitely generated $A_K^\dagger$-module. Since M is finitely generated, there is a finitely generated $A_{\lambda,K}$-module M_λ with a connection $\nabla_\lambda : M_\lambda \to M_\lambda \otimes_A \Omega^1_{A/V}$ for any $\lambda > 1$ sufficiently close to 1 such that $(M_\lambda, \nabla_\lambda) \otimes_{A_{\lambda,K}} A_{\lambda',K} \cong (M_{\lambda'}, \nabla_{\lambda'})$ for $1 < \lambda' < \lambda$ and $\lim_{\lambda \to 1^+} (M_\lambda, \nabla_\lambda) \cong (M, \nabla)$. We say that the connection $\nabla : M \to M \otimes_A \Omega^1_{A/V}$ is overconvergent if it is integrable and, for any $\eta < 1$, there exists $\lambda > 1$ such that

$$\left| \frac{1}{\underline{i}!} \nabla_\lambda(\underline{\partial}^{\underline{i}})(m) \right|_\lambda \eta^{|\underline{i}|} \to 0 \qquad (|\underline{i}| \to \infty)$$

for any $m \in M_\lambda$. Here, $|\ |_\lambda$ is a quotient norm of M_λ which is determined by the fixed presentation of M_λ over $A_{\lambda,K}$, $\underline{i}! = i_1! \cdots i_n!$ and $\underline{\partial}^{\underline{i}} = \partial_1^{i_1} \cdots \partial_n^{i_n}$. The condition of overconvergence is independent of the choices of the presentation of M over $A_K^\dagger$ and the basis of the derivation $Der(\mathscr{X}/\mathrm{Spec}\, V)$. A morphism of $A_K^\dagger$-modules with overconvergent connection is a horizontal $A_K^\dagger$-homomorphism. We denote by $\mathrm{Conn}^\dagger(\mathscr{X}/K)$ the category of finitely generated $A_K^\dagger$-modules with overconvergent connection. The category of $\mathrm{Conn}^\dagger(\mathscr{X}/K)$ is independent of the choices of the affine smooth lift $\mathscr{X}$ of X and the presentation of A over V up to canonical isomorphisms [5, Proposition 2.5.2]. If (M, ∇) is an object in $\mathrm{Conn}^\dagger(\mathscr{X}/K)$, M is projective over $A_K^\dagger$.

Let φ be a Frobenius on $A^\dagger$ and let a be a positive integer. For an object (M, ∇) in $\mathrm{Conn}^\dagger(\mathscr{X}/K)$, a horizontal isomorphism $\Phi : (\varphi^a)^* M \to M$ is called a Frobenius structure on (M, ∇) with respect to φ^a. A morphism of $A_K^\dagger$-modules with overconvergent connection and Frobenius structure is a horizontal $A_K^\dagger$-homomorphism which commutes with Frobenius structures. We denote by $F\text{-}\mathrm{Conn}^\dagger(\mathscr{X}/K, \varphi^a)$ the category of finitely generated $A_K^\dagger$-modules with overconvergent connection and Frobenius structure. The category of $F\text{-}\mathrm{Conn}^\dagger(\mathscr{X}/K, \varphi^a)$ is independent of the choices of the affine smooth lift $\mathscr{X}$ of X, the presentation of A and the Frobenius φ on $A^\dagger$ up to canonical isomorphisms [5, Théorème 2.5.7].

For an object (M, ∇) in $\mathrm{Conn}^\dagger(\mathscr{X}/K)$, we define a de Rham complex $DR^\bullet(M)$ of K-vector spaces by

$$\cdots \to 0 \to M \xrightarrow{\nabla} M \otimes_A \Omega^1_{A/V} \xrightarrow{\nabla} M \otimes_A \Omega^2_{A/V} \xrightarrow{\nabla} \cdots,$$

where we put M at the degree 0. We denote by $H^l_{MW}(X/K, M)$ the Monsky-Washnitzer cohomology $H^l(DR^\bullet(M))$. For an object (M, ∇, Φ) in $F\text{-}\mathrm{Conn}^\dagger(\mathscr{X}/K, \varphi^a)$, the Frobenius structure Φ on M induces the Frobenius structure on $H^l_{MW}(X/K, M)$ and we also denote this Frobenius structure by Φ.

PROPOSITION 2.2.1. (1) [5, Proposition 2.5.2, Théorème 2.5.7] *The functor* $\Gamma(]\overline{X}[_{\hat{\mathscr{X}}}, ?)$ *gives canonical equivalences*

$$\mathrm{Isoc}^\dagger(X/K) \to \mathrm{Conn}^\dagger(\mathscr{X}/K)$$

$$F\text{-}\mathrm{Isoc}^\dagger(X/K, \sigma^a) \to F\text{-}\mathrm{Conn}^\dagger(\mathscr{X}/K, \varphi^a)$$

of categories.

(2) [6, Proposition 1.10] *For an object* $(\mathscr{M}, \nabla)$ *in* $\mathrm{Isoc}^\dagger(X/K)$, *if we put* $M = \Gamma(]\overline{X}[_{\hat{\mathscr{X}}}, \mathscr{M})$, *then the functor* $\Gamma(]\overline{X}[_{\hat{\mathscr{X}}}, ?)$ *induces the canonical isomorphism*

$$DR^\bullet(M) \to \mathbf{R}\Gamma_{rig}(X/K, \mathscr{M})$$

in the derived category of complexes of K-vector spaces.

For an object in $F\text{-}\mathrm{Isoc}^\dagger(X/K, \sigma^a)$, *the isomorphism* $H^l_{MW}(X/K, M) \to H^l_{rig}(X/K, \mathscr{M})$ *commutes with Frobenius structures.*

(2.3) Keep the notation in 2.2. Let $f : \mathscr{Y} \to \mathscr{X}$ be an etale morphism of affine smooth V-schemes of finite type such that f is surjective on the special fiber, and put $A = \Gamma(\mathscr{X}, O_{\mathscr{X}})$ and $B = \Gamma(\mathscr{Y}, O_{\mathscr{Y}})$. For an object (M, ∇) in $\mathrm{Conn}^\dagger(\mathscr{X}/K)$, we define a double complex $DR^\bullet(\mathscr{Y}^\bullet/K, M)$ of K-vector spaces by the Čech complex

$$DR^\bullet(f^\dagger M) \to DR^\bullet((f^2)^\dagger M) \to DR^\bullet((f^3)^\dagger M) \to \cdots$$

for the hypercovering induced by f, where $(f^\nu)^\dagger M = M \otimes_{A_K^\dagger} (B \otimes_A \cdots \otimes_A B)^\dagger_K$ (ν times) and $f^\dagger M$ is of bidegree $(0,0)$. For an object (M, ∇, Φ) in F-$\mathrm{Conn}^\dagger(\mathscr{X}/K, \varphi^a)$, the Frobenius structure Φ induces the Frobenius structure on the double complex $DR^\bullet(\mathscr{Y}^\bullet/K, M)$.

PROPOSITION 2.3.1. *With the notation as above, if* (M, ∇) *is an object in* $\mathrm{Conn}^\dagger(\mathscr{X}/K)$, *then the natural homomorphism*

$$DR^\bullet(M) \to Tot(DR^\bullet(\mathscr{Y}^\bullet/K, M))$$

of complexes of K-vector spaces is a quasi-isomorphism. Here $Tot(DR^\bullet(\mathscr{Y}^\bullet/K, M))$ *is the total complex of the double complex* $DR^\bullet(\mathscr{Y}^\bullet/K, M)$. *For an object in* $F\text{-}\mathrm{Conn}^\dagger(\mathscr{X}/K, \varphi^a)$, *the induced homomorphism of cohomologies commutes with Frobenius structures.*

Note that $f^\dagger : A^\dagger \to B^\dagger$ is faithfully flat. Indeed, the p-adic completion $\hat{A}$ (resp. $\hat{B}$) of A (resp. B) is faithfully flat over the weak completion $A^\dagger$ (resp. $B^\dagger$) (See the proof of [6, Proposition 3.6].) and $\hat{B}$ is faithfully flat over $\hat{A}$ since $B/\boldsymbol{m}^l B$ is faithfully flat over $A/\boldsymbol{m}^l A$ for any l.

Since M is projective over $A_K^\dagger$, Proposition 2.3.1 easily follows from Lemma 2.3.2 below.

Lemma 2.3.2. *With the notation as above, the Čech complex*

$$0 \to A^\dagger \to B^\dagger \to (B \otimes_A B)^\dagger \to (B \otimes_A B \otimes_A B)^\dagger \to \cdots$$

of $A^\dagger$-modules is exact.

Proof. By [5, Proposition 2.1.8] the assertion is local, hence we may assume that $\mathscr{X}$ is a standard etale extension over affine space on Spec V and that $\mathscr{Y}$ is a finite disjoint sum of standard etale extensions over $\mathscr{X}$, that is, $A = V[\underline{x}][y,z]/(s(y), t(y)z - 1)$ (resp. $\mathscr{Y} = \coprod_i \operatorname{Spec} B_i, B_i = A[u_i, v_i]/(p_i(u_i), q_i(u_i)v_i - 1)$), where $s(y)$ (resp. $p_i(u_i)$) is a monic irreducible polynomial over $V[\underline{x}]$ (resp. A) which is separable over the field of fraction of $V[\underline{x}]$ (resp. A), $t(y)$ (resp. $q_i(u_i)$) is a non-zero polynomial over $V[\underline{x}]$ (resp. A) such that $s'(y)$ (resp. $p_i'(u_i)$) is invertible in A (resp. B_i). Denote by d_i (resp. e_i) the degree of $p_i(u_i)$ (resp. $q_i(u_i)$). Fix a lift $\tilde{p}_i(u_i)$ (resp. $\tilde{q}_i(u_i)$) of polynomial in $V[\underline{x}, y, z, u_i]$ of degree d_i (resp. e_i) on u_i such that $\tilde{p}_i(u_i)$ is monic. Then we have a compatible system of presentations

$$\begin{array}{ccc} V[\underline{x}, y, z] & \longrightarrow & A \\ \downarrow & & \downarrow \\ \prod_i V[\underline{x}, y, z, u_i, v_i] & \longrightarrow & B \end{array}$$

of A and B as V-algebras and also a compatible presentation $(\prod_i V[\underline{x}, y, z, u_i, v_i])^{\otimes r} \to B^{\otimes r}$ for any r, where $(\prod_i V[\underline{x}, y, z, u_i, v_i])^{\otimes r}$ is the tensor product of r copies of $\prod_i V[\underline{x}, y, z, u_i, v_i]$ over $V[\underline{x}, y, z]$. For $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ ($\lambda_j > 1$), define a V-subalgebra I_λ^0 (resp. I_λ^r ($r \in \mathbf{Z}_{\geqq 1}$)) of A_λ (resp. $(B^{\otimes r})_\lambda$) which consists of elements a with $\|a\|_\lambda \leqq 1$. Here the norm $\|\ \|_\lambda$ is defined as in 2.2 using the triple λ for the coordinate $(\underline{x}, (y,z), (\underline{u}, \underline{v}))$, respectively. Let $\mathscr{C}^\bullet$, $\mathscr{C}_\lambda^\bullet$ and $\mathscr{I}_\lambda^\bullet$ be the complex in the assertion,

$$0 \to A_\lambda \to B_\lambda \to (B \otimes_A B)_\lambda \to (B \otimes_A B \otimes_A B)_\lambda \to \cdots,$$

$$0 \to I_\lambda^0 \to I_\lambda^1 \to I_\lambda^2 \to I_\lambda^3 \to \cdots,$$

respectively, which is induced by the Čech complex

$$0 \to V[\underline{x}, y, z] \to \prod_i V[\underline{x}, y, z, u_i, v_i]$$
$$\to \left(\prod_i V[\underline{x}, y, z, u_i, v_i]\right) \otimes_{V[\underline{x}, y, z]} \left(\prod_i V[\underline{x}, y, z, u_i, v_i]\right) \to \cdots.$$

Here we put $V[\underline{x}, y, z]$ at degree 0.

Choose rational numbers λ_j ($j = 1, 2, 3$) which are greater than 1 such that, if we fix elements π_j ($j = 1, 2, 3$) in the algebraic closure of K with $|\pi_j| = \lambda_j^{-1}$ and $V_\lambda = V[\pi_1, \pi_2, \pi_3]$, $\pi_2^{deg(s(y))} s(\pi_1^{-1}\underline{x}, \pi_2^{-1}y)$ (resp. $\pi_3^{d_i}\tilde{p}_i(\pi_1^{-1}\underline{x}, \pi_2^{-1}(y,z), \pi_3^{-1}u_i))$ is a monic polynomial in $V_\lambda[\underline{x}, y]$ (resp. $V_\lambda[\underline{x}, y, z, u_i]$) of degree $deg(s(y))$ (resp. d_i) on y (resp. u_i) whose reduction modulo the maximal ideal $\boldsymbol{m}_\lambda$ of V_λ is a monomial, and $\alpha t(\pi_1^{-1}\underline{x}, \pi_2^{-1}y)$ (resp. $\alpha_i\tilde{q}_i(\pi_1^{-1}\underline{x}, \pi_2^{-1}(y,z), \pi_3^{-1}u_i))$ is a polynomial in $V_\lambda[\underline{x}, y]$ (resp. $V_\lambda[\underline{x}, y, z, u_i]$) whose reduction modulo $\boldsymbol{m}_\lambda$ is a non-zero monomial of degree $deg(t(y))$ on y (resp. e_i on u_i) for some element α (resp. α_i) in V_λ which is contained in $\boldsymbol{m}_\lambda$. Such λ exists if we take $\lambda_1 \ll \lambda_2 \ll \lambda_3$ for any λ_3. We define V_λ-algebras

$$\tilde{A}_\lambda = V_\lambda[\underline{x}, y, z]/(\pi_2^{deg(s(y))} s(\pi_1^{-1}\underline{x}, \pi_2^{-1}y), \alpha t(\pi_1^{-1}\underline{x}, \pi_2^{-1}y)z - \pi_2\alpha)$$

$$\tilde{B}_{i,\lambda} = \tilde{A}_\lambda[u_i, v_i]/(\pi_3^{d_i}\tilde{p}_i(\pi_1^{-1}\underline{x}, \pi_2^{-1}(y,z), \pi_3^{-1}u_i), \alpha_i\tilde{q}_i(\pi_1^{-1}\underline{x}, \pi_2^{-1}(y,z), \pi_3^{-1}u_i)v_i - \pi_3\alpha_i),$$

and put $\tilde{B}_\lambda = \prod_i \tilde{B}_{i,\lambda}$. We denote by $\hat{\tilde{A}}_\lambda$ (resp. $\hat{\tilde{B}}_\lambda^{\otimes r}$) the p-adic completion of $\tilde{A}_\lambda$ (resp. $\tilde{B}_\lambda^{\otimes r}$) modulo $\boldsymbol{m}_\lambda$-power torsions and by $\hat{\tilde{\mathscr{C}}}_\lambda^\bullet$ the Čech complex

$$0 \to \hat{\tilde{A}}_\lambda \to \hat{\tilde{B}}_\lambda \to (\tilde{B}_\lambda \otimes_{\tilde{A}_\lambda} \tilde{B}_\lambda)\hat{} \to (\tilde{B}_\lambda \otimes_{\tilde{A}_\lambda} \tilde{B}_\lambda \otimes_{\tilde{A}_\lambda} \tilde{B}_\lambda)\hat{} \to \cdots,$$

where we put $\hat{\tilde{A}}_\lambda$ at degree 0. Since there is a section $\tilde{B}_\lambda/\boldsymbol{m}_\lambda\tilde{B}_\lambda \to \tilde{A}_\lambda/\boldsymbol{m}_\lambda\tilde{A}_\lambda$, one gets $H^l(\hat{\tilde{\mathscr{C}}}_\lambda^\bullet/\boldsymbol{m}_\lambda\hat{\tilde{\mathscr{C}}}_\lambda^\bullet) = 0$ for any l. Since $\hat{\tilde{A}}_\lambda$ (resp. $\hat{\tilde{B}}_\lambda^{\otimes r}$) is free over V_λ, we have $H^l(\hat{\tilde{\mathscr{C}}}_\lambda^\bullet/\boldsymbol{m}_\lambda^n\hat{\tilde{\mathscr{C}}}_\lambda^\bullet) = 0$ for any n and l. Hence, we have

$$H^l(\hat{\tilde{\mathscr{C}}}_\lambda^\bullet) \cong \varprojlim_n H^l(\hat{\tilde{\mathscr{C}}}_\lambda^\bullet/\boldsymbol{m}_\lambda^n\hat{\tilde{\mathscr{C}}}_\lambda^\bullet) = 0.$$

Since $H^l(\mathscr{C}^\bullet) \cong \varinjlim H^l(\mathscr{C}_\lambda^\bullet)$, it is sufficient to prove $H^l(\mathscr{C}_\lambda^\bullet) = 0$ for any l. Here we take the direct limit above by $\max_i\{\lambda_i\} \to 1$. Since V_λ over V is a finite extension of complete discrete valuation rings, $H^l(\mathscr{C}_\lambda^\bullet \otimes_V V_\lambda) = 0$ implies $H^l(\mathscr{C}_\lambda^\bullet) = 0$. So we may assume that $V = V_\lambda$. Then, there is an isomorphism $\hat{\tilde{A}}_\lambda \to I_\lambda^0$ (resp. $\hat{\tilde{B}}_\lambda^{\otimes r} \to I_\lambda^r$) defined by $\underline{x} \mapsto \pi_1\underline{x}, (y,z) \mapsto \pi_2(y,z), (\underline{u}, \underline{v}) \mapsto \pi_3(\underline{u}, \underline{v})$ by the universality of tensor products and inverse limits. This map induces an isomorphism $\hat{\tilde{\mathscr{C}}}_\lambda^\bullet \to \mathscr{I}_\lambda^\bullet$ of complexes. Hence we have $H^l(\mathscr{I}_\lambda^\bullet) = 0$.

Now we consider the exact sequence $0 \to \mathscr{C}_\lambda^\bullet \xrightarrow{p} \mathscr{C}_\lambda^\bullet \to \mathscr{C}_\lambda^\bullet/p\mathscr{C}_\lambda^\bullet \to 0$ of complexes of A_λ-modules. Since $A_\lambda/pA_\lambda = A/pA$ (resp. $(B^{\otimes r})_\lambda/p(B^{\otimes r})_\lambda = (B^{\otimes r})/p(B^{\otimes r})$) and f is surjective on the special fiber (hence, B/pB is faithfully flat over A/pA), we have $H^l(\mathscr{C}_\lambda^\bullet/p\mathscr{C}_\lambda^\bullet) = 0$. In other words, the multiplication p map on $H^l(\mathscr{C}_\lambda^\bullet)$ is bijective. Since any element of $(B^r)_\lambda/I_\lambda^r$ is p-power torsion, any element is so in $H^l(\mathscr{C}_\lambda^\bullet/\mathscr{I}_\lambda^\bullet)$. From the exact sequence $0 \to \mathscr{I}_\lambda^\bullet \to \mathscr{C}_\lambda^\bullet \to \mathscr{C}_\lambda^\bullet/\mathscr{I}_\lambda^\bullet \to 0$, we have an isomorphism

$$H^l(\mathscr{C}_\lambda^\bullet) \cong H^l(\mathscr{C}_\lambda^\bullet/\mathscr{I}_\lambda^\bullet).$$

Hence, we have $H^l(\mathscr{C}_\lambda^\bullet) = 0$. This completes the proof. □

By Proposition 2.2.1, 2.3.1 we have

COROLLARY 2.3.3. *With the notation as above, if* $(\mathcal{M}, \nabla)$ *is an object in* $\mathrm{Isoc}^\dagger(X/K)$ *and if we put* $M = \Gamma(]\bar{X}[_{\hat{\bar{\mathscr{X}}}}, \mathcal{M})$, *then there is an isomorphism*

$$\mathbf{R}\Gamma_{rig}(X/K, \mathcal{M}) \cong Tot(DR^\bullet(\mathscr{Y}^\bullet/K, M))$$

in the derived categories of complexes of K-vector spaces. Moreover, for an object in F-$\mathrm{Isoc}^\dagger(X/K, \varphi^a)$, *the induced homomorphism of cohomologies commutes with Frobenius structures.*

(2.4) Let Z (resp. $\bar{Z}$) be a closed subscheme of X over $\mathrm{Spec}\, k$ (resp. the closure of Z in $\bar{X}$) and put $\bar{i} : \bar{Z} \to \bar{X}$ (resp. $j_Z : Z \to \bar{Z}$) to be the correspondent closed (resp. open) immersion. We define functors

$$]\bar{i}[^*\colon \mathrm{Isoc}^\dagger(X/K) \to \mathrm{Isoc}^\dagger(Z/K)$$

$$]\bar{i}[^*\colon F\text{-}\mathrm{Isoc}^\dagger(X/K, \sigma^a) \to F\text{-}\mathrm{Isoc}^\dagger(Z/K, \sigma^a)$$

of the inverse image as follows. For an object $(\mathcal{M}, \nabla)$, we put $]\bar{i}[^*\mathcal{M} =]\bar{i}[^{-1}\mathcal{M} \otimes_{]\bar{i}[^{-1}j^\dagger O_{]\bar{X}[}} j_Z^\dagger O_{]\bar{Z}[}$. Put $]X[_{\hat{\mathscr{P}}^2}$ to be the tubular neighbourhood of the diagonal embedding of X in $\hat{\mathscr{P}}^2$ and denote by $pr_i :]\bar{X}[_{\hat{\mathscr{P}}^2} \to]\bar{X}[_{\hat{\mathscr{P}}}$ the natural projection of tubes for $i = 1, 2$. Since ∇ is overconvergent, the stratification $\varepsilon : pr_1^*\mathcal{M} \cong pr_2^*\mathcal{M}$ which is induced from the connection ∇ extends on a strict neighbourhood of $]X[_{\hat{\mathscr{P}}^2}$. Hence, the extension of ε determines a stratification on the strict neighbourhood of $]Z[_{\hat{\mathscr{P}}^2}$ since the strict neighbourhood of $]X[$ includes the strict neighbourhood of $]Z[$. The functor $]\bar{i}[^*$ is independent of the choice of the formal scheme $\hat{\mathscr{P}}$ and commutes with tensor products and duals.

Now we assume that both X and Z are affine smooth and there exist an affine smooth scheme $\mathscr{X} = \mathrm{Spec}\, A$ of finite type over $\mathrm{Spec}\, V$ and an affine smooth closed subscheme $\mathscr{Z} = \mathrm{Spec}\, C$ of $\mathscr{X}$ over $\mathrm{Spec}\, V$ such that $X = \mathscr{X} \times_{\mathrm{Spec}\, V} \mathrm{Spec}\, k$ and $Z = \mathscr{Z} \times_{\mathrm{Spec}\, V} \mathrm{Spec}\, k$. We fix $\hat{\bar{\mathscr{X}}}$ and $\bar{X}$ (resp. $\hat{\bar{\mathscr{Z}}}$ and $\bar{Z}$) as in 2.2. Let (M, ∇) be an object in $\mathrm{Conn}^\dagger(\mathscr{X}/K)$. If $u \in A$ vanishes in C, then the image of du under the projection $\Omega^1_{A/V} \to \Omega^1_{C/V}$ vanishes and ∇ induces a connection $i^\dagger\nabla$ on $i^\dagger M = M \otimes_{A_K^\dagger} C_K^\dagger$. If we fix a presentation of A over V, then this presentation determines a presentation of C and $\|i^\dagger(u)\|_{\mathscr{Z}_\lambda} \leqq \|u\|_{\mathscr{X}_\lambda}$ for any $\lambda > 1$. Hence, the connection $i^\dagger\nabla$ is overconvergent. We define a functor

$$i^\dagger : \mathrm{Conn}^\dagger(\mathscr{X}/K) \to \mathrm{Conn}^\dagger(\mathscr{Z}/K)$$

by $i^\dagger(M, \nabla) = (M \otimes_{A_K^\dagger} C_K^\dagger, i^\dagger\nabla)$.

If φ is a Frobenius on $A^\dagger$ such that φ induces a Frobenius on $C^\dagger$, then one can easily see that the functor $i^\dagger$ induces the functor

$$i^\dagger : \mathrm{Conn}^\dagger(\mathscr{X}/K, \varphi^a) \to \mathrm{Conn}^\dagger(\mathscr{Z}/K, \varphi^a).$$

By definition, we have

PROPOSITION 2.4.1. *Under the assumption as above, the diagram*

$$\begin{tikzcd} \mathrm{Isoc}^\dagger(X/K) \arrow[r, "]\bar{i}[^*"] \arrow[d, "\Gamma_{rig}(]\bar{X}[_{\hat{\mathscr{X}}},?)"'] & \mathrm{Isoc}^\dagger(Z/K) \arrow[d, "\Gamma_{rig}(]\bar{Z}[_{\hat{\mathscr{X}}},?)"] \\ \mathrm{Conn}^\dagger(\mathscr{X}/K) \arrow[r, "i^\dagger"] & \mathrm{Conn}^\dagger(\mathscr{Z}/K) \end{tikzcd}$$

of categories is commutative. The same holds for overconvergent F-isocrystals.

(2.5) We recall the definition of rigid cohomology with compact supports in [4, Sect. 3, 4.2]. Let $\imath :]\bar{X} - X[_{\hat{\mathscr{P}}} \to]\bar{X}[_{\hat{\mathscr{P}}}$ be the corresponding immersion. For a sheaf $\mathscr{E}$ of abelian groups on $]\bar{X}[_{\hat{\mathscr{P}}}$, we define a sheaf on $]\bar{X}[_{\hat{\mathscr{P}}}$ by

$$\underline{\Gamma}_{]X[_{\hat{\mathscr{P}}}}(\mathscr{E}) = \ker(\mathscr{E} \to \imath_* \imath^* \mathscr{E}).$$

Let $(\mathscr{M}, \nabla)$ be an object in $\mathrm{Isoc}^\dagger(X/K)$ and let W be a strict neighbourhood of $]X[_{\hat{\mathscr{P}}}$ in $]\bar{X}[_{\hat{\mathscr{P}}}$ such that there exists a coherent O_W-module $\mathscr{M}_W$ and a connection ∇_W on $\mathscr{M}_W$ with $j_W^\dagger(\mathscr{M}_W, \nabla_W) \cong (\mathscr{M}, \nabla)$. Here we denote by $j_W : W \to]\bar{X}[_{\hat{\mathscr{P}}}$ the open immersion. We define a complex

$$\mathbf{R}\Gamma_{c,rig}(X/K, \mathscr{M}) = \mathbf{R}\Gamma(]\bar{X}[_{\hat{\mathscr{P}}}, \underline{\Gamma}_{]X[_{\hat{\mathscr{P}}}}(DR^\bullet(\mathscr{M}_W)))$$

in the derived category of complexes of K-vector spaces bounded below. The complex above is independent of the choices of W, $\bar{X}$ and $\hat{\mathscr{P}}$ up to the canonical isomorphism. The rigid cohomology with compact supports for $(\mathscr{M}, \nabla)$ is defined by

$$H^l_{c,rig}(X/K, \mathscr{M}) = \mathbf{R}^l\Gamma_{c,rig}(X/K, \mathscr{M}).$$

If $\mathscr{E}$ is a sheaf of coherent $O_{]\bar{X}[_{\hat{\mathscr{P}}}}$-module, then $R^l \imath_* \imath^* \mathscr{E} = 0$ for $l \neq 0$. Hence, for a short exact sequence

$$0 \to (\mathscr{M}_1, \nabla_1) \to (\mathscr{M}_2, \nabla_2) \to (\mathscr{M}_3, \nabla_3) \to 0$$

in $\mathrm{Isoc}^\dagger(X/K)$, there exists a distinguished triangle

$$\mathbf{R}\Gamma_{c,rig}(X/K, \mathscr{M}_1) \longrightarrow \mathbf{R}\Gamma_{c,rig}(X/K, \mathscr{M}_2) \longrightarrow \mathbf{R}\Gamma_{c,rig}(X/K, \mathscr{M}_3) \xrightarrow{+1} .$$

The natural homomorphism $\underline{\Gamma}_{]X[_{\hat{\mathscr{P}}}}(\mathscr{E}) \to \mathscr{E}$ of complexes of sheaves on $]\bar{X}[_{\hat{\mathscr{P}}}$ induces a homomorphism

$$\mathbf{R}\Gamma_{c,rig}(X/K, \mathscr{M}) \to \mathbf{R}\Gamma_{rig}(X/K, \mathscr{M})$$

of complexes of K-vector spaces for an object $(\mathscr{M}, \nabla)$ in $\mathrm{Isoc}^\dagger(X/K)$. In the case where $\bar{X} = X$ the homomorphism above is an isomorphism by definition.

Let Z (resp. $\bar{Z}$, resp. U) be a closed subscheme of X over $\mathrm{Spec}\, k$ (resp. the closure of Z in $\bar{X}$, resp. $U = X - Z$) and put $\bar{i} : \bar{Z} \to \bar{X}$ (resp. $j_Z : Z \to \bar{Z}$, resp. $j_U : U \to \bar{X}$) to be the corresponding closed immersion (resp. open immersions).

Proposition 2.5.1. *For an object* $(\mathscr{M}, \nabla)$ *in* $\mathrm{Isoc}^{\dagger}(X/K)$, *there is a canonical distinguished triangle*

$$\mathbf{R}\Gamma_{c,rig}(U/K, j_U^{\dagger}\mathscr{M}) \longrightarrow \mathbf{R}\Gamma_{c,rig}(X/K, \mathscr{M}) \longrightarrow \mathbf{R}\Gamma_{c,rig}(Z/K,]\bar{i}[^{*}\mathscr{M}) \xrightarrow{+1}.$$

We denote the triangle above by $\Delta_{c,rig}(X, Z, \mathscr{M})$.

Proof. Let W be a strict neighbourhood of $]X[_{\hat{\mathscr{P}}}$ in $]\bar{X}[_{\hat{\mathscr{P}}}$ such that $j_W^{\dagger}(\mathscr{M}_W, \nabla) \cong (\mathscr{M}, \nabla)$ with a coherent O_W-module $\mathscr{M}_W$. Since $]\bar{i}[_{*}]\bar{i}[^{*}\mathscr{M}_W \cong (\mathscr{M}_W)|_{W-]X[_{\hat{\mathscr{P}}}}$ and $]\bar{i}[_{*}$ is exact,

$$0 \to \underline{\Gamma}_{]U[}(j_U^{\dagger}\mathscr{M}) \to \underline{\Gamma}_{]X[}(\mathscr{M}) \to]\bar{i}[_{*}\underline{\Gamma}_{]Z[}(]\bar{i}[^{*}\mathscr{M}) \to 0$$

is an exact sequence of sheaves of $O_{]\bar{X}[_{\hat{\mathscr{P}}}}$-modules. This completes the proof. □

Proposition 2.5.2. *With the notation as in Proposition* 2.1.2, *if* K' *is a finite extension of* K, *then the morphism*

$$\mathbf{R}\Gamma_{c,rig}(X/K, \mathscr{M}) \otimes_K K' \to \mathbf{R}\Gamma_{c,rig}(X'/K', \mathscr{M}')$$

induced by $\tau_{K'/K}^{-1}\underline{\Gamma}_{]X[_{\hat{\mathscr{P}}}}(\mathscr{M}) \to \underline{\Gamma}_{]X'[_{\hat{\mathscr{P}}'}}(\mathscr{M}')$ *is an isomorphism in the derived category of complexes of* K'*-vector spaces.*

Proof. Considering the Čech cohomology, the assertion follows easily from the fact that, if we choose a strict neighbourhood W of $]X[$ where $\mathscr{M}$ is defined, then $\Gamma(\tau_{K'/K}^{-1}(U), \underline{\Gamma}_{]X'[}(O_{]\bar{X}'[})) = \Gamma(U, \underline{\Gamma}_{]X[}(O_{]\bar{X}[})) \otimes_K K'$ and that $H^l(U, \underline{\Gamma}_{]X[}(\mathscr{M}_W)) = H^l(\tau_{K'/K}^{-1}(U), \underline{\Gamma}_{]X'[}(\mathscr{M}'_{\tau_{K'/K}^{-1}(W)})) = 0$ for $l \neq 0$ and any admissible affinoid subspace U of $]X[$. □

Let φ be a lift of Frobenius on $\hat{\mathscr{P}}$ with respect to σ. For a strict neighbourhood W of $]X[_{\hat{\mathscr{P}}}$, if we choose a sufficiently small strict neighbourhood W' of $]X[_{\hat{\mathscr{P}}}$, then φ induces a map $\tilde{\varphi} : W' \to W$ [5, 2.4.1.3]. There is a Frobenius structure on $\Gamma_{]X[}(\mathscr{M})$ for an overconvergent F-isocrystal $(\mathscr{M}, \nabla, \Phi)$ and all induced homomorphisms of cohomologies with compact supports above commute with Frobenius structures.

(2.6) We discuss on the relative cases of rigid cohomologies. Let

$$\begin{array}{ccccc} Y & \xrightarrow{j_Y} & \bar{Y} & \xrightarrow{i_Y} & \hat{\mathscr{Q}} \\ {\scriptstyle f}\downarrow & & {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle w} \\ X & \xrightarrow[j_X]{} & \bar{X} & \xrightarrow[i_X]{} & \hat{\mathscr{P}} \end{array} \tag{2.6.1}$$

be a commutative diagram which satisfies the following conditions: X and Y are separated schemes of finite type over $\operatorname{Spec} k$, $\overline{X}$ (resp. $\overline{Y}$) is a compactification of X (resp. Y) over $\operatorname{Spec} k$ with an open immersion j_X (resp. j_Y), $\hat{\mathscr{P}}$ (resp. $\hat{\mathscr{Q}}$) is a formal scheme of finite type over $\operatorname{Spf} V$, i_X (resp. i_Y) is a closed immersion, $\hat{\mathscr{P}}$ (resp. $\hat{\mathscr{Q}}$) is smooth around X (resp Y), f is smooth and w is smooth around Y.

Denote by $\tilde{w}_K :]\overline{Y}[_{\hat{\mathscr{Q}}} \to]\overline{X}[_{\hat{\mathscr{P}}}$ the induced morphism of analytic spaces by w. Since w is smooth around Y, there is a strict neighbourhood U (resp. W) of $]X[_{\hat{\mathscr{P}}}$ in $]\overline{X}[_{\hat{\mathscr{P}}}$ (resp. $]Y[_{\hat{\mathscr{Q}}}$ in $]\overline{Y}[_{\hat{\mathscr{Q}}}$) such that $\tilde{w}_K(W) \subset U$ and $\tilde{w}_K$ is smooth on W by [5, Proposition 1.2.7]. Then the sequence

$$0 \to (\tilde{w}_K|_W)^*\Omega^1_{U/K} \to \Omega^1_{W/K} \to \Omega^1_{W/U} \to 0$$

of sheaves of O_W-modules is exact. Let (M, ∇) be an object in $\mathrm{Isoc}^\dagger(Y/K)$ such that there exists a sheaf $\mathscr{M}_W$ of coherent O_W-module with an integrable connection $\nabla_W : \mathscr{M}_W \to \mathscr{M}_W \otimes_{O_W} \Omega^1_{W/K}$ and that $j^\dagger_W(\mathscr{M}_W, \nabla_W) \cong (\mathscr{M}, \nabla)$, where $j_W :]Y[_{\hat{\mathscr{Q}}} \to W$ is the corresponding open immersion. Then the connection ∇_W on $\mathscr{M}_W$ induces a relative integrable connection

$$\nabla_{W/U} : \mathscr{M}_W \to \mathscr{M}_W \otimes_{O_W} \Omega^1_{W/U}.$$

We denote by $\mathscr{M}_W \otimes_{O_W} \Omega^\bullet_{W/U}$ the induced relative de Rham complex of sheaves of $\tilde{w}_K|_W^{-1} O_U$-modules

$$\cdots \longrightarrow 0 \longrightarrow \mathscr{M}_W \xrightarrow{\nabla_{W/U}} \mathscr{M}_W \otimes_{O_W} \Omega^1_{W/U} \xrightarrow{\nabla_{W/U}} \mathscr{M}_W \otimes_{O_W} \Omega^2_{W/U} \xrightarrow{\nabla_{W/U}} \cdots,$$

where we put $\mathscr{M}_W$ at the degree 0. By the similar proof as in Theorem 1 and Theorem 2 in [4, Sect. 2] we have

PROPOSITION 2.6.2. *Under the assumption as above, let*

$$\begin{tikzcd} & \overline{Y}' \arrow[r, "i_{Y'}"] \arrow[d, "\bar{g}"'] & \hat{\mathscr{Q}}' \arrow[d, "v"] \\ Y \arrow[ur, "j_{Y'}"] \arrow[r, "j_Y"'] & \overline{Y} \arrow[r, "i_Y"'] & \hat{\mathscr{Q}} \end{tikzcd}$$

be a commutative diagram such that $j_{Y'}$ is an open immersion, $\bar{g}$ is proper, $\overline{Y}'$ is a closed subscheme of the formal scheme $\hat{\mathscr{Q}}'$ of finite type over $\operatorname{Spf} V$ *and v is smooth around Y. If we put $\tilde{v}_K :]\overline{Y}'[_{\hat{\mathscr{Q}}'} \to]\overline{Y}[_{\hat{\mathscr{Q}}}$ to be the induced morphism of analytic spaces, $W' = \tilde{v}_K^{-1}(W)$ and $\mathscr{M}_{W'} = \tilde{v}_K^{-1}\mathscr{M}_W \otimes_{\tilde{v}_K^{-1} O_W} O_{W'}$, then the natural morphism $j_{W*} \to (\tilde{v}_K|_{W'})_* j_{W'*}(\tilde{v}_K|_{W'})^*$ induces an isomorphism*

$$\mathscr{M}_W \otimes_{O_W} \Omega^\bullet_{W/U} \to \mathbf{R}(\tilde{v}_K|_{W'})_*(\mathscr{M}_{W'} \otimes_{O_{W'}} \Omega^\bullet_{W'/U})$$

in the derived category of complexes of sheaves of $\tilde{w}_K|_W^{-1} O_U$-modules bounded below.

If U_1 is a strict neighbourhood of $]X[_{\hat{\mathscr{P}}}$ in U and if we put $W_1 = \tilde{w}_K^{-1}(U_1) \cap W$, then W_1 is a strict neighbourhood of $]Y[_{\hat{\mathscr{P}}}$ and there is a canonical morphism

$$j_{U_1}^{-1}\mathbf{R}(\tilde{w}_K|_W)_*(\mathscr{M}_W \otimes_{O_W} \Omega^\bullet_{W/U}) \to \mathbf{R}(\tilde{w}_K|_{W_1})_*(\mathscr{M}_{W_1} \otimes_{O_{W_1}} \Omega^\bullet_{W_1/U_1}).$$

Here $j_{U_1} : U_1 \to]\overline{X}[_{\hat{\mathscr{P}}}$. Now we define a complex

$$(\mathbf{R}f_{rig*}\mathscr{M})_{\hat{\mathscr{P}}} = \varinjlim_{U_1} j_{U_1*}\mathbf{R}(\tilde{w}_K|_{W_1})_*(\mathscr{M}_{W_1} \otimes_{O_{W_1}} \Omega^\bullet_{W_1/U_1})$$

of $j_X^\dagger O_{]\overline{X}[_{\hat{\mathscr{P}}}}$-modules. Here U_1 runs over all strict neighbourhood of $]X[_{\hat{\mathscr{P}}}$ in $]\overline{X}[_{\hat{\mathscr{P}}}$ and $W_1 = \tilde{w}_K^{-1}(U_1) \cap W$. The complex $(\mathbf{R}f_{rig*}\mathscr{M})_{\hat{\mathscr{P}}}$ is independent of the choice of $\overline{Y}$ and $\hat{\mathscr{Q}}$ by Proposition 2.6.2.

We define a decreasing filtration

$$Fil^r(\mathscr{M}_W \otimes_{O_W} \Omega^\bullet_{W/K})$$
$$= \mathrm{Image}\left(\sum_{s=0}^{r} \mathscr{M}_W \otimes_{O_W} \Omega^{\bullet-s}_{W/K} \otimes_{\tilde{w}_K|_W^{-1} O_U} \tilde{w}_K|_W^{-1}\Omega^s_{U/K} \to \mathscr{M}_W \otimes_{O_W} \Omega^\bullet_{W/K}\right)$$

of $\mathscr{M}_W \otimes_{O_W} \Omega^\bullet_{W/K}$. Since both W/U and U/K are smooth, we have

$$gr^r_{Fil}(\mathscr{M}_W \otimes_{O_W} \Omega^\bullet_{W/K}) = \mathscr{M}_W \otimes_{O_W} \Omega^{\bullet-r}_{W/U} \otimes_{\tilde{w}_K|_W^{-1} O_U} \tilde{w}_K|_W^{-1}\Omega^r_{U/K}.$$

The edge morphism induces an integrable connection

$$\nabla_U^{GM} : \mathbf{R}^l(\tilde{w}_K|_W)_*(\mathscr{M}_W \otimes_{O_W} \Omega^\bullet_{W/U}) \to \mathbf{R}^l(\tilde{w}_K|_W)_*(\mathscr{M}_W \otimes_{O_W} \Omega^\bullet_{W/U}) \otimes_{O_U} \Omega^1_{U/K}.$$

Since $\tilde{w}_K|_{W_1} = j_{U_1} \circ \tilde{w}_K|_W$ for a strict neighbourhood U_1 of $]X[_{\hat{\mathscr{P}}}$ in U and $W_1 = \tilde{w}_K^{-1}(U_1) \cap W$, we have a Gauss-Manin connection

$$\nabla^{GM} : (\mathbf{R}^l f_{rig*}\mathscr{M})_{\hat{\mathscr{P}}} \to (\mathbf{R}^l f_{rig*}\mathscr{M})_{\hat{\mathscr{P}}} \otimes_{O_U} \Omega^1_{U/K}.$$

We fix a Frobenius $\varphi_{\hat{\mathscr{P}}}$ (resp. $\varphi_{\hat{\mathscr{Q}}}$) on $\hat{\mathscr{P}}$ (resp. $\hat{\mathscr{Q}}$) with $w \circ \varphi_{\hat{\mathscr{Q}}} = \varphi_{\hat{\mathscr{P}}} \circ w$. We may assume that such Frobenius always exists since w is smooth around Y. Let $(\mathscr{M}, \nabla, \Phi)$ be an object in $F\text{-Isoc}^\dagger(Y/K, \sigma^a)$. If we choose a sufficiently small strict neighbourhood U_1 of $]X[_{\hat{\mathscr{P}}}$ in U and put $W_1 = \tilde{w}_K^{-1}(U_1) \cap W$, then $\varphi_{\hat{\mathscr{Q}}}$ induces a σ-linear homomorphism $\tilde{\varphi}^*_{\hat{\mathscr{Q}}} : j_{W_1}^{-1}(\Omega^{\bullet-s}_{W/K} \otimes_{\tilde{w}_K|_W^{-1} O_U} \tilde{w}_K|_W^{-1}\Omega^s_{U/K}) \to \Omega^{\bullet-s}_{W_1/K} \otimes_{\tilde{w}_K|_{W_1}^{-1} O_{U_1}} \tilde{w}_K|_{W_1}^{-1}\Omega^s_{U_1/K}$. The Frobenius structure

$$\Phi : j_{W_1}^{-1}(\mathscr{M}_W \otimes_{O_W} \Omega^\bullet_{W/U}) \to \mathscr{M}_{W_1} \otimes_{O_{W_1}} \Omega^\bullet_{W_1/U_1}$$

induces a σ^a-linear homomorphism Φ^{GM} on $(f_{rig*}\mathscr{M})_{\hat{\mathscr{P}}}$.

THEOREM 2.6.3. *With the notation as above, assume furthermore that X is smooth over* Spec k *and that f is finite etale. Then we have*

(1) $(\mathbf{R}^l f_{rig*}\mathscr{M})_{\hat{\mathscr{P}}} = 0$ *for* $l \neq 0$ *and* $(f_{rig*}\mathscr{M})_{\hat{\mathscr{P}}}$ *is a sheaf of coherent* $j_X^\dagger O_{]\bar{X}[_{\hat{\mathscr{P}}}}$*-module.*

(2) *If we denote by ∇^{GM} the Gauss-Manin connection on $(f_{rig*}\mathscr{M})_{\hat{\mathscr{P}}}$, then ∇^{GM} is overconvergent. We denote by $f_{rig*}(\mathscr{M},\nabla)$ or $f_{rig*}\mathscr{M}$ the corresponding object $((f_{rig*}\mathscr{M})_{\hat{\mathscr{P}}}, \nabla^{GM})$ in the category* $\mathrm{Isoc}^\dagger(X/K)$.

(3) *If $(\mathscr{M},\nabla,\Phi)$ is an object in F-$\mathrm{Isoc}^\dagger(Y/K,\sigma^a)$, then the induced σ^a-linear map Φ^{GM} on $(f_{rig*}\mathscr{M})_{\hat{\mathscr{P}}}$ is a Frobenius structure. Moreover, if Φ is unit-root, then the induced Frobenius structure Φ^{GM} is also unit-root. We denote by $f_{rig*}(\mathscr{M},\nabla,\Phi)$ or $f_{rig*}\mathscr{M}$ the corresponding object $((f_{rig*}\mathscr{M})_{\hat{\mathscr{P}}}, \nabla^{GM}, \Phi^{GM})$ in the category F-$\mathrm{Isoc}^\dagger(X/K,\sigma^a)$.*

PROOF. Since the assertion is local on X and f is finite etale, we may assume that both X and Y are affine integral. By Proposition 2.6.1 we may choose $\hat{\mathscr{P}}$ and $\hat{\mathscr{Q}}$ as follows. We choose a smooth integral affine lift $\mathscr{X}$ of X of finite type over Spec V by [13, Théorème 6], embed $\mathscr{X}$ into a projective space over Spec V and denote by $\mathscr{P}$ (resp. $\bar{X}$, resp. $\hat{\mathscr{P}}$) the Zariski closure of $\mathscr{X}$ in the projective space (resp. the Zariski closure of X in $\mathscr{P}$, resp. the p-adic completion of $\mathscr{P}$). By our assumption there is a finite integral closed affine scheme $\mathscr{Y}$ over $\mathscr{X}$ such that $Y = \mathscr{Y} \times_{\mathrm{Spec}\, V} \mathrm{Spec}\, k$. We denote by $\mathscr{Q}$ (resp. $\bar{Y}$, resp. $\hat{\mathscr{Q}}$) the normalization of $\mathscr{P}$ in $\mathscr{Y}$ (resp. the Zariski closure of Y in $\mathscr{Q}$, resp. the p-adic completion of $\mathscr{Q}$). Since Y is etale over X, $\mathscr{Q}$ is finite over $\mathscr{P}$. Hence, $\hat{\mathscr{Q}}$ is finite over $\hat{\mathscr{P}}$.

For an object $(\mathscr{M},\nabla)$ in $\mathrm{Isoc}^\dagger(Y/K)$, we can choose a strict neighbourhood U in $]X[_{\hat{\mathscr{P}}}$ such that, if we put $W = \tilde{w}_K^{-1}(U)$, there are a coherent O_W-module $\mathscr{M}_W$ and an integrable connection ∇_W on $\mathscr{M}_W$ with $j_W^\dagger(\mathscr{M}_W,\nabla_W) \cong (\mathscr{M},\nabla)$. Since $(\tilde{w}_K|_W)_*O_W$ is finite over O_U, $(\tilde{w}_K|_W)_*\mathscr{M}_W$ is a coherent O_U-module. If we choose a sufficiently small U, then $\Omega^s_{W/U} = 0$ for any $s > 0$ since the etaleness is an open condition. Hence, we have the assertion (1).

Put $]\bar{X}[_{\hat{\mathscr{P}}^2}$ (resp. $]\bar{Y}[_{\hat{\mathscr{Q}}^2}$) to be the tubular neighbourhood of the diagonal embedding of $\bar{X}$ (resp. $\bar{Y}$) in $\hat{\mathscr{P}}^2$ (resp. $\hat{\mathscr{P}}^2$) and denote by $pr_X^i :]\bar{X}[_{\hat{\mathscr{P}}^2} \to]\bar{X}[_{\hat{\mathscr{P}}}$ (resp. $pr_Y^i :]\bar{Y}[_{\hat{\mathscr{Q}}^2} \to]\bar{Y}[_{\hat{\mathscr{Q}}}$) the natural projection of tubes for $i = 1,2$. Since the connection ∇_W of $\mathscr{M}_W$ is overconvergent and $\tilde{w}_K^2 : ((pr_Y^1)^{-1}(W) \cap (pr_Y^2)^{-1}(W)) \to ((pr_X^1)^{-1}(U) \cap (pr_X^2)^{-1}(U))$ is finite etale [5, Proposition 1.2.10], there is a strict neighbourhood U_1 of $]X[_{\hat{\mathscr{P}}^2}$ such that (i) the strict neighborhood $W_1 = (\tilde{w}_K^2)^{-1}U_1$ of $]Y[_{\hat{\mathscr{Q}}^2}$ is included in $(pr_Y^1)^{-1}(W) \cap (pr_Y^2)^{-1}(W)$, (ii) there exists an isomorphism $\varepsilon : (pr_Y^1|_{W_1})^*\mathscr{M}_W \cong (pr_Y^2|_{W_1})^*\mathscr{M}_W$ which satisfies the usual cocycle condition and (iii) ε induces the connection ∇_W of $\mathscr{M}_W$ by [5, Proposition 2.2.6]. Since $\mathscr{M}_W$ is coherent, $\tilde{w}_K|_W$ is finite and

$pr_X^l|_{U_1}$ $(l = 1, 2)$ is flat, ε induces the isomorphism

$$\begin{aligned}(pr_X^1|_{U_1})^*(\tilde{w}_K|_W)_*\mathcal{M}_W &\cong (\tilde{w}_K^2|_{W_1})_*(pr_Y^1|_{W_1})^*\mathcal{M}_W \\ &\cong (\tilde{w}_K^2|_{W_1})_*(pr_Y^2|_{W_1})^*\mathcal{M}_W \\ &\cong (pr_X^2|_{U_1})^*(\tilde{w}_K^2|_W)_*\mathcal{M}_W\end{aligned}$$

by Lemma 2.6.4 below. One can check that the isomorphism above satisfies the cocycle condition by the same method and this isomorphism induces the overconvergent connection ∇^{GM} on $(f_{rig*}\mathcal{M})_{\hat{\mathcal{P}}}$. Hence, we have the assertion (2).

The assertion (3) is easy. □

Lemma 2.6.4. *With the notation as in the proof of Theorem* 2.6.3, *the commutative diagram*

$$\begin{tikzcd} W_1 \arrow[r, "pr_Y^l"] \arrow[d, "\tilde{w}_K^2|_{W_1}"'] & W \arrow[d, "\tilde{w}_K|_W"] \\ U_1 \arrow[r, "pr_X^l"'] & U \end{tikzcd}$$

is cartesian for $l = 1, 2$.

Proof. The proof is similar as in [10, 1.7]. Consider the commutative diagram

$$\begin{tikzcd}]Y[_{\hat{\mathcal{Q}}^2} \arrow[r] \arrow[d, equal] & W_1 \arrow[r] \arrow[d, "((\tilde{w}_K^2|_{W_1}, pr_Y^l)"] &]\overline{Y}[^2_{\hat{\mathcal{Q}}} \arrow[d, "(\tilde{w}_K, \mathrm{id})"] \\ ]Y[_{\hat{\mathcal{Q}}^2} \arrow[r, "(\tilde{w}_K^2, pr_Y^l)"'] & U_1 \times_{pr_X^l} W \arrow[r] &]\overline{X}[_{\hat{\mathcal{P}}} \times]\overline{Y}[_{\hat{\mathcal{Q}}}. \end{tikzcd}$$

Here $U_1 \times_{pr_X^l} W$ means the fiber product for the map $pr_X^l : U_1 \to U$. Since $\tilde{w}_K|_W$ is finite etale, $(\tilde{w}_K^2, \mathrm{id})$ induces an isomorphism between W_1 and $U_1 \times_{pr_X^l} W$ by [5, Théorème 1.3.5]. □

Corollary 2.6.5. *Under the same assumption as in Theorem* 2.6.3, *let* Z_X *be a closed subscheme in* X *and put* $Z_Y = f^{-1}Z_X$. *Then, for an object* $(\mathcal{M}, \nabla)$ *in* $\mathrm{Isoc}^\dagger(Y/K)$, *we have a natural isomorphism*

$$H^l_{Z_X, rig}(X/K, f_{rig*}\mathcal{M}) \cong H^l_{Z_Y, rig}(Y/K, \mathcal{M})$$

of K-vector spaces for any l. For an overconvergent F-isocrystal, the isomorphism above commutes with Frobenius structures.

COROLLARY 2.6.6. *Under the same assumption as in Theorem* 2.6.3, *we have a natural isomorphism*

$$H^l_{c,rig}(X/K, f_{rig*}\mathscr{M}) \cong H^l_{c,rig}(Y/K, \mathscr{M})$$

of K-vector spaces for any object $(\mathscr{M}, \nabla)$ *in* $\mathrm{Isoc}^\dagger(Y/K)$ *and any l. For an overconvergent F-isocrystal, the isomorphisms above commutes with Frobenius structures.*

(2.7) Assume that X is smooth over $\mathrm{Spec}\, k$ and that f is finite etale in the diagram 2.6.1. Denote by $f^*_{rig} : \mathrm{Isoc}^\dagger(X/K) \to \mathrm{Isoc}^\dagger(Y/K)$ (resp. f^*_{rig} : F-$\mathrm{Isoc}^\dagger(X/K, \sigma^a) \to F$-$\mathrm{Isoc}^\dagger(Y/K, \sigma^a)$) the inverse image functor as overconvergent isocrystals (resp. as overconvergent F-isocrystals).

Let $(\mathscr{M}, \nabla)$ be an object in $\mathrm{Isoc}^\dagger(X/K)$ (resp. F-$\mathrm{Isoc}^\dagger(X/K)$). We define an adjoint map

$$\mathrm{ad} : \mathscr{M} \to f_{rig*}f^*_{rig}\mathscr{M}$$

by $m \mapsto 1 \otimes m$ for $m \in \mathscr{M}$. Then, one can easily check that the adjoint map ad is a morphism in $\mathrm{Isoc}^\dagger(X/K)$ (resp, F-$\mathrm{Isoc}^\dagger(X/K, \sigma^a)$) and that f^*_{rig} and f_{rig*} are adjoint each other by the adjoint map ad.

We define a trace map

$$tr : f_{rig*}f^*_{rig}\mathscr{M} \to \mathscr{M}$$

which is a morphism in $\mathrm{Isoc}^\dagger(X/K)$ (resp. F-$\mathrm{Isoc}^\dagger(X/K, \sigma^a)$) as follows. In general, the construction of the trace map is a local problem. Hence, we may assume the local situation as in the proof of Theorem 2.6.3. Since W is finite etale over U, we can define a trace map

$$tr_U : (\tilde{w}_K|_W)_* O_W \to O_U$$

and define a trace map $tr_U : (\tilde{w}_K|_W)_*(\tilde{w}_K|_W)^*\mathscr{M}_U \to \mathscr{M}_U$ by

$$\begin{aligned}(\tilde{w}_K|_W)_*(\tilde{w}_K|_W)^*\mathscr{M}_U &\cong (\tilde{w}_K|_W)_*(O_W \otimes_{(\tilde{w}_K|_W)^{-1}O_U} (\tilde{w}_K|_W)^{-1}\mathscr{M}_U) \\ &\cong ((\tilde{w}_K|_W)_* O_W) \otimes_{O_U} \mathscr{M}_U \\ &\xrightarrow{tr_U \otimes \mathrm{id}} \mathscr{M}_U.\end{aligned}$$

One can easily check that the trace map tr commutes with connections. If we denote by r the degree of Y over X, then the composition

$$\mathscr{M} \xrightarrow{\mathrm{ad}} f_{rig*}f^*_{rig}\mathscr{M} \xrightarrow{tr} \mathscr{M}$$

of the adjoint map and the trace map is $r\,\mathrm{id}_{\mathscr{M}}$, where $\mathrm{id}_{\mathscr{M}}$ is the identity map on $\mathscr{M}$. One can easily see that the trace map tr commutes with Frobenius structures for F-isocrystals.

3. Local comparison theorem

(3.1) First we fix our situation. Let $\mathscr{X} = \operatorname{Spec} A$ be an affine smooth scheme of finite type over $\operatorname{Spec} V$. We suppose that

(3.1.1) there exists a system $t_1, t_2, \dots, t_n \in A$ of coordinates of $\mathscr{X}$ over $\operatorname{Spec} V$.

In other words, the V-morphism

$$\mathscr{X} \to \mathbf{A}_V^n$$

which is defined by the system $\{t_1, \dots, t_n\}$ is etale. Let d be a nonnegative integer $\leqq n$. We denote by $\mathscr{Y}$ (resp. $\mathscr{Y}_\mu$) the open subscheme $\operatorname{Spec} B = \operatorname{Spec} A\left[\dfrac{1}{t_1 \cdots t_d}\right]$ (resp. the open subscheme $\operatorname{Spec} B_\mu = \operatorname{Spec} A\left[\dfrac{1}{t_\mu}\right]$) of $\mathscr{X}$ and put $j_Y : \mathscr{Y} \to \mathscr{X}$ (resp. $j_\mu : \mathscr{Y}_\mu \to \mathscr{X}$, resp. $j'_\mu : \mathscr{Y} \to \mathscr{Y}_\mu$) to be the corresponding open immersion (resp. for $1 \leqq \mu \leqq d$). We also denote by $\mathscr{D}$ (resp. $\mathscr{D}_\mu$) the divisor of $\mathscr{X}$ which is defined by the equation $t_1 \cdots t_d = 0$ (resp. by the equation $t_\mu = 0$). We put X, Y, D, Y_μ and D_μ to be the special fiber of $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{D}$, $\mathscr{Y}_\mu$ and $\mathscr{D}_\mu$, respectively.

Keep the notation as in 2.2. Now we fix a presentation

$$V[x_1, \dots, x_N]/I \cong A$$

of the V-algebra A with $x_\mu \mapsto t_\mu$ $(1 \leqq \mu \leqq n)$. For $\lambda > 1$, we define V-algebras by

$$\begin{aligned} A_\lambda &= V[\underline{x}]_\lambda / IV[\underline{x}]_\lambda \\ B_\lambda &= V[x_0, \underline{x}]_\lambda / (I, (x_0 x_1 \cdots x_d - 1)) V[x_0, \underline{x}]_\lambda, \\ B_{\mu,\lambda} &= V[x_{0\mu}, \underline{x}]_\lambda / (I, (x_{0\mu} x_\mu - 1)) V[x_{0\mu}, \underline{x}]_\lambda. \end{aligned}$$

We denote by $\| \; \|_{\mathscr{X},\lambda}$ (resp. $\| \; \|_{\mathscr{Y},\lambda}$, resp. $\| \; \|_{\mathscr{Y}_\mu,\lambda}$) the quotient norm as in 2.2. If we define a homomorphism $j_{Y,\lambda} : A_\lambda \to B_\lambda$ (resp. $j_{\mu,\lambda} : A_\lambda \to B_{\mu,\lambda}$, resp. $j'_{\mu,\lambda} : B_{\mu,\lambda} \to B_\lambda$) of V-algebras by the natural injection (resp. by the natural injection, resp. by $j'_\mu(x_{0\mu}) = x_0 x_1 x_2 \cdots x_d / x_\mu$ and $j'_\mu(x_\nu) = x_\nu$ $(1 \leqq \nu \leqq N)$). Then, $j_{Y,\lambda}$ (resp. $j_{\mu,\lambda}$, resp. $j'_{\mu,\lambda}$) commutes with the Banach norms, that is, $\|j_{Y,\lambda}(a)\|_{\mathscr{Y},\lambda} \leqq \|a\|_{\mathscr{X},\lambda}$ for $a \in A_\lambda$. We also denote by $\| \; \|_{\mathscr{X}}$ (resp. $\| \; \|_{\mathscr{Y}}$, resp. $\| \; \|_{\mathscr{Y}_\mu}$) the norm on $A^\dagger$ (resp. $B^\dagger$, resp. $B_\mu^\dagger$) as in 2.2. Then, $j_U^\dagger$ (resp. $j_\mu^\dagger$, resp. $(j'_\mu)^\dagger$) commutes with the norms.

Define a sheaf of differential module $\Omega^l_{\mathscr{X}/\operatorname{Spec} V}(\mathscr{D})$ on $\mathscr{X}$ over Spec V with logarithmic poles along $\mathscr{D}$ by an $O_{\mathscr{X}}$-submodule of $\Omega^s_{\mathscr{Y}/\operatorname{Spec} V}$ which is generated by

$$\frac{dt_{j_1}}{t_{j_1}} \wedge \cdots \wedge \frac{dt_{j_s}}{t_{j_s}} \wedge dt_{j_{s+1}} \wedge \cdots \wedge dt_{j_l}$$

for $s \leqq \min\{l, d\}$, $1 \leqq j_1 < \cdots < j_s \leqq d$ and $d + 1 \leqq j_{s+1} < \cdots < j_l \leqq n$. By the assumption 3.1.1, $\Omega^s_{A/V}(\mathscr{D}) = \Gamma(\mathscr{X}, \Omega^l_{\mathscr{X}/\operatorname{Spec} V}(\mathscr{D}))$ is a free A-module of finite rank. We put $\Omega^l_{A_K^\dagger/K}(\mathscr{D}) = A_K^\dagger \otimes_A \Omega^l_{A/V}(\mathscr{D})$ and denote by $d : A_K^\dagger \to \Omega^1_{A_K^\dagger/K}(\mathscr{D})$ the natural K-derivation.

We denote by $\partial_\mu = \dfrac{\partial}{\partial t_\mu}$ the dual differential operator of dt_μ of A and put

$$\delta_\mu^{[0]} = 1 \qquad (1 \leqq \mu \leqq n)$$

$$\delta_\mu^{[i]} = \begin{cases} \dfrac{1}{i}(t_\mu \partial_\mu - (i-1))\delta_\mu^{[i-1]} & 1 \leqq \mu \leqq d \\ \dfrac{1}{i}\partial_\mu \delta_\mu^{[i-1]} & \mu \geq d+1 \end{cases}$$

for any nonnegative integer i. By the condition 3.1.1, we have

LEMMA 3.1.2. *Let* $\mathscr{X}^{(1)} = \operatorname{Spec} A^{(1)}$ *(resp.* $\mathscr{X}^{(2)} = \operatorname{Spec} A^{(2)}$*) be a smooth affine scheme of finite type over* Spec V *which satisfies the condition* 3.1.1 *and let* $t_1^{(1)}, \dots, t_n^{(1)}$ *(resp.* $t_1^{(2)}, \dots, t_n^{(2)}$*) be the fixed system of parameter of* $\mathscr{X}^{(1)}$ *(resp.* $\mathscr{X}^{(2)}$*). If there is an isomorphism* $\iota : \mathscr{X}^{(1)} \times_{\operatorname{Spec} V} \operatorname{Spec} k \to \mathscr{X}^{(2)} \times_{\operatorname{Spec} V} \operatorname{Spec} k$ *of k-algebra with* $\iota(t_\mu^{(1)} \pmod{\boldsymbol{m}A^{(1)}}) = t_\mu^{(2)} \pmod{\boldsymbol{m}A^{(2)}}$ $(1 \leqq \mu \leqq d)$, *then there exists a unique V-algebra isomorphism*

$$\iota^\dagger : A^{(1)\dagger} \to A^{(2)\dagger}$$

such that $\iota(t_\mu^{(1)}) = t_\mu^{(2)}$ *for any* μ *and that the diagram*

$$\begin{array}{ccc} A_K^{(1)\dagger} & \xrightarrow{d} & \Omega^1_{A_K^{(1)\dagger}/K}(\mathscr{D}) \\ \downarrow{\scriptstyle \iota^\dagger} & & \downarrow{\scriptstyle \iota^\dagger} \\ A_K^{(2)\dagger} & \xrightarrow{d} & \Omega^1_{A_K^{(2)\dagger}/K}(\mathscr{D}) \end{array}$$

is commutative.

We define a Frobenius on $A^\dagger$ as in 2.2. Later we use a Frobenius φ on $A^\dagger$ which satisfies the condition

(3.1.3) $\qquad \varphi(t_\mu) = t_\mu^p u_\mu \qquad$ for some $u_\mu \in 1 + \boldsymbol{m}A^\dagger \qquad (1 \leqq \mu \leqq d).$

(Note that u_μ is a unit in $A^\dagger$.) Then φ induces a σ-linear homomorphism $\varphi : \Omega^1_{A_K^\dagger/K}(\mathscr{D}) \to \Omega^1_{A_K^\dagger/K}(\mathscr{D})$ with $\varphi\left(\frac{dt_\mu}{t_\mu}\right) = p\frac{dt_\mu}{t_\mu} + \frac{du_\mu}{u_\mu}$ $(1 \leqq \mu \leqq d)$ and the diagram

$$\begin{array}{ccc} A_K^\dagger & \xrightarrow{d} & \Omega^1_{A_K^\dagger/K}(\mathscr{D}) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ A_K^\dagger & \xrightarrow{d} & \Omega^1_{A_K^\dagger/K}(\mathscr{D}) \end{array}$$

commutes. By [8, Lemma 3.1.1] there always exists a unique Frobenius on $A^\dagger$ with $\varphi(t_i) = t_i^p$ $(1 \leqq i \leqq n)$ under our condition 3.1.1.

(3.2) We define a logarithmic overconvergent connection on $A_K^\dagger$. In the case where $d = 0$, a logarithmic overconvergent connection is a usual overconvergent connection in [5, 2.5] (See 2.2.).

Definition 3.2.1. (1) Let M be an $A_K^\dagger$-module. A K-homomorphism $\nabla : M \to M \otimes_{A_K^\dagger} \Omega^1_{A_K^\dagger/K}(\mathscr{D})$ is a connection with logarithmic poles along $\mathscr{D}$ if and only if ∇ is additive and satisfies the relation $\nabla(am) = a\nabla(m) + m \otimes da$ for $m \in M$ and $a \in A_K^\dagger$. A connection ∇ is integrable if and only if $\nabla^2 = 0$, where we define $\nabla : M \otimes_{A_K^\dagger} \Omega^s_{A_K^\dagger/K}(\mathscr{D}) \to M \otimes_{A_K^\dagger} \Omega^{s+1}_{A_K^\dagger/K}(\mathscr{D})$ by $\nabla(m \otimes \omega) = \nabla(m) \wedge \omega + m \otimes d\omega$. A morphism of $A_K^\dagger$-modules with a logarithmic connection along $\mathscr{D}$ is a horizontal $A_K^\dagger$-homomorphism.

(2) Let M be a finitely generated $A_K^\dagger$-module with a logarithmic connection ∇ along $\mathscr{D}$ and choose a real number $\lambda_1 > 1$ such that there exists a pair (M_{λ_1}, ∇) of an $A_{\lambda_1,K}$-module of finite presentation and a logarithmic connection with $(M, \nabla) \cong (M_{\lambda_1}, \nabla) \otimes_{A_{\lambda_1,K}} A_K^\dagger$. We fix a presentation of M_{λ_1} over A_{λ_1} and denote by $|\ |_\lambda$ the quotient norm on $M_\lambda \cong M_{\lambda_1} \otimes_{A_{\lambda_1,K}} A_{\lambda,K}$ which is determined by the fixed presentation for $1 < \lambda \leqq \lambda_1$. The connection ∇ is overconvergent if and only if it is integrable and, for any $\eta < 1$, there exists $\lambda > 1$ such that

$$|\nabla(\underline{\partial}^{[\underline{i}]})(m)|_\lambda \eta^{|\underline{i}|} \to 0 \qquad (|\underline{i}| \to \infty)$$

for any $m \in M_\lambda$. Here $\underline{\partial}^{[\underline{i}]} = \partial_1^{[i_1]} \cdots \partial_n^{[i_n]}$. We denote by $\mathrm{Conn}^\dagger((\mathscr{X}, \mathscr{D})/K)$ the full subcategory of $A_K^\dagger$-modules with a logarithmic connection along $\mathscr{D}$ which consists of overconvergent objects.

(3) Let φ be a Frobenius on $A_K^\dagger$ which satisfies the condition 3.1.3 and let a be a positive integer. For an $A_K^\dagger$-module M with an integrable logarithmic connection ∇ along $\mathscr{D}$, we say that an $A_K^\dagger$-homomorphism

$$\Phi : (\varphi^a)^*(M, \nabla) \to (M, \nabla)$$

is a Frobenius structure with respect to φ^a if and only if Φ is a horizontal isomorphism. Here $(\varphi^a)^*(M,\nabla)$ is the induced logarithmic connection by the scalar extension $\varphi^a : A_K^\dagger \to A_K^\dagger$. A morphism of $A_K^\dagger$-modules with a logarithmic connection along $\mathscr{D}$ and a Frobenius structure is a horizontal $A_K^\dagger$-homomorphism which commutes with Frobenius structures. We denote by $F\text{-Conn}^\dagger((\mathscr{X},\mathscr{D})/K,\varphi^a)$ the category of $A_K^\dagger$-modules with an overconvergent logarithmic connection along $\mathscr{D}$ and a Frobenius structure with respect to φ^a.

In our definition the finitely generated $A_K^\dagger$-module with integrable logarithmic connection is not always projective. For example, if $d \geqq 1$, then $M = A_K^\dagger/t_1A_K^\dagger$ with a connection $t_1\partial_1$ is an object in $\text{Conn}^\dagger((\mathscr{X},\mathscr{D})/K)$.

It is clear that the category $\text{Conn}^\dagger((\mathscr{X},\mathscr{D})/K)$ (resp. $F\text{-Conn}^\dagger((\mathscr{X},\mathscr{D})/K,\varphi^a))$ is abelian and it has tensor products. We define the dual $(M,\nabla,(\Phi))^\vee = (M^\vee,\nabla^\vee,(\Phi^\vee))$ of $(M,\nabla,(\Phi))$ by

$$M^\vee = Hom_{A_K^\dagger}(M,A_K^\dagger)$$

$$(\nabla^\vee(\delta_\mu)(f))(m) = \delta_\mu(f(m)) - f(\nabla(\delta_\mu)(m)) \qquad \text{for } 1 \leqq \mu \leqq n, f \in M^\vee, m \in M$$

$$\Phi^\vee(f) = (\mathrm{id}_{A_K^\dagger} \otimes \sigma^a) \circ (\mathrm{id}_{A_K^\dagger} \otimes f) \circ \Phi^{-1} \qquad \text{for } f \in M^\vee.$$

It is clear that, if M is projective over $A_K^\dagger$, we have $(M^\vee)^\vee \cong M$.

By Lemma 3.1.2 we have

PROPOSITION 3.2.2. (1) *The category* $\text{Conn}^\dagger((\mathscr{X},\mathscr{D})/K)$ *depends only on* X *and* D.

(2) *The category* $F\text{-Conn}^\dagger((\mathscr{X},\mathscr{D})/K,\varphi^a)$ *depends only on* X *and* D *and it is independent of the choice of Frobenius* φ *which satisfies the condition* 3.1.3.

PROOF. The assertion (1) follows from Lemma 3.1.2. (2) It is sufficient to see the independence on the choice of Frobenius by Lemma 3.1.2. Let φ_1 and φ_2 be Frobenius on $A_K^\dagger$ which satisfy the condition 3.1.3 and put

$$v_\mu = \begin{cases} \dfrac{\varphi_2^a(t_\mu)}{\varphi_1^a(t_\mu)} - 1 & 1 \leqq \mu \leqq d \\ \varphi_2^a(t_\mu) - \varphi_1^a(t_\mu) & \mu \geqq d+1. \end{cases}$$

We define a functor

$$\alpha(\varphi_1^a,\varphi_2^a)^* : F\text{-Conn}^\dagger((\mathscr{X},\mathscr{D})/K,\varphi_1^a) \to F\text{-Conn}^\dagger((\mathscr{X},\mathscr{D})/K,\varphi_2^a)$$

as follows. Let (M,∇,Φ) be an object in $F\text{-Conn}^\dagger((\mathscr{X},\mathscr{D})/K,\varphi_1^a)$. We define an $A_K^\dagger$-linear homomorphism

$$\alpha(\varphi_1^a,\varphi_2^a) : (\varphi_2^a)^*M \to (\varphi_1^a)^*M$$

by Taylor's series

$$\alpha(\varphi_1^a, \varphi_2^a)(m \otimes 1) = \sum_{\underline{i} \in \mathbf{N}^n} \nabla(\underline{\delta}^{[\underline{i}]})(m) \otimes \underline{v}^{\underline{i}}.$$

$\alpha(\varphi_1^a, \varphi_2^a)$ is well-defined since ∇ is overconvergent. One can check that $\alpha(\varphi_1^a, \varphi_2^a)$ commutes with connections, $\alpha(\varphi^a, \varphi^a) = 1$ and $\alpha(\varphi_1^a, \varphi_3^a) = \alpha(\varphi_2^a, \varphi_3^a)\alpha(\varphi_1^a, \varphi_2^a)$ by explicit calculations. Moreover, $\Phi \circ \alpha(\varphi_1^a, \varphi_2^a)$ is a Frobenius structure on (M, ∇) with respect to the Frobenius φ_2^a. Now we define the functor by $\alpha(\varphi_1^a, \varphi_2^a)^*(M, \nabla, \Phi) = (M, \nabla, \Phi \circ \alpha(\varphi_1^a, \varphi_2^a))$. Then $\alpha(\varphi_1^a, \varphi_2^a)^*$ is an equivalence of categories. □

(3.3) For an object (M, ∇) in $\mathrm{Conn}^\dagger((\mathscr{X}, \mathscr{D})/K)$, we denote by $DR^\bullet((\mathscr{X}, \mathscr{D})/K, M)$ the complex

$$\cdots \to 0 \to M \xrightarrow{\nabla} M \otimes_{A_K^\dagger} \Omega^1_{A_K^\dagger/K}(\mathscr{D}) \xrightarrow{\nabla} M \otimes_{A_K^\dagger} \Omega^2_{A_K^\dagger/K}(\mathscr{D}) \to \cdots$$

of K-vector spaces, where we put M at the degree 0. We define the logarithmic Monsky-Washnitzer cohomology $H^l_{MW}((X, D)/K, M)$ by the cohomology of the complex $DR^\bullet((\mathscr{X}, \mathscr{D})/K, M)$. The logarithmic Monsky-Washnitzer cohomology is functorial for (M, ∇) and $H^l_{MW}((X, D)/K, M) = 0$ for $l < 0$ and $l > n$ by definition. For any short exact sequence in $\mathrm{Conn}^\dagger((\mathscr{X}, \mathscr{D})/K)$, we have a long exact sequence of K-vector spaces as usual. In general, the K-vector space $H^l_{MW}((X, D)/K, M)$ is not of finite dimension over K.

By Lemma 3.1.2 and Proposition 3.2.2 we have

Proposition 3.3.1. *The logarithmic Monsky-Washnitzer cohomology $H^l_{MW}((X, D)/K, M)$ depends only on X and D.*

Now we fix a Frobenius φ on $A^\dagger$ which satisfies the condition 3.1.3. For an object (M, ∇, Φ) in $F\text{-}\mathrm{Conn}^\dagger((\mathscr{X}, \mathscr{D})/K, \varphi^a)$, we define a σ^a-linear endomorphism

$$\Phi : H^l_{MW}((X, D)/K, M) \to H^l_{MW}((X, D)/K, M)$$

by $m \otimes \omega \mapsto \Phi(m) \otimes \phi^a(\omega)$ for $m \in M$ and $\omega \in \Omega^l_{A_K^\dagger/K}$.

Proposition 3.3.2. *With the notation as above, the σ^a-linear endomorphism Φ on $H^l_{MW}((X, D)/K, M)$ is independent of the choice of the Frobenius on $A^\dagger$ which satisfies the condition* 3.1.3 *under the canonical equivalence of categories in Proposition* 3.2.2.

Proof. The proof is the same as in the case without logarithmic structures. [17, Sect. 5] Let φ_1 and φ_2 be Frobenius on $A^\dagger$ which satisfy the condition 3.1.3. We keep the notation as in the proof of Proposition 3.2.2. Define a K-homomorphism $h_\mu : M \to (\varphi_1^a)^*M$ by

$$h_\mu(m) = \begin{cases} \displaystyle\sum_{0 \leqq i \leqq l < \infty} (-1)^{l-i} \nabla(\delta_\mu^{[i]})(m) \otimes \frac{v_\mu^{l+1}}{l+1} & \mu \leqq d \\ \displaystyle\sum_{i=0}^{\infty} \nabla(\delta_\mu^{[i]})(m) \otimes \frac{v_\mu^{i+1}}{i+1} & \mu \geqq d+1 \end{cases}$$

Since the connection ∇ is overconvergent, the infinite sums are convergent in M. We define a K-homomorphism

$$H : DR^\bullet((\mathscr{X}, \mathscr{D})/K, M) \to DR^\bullet((\mathscr{X}, \mathscr{D})/K, (\varphi_1^a)^*M)$$

of degree -1 by $H(m \otimes (\bigwedge_{s=1}^{l} \omega_{\mu_s})) = \sum_{s=1}^{l} (-1)^{s-1} h_{\mu_s}(m) \otimes (\bigwedge_{i \neq s} \omega_{\mu_i})$, where $\omega_\mu = \dfrac{dx_\mu}{x_\mu}$ for $\mu \leqq d$ and $\omega_\mu = dx_\mu$ for $\mu \geqq d+1$. One can see $\alpha(\varphi_1^a, \varphi_2^a)^* \circ (\varphi_2^a)^* - (\varphi_1^a)^* = H \circ \nabla + (\varphi_1^a)^* \nabla \circ H$. Hence, H gives a homotopy. This completes the proof. $\square$

(3.4) We define a functor

$$j_Y^{log} : \mathrm{Conn}^\dagger(\mathscr{X}/K) \to \mathrm{Conn}^\dagger((\mathscr{X}, \mathscr{D})/K)$$

as follows. For an object (M, ∇) in $\mathrm{Conn}^\dagger(\mathscr{X}/K)$, we put $j_Y^{log} M = M$ and $\nabla(t_\mu \partial_\mu)(m) = t_\mu \nabla(\partial_\mu)(m)$ $(1 \leqq \mu \leqq d)$. For $\eta < 1$, if we choose $\lambda > 1$ with $\left|\dfrac{1}{\underline{i}!} \nabla(\underline{\partial}^{[\underline{i}]})(m)\right|_\lambda \eta^{|\underline{i}|/2} \to 0$ $(|\underline{i}| \to \infty)$ for any $m \in M_\lambda$, then we have

$$|\nabla(\underline{\delta}^{[\underline{i}]})(m)|_{\min\{\lambda, \eta^{-1/2}\}} \eta^{|\underline{i}|} \to 0 \qquad (|\underline{i}| \to \infty)$$

since $\delta_\mu^{[i]} = \dfrac{1}{i!} t_\mu^i \partial_\mu^i$ $(1 \leqq \mu \leqq d)$. Hence, the connection $j_Y^{log} \nabla$ is overconvergent. It is clear that the functor j_Y^{log} is fully faithful.

We define a functor

$$j_Y^\dagger : \mathrm{Conn}^\dagger((\mathscr{X}, \mathscr{D})/K) \to \mathrm{Conn}^\dagger(\mathscr{Y}/K)$$

$$j_\mu^\dagger : \mathrm{Conn}^\dagger((\mathscr{X}, \mathscr{D})/K) \to \mathrm{Conn}^\dagger((\mathscr{Y}_\mu, j_\mu^{-1}\mathscr{D})/K)$$

by the extension $j_Y^\dagger : A_K^\dagger \to B_K^\dagger$ (resp. $j_\mu^\dagger : A_K^\dagger \to B_{\mu,K}^\dagger$) of the scalar. Let M be an object in $\mathrm{Conn}^\dagger((\mathscr{X}, \mathscr{D})/K)$. For $\eta < 1$, if we choose $\lambda > 1$ with $|\nabla(\underline{\delta}^{[\underline{i}]})(m)|_\lambda \eta^{|\underline{i}|/2} \to 0$ $(|\underline{i}| \to \infty)$ for any $m \in M_\lambda$, then we have

$$\begin{aligned} &|\nabla(\underline{\delta}^{[\underline{i}]})(m)|_{\min\{\lambda, \eta^{-(1/(2(d+1)))}\}} \eta^{|\underline{i}|} \\ &\qquad < |(x_0 x_1 \cdots x_d)^{|\underline{i}|} \nabla(\underline{\delta}^{[\underline{i}]})(m)|_{\min\{\lambda, \eta^{-(1/(2(d+1)))}\}} \eta^{|\underline{i}|} \to 0 \qquad (|\underline{i}| \to \infty) \end{aligned}$$

for any $m \in M_{\min\{\lambda, \eta^{-(1/(2(d+1)))}\}}$ since $x_0 x_1 \cdots x_d = 1$ and $\delta_\mu^{[i]} = x_\mu^i \partial_\mu^i$ $(1 \leqq \mu \leqq d)$. Hence, the connection $j_Y^\dagger \nabla$ is overconvergent. Similarly, one can see that the connection $j_\mu^\dagger \nabla$ is overconvergent. The functor $j_Y^\dagger$ (resp. $j_\mu^\dagger$) is neither faithful nor full. By definition we have $j_Y^\dagger = (j'_\mu)^\dagger j_\mu^\dagger$, where $(j'_\mu) : \mathscr{Y} \to \mathscr{Y}_\mu$.

Let φ be a Frobenius on $A^\dagger$ which satisfies the condition 3.1.3. One can easily see that the functors j_Y^{log}, $j_Y^\dagger$ and $j_\mu^\dagger$ induce the functors

$$\begin{aligned}
j_Y^{log} &: F\text{-Conn}^\dagger(\mathscr{X}/K, \varphi^a) \qquad \to F\text{-Conn}^\dagger((\mathscr{X}, \mathscr{D})/K, \varphi^a) \\
j_Y^\dagger &: F\text{-Conn}^\dagger((\mathscr{X}, \mathscr{D})/K, \varphi^a) \to F\text{-Conn}^\dagger(\mathscr{Y}/K, \varphi^a) \\
j_\mu^\dagger &: F\text{-Conn}^\dagger((\mathscr{X}, \mathscr{D})/K, \varphi^a) \to F\text{-Conn}^\dagger((\mathscr{Y}_\mu, j_\mu^{-1}\mathscr{D})/K, \varphi^a).
\end{aligned}$$

It is clear the functor j_Y^{log} is fully faithful.

(3.5) Let (M, ∇) be an object in $\text{Conn}^\dagger((\mathscr{X}, \mathscr{D})/K)$. The natural homomorphism

$$\begin{aligned}
DR^\bullet((\mathscr{X}, \mathscr{D})/K, M) &\to DR^\bullet(\mathscr{Y}/K, j_Y^\dagger M) \\
DR^\bullet((\mathscr{X}, \mathscr{D})/K, M) &\to DR^\bullet((\mathscr{Y}_\mu, j_\mu^{-1}\mathscr{D})/K, j_\mu^\dagger M)
\end{aligned}$$

of complexes of K-vector spaces induces a K-linear homomorphism

$$\begin{aligned}
(j_Y^\dagger)^* &: H_{MW}^l((X, D)/K, M) \to H_{MW}^l(Y/K, j_Y^\dagger M) \\
(j_\mu^\dagger)^* &: H_{MW}^l((X, D)/K, M) \to H_{MW}^l((Y_\mu, j_\mu^{-1}D)/K, j_\mu^\dagger M).
\end{aligned}$$

By the construction we have $(j_Y^\dagger)^* = ((j'_\mu)^\dagger)^*(j_\mu^\dagger)^*$. If (M, ∇, Φ) is an object in $F\text{-Conn}^\dagger((\mathscr{X}, \mathscr{D})/K, \varphi^a)$, the transformation $(j_Y^\dagger)^*$ (resp. $(j_\mu^\dagger)^*$) above commutes with σ^a-linear endomorphisms Φ of both sides.

Theorem 3.5.1. *Let* (M, ∇) *be an object in* $\text{Conn}^\dagger(\mathscr{X}/K)$. *Then the natural transformation*

$$(j_Y^\dagger)^* : H_{MW}^l((X, D)/K, j_Y^{log} M) \to H_{MW}^l(Y/K, j_Y^\dagger M).$$

is bijective.

When M is algebraic, the assertion has been proved in more general situations in [1].

In the case where $d = 0$, there is nothing to prove since $\mathscr{X} = \mathscr{Y}$. Since $j_1^\dagger j_Y^{log} M$ arises from an object in $\text{Conn}^\dagger(\mathscr{Y}_1/K)$ canonically, Theorem 3.5.1 follows from Theorem 3.5.2 below by the induction on d.

Theorem 3.5.2. *Let* (M, ∇) *be an object in* $\text{Conn}^\dagger(\mathscr{X}/K)$. *Then the natural homomorphism*

$$DR^\bullet((\mathscr{X},\mathscr{D})/K, j_Y^{log} M) \to DR^\bullet((\mathscr{Y}_1, j_1^{-1}\mathscr{D})/K, j_1^\dagger j_Y^{log} M)$$

of complexes of K-vector spaces is a quasi-isomorphism.

Corollary 3.5.3. *For an object* (M,∇,Φ) *in* $F\text{-}\mathrm{Conn}^\dagger(\mathscr{X}/K,\varphi^a)$, *the induced K-homomorphism*

$$(j_Y^\dagger)^* : H^l_{MW}((X,D)/K, j_Y^{log} M) \to H^l_{MW}(Y/K, j_Y^\dagger M).$$

is bijective and commutes with Frobenius structures.

We prove Theorem 3.5.2 in the rest of this section.

(3.6) To prove Theorem 3.5.2, one may assume the following conditions (1) (2) simultaneously. (1) $\mathscr{D}_1$ is connected and there is a smooth morphism $g : \mathscr{X} \to \mathscr{D}_1$ such that the diagram

$$\begin{tikzcd} \mathscr{D}_1 \arrow[r] \arrow[dr, "\mathrm{id}_{\mathscr{D}_1}"'] & \mathscr{X} \arrow[r] \arrow[d, "g"] & \mathbf{A}_V^n \arrow[d] \\ & \mathscr{D}_1 \arrow[r] & \mathbf{A}_V^{n-1} \end{tikzcd}$$

is commutative. Here the morphism $\mathscr{D}_1 \to \mathbf{A}_V^{n-1}$ (resp. $\mathbf{A}_V^n \to \mathbf{A}_V^{n-1}$) is determined by the system $t_2, \dots, t_n$ of coordinates. (2) M is a free $A_K^\dagger$-module.

Indeed, one may choose a union $\mathscr{W}$ of open affine smooth V-subschemes of $\mathscr{X} \times_{\mathbf{A}_V^{n-1}} \mathscr{D}_1 \coprod (\mathscr{X} - \mathscr{D}_1)$ such that, if we denote by $f : \mathscr{W} \to \mathscr{X}$ the etale structure morphism, then (i) $f^{-1}\mathscr{D}_1 \cong \mathscr{D}_1$ and f is surjective on the special fiber, (ii) if the intersection between a connected component of $\mathscr{W}$ and $f^{-1}\mathscr{D}_1$ is not empty, the restriction of the divisor in the connected component is a section as in the assumption (1) and (iii) the inverse image $f^\dagger M$ is free over $\Gamma(\mathscr{W}, \mathcal{O}_{\mathscr{W}})_K^\dagger$. Note that M is free over $A_K^\dagger$ if and only if $M \otimes_{A_K^\dagger} \hat{A}_K$ is free over $\hat{A}_K$ since $\hat{A}$ is faithfully flat over $A^\dagger$ (see 2.3), where $\hat{A}$ is the p-adic completion of A. Since M is projective, we can choose such $\mathscr{W}$ as in the condition (iii).

We put $\mathscr{W}_1 = \mathscr{W} - f^{-1}\mathscr{D}_1$ and $\mathscr{W}^r = \mathscr{W} \times_{\mathscr{X}} \cdots \times_{\mathscr{X}} \mathscr{W}$ (r times). One can easily see that the triple $(\mathscr{W}^r, (f^r)^{-1}\mathscr{D}, (f^r)^\dagger j_Y^{log}(M,\nabla))$ satisfies the assumptions (1) (2) simultaneously for any r. We define a double complex $DR^\bullet((\mathscr{W}^\bullet, f^{-1}\mathscr{D})/K, j_Y^{log} M)$ of K-vector spaces by

$$DR^\bullet((\mathscr{W}, f^{-1}\mathscr{D})/K, f^\dagger j_Y^{log} M) \to DR^\bullet((\mathscr{W} \times_{\mathscr{X}} \mathscr{W}, (f^2)^{-1}\mathscr{D})/K, (f^2)^\dagger j_Y^{log} M)$$

$$\to DR^\bullet((\mathscr{W} \times_{\mathscr{X}} \mathscr{W} \times_{\mathscr{X}} \mathscr{W}, (f^3)^{-1}\mathscr{D})/K, (f^3)^\dagger j_Y^{log} M) \to \cdots,$$

where we put $f^\dagger j_Y^{log} M$ at the bidegree $(0,0)$ and we define the derivation of the double complex as usual. Then the natural injection induces a commutative diagram

$$\begin{array}{ccc} DR^{\bullet}((\mathscr{X},\mathscr{D})/K, j_Y^{log}M) & \longrightarrow & Tot(DR^{\bullet}((\mathscr{W}^{\bullet}, f^{-1}\mathscr{D})/K, j_Y^{log}M)) \\ \downarrow & & \downarrow \\ DR^{\bullet}((\mathscr{Y}_1, j_1^{-1}\mathscr{D})/K, j_1^{\dagger}j_Y^{log}M) & \longrightarrow & Tot(DR^{\bullet}((\mathscr{W}_1^{\bullet}, (f|_{\mathscr{W}_1})^{-1}\mathscr{D})/K, j_1^{\dagger}f^{\dagger}j_Y^{log}M)) \end{array}$$

of complexes of K-vector spaces. Both horizontal arrows are quasi-isomorphisms by Lemma 3.6.1 below and the right vertical arrow is a quasi-isomorphism by the assumption. Hence, the left vertical arrow is a quasi-isomorphism. Therefore, we may assume the situations (1) (2) above simultaneously.

By Lemma 2.3.2 we have

Lemma 3.6.1. *Let $f : \mathscr{W} \to \mathscr{X}$ be an etale morphism of affine V-schemes of finite type such that $f \times_{\mathrm{Spec}\, V} \mathrm{Spec}\, k : \mathscr{W} \times_{\mathrm{Spec}\, V} \mathrm{Spec}\, k \to \mathscr{W} \otimes_{\mathrm{Spec}\, V} \mathrm{Spec}\, k$ is surjective. For an object (M, ∇) in $\mathrm{Conn}^{\dagger}((\mathscr{X},\mathscr{D})/K)$, the natural homomorphism*

$$DR^{\bullet}((\mathscr{X},\mathscr{D})/K, M) \to Tot(DR^{\bullet}((\mathscr{W}^{\bullet}, f^{-1}\mathscr{D})/K, M))$$

of complexes of K-vector spaces is a quasi-isomorphism. Here $Tot(DR^{\bullet}((\mathscr{W}^{\bullet}, f^{-1}\mathscr{D})/K, M))$ is the total complex of the double complex $DR^{\bullet}((\mathscr{W}^{\bullet}, f^{-1}\mathscr{D})/K, M)$.

(3.7) We continue the proof of Theorem 3.5.2. Put $i_1 : A_1 \to A/t_1A = C_1$ to be the natural projection. By our assumptions (1) in 3.6 there is a smooth homomorphism $g : C_1 \to A$ of smooth V-domains such that the diagram

$$\begin{array}{ccc} C_1 & \xleftarrow{i_1} & A \\ & \overset{\mathrm{id}}{\searrow} & \uparrow g \\ & & C_1 \end{array}$$

is commutative. We fix a presentation of V-algebra A as follows; first we fix a presentation

$$V[x_2, x_3, \ldots, x_{N'}] \to C_1$$

with $x_j \mapsto t_j$ $(2 \leqq j \leqq n)$ and then we fix a presentation

$$V[x_1, x_2, \ldots, x_{N'}, x_{N'+1}, \ldots, x_N] \to A$$

such that $x_1 \mapsto t_1$, the value of x_j $(2 \leqq j \leqq N')$ is determined by the presentation of C_1 above and that x_j $(j \geqq N'+1)$ goes to 0 in C_1. Then one can easily see that, for any $\lambda > 1$, $C_{1,\lambda}$ and the Banach norm of $C_{1,\lambda,K}$ is independent of the choice of two presentations above. We denote by $\|\ \|_{\mathscr{D}_1,\lambda}$ (resp. $\|\ \|_{\mathscr{D}_1}$) this Banach norm on $C_{1,\lambda,K}$ (resp. the limit norm on $C^\dagger_{1,K}$). We put $g^\dagger : C^\dagger_{1,K} \to A^\dagger_K$ (resp. $i^\dagger_1 : A^\dagger_K \to C^\dagger_{1,K}$, resp. $i_{1,\lambda} : A_{\lambda,K} \to C_{1,\lambda,K}$) to be the induced K-algebra homomorphism from g (resp. i_1).

Lemma 3.7.1. (1) *There exists a positive integer α which is independent of the choice of λ such that*

$$\left\|\frac{1}{i!} i_{1,\lambda}(\partial_1^i a)\right\|_{\mathscr{X},\lambda} \leqq \|a\|_{\mathscr{X},\lambda}\lambda^{i\alpha}$$

for $a \in A_{\lambda,K}$ and for any nonnegative integer i.

(2) *Let β be a positive integer. For $a \in A^\dagger_K$, $a \in t_1^\beta A^\dagger_K$ if and only if $i^\dagger_1(\partial_1^i a) = 0$ for $0 \leqq i < \beta$. Moreover, $a = 0$ if and only if $i^\dagger_1(\partial_1^i a) = 0$ for all $i \geqq 0$.*

Proof. (1) By Leipnitz's rule it is sufficient to see that there exists a positive integer α which does not depend on λ such that $\left\|\frac{1}{i!}\partial_1^i(\bar{x}_\mu)\right\|_{\mathscr{X},\lambda} \leqq \lambda^{i\alpha}$ for $n+1 \leqq \mu \leqq N$, where $\bar{x}_\mu$ is the image of x_μ in A. Let $F_\nu(\underline{x}) = 0$ $(n+1 \leqq \nu \leqq N)$ be a system of equation of A in $V[\underline{x}]$. Since $\mathscr{X}$ is etale over $\mathbf{A}^n_V$, the image c of the matrix $\left(\frac{\partial F_\nu}{\partial x_\mu}\right)_{n+1 \leqq \mu,\nu \leqq N}$ in $M_{N-n}(A)$ is invertible. We denote by γ the maximum of the total degree of the presentations of the entries of c^{-1} in $V[\underline{x}]$ and the total degree of F_ν $(n+1 \leqq \nu \leqq N)$. By careful calculations of $\partial_1^i F_\nu$ the sum

$$\sum \frac{i!}{l_1! \prod_\mu \prod_j m_j^\mu!}\left(\frac{\partial^{l_1}}{\partial x_1^{l_1}} \prod_\mu \frac{\partial^{s_\mu}}{\partial x_\mu^{s_\mu}}\right)(F_\nu(\underline{x})) \prod_\mu (\partial_1^{m_1^\mu}(\bar{x}_\mu)\cdots\partial_1^{m_{s_\mu}^\mu}(\bar{x}_\mu))$$

with $l_1 + (m_1^{n+1} + \cdots + m_{s_{n+1}}^{n+1}) + \cdots + (m_1^N + \cdots + m_{s_N}^N) = i$ and $m_1^\mu \leqq \cdots \leqq m_{s_\mu}^\mu$ is 0 in A for any positive integer i. We have

$$\left\|\frac{1}{i!}\partial_1^i(\bar{x}_\mu)\right\|_{\mathscr{X},\lambda} \leqq \lambda^{(4i-2)\gamma}$$

inductively. Hence, it is sufficient to take $\alpha = 4\gamma$.

The assertion (2) follows from the fact t_1 is a prime divisor of $A^\dagger_K$ and $\bigcap_{\beta \geqq 0} t_1^\beta A^\dagger_K = 0$. □

We define $C^\dagger_{1,K}$-algebras

$$\mathscr{S}=\left\{\sum_{i=0}^{\infty}a_it^i \;\middle|\; \begin{array}{l}\text{there exists } \lambda>1 \text{ such that } a_i\in C_{1,\lambda,K} \text{ for all } i\\ \text{and that, for any } \eta<1,\text{ there exists } 1<\lambda'\leqq\lambda\\ \text{with } \|a_i\|_{\mathscr{D}_1,\lambda'}\eta^i\to 0\ (i\to\infty)\end{array}\right\}$$

$$\mathscr{R}=\left\{\sum_{i=-\infty}^{\infty}a_it^i \;\middle|\; \begin{array}{l}\text{there exists } \lambda>1 \text{ such that } a_i\in C_{1,\lambda,K} \text{ for all } i,\\ \sum_{i=0}^{\infty}a_it^i\in\mathscr{S} \text{ and that there exists } \eta<1\\ \text{with } \|a_i\|_{\mathscr{D}_1,\lambda}\eta^i\to 0\ (i\to-\infty).\end{array}\right\}$$

$$\mathscr{T}=\left\{\sum_{i=-\infty}^{\infty}a_it^i\in\mathscr{R} \;\middle|\; \sup_i\|a_i\|_{\mathscr{D}_1}<\infty\right\}$$

Since $C_{1,\lambda,K}$ is complete under the norm $\| \ \|_{\mathscr{D}_1,\lambda}$ and since $\|a\|_{\mathscr{D}_1,\lambda'}\leqq\|a\|_{\mathscr{D}_1,\lambda}$ for $a\in C_{1,\lambda,K}$ if $\lambda'\leqq\lambda$, the multiplication of $\mathscr{R}$ (resp. $\mathscr{T}$) is well-defined.

Define a map

$$| \ |_{\mathscr{T}}:\mathscr{T}\to\mathbf{R}_{\geqq 0}$$

by $|\sum_{i=-\infty}^{\infty}a_it^i|_{\mathscr{T}}=\sup_i\|a_i\|_{\mathscr{D}_1}$. Then, $| \ |_{\mathscr{T}}$ is a norm on $\mathscr{T}$. We also define $\partial_t(\sum a_it^i)=\sum ia_it^{i-1}$. Then ∂_t is a $C^\dagger_{1,K}$-derivation on $\mathscr{S}$ (resp. $\mathscr{R}$, resp. $\mathscr{T}$).

We define a map

$$\imath : A^\dagger_K\to\mathscr{T}$$

by $\imath(a)=\sum_{i=0}^{\infty}i_1^\dagger\left(\frac{1}{i!}(\partial_1^ia)\right)t^i$. By Lemma 3.7.1 $\imath$ is well-defined and we have

LEMMA 3.7.2. *$\imath$ is an injective homomorphism of $C^\dagger_{1,K}$-algebras such that $|\imath(a)|_{\mathscr{T}}\leqq\|a\|_{\mathscr{X}}$ and $\imath(\partial_1a)=\partial_t\imath(a)$ for $a\in A^\dagger_K$.*

Now we will extend the map $\imath$ above to the map

$$\imath : B^\dagger_{1,K}\to\mathscr{T}.$$

For $a\in B_{1,K}$, there exists a non-negative integer β with $t_1^\beta a\in A_K$ and we define the extension of $\imath$ by $a\mapsto t^{-\beta}\imath(t_1^\beta a)$. This definition does not depend on the choice of β. Let $a\in B_{1,\lambda,K}$ for $\lambda>1$ sufficiently close to 1. Fix a lift $\sum_{i_0\underline{i}}a_{i_0,\underline{i}}x_0^{i_0}\underline{x}^{\underline{i}}$ of a under the presentation of $B_{1,\lambda,K}$ in $V[x_0,\underline{x}]_{\lambda,K}$ and, for a non-negative integer β, put $a^{(\beta)}$ to be the image of $\sum_{|(i_0,\underline{i})|\leqq\beta}a_{i_0,\underline{i}}x_0^{i_0}\underline{x}^{\underline{i}}$ in $B_{1,K}$. Here $|(i_0,\underline{i})|$ means the sum of i_0 and all indices of $\underline{i}$. Then, $t_1^\beta a^{(\beta)}\in A_K$ and $\|t_1^\beta a^{(\beta)}\|_{\mathscr{X},\lambda'}\leqq\|\sum_{i_0,\underline{i}}a_{i_0,\underline{i}}x_0^{i_0}\underline{x}^{\underline{i}}\|_{\lambda'}(\lambda')^\beta$ for any $\lambda'\leqq\lambda$, where $\| \ \|_{\lambda'}$ is the Banach norm on $V[x_0,\underline{x}]_{\lambda',K}$. Define $a^{(\beta)}$ $(i\geqq\beta)$ by $\imath(a^{(\beta)})=\sum_{i=-\beta}^{\infty}a_i^{(\beta)}t^i$. Since $\dfrac{1}{i+\beta}\partial_t^{1+\beta}\imath(t_1^\beta a^{(\beta)})=a_i^{(\beta)}+(\text{higher terms on } t)$, we have

$$\|a_i^{(\beta)}-a_i^{(\beta-1)}\|_{\mathscr{D}_1,\lambda'}\leqq\|t_1^\beta(a^{(\beta)}-a^{(\beta-1)})\|_{\mathscr{X},\lambda'}(\lambda')^{(i+\beta)\alpha}\leqq\sup_{|(i_0,\underline{i})|=\beta}|a_{i_0,\underline{i}}|(\lambda')^{(\alpha+2)\beta+i\alpha}$$

for any $\lambda' \leqq \lambda$ by Lemma 3.7.1 and 3.7.2. Here α is as in Lemma 7.3.1. We choose a real number $\lambda_1 > 1$ with $\lambda_1^{\alpha+2} \leqq \lambda$. Then, if we fix i, the sequence of $\{a_i^{(\beta)}\}_{\beta \geqq i}$ is convergent in $C_{1,\lambda_1,K}$ for $\beta \to \infty$ since $|a_{i_0,\underline{i}}|\lambda^{|(i_0,\underline{i})|} \to 0$ ($|(i_0,\underline{i})| \to \infty$), and we denote the limit in $C_{1,\lambda_1,K}$ by a_i. Then one gets

$$(3.7.3) \qquad \|a_i\|_{\mathscr{D}_1,\lambda_1} \leqq \max_{\beta \geqq i}\{\|a_i^{(\beta)} - a_i^{(\beta-1)}\|_{\mathscr{D}_1,\lambda_1}\} \leqq \left\| \sum_{i_0,\underline{i}} a_{i_0,\underline{i}} x_0^{i_0} \underline{x}^{\underline{i}} \right\|_\lambda \lambda_1^{i\alpha},$$

where we put $a_i^{(i-1)} = 0$. Hence, $\sum_i a_i t^i$ is an element in $\mathscr{T}$. We define the extension $\iota : B_{1,K}^\dagger \to \mathscr{T}$ by $\iota(a) = \sum_i a_i t^i$.

We check the well-definedness of the extension ι. If $\sum_{\underline{i}} a_{\underline{i}} \underline{x}^{\underline{i}}$ is contained in the kernel of the surjection $V[x_0, \underline{x}]_{\lambda,K} \to B_{1,\lambda,K}$, then $|\iota(t^\beta a^{(\beta)})|_{\mathscr{T}} \to 0$ for $\beta \to \infty$ since $\|a^{(\beta)}\|_{\mathscr{X},\lambda} \leqq \|\sum_{|\underline{i}|>\beta} a_{\underline{i}} \underline{x}^{\underline{i}}\|_\lambda$. Hence, all coefficients a_i of $\iota(a)$ are 0 and our definition is independent of the choice of the lifting in $V[x_0, \underline{x}]_{\lambda,K}$. The independence of the choices of λ and λ' is trivial.

By the relation 3.7.3 and Lemma 3.7.2, 3.7.1 we have

Lemma 3.7.4. *The extension $\iota : B_{1,K}^\dagger \to \mathscr{T}$ is an injective homomorphism of $C_{1,K}^\dagger$-algebras such that $|\iota(a)|_{\mathscr{T}} \leqq \|a\|_{\mathscr{D}_1}$ and $\iota(\partial_1 a) = \partial_t \iota(a)$ for $a \in B_{1,K}^\dagger$.*

Note that $\iota(A_K^\dagger)$ is contained in $\mathscr{T} \cap \mathscr{S}$.

Lemma 3.7.5. *The natural $A_K^\dagger$-homomorphism*

$$\bar{\iota} : B_{1,K}^\dagger / A_K^\dagger \to \mathscr{T}/(\mathscr{T} \cap \mathscr{S})$$

which is induced by ι is an isomorphism.

Proof. Let $\sum_{t=-\infty}^{-1} a_i t^i \in \mathscr{T}$. Then there is a $\lambda > 1$ sufficiently close to 1 such that $a_i \in C_{1,\lambda,K}$ and that $\|a_i\|_{\mathscr{D}_1,\lambda} \lambda^{-i} \to 0$ $(i \to -\infty)$. Then one can easily see that $a = \sum_{t=-\infty}^{-1} g_\lambda(a_i) t_1^i$ is convergent in $B_{1,\lambda,K}$ and $\iota(a) = \sum_{t=-\infty}^{-1} a_i t^i$. Hence, $\bar{\iota}$ is surjective. To prove the injectivity of $\bar{\iota}$, it is sufficient to see that, if $a \in B_1^\dagger - (\boldsymbol{m} B_1^\dagger \cup A^\dagger)$, $\iota(a) \notin \mathscr{T} \cap \mathscr{S}$. Let $a \in B_1^\dagger - (\boldsymbol{m} B_1^\dagger \cup A^\dagger)$. Then $a = t_1^\beta a_0 + a_1$ for some $a_0 \in A^\dagger, a_1 \in \boldsymbol{m} B_1^\dagger$ and some negative integer β with $a_0 \not\equiv 0$ $(\mathrm{mod}\, ((t_1) + \boldsymbol{m}) A^\dagger)$. By definition and the relation 3.7.3 the coefficient of t^β is a_0 modulo $((t_1) + \boldsymbol{m}) A^\dagger$. Hence, $\bar{\iota}$ is injective. $\square$

Since $\mathscr{R}/\mathscr{S} = \mathscr{T}/(\mathscr{T} \cap \mathscr{S})$ by definition, we have

Corollary 3.7.6. *With the notation as above, ι induces the commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_K^\dagger & \longrightarrow & B_{1,K}^\dagger & \longrightarrow & B_{1,K}^\dagger / A_K^\dagger & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{S} & \longrightarrow & \mathscr{R} & \longrightarrow & \mathscr{R}/\mathscr{S} & \longrightarrow & 0
\end{array}$$

of $A_K^\dagger$-modules such that two horizontal rows are exact, the first and the second vertical arrows are injective and the third vertical arrow is bijective. Moreover, all arrows commute with derivations.

(3.8) We continue to assume the situation in 3.7. Let (M, ∇) be an object in $\mathrm{Conn}^\dagger(\mathscr{X}/K)$ such that M is free over $A_K^\dagger$. We put Ω' (resp. Ω'') to be a sub-$A_K^\dagger$-module of $\Omega^1_{A_K^\dagger/K}$ which is generated by $\dfrac{dt_1}{t_1}$ $\left(\text{resp. } \dfrac{dt_2}{t_2}, \dots, \dfrac{dt_d}{t_d}, dt_{d+1}, \dots, dt_n\right)$. Define a connection

$$\nabla' : M \to M \otimes_{A_K^\dagger} \Omega'$$

$$(\text{resp. } \nabla'' : M \to M \otimes_{A_K^\dagger} \Omega'')$$

by $\nabla'(m) = \nabla(t_1\partial_1)(m) \otimes \dfrac{dt_1}{t_1}$ (resp. $\nabla''(m) = \sum_{\mu=2}^{d} \nabla(t_\mu\partial_\mu)(m) \otimes \dfrac{dt_\mu}{t_\mu} + \sum_{\mu=d+1}^{n} \nabla(\partial_\mu)(m) \otimes dt_\mu$). Then the complex $DR^\bullet((\mathscr{X}, \mathscr{D})/K, j_Y^{log} M)$ is naturally quasi-isomorphic to the total complex of the double complex

$$\begin{array}{ccccccc}
M & \longrightarrow & M \otimes_{A_K^\dagger} \Omega'' & \longrightarrow \cdots \longrightarrow & M \otimes_{A_K^\dagger} (\bigwedge^{n-1}_{A_K^\dagger} \Omega'') \\
\downarrow & & \downarrow & & \downarrow \\
M \otimes_{A_K^\dagger} \Omega' & \longrightarrow & M \otimes_{A_K^\dagger} \Omega' \otimes_{A_K^\dagger} \Omega'' & \longrightarrow \cdots \longrightarrow & M \otimes_{A_K^\dagger} \Omega' \otimes_{A_K^\dagger} (\bigwedge^{n-1}_{A_K^\dagger} \Omega'')
\end{array}$$

as complexes of K-vector spaces. The same holds for $DR^\bullet((\mathscr{Y}_1, j_1^{-1}\mathscr{D})/K, j_1^\dagger j_Y^{log} M)$. To see that the natural morphism

$$DR^\bullet((\mathscr{X}, \mathscr{D})/K, M) \to DR^\bullet((\mathscr{Y}_1, j_1^{-1}\mathscr{D})/K, j_1^\dagger M)$$

is a quasi-isomorphism, it is sufficient to prove that the natural inclusion

$$[M \to M \otimes_{A_K^\dagger} \Omega'] \to [M \otimes_{A_K^\dagger} B_{1,K}^\dagger \to (M \otimes_{A_K^\dagger} B_{1,K}^\dagger) \otimes_{A_K^\dagger} \Omega']$$

is a quasi-isomorphism of complexes of K-vector spaces by the argument of spectral sequences.

Put $M_{\mathscr{S}} = M \otimes_{A_K^\dagger} \mathscr{S}$ (resp. $M_{\mathscr{R}} = M \otimes_{A_K^\dagger} \mathscr{R}$) and define a connection on $M_{\mathscr{S}}$ (resp. $M_{\mathscr{R}}$) by $t\partial_t(m \otimes a) = \nabla(t_1\partial_1)(m) \otimes a + m \otimes t\partial_t(a)$ for $m \in M$ and $a \in \mathscr{S}$ (resp. $a \in \mathscr{R}$). Since M is free over $A_K^\dagger$, the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & M \otimes_{A_K^\dagger} B_{1,K}^\dagger & \longrightarrow & M \otimes_{A_K^\dagger} B_{1,K}^\dagger / A_K^\dagger & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{\mathscr{S}} & \longrightarrow & M_{\mathscr{R}} & \longrightarrow & M_{\mathscr{R}} / M_{\mathscr{S}} & \longrightarrow & 0
\end{array}$$

is commutative such that two horizontal rows are exact, the first and the second vertical arrows are injective, the third vertical arrow is bijective by Corollary 3.7.6 and each arrow commutes with connections. Hence, we obtain

LEMMA 3.8.1. *The following two conditions are equivalent*:
(i) *the natural morphism*

$$[M \to M \otimes_{A_K^\dagger} \Omega'] \to [M \otimes_{A_K^\dagger} B_{1,K}^\dagger \to (M \otimes_{A_K^\dagger} B_{1,K}^\dagger) \otimes_{A_K^\dagger} \Omega']$$

is a quasi-isomorphism of complexes of K-vector spaces;
(ii) *the natural morphism*

$$[M_{\mathscr{S}} \to M_{\mathscr{S}} \otimes_{A_K^\dagger} \Omega'] \to [M_{\mathscr{R}} \to M_{\mathscr{R}} \otimes_{A_K^\dagger} \Omega']$$

is a quasi-isomorphism of complexes of K-vector spaces.

Therefore, Theorem 3.5.2 follows from Lemma 3.8.2 below.

LEMMA 3.8.2. (1) *There is a basis* $e_1, \dots, e_r$ *of* $M_{\mathscr{S}}$ *such that*

$$t\partial_t(e_1, \dots, e_r) = 0;$$

(2) *The natural morphism*

$$[M_{\mathscr{S}} \to M_{\mathscr{S}} \otimes_{A_K^\dagger} \Omega'] \to [M_{\mathscr{R}} \to M_{\mathscr{R}} \otimes_{A_K^\dagger} \Omega']$$

of complexes of K-vector spaces is a quasi-isomorphism.

PROOF. (1) Let $e_1, \dots, e_r$ be a basis of M over $A_K^\dagger$ and let G be a matrix in $M_r(A_K^\dagger)$ such that $\nabla(\partial_1)(e_1, \dots, e_r) = (e_1, \dots, e_r)G$. Then the entries of G are contained in $A_{\lambda,K}$ for some $\lambda > 1$. Define matrices $G_i \in M_r(A_{\lambda,K})$ by $G_0 = 1_r$ and $G_i = \frac{1}{i}(\partial_1(G_i) - G_iG)$ for $i \geqq 1$, where 1_r is a unit matrix. Then the matrix $Q = \sum_{i=0}^{\infty} i_1^\dagger(G_i)t^i$ satisfies the relation $\partial_t(Q) + GQ = 0$ in $M_r(C_{1,K}^\dagger[[t]])$. Let $M^\vee$ (resp. $e_1^\vee, \dots, e_r^\vee$) be the dual of M in $\mathrm{Conn}^\dagger(\mathscr{X}/K)$ (resp. the dual basis of $e_1, \dots, e_r$). Then, $\frac{1}{i!}\nabla(\partial_1^i)(e_1, \dots, e_r) = (e_1, \dots, e_r){}^tG_i$ for any i. Hence, for any $\eta < 1$, there exists some $\lambda' > 1$ such that $\|G_i\|_{\mathscr{X},\lambda'}\eta^i \to 0$ $(i \to \infty)$ and the entries of Q are contained in $\mathscr{S}$. By the existence of the solution of the dual $M_{\mathscr{S}}^\vee$, Q is invertible in $M_r(\mathscr{S})$.

(2) By (1) $M_{\mathscr{S}}$ is isomorphic to $\mathscr{S}^r$ as $\mathscr{S}[\partial_t]$. So we have only to show that the $C_{1,K}^\dagger$-homomorphism

$$h : (\mathscr{R}/\mathscr{S})^r \to (\mathscr{R}/\mathscr{S})^r$$

which is defined by $\boldsymbol{a} \mapsto t\partial_t(\boldsymbol{a}) (\boldsymbol{a} \in (\mathscr{R}/\mathscr{S})^r)$ is bijective. The injectivity is trivial. Since $|i^{-1}|\eta^{-i} \to 0$ $(i \to -\infty)$ for any $\eta < 1$, h is surjective. □

(3.9) We globalize our local result. Let $\mathscr{X}$ be a quasi-projective smooth scheme of finite type over Spec V and let $\mathscr{D}$ be a relative normal crossing divisor over Spec V, that is, any intersection of irreducible components is smooth over Spec V after taking an etale covering of $\mathscr{X}$. We fix a completion $\bar{\mathscr{X}}$ of $\mathscr{X}$ over Spec V and put $\hat{\bar{\mathscr{X}}}$ to be the p-adic completion of $\bar{\mathscr{X}}$. Let X and D (resp. $\bar{X}$) be the special fiber of $\mathscr{X}$ and $\mathscr{D}$ (resp. the Zariski closure of X in $\bar{\mathscr{X}}$) and put $U = X - D$ with the open immersion $j_U : U \to \bar{X}$. Denote by $\Omega^l_{\mathscr{X}/\mathrm{Spec}\,V}(\mathscr{D})$ the l-th differential module of $\mathscr{X}$ over Spec V with logarithmic poles along $\mathscr{D}$ as in 2.1.

For an object $(\mathscr{M}, \nabla)$ in $\mathrm{Isoc}^{\dagger}(X/K)$, we define a complex $DR^{\bullet}((X,D)_{\hat{\bar{\mathscr{X}}}}/K, \mathscr{M})$ of K-sheaves on $]\bar{X}[_{\hat{\bar{\mathscr{X}}}}$ by

$$\cdots \to 0 \to \mathscr{M} \xrightarrow{\nabla} \mathscr{M} \otimes_{O_{\mathscr{X}}} \Omega^1_{\mathscr{X}/\mathrm{Spec}\,V}(\mathscr{D}) \xrightarrow{\nabla} \mathscr{M} \otimes_{O_{\mathscr{X}}} \Omega^2_{\mathscr{X}/\mathrm{Spec}\,V}(\mathscr{D}) \to \cdots.$$

Here we put $\mathscr{M}$ at the degree 0. We define

$$H^l_{rig}((X,D)_{\hat{\bar{\mathscr{X}}}}/K, \mathscr{M}) = \mathbf{R}^l\Gamma(]\bar{X}[_{\hat{\bar{\mathscr{X}}}}, DR^{\bullet}((X,D)_{\hat{\bar{\mathscr{X}}}}/K, \mathscr{M}))$$

for any l. In the case that $\mathscr{X}$ is affine, the cohomology above coincides with the logarithmic Monsky-Washnitzer cohomology.

Theorem 3.9.1. *With the notation as above, the natural morphism*

$$DR^{\bullet}((X,D)_{\hat{\bar{\mathscr{X}}}}/K, \mathscr{M}) \to DR^{\bullet}(j^{\dagger}_U \mathscr{M})$$

of complexes of K-sheaves on $]\bar{X}[_{\hat{\bar{\mathscr{X}}}}$ induces a K-isomorphism

$$H^l_{rig}((X,D)_{\hat{\bar{\mathscr{X}}}}/K, \mathscr{M}) \cong H^l_{rig}(U/K, j^{\dagger}_U \mathscr{M})$$

for any l.

Proof. Take a hyper etale covering $f : \mathscr{Y}^{\bullet} \to \mathscr{X}$ such that each piece of the pair $(\mathscr{Y}^{\bullet}, f^*\mathscr{D})$ satisfies the assumption of Theorem 3.5.1. By the similar argument of the proof of Proposition 2.3.1 and Lemma 3.6.1, the assertion follows from Proposition 3.2.2 and Theorem 3.5.1. □

Remark 3.9.2. It is expected to define the logarithmic rigid cohomology. If one uses such cohomology theory, the statement of Theorem 3.9.1 will become more functorial.

4. The Gysin isomorphism

(4.1) We keep the situation as in 3.1. For a subset $\underline{\mu} = \{\mu_1, \ldots, \mu_s\}$ of $\mathbf{Z}^s$ with $1 \leqq \mu_1 < \cdots < \mu_s \leqq d$, we put $\mathscr{D}_{\underline{\mu}} = \sum_{l=1}^{s} \mathscr{D}_{\mu_l}$ (resp. $\mathscr{U}_{\underline{\mu}} = \mathscr{X} - \mathscr{D}_{\underline{\mu}}$) to be a divisor (resp. an open subscheme) of $\mathscr{X}$ and denote by $j_{\underline{\mu}} : \mathscr{U}_{\underline{\mu}} \to \mathscr{X}$ the corresponding open immersion. We put $\mathscr{Z} = \mathrm{Spec}\, C = \mathrm{Spec}\, A/(t_1, \ldots, t_d)A$,

$\mathscr{U} = \mathscr{X} - \mathscr{Z}$, the closed immersion $i : \mathscr{Z} \to \mathscr{X}$, $C^\dagger = A^\dagger/(t_1, \dots, t_d)A^\dagger$ to be the weak completion of C over V and the natural surjection $i^\dagger : A^\dagger \to C^\dagger$. We denote the special fiber of $\mathscr{X}$, $\mathscr{U}_{\underline{\mu}}$, $\mathscr{Z}$ and $\mathscr{U}$ by X, $U_{\underline{\mu}}$, Z and U, respectively.

Let $\mathscr{X} \to \mathbf{P}_V^N$ be the immersion which is determined by the fixed presentation of $\mathscr{X}$ over Spec V as in 2.2. We denote by $\bar{\mathscr{X}}$ (resp. $\bar{\mathscr{Z}}$) the Zariski closure of $\mathscr{X}$ (resp. $\mathscr{Z}$) in $\mathbf{P}_V^N$ and put $\hat{\bar{\mathscr{X}}}$ (resp. $\hat{\bar{\mathscr{Z}}}$) to be the p-adic completion of $\bar{\mathscr{X}}$ (resp. $\bar{\mathscr{Z}}$). We put $\bar{X} = \bar{\mathscr{X}} \times_{\mathrm{Spec}\, V} \mathrm{Spec}\, k$ and $\bar{Z} = \bar{\mathscr{Z}} \times_{\mathrm{Spec}\, V} \mathrm{Spec}\, k$ and use the notation $j_U : U \to \bar{X}$ (resp. $j_{\underline{\mu}} : U_{\underline{\mu}} \to \bar{X}$, resp. $\bar{i} : \bar{Z} \to \bar{X}$) for the corresponding structure map.

In this section we define, for an object $(\mathscr{M}, \nabla)$ in $\mathrm{Isoc}^\dagger(X/K)$, a Gysin morphism

$$G_{Z/X} : \mathbf{R}\Gamma_{rig}(Z/K,]\bar{i}[^*\mathscr{M}) \to \mathbf{R}\Gamma_{Z,rig}(X/K, \mathscr{M})[2d]$$

in the derived category of complexes of K-vector spaces and prove the Theorem 4.1.1 below. Here $]\bar{i}[^*\mathscr{M}$ is the inverse image of $\mathscr{M}$ in $\mathrm{Isoc}^\dagger(Z/K)$ defined in 2.4.

Theorem 4.1.1. *With the notation as above, the Gysin morphism $G_{Z/X}$ is an isomorphism. In other words, the induced K-homomorphism*

$$G_{Z/X} : H^l_{rig}(Z/K,]\bar{i}[^*\mathscr{M}) \to H^{l+2d}_{Z,rig}(X/K, \mathscr{M}).$$

is an isomorphism. Moreover, if $(\mathscr{M}, \nabla, \Phi)$ is an object in F-$\mathrm{Isoc}^\dagger(X/K, \sigma^a)$, the Gysin morphism induces the isomorphism

$$G_{Z/X} : H^l_{rig}(Z/K,]\bar{i}[^*\mathscr{M}) \to H^{l+2d}_{Z,rig}(X/K, \mathscr{M})(d)$$

with Frobenius structure for any l. Here (d) means the d-th twist of the Frobenius structure, that is, the multiplication of the Frobenius structure with p^{-ad}.

Theorem 4.1.1 follows from Corollary 4.2.3 and Proposition 4.3.1 below. We will construct the Gysin isomorphism for unit-root objects in general cases using Poincaré duality in 6.2. We also prove that our Gysin morphism coincides with the one in [4, Sect. 5] in 6.2.

Corollary 4.1.2. *Let X be a smooth scheme of finite type and pure of dimension n over* Spec k *and let Z be a closed k-subscheme of codimension $\geqq d$ in X. If $(\mathscr{M}, \nabla)$ is an object in* $\mathrm{Isoc}^\dagger(X/K)$, *then $H^l_{Z,rig}(X/K, \mathscr{M}) = 0$ for $l < 2d$ and for $l > 2n$.*

Proof. We prove the assertion by induction on $n - d$. Since the rigid cohomology with supports in Z does not change if we replace Z into the reduced subscheme Z^{red} of Z, we may assume that Z is smooth and connected over Spec k by Proposition 2.1.1, 2.1.2 and the hypothesis of induction. If one takes an affine open subscheme Z' of Z over Spec k, then the codimension of $Z - Z'$ in $X - Z'$ is greater than d. So we can assume the situation as in

Theorem 4.1.1. Therefore, the assertion follows from the Gysin isomorphism

$$G_{Z/X} : H^l_{rig}(Z/K,]\bar{i}[^*\mathcal{M}) \to H^{l+2d}_{Z,rig}(X/K, \mathcal{M})$$

and the fact $H^l_{rig}(Z/K, \mathcal{M}) = 0$ for $l < 0$ and for $l > n$ since Z is affine. $\square$

(4.2) We define a double complex $DR^\bullet(j^\dagger_\bullet \mathcal{M})$ of sheaves of K-spaces on $]\bar{X}[_{\hat{\mathscr{X}}}$ by the Čech complex

$$\prod_{\mu_1} DR^\bullet(j^\dagger_{\mu_1}\mathcal{M}) \to \prod_{\mu_1<\mu_2} DR^\bullet(j^\dagger_{\mu_1\mu_2}\mathcal{M}) \to \cdots \to DR^\bullet(j^\dagger_{12\cdots d}\mathcal{M})$$

for the covering $\{U_\bullet\}$ of U, where we put $\prod_{\mu_1} j^\dagger_{\mu_1}\mathcal{M}$ at the bidegree $(0,0)$. By [5, Proposition 2.1.8] we have

Proposition 4.2.1. *The natural morphism $DR^\bullet(j^\dagger_U\mathcal{M}) \to DR^\bullet(j^\dagger_\bullet\mathcal{M})$ of complexes of sheaves on $]\bar{X}[_{\hat{\mathscr{X}}}$ induces an isomorphism*

$$\mathbf{R}\Gamma_{rig}(U/K, j^\dagger_U\mathcal{M}) \to \mathbf{R}\Gamma(]\bar{X}[_{\hat{\mathscr{X}}}, Tot(\mathscr{DR}^\bullet(j^\dagger_\bullet\mathcal{M}))).$$

in the derived category of complexes of K-vector spaces.

Let $(\mathcal{M}, \nabla)$ be an object in $\mathrm{Isoc}^\dagger(X/K)$ and put $(M, \nabla) = \Gamma(]\bar{X}[_{\hat{\mathscr{X}}}, (\mathcal{M}, \nabla))$. We define a double complex $DR^\bullet(j^{log}_\bullet M)$ of K-vector spaces which corresponds to the Čech complex for the covering $\mathscr{U}_\bullet$ of $\mathscr{U}$:

$$\prod\nolimits_{\mu_1} DR^\bullet((\mathscr{X}, \mathscr{D}_{\mu_1})/K, j^{log}_{\mu_1}M) \to \prod\nolimits_{\mu_1<\mu_2} DR^\bullet((\mathscr{X}, \mathscr{D}_{\mu_1\mu_2})/K, j^{log}_{\mu_1\mu_2}M) \to$$
$$\cdots \to DR^\bullet((\mathscr{X}, \mathscr{D})/K, j^{log}M),$$

where we put $\prod_{\mu_1} j^{log}_{\mu_1}M$ at the bidegree $(0,0)$. We denote by $H^l_{MW}((U_\bullet, D_\bullet)/K, M)$ the l-th cohomology of the total complex of $DR^\bullet(j^{log}_\bullet M)$. If φ is a Frobenius on $A^\dagger$ which satisfies the condition 3.1.3, the Frobenius structure Φ on (M, ∇) induces the Frobenius structure on $H^l_{MW}((U_\bullet, D_\bullet)/K, M)$ for any object $(\mathcal{M}, \nabla, \Phi)$ in $F\text{-}\mathrm{Isoc}^\dagger(X/K, \sigma^a)$.

Theorem 4.2.2. *With the notation as above, there is a natural isomorphism*

$$\mathbf{R}\Gamma_{rig}(U/K, j^\dagger_U\mathcal{M}) \to Tot(DR^\bullet(j^{log}_\bullet M))$$

in the derived category of complexes of K-vector spaces such that the induced diagram

$$\begin{array}{ccc} H^l_{rig}(X/K, \mathcal{M}) & \longrightarrow & H^l_{rig}(U/K, j^\dagger_U\mathcal{M}) \\ \downarrow & & \uparrow \\ H^l_{MW}(X/K, M) & \longrightarrow & H^l_{MW}((U_\bullet, D_\bullet)/K, M) \end{array}$$

is commutative. Here the top horizontal arrow is the restriction, the bottom horizontal arrow is defined by the natural inclusion $DR^\bullet(M) \to DR^\bullet(j_\bullet^{log} M)$ *of complexes and the left vertical arrow is the comparison isomorphism between the rigid cohomology and the Monsky-Washnitzer cohomology in [5, Proposition 2.5.2]. For an object in* F-$\mathrm{Isoc}^\dagger(X/K, \sigma^a)$, *the commutative square above commutes with Frobenius structures.*

Proof. By Proposition 2.2.1 there exists a canonical isomorphism

$$\mathbf{R}\Gamma_{rig}(U_{\underline{\mu}}/K, j_{\underline{\mu}}^\dagger \mathscr{M}) \to DR^\bullet(j_{\underline{\mu}}^\dagger M)$$

for any multi index $\underline{\mu}$. The existence of the canonical isomorphism follows from Theorem 3.5.1 and Proposition 4.2.1. The commutativity of the Frobenius structures follows from the fact that the Frobenius structure on the rigid cohomology is independent of the choice of the embedding into formal schemes and the lift of Frobenius. □

Corollary 4.2.3. *The isomorphism in Theorem* 4.2.2 *induces an isomorphism*

$$\mathbf{R}\Gamma_{Z,rig}(X/K, \mathscr{M}) \to Cone(DR^\bullet(M) \to Tot(DR^\bullet(j_\bullet^\dagger M)))[-1]$$

in the derived category of complexes of K*-vector spaces. For an object in* F-$\mathrm{Isoc}(X/K, \sigma^a)$, *the induced isomorphism of cohomologies commutes with the Frobenius structures of both sides.*

(4.3) Let (M, ∇) be an object in $\mathrm{Conn}^\dagger(\mathscr{X}/K)$. We define a morphism

$$Res_{\mathscr{Y}/\mathscr{X}} : DR^\bullet(j_\bullet^{log} M) \to DR^\bullet(i^\dagger M)[-d]$$

of complexes of K-vector spaces by 0 at degree $l < d$ and by

$$\sum_{\mu_1 < \cdots < \mu_l} m_{\mu_1 \cdots \mu_l} \otimes \omega_{\mu_1 \cdots \mu_l} \mapsto (-1)^{(l-d)d} \sum_{\mu_i = i\ (1 \leqq i \leqq d)} i^\dagger(m_{12 \cdots d \mu_{l_{d+1}} \cdots \mu_l}) \otimes \omega_{\mu_{l_{d+1}} \cdots \mu_l}$$

at degree $l \geqq d$. Here $i^\dagger : M \to i^\dagger M$ is the projection, $\omega_\mu = \dfrac{dt_\mu}{t_\mu}$ for $\mu \leqq d$, $\omega_\mu = dt_\mu$ for $\mu > d$ and $\omega_{\mu_1 \cdots \mu_l} = \omega_{\mu_1} \wedge \cdots \wedge \omega_{\mu_l}$. Note that $Res_{\mathscr{Y}/\mathscr{X}} = 0$ at degree $l < d$ and at degree $l > n$. One can easily check that $Res_{\mathscr{Y}/\mathscr{X}}$ is a morphism of complexes of K-vector spaces. If φ is a Frobenius on $A^\dagger$ which satisfies the condition 3.1.3, then $Res_{\mathscr{Y}/\mathscr{X}}$ induces a morphism

$$Res_{\mathscr{Y}/\mathscr{X}} : DR^\bullet(j_\bullet^{log} M) \to DR^\bullet(i^\dagger M)[-d](-d)$$

of complexes which commutes with Frobenius structure Φ, where $(-d)$ means the twist of Frobenius structure $i^\dagger\Phi$ by $\dfrac{(\varphi^a)^*\omega_{12\cdots d}}{\omega_{12\cdots d}} \in A^\dagger$. Note that, if $\varphi(t_\mu) = t_\mu^p$ for $1 \leqq \mu \leqq d$, then $\dfrac{(\varphi^a)^*\omega_{12\cdots d}}{\omega_{12\cdots d}} = p^{ad}$.

Proposition 4.3.1. *With the notation as above, if (M, ∇) is an object in* $\mathrm{Conn}^\dagger(\mathscr{X}/K)$, *then $Res_{\mathscr{Y}/\mathscr{X}}$ induces a quasi-isomorphism*

$$Cone(DR^\bullet(M) \longrightarrow Tot(DR^\bullet(j_\bullet^{log}M)))[-1] \xrightarrow{Res[-d]} DR^\bullet(i^\dagger M)[-2d]$$

of complexes of K-vector spaces. If φ is a Frobenius on $A^\dagger$ which satisfies the condition 3.1.3, *then the quasi-isomorphism above commutes with Frobenius structures for any object in F-*$\mathrm{Conn}^\dagger(\mathscr{X}/K, \varphi^a)$.

Proposition 4.3.1 follows easily from Lemma 4.3.2 below.

Lemma 4.3.2. *The sequence*

$$0 \longrightarrow \Omega^l_{A_K^\dagger/K} \longrightarrow \Pi_{1\leqq\mu_1\leqq d}\Omega^l_{A_K^\dagger/K}(\mathscr{D}_{\mu_1}) \longrightarrow \Pi_{1\leqq\mu_1<\mu_2\leqq d}\Omega^l_{A_K^\dagger/K}(\mathscr{D}_{\mu_1\mu_2}) \longrightarrow \cdots$$
$$\longrightarrow \Omega^l_{A_K^\dagger/K}(\mathscr{D}_{12\cdots d}) \xrightarrow{Res^l} \Omega^{l-d}_{C_K^\dagger/K} \longrightarrow 0$$

is exact for any l.

Proof. Denote by $E(d,l)$ $(d \geqq 0)$ the complex

$$0 \to \Omega^l_{A_K^\dagger/K} \to \coprod_{1\leqq\mu_1\leqq d}\Omega^l_{A_K^\dagger/K}(\mathscr{D}_{\mu_1}) \to \coprod_{1\leqq\mu_1<\mu_2\leqq d}\Omega^l_{A_K^\dagger/K}(\mathscr{D}_{\mu_1\mu_2}) \to \cdots$$
$$\to \Omega^l_{A_K^\dagger/K}(\mathscr{D}_{12\cdots d}) \to 0,$$

where $\Omega^l_{A_K^\dagger/K}(\mathscr{D}_{12\cdots d})$ is at the degree 0. We prove that $H^0(E(d,l)) = \Omega^{l-d}_{C_K^\dagger/K}$ and $H^m(E(d,l)) = 0$ for any $m \neq 0$ by induction on d. One can easily see that there is a natural exact sequence

$$0 \to E(d-1,l) \to E(d,l) \to E(d-1,l)[1] \to 0$$

of complexes of $A_K^\dagger$-modules, where the first map is defined by $dt_d \mapsto \dfrac{dt_d}{t_d}$ and the second map is defined by the projection. If we denote by $C' = A/(t_1,\ldots,t_{d-1})A$, then the connecting homomorphism $H^{-1}(E(d-1,l)[1]) \to H^0(E(d-1,l))$ is a homomorphism

$$\Omega^l_{(C')_K^\dagger/K} \to \Omega^l_{(C')_K^\dagger/K}$$

given by $dt_d \mapsto t_d dt_d$ and $dt_\mu \mapsto dt_\mu$ $(\mu \neq d)$ from the hypothesis of induction. This completes the proof. $\square$

5. Comparison theorem between the crystalline cohomology and the rigid cohomology

(5.1) Let $j : X \to \overline{X}$ be an open immersion of separated k-scheme of finite type and let $\overline{X} \to \hat{\mathscr{P}}$ be a closed immersion with a formal V-scheme $\hat{\mathscr{P}}$ of finite type such that $\hat{\mathscr{P}}$ is smooth over Spf V around X.

Let K' be a finite extension of K and keep the notation as in Proposition 2.1.2. Since $\tau_{K'/K} :]\overline{X}'[_{\hat{\mathscr{P}}'} \to]\overline{X}[_{\hat{\mathscr{P}}}$ is finite etale as rigid spaces, the Proposition below can be proved using the similar methods as in Theorem 2.6.3 and 2.7.

Proposition 5.1.1. (1) *$\mathbf{R}^l \tau_{K'/K*}\mathscr{M} = 0$ for any $l \neq 0$.*

(2) *$\tau_{K'/K}$ induces the direct image functor*

$$\tau_{K'/K*} : \mathrm{Isoc}^\dagger(X'/K') \to \mathrm{Isoc}^\dagger(X/K)$$

and it is a right adjoint of $\tau^*_{K'/K}$.

(3) *If $\sigma'; K' \to K'$ is the extension of the Frobenius σ, then $\tau_{K'/K}$ induces the direct image functor*

$$\tau_{K'/K*} : F\text{-}\mathrm{Isoc}^\dagger(X'/K', (\sigma')^a) \to F\text{-}\mathrm{Isoc}^\dagger(X/K, \sigma^a)$$

and it is a right adjoint of $\tau^*_{K'/K}$.

Corollary 5.1.2. *Let $(\mathscr{M}, \nabla)$ be an object in* $\mathrm{Isoc}^\dagger(X'/K')$.

(1) *If X is smooth over* Spec k *and Z is a closed subscheme of X over* Spec k, *we have a canonical isomorphism*

$$\mathbf{R}\Gamma_{Z,rig}(X/K, \tau_{K'/K*}\mathscr{M}) \to \mathbf{R}\Gamma_{Z,rig}(X'/K', \mathscr{M})$$

of K-complexes.

(2) *We have a canonical isomorphism*

$$\mathbf{R}\Gamma_{c,rig}(X/K, \tau_{K'/K*}\mathscr{M}) \to \mathbf{R}\Gamma_{c,rig}(X'/K', \mathscr{M})$$

of K-complexes.

Moreover, for any object $(\mathscr{M}, \nabla, \Phi)$ in F-$\mathrm{Isoc}^\dagger(X/K, (\sigma')^a)$, the isomorphisms in (1) *and* (2) *induce isomorphisms of K-vector spaces with Frobenius structures with respect to σ^a on each cohomology group.*

(5.2) We denote by K_0 (resp. V_0, resp. e) an absolutely unramified subfield of K with the residue field k, i.e., K_0 is the maximal subfield such that p is a uniformizer (resp. the integer ring of K_0, resp. the ramification index $e = [K : K_0] < \infty$). We assume that the Frobenius σ on K is an extension of a Frobenius σ_0 on K_0.

Theorem 5.2.1. *With the notation as above, assume furthermore that X is proper smooth over* Spec k. *For an object $(\mathscr{M}, \nabla, \Phi)$ in F-$\mathrm{Isoc}(X/K, (\sigma)^a)$, there* *exists a non-degenerated F-crystal* $(\mathscr{L}, \Phi)$ *on* X/V_0 *with respect to* σ_0^a *and a non-negative integer* s *such that* $\tau_{K/K_0*}(\mathscr{M}, \nabla, \Phi) \cong (\mathscr{L}, \Phi')^{an}(s)$. *Here* (s) *is the s-th Tate twist, that is, the Frobenius acts by* $p^{-s}\Phi'$. *If we choose such* $(\mathscr{L}, \Phi')$ *and* s, *then we have a* K_0*-isomorphism*

$$\mathbf{R}\Gamma_{rig}(X/K_0, \tau_{K/K_0*}\mathscr{M}) \to \mathbf{R}\Gamma_{crys}(X/V_0, \mathscr{L}) \otimes_{V_0} K_0.$$

Moreover, the induced morphism

$$H^l_{rig}(X/K_0, \tau_{K/K_0*}\mathscr{M}) \to H^l_{crys}(X/V_0, \mathscr{L}) \otimes_{V_0} K_0(s)$$

is a K_0*-isomorphism with Frobenius structures with respect to* σ_0^a *for any* l.

Proof. By Proposition 5.1.1 and Corollary 5.1.2 we may assume that $K = K_0$. The existence of $(\mathscr{L}, \Phi')$ and s follows from [5, Théorème 2.4.2]. If we denote by $\hat{\mathscr{P}}_K$ (resp. $\mathrm{sp} : \hat{\mathscr{P}}_K \to \hat{\mathscr{P}}$) the rigid analytic space over K associated to $\hat{\mathscr{P}}$ in the sense of Raynaud (resp. the specialization morphism), then there is a natural isomorphism

$$\begin{aligned}\mathbf{R}\Gamma_{rig}(X/K, \mathscr{M}) &\cong \mathbf{R}\Gamma(\hat{\mathscr{P}}, \mathbf{R}\mathrm{sp}_*(DR^\bullet(\mathscr{M}))) \\ &\cong \mathbf{R}\Gamma(\hat{\mathscr{P}}, \mathrm{sp}_*\mathscr{M} \otimes_{O_{\hat{\mathscr{P}}}} \Omega^\bullet_{\hat{\mathscr{P}}/\mathrm{Spf}\, V}).\end{aligned}$$

Let $\hat{\mathscr{P}}^{PD}$ be the p-adic completion of the divided power envelope of $\hat{\mathscr{P}}$ by the ideal of definition of X. If we denote by $u_{X/V} : (X/V)^\sim_{crys} \to X^\sim_{Zar}$ the canonical morphism from the crystalline topos to the Zariski topos, then there is a natural isomorphism

$$\begin{aligned}\mathbf{R}\Gamma_{crys}(X/V, \mathscr{L}) &\cong \mathbf{R}\Gamma(X, \mathbf{R}u_{X/V*}\mathscr{L}) \\ &\cong \mathbf{R}\Gamma(X, u_{X/V*}\mathscr{L} \otimes_{O_{\hat{\mathscr{P}}}} \Omega^\bullet_{\hat{\mathscr{P}}^{PD}/\mathrm{Spf}\, V}).\end{aligned}$$

by [2, Chapitre V, Théorème 2.3.2]. The comparison morphism is induced by the canonical morphism $\mathrm{sp}_*\mathscr{M} \to u_{X/V*}\mathscr{L} \otimes_V K$ of sheaves on X which commutes with connections and Frobenius structures in [5, 2.4]. It is isomorphic by the argument of the spectral sequence for Čech covering of X. (See [6, Theorem 1.9].) □

6. The finiteness theorem for overconvergent unit-root F-isocrystals

(6.1) We prove the finiteness theorem of rigid cohomologies for overconvergent unit-root F-isocrystals. In the case of the constant coefficient it was proved in [6] and in the case of curves it was proved in [10].

Let $j : X \to \bar{X}$ be an open immersion of separated k-scheme of finite type of dimension n. Let Z (resp. $\bar{Z}$) be a closed subscheme of X over $\mathrm{Spec}\, k$ (resp.

the closure of Z in $\overline{X}$) and denote by $i : Z \to X$ and $\overline{i} : \overline{Z} \to \overline{X}$ the closed immersion, respectively.

Let a be a positive integer. We say that an object $(\mathcal{M}, \nabla, \Phi)$ in $F\text{-Isoc}^{\dagger}(X/K, \sigma^a)$ is unit-root if and only if, for any geometrically closed point $i_{\bar{s}} : \bar{s} \to X$, there is a basis $\{e_1, e_2, \dots, e_r\}$ of $i_{\bar{s}}^* \mathcal{M}$ such that $i_{\bar{s}}^* \Phi(1 \otimes e_v) = e_v$. We denote the category of overconvergent unit-root F-isocrystals on X/K with respect to σ^a by $F\text{-Isoc}^{\dagger}(X/K, \sigma^a)^0$.

THEOREM 6.1.1. *With the notation as above, assume furthermore that k is perfect, X is smooth over* $\operatorname{Spec} k$ *and $(\mathcal{M}, \nabla, \Phi)$ is an object in* $F\text{-Isoc}^{\dagger}(X/K, \sigma^a)^0$.

(1) *The rigid cohomology $H^l_{Z,rig}(X/K, \mathcal{M})$ with supports in Z is of finite dimension over K for any l.*

(2) *With the notation as in* 2.1.2, *if K'/K is an extension of complete discrete valuation fields (possibly infinite) and the Frobenius σ extends on K', the induced homomorphism*

$$\tau^*_{K'/K} : H^l_{Z,rig}(X/K, \mathcal{M}) \otimes_K K' \to H^l_{Z',rig}(X'/K', \mathcal{M}')$$

is an isomorphism of K'-vector spaces with Frobenius structures.

PROOF. (1) The argument of the proof is the same as in [6, Théorème 3.1]. We prove two assertions;

$(a)_d$: $H^l_{rig}(X/K, \mathcal{M})$ is of finite dimension over K for the dimension $X \leqq d$;

$(b)_d$: $H^l_{Z,rig}(X/K, \mathcal{M})$ is of finite dimension over K for the dimension $Z \leqq d$;

by induction on d simultaneously. The assertion $(a)_0$ is trivial.

We prove $(a)_d \Rightarrow (b)_d$. Since the rigid cohomology with supports in Z does not change if we replace Z into the reduced subscheme Z^{red} of Z, we may assume that Z is smooth over $\operatorname{Spec} k$ by Proposition 2.1.1, 2.1.2 and the hypothesis of induction. We can also assume the situation of the pair (X, Z) as in Theorem 4.1.1. Therefore, the assertion follows from the Gysin isomorphism.

We prove $(b)_d \Rightarrow (a)_{d+1}$. By [20, Theorem 1.3.1] one can find a smooth scheme X' over $\operatorname{Spec} k$ with a smooth compactification $j' : X' \to \overline{X}'$ and a generically etale proper surjective morphism $f : X' \to X$ and find a convergent unit-root F-isocrystal $\mathcal{N}$ on $\overline{X}'/K$ with respect to σ^a such that $f^*_{rig}\mathcal{M} \cong (j')^{\dagger}\mathcal{N}$. Since the crystalline cohomology is of finite dimension [2, Chapitre VII, Corollaire 1.1.2], the assertion $(b)_d \Rightarrow (a)_{d+1}$ follows from Proposition 2.1.1, 2.1.2, 2.6.5, Corollary 5.1.2, Theorem 5.2.1 and the hypothesis of induction.

(2) The assertion follows from the same argument as in the proof of (1) and the fact that the crystalline cohomology commutes with the arbitrary extension of the base field [2, Chapitre VII, Proposition 1.1.8]. □

Theorem 6.1.2. *With the notation as above, assume furthermore that k is perfect and let $(\mathcal{M}, \nabla, \Phi)$ be an object in F-$\mathrm{Isoc}^{\dagger}(X/K, \sigma^a)^0$.*

(1) *The rigid cohomology $H^l_{c,rig}(X/K, \mathcal{M})$ with compact supports is of finite dimension over K for any l.*

(2) *With the notation as in 2.5.2, if K'/K is an extension of complete discrete valuation fields (possibly infinite), the induced homomorphism*

$$H^l_{c,rig}(X/K, \mathcal{M}) \otimes_K K' \to H^l_{c,rig}(X'/K', \mathcal{M}')$$

is an isomorphism of K'-vector spaces with Frobenius structures.

Proof. We prove the finiteness $H^l_{c,rig}(X/K, \mathcal{M})$ by induction of the dimension of X. The rigid cohomology with compact supports is the same if we replace X into the reduced subscheme X^{red} in X. By Proposition 2.5.1 and 2.5.2. we may assume that X is smooth. By [20, Theorem 1.3.1], Proposition 2.5.1 and 2.6.6 we may assume that X is proper. The assertion follows from Corollary 5.1.2, Theorem 5.2.1 and the finiteness of the crystalline cohomology. The rest is the same as in Theorem 6.1.1. □

(6.2) We study Poincaré duality of the rigid cohomology. In the case of the constant coefficient it was proved in [7] and in the case of curves it was proved in [10]. First we recall the definition of the pairing in [7, Sect. 3]. Keep the notation in 6.1 and assume that X is pure of dimension n over $\mathrm{Spec}\, k$. We have $H^l_{c,rig}(X/K, j^{\dagger}O_{]\bar{X}[}) = 0$ for $l > 2n$ and there is a canonical trace map

$$Tr_X : H^{2n}_{c,rig}(X/K, j^{\dagger}O_{]\bar{X}[}) \to K$$

by [7, Proposition 2.1, 2.6]. If we also consider Frobenius structures, the trace map

$$Tr_X : H^{2n}_{c,rig}(X/K, j^{\dagger}O_{]\bar{X}[}) \to K(-n)$$

commutes with the Frobenius structures with respect to σ^a by the theorem of alteration [15, Theorem 3.1], Proposition 2.5.1, Corollary 2.6.6 and Theorem 5.2.1. Here $K(-n)$ is the one dimensional K-vector space with the Frobenius structure $\Phi_{K(-n)} = p^{an}\sigma^a$.

Let $(\mathcal{M}, \nabla)$ be an object in $\mathrm{Isoc}^{\dagger}(X/K)$ and let $(\mathcal{M}^{\vee}, \nabla^{\vee})$ be the dual of $(\mathcal{M}, \nabla)$. The morphism

$$\underline{\Gamma}^{\dagger}_{]Z[}(\mathcal{M}) \otimes_K \underline{\Gamma}_{]Z[}(]\bar{i}[^* \mathcal{M}^{\vee}) \to \underline{\Gamma}_{]X[}(j^{\dagger}O_{]\bar{X}[})$$

of sheaves on $]\bar{X}[_{\mathscr{P}}$ which is defined by the multiplication induces a pairing

$$\mathbf{R}\Gamma_{Z,rig}(X/K, \mathcal{M}) \otimes_K \mathbf{R}\Gamma_{c,rig}(Z/K,]\bar{i}[^* \mathcal{M}) \to \mathbf{R}\Gamma_{c,rig}(X/K, j^{\dagger}O_{\bar{X}}).$$

in the derived category of complexes of K-vector spaces. The induced morphisms of rigid cohomology groups commute with Frobenius structures. Composing with the trace map Tr_X and by Corollary 4.1.2, we can define a morphism

$$\eta_{Z,X} : \mathbf{R}\Gamma_{Z,rig}(X/K, \mathcal{M}) \to \mathbf{R}\operatorname{Hom}_K(\mathbf{R}\Gamma_{c,rig}(Z/K,]\bar{i}[^*\mathcal{M}), K)[-2n]$$

in the derived category of complexes of K-vector spaces bounded above.

If we put $U = X - Z$ (resp. $j_U : U \to \overline{X}$), then the trace maps Tr_X and Tr_U commute with the natural map $H^{2n}_{c,rig}(U/K, j_U^\dagger O_{]\overline{X}[}) \to H^{2n}_{c,rig}(X/K, j^\dagger O_{]\overline{X}[})$. Hence, we have

LEMMA 6.2.1. *With the notation as above, there is a morphism*

$$(\eta_{Z,X}, \eta_{X,X}, \eta_{U,U}) : \Delta_{rig}(X, Z, \mathcal{M}) \to \mathbf{R}\operatorname{Hom}_K(\Delta_{c,rig}(X, Z, \mathcal{M}), K)[-2n]$$

of distinguished triangles.

LEMMA 6.2.2. *With the notation as above, assume that X is smooth over* $\operatorname{Spec} k$. *Let $f : Y \to X$ be a finite etale morphism of degree r and put $\overline{Y}$ (resp. $j_Y : Y \to \overline{Y}$) to be the normalization of $\overline{X}$ in Y (resp. the open immersion).*

(1) *For any object $\mathcal{N}$ in* $\operatorname{Isoc}^\dagger(Y/K)$, *the natural morphism*

$$f_{rig*}\mathcal{N} \otimes f_{rig*}\mathcal{N}^\vee \to f_{rig*}(\mathcal{N} \otimes \mathcal{N}^\vee) \to f_{rig*}j_Y^\dagger O_{]\overline{Y}[} \xrightarrow{tr} j_X^\dagger O_{]\overline{X}[}$$

induces a duality $(f_{rig*}\mathcal{N})^\vee \cong f_{rig*}\mathcal{N}^\vee$ in $\operatorname{Isoc}^\dagger(Y/K)$ *and a commutative diagram*

$$\begin{array}{ccc} \mathbf{R}\Gamma_{rig}(X/K, f_{rig*}\mathcal{N}) & \xrightarrow{\eta_{X,X}} & \mathbf{R}\operatorname{Hom}_K(\mathbf{R}\Gamma_{c,rig}(X/K, f_{rig*}\mathcal{N}^\vee), K)[-2n] \\ \downarrow & & \uparrow \\ \mathbf{R}\Gamma_{rig}(Y/K, \mathcal{N}) & \xrightarrow{\eta_{Y,Y}} & \mathbf{R}\operatorname{Hom}_K(\mathbf{R}\Gamma_{c,rig}(Y/K, \mathcal{N}^\vee), K)[-2n], \end{array}$$

where the vertical arrows are the isomorphisms in Theorem 2.6.3.

(2) *For any object $\mathcal{N}$ in* $\operatorname{Isoc}^\dagger(Y/K)$, *the adjoint map* $\mathrm{ad} : \mathrm{id} \to f_{rig*}f^*_{rig}$ *and the trace map $tr : f_{rig*}f^*_{rig} \to$* id *in* 2.7 *induce commutative diagrams*

$$\begin{array}{ccc} \mathbf{R}\Gamma_{rig}(X/K, \mathcal{M}) & \xrightarrow{\eta_{X,X}} & \mathbf{R}\operatorname{Hom}_K(\mathbf{R}\Gamma_{c,rig}(X/K, \mathcal{M}^\vee), K)[-2n] \\ {\scriptstyle \mathrm{ad}}\downarrow & & \downarrow{\scriptstyle {}^t tr} \\ \mathbf{R}\Gamma_{rig}(X/K, f_{rig*}f^*_{rig}\mathcal{M}) & \xrightarrow{\eta_{X,X}} & \mathbf{R}\operatorname{Hom}_K(\mathbf{R}\Gamma_{c,rig}(X/K, f_{rig*}f^*_{rig}\mathcal{M}^\vee), K)[-2n] \end{array}$$

$$\begin{array}{ccc} \mathbf{R}\Gamma_{rig}(X/K, f_{rig*}f^*_{rig}\mathcal{M}) & \xrightarrow{\eta_{X,X}} & \mathbf{R}\operatorname{Hom}_K(\mathbf{R}\Gamma_{c,rig}(X/K, f_{rig*}f^*_{rig}\mathcal{M}^\vee), K) \\ {\scriptstyle tr}\downarrow & & \downarrow{\scriptstyle {}^t\mathrm{ad}} \\ \mathbf{R}\Gamma_{rig}(X/K, \mathcal{M}) & \xrightarrow{\eta_{X,X}} & \mathbf{R}\operatorname{Hom}_K(\mathbf{R}\Gamma_{c,rig}(X/K, \mathcal{M}^\vee), K). \end{array}$$

*Here ${}^t tr$ (resp. ^{t}*ad*) is the transpose of tr (resp.* ad*).*

(3) *Let K' be a finite extension over K. For an object $(\mathscr{M}, \nabla)$ in* $\mathrm{isoc}^{\dagger}(X/K)$ *(resp. $(\mathscr{M}', \nabla)$ in* $\mathrm{isoc}^{\dagger}(X'/K')$*), the diagram*

$$\begin{tikzcd}
\mathbf{R}\Gamma_{rig}(X/K, \mathscr{M}) \otimes_K K' \arrow[r, "\eta_{X,X}"] \arrow[d] & \mathbf{R}\,\mathrm{Hom}_{K'}(\mathbf{R}\Gamma_{c,rig}(X/K, \mathscr{M}^{\vee}) \otimes_K K', K')[-2n] \arrow[d] \\
\mathbf{R}\Gamma_{rig}(X'/K', \tau_{K'/K}^{*}\mathscr{M}) \arrow[r, "\eta_{X,X}"] & \mathbf{R}\,\mathrm{Hom}_{K'}(\mathbf{R}\Gamma_{c,rig}(X'/K', \tau_{K'/K}^{*}\mathscr{M}^{\vee}), K')[-2n]
\end{tikzcd}$$

(resp.

$$\begin{tikzcd}
\mathbf{R}\Gamma_{rig}(X/K, \tau_{K'/K*}\mathscr{M}') \arrow[r, "\eta_{X,X}"] \arrow[d] & \mathbf{R}\,\mathrm{Hom}_{K}(\mathbf{R}\Gamma_{c,rig}(X/K, \tau_{K'/K*}(\mathscr{M}')^{\vee}), K) \\
\mathbf{R}\Gamma_{rig}(X'/K', (\mathscr{M}')) \arrow[r, "\eta_{X,X}"] & \mathbf{R}\,\mathrm{Hom}_{K'}(\mathbf{R}\Gamma_{c,rig}(X'/K', (\mathscr{M}')^{\vee}), K')) \arrow[u, "tr_{K'/K}"']
\end{tikzcd}$$

is commutative. Here the vertical arrows are defined by the morphism in 5.1 *(resp. by the morphism in* 5.1 *and the trace map $tr_{K'/K} : K' \to K$).*

Proof. The assertions (1) and (2) follow from the commutativities $Tr_Y = Tr_X \circ tr$ and $tr \circ \mathrm{ad} = r\mathrm{id}$. The assertion (3) follows from the commutativity $tr_{K'/K} \circ Tr_{X/K'} = Tr_{X/K} \circ \tau_{K'/K*}$. $\square$

Lemma 6.2.3. *With the notation as above, assume furthermore that there is an affine smooth lift of the pair (X, Z) over* Spec V *which satisfies the situation in* 4.1. *If we denote by d the codimension of Z in X, then the diagram*

$$\begin{tikzcd}
\mathbf{R}\Gamma_{rig}(Z/K,]\bar{i}[^{*}\mathscr{M}) \arrow[r, "\eta_{Z,Z}"] \arrow[d, "G_{Z/X}"'] & \mathbf{R}\,\mathrm{Hom}_{K}(\mathbf{R}\Gamma_{c,rig}(Z/K,]\bar{i}[^{*}\mathscr{M}^{\vee}), K)[-2(n-d)] \arrow[d] \\
\mathbf{R}\Gamma_{Z,rig}(X/K, \mathscr{M})[2d] \arrow[r, "\eta_{Z,X}"] & \mathbf{R}\,\mathrm{Hom}_{K}(\mathbf{R}\Gamma_{c,rig}(Z/K,]\bar{i}[^{*}\mathscr{M}^{\vee}), K)[-2(n-d)]
\end{tikzcd}$$

is commutative, where the left vertical arrow is the Gysin morphism and the right vertical arrow is the identity.

Proof. Put $M = \Gamma(]\bar{X}[_{\hat{\mathscr{P}}}, \mathscr{M})$ and $M^{\vee} = \Gamma(]\bar{X}[_{\hat{\mathscr{P}}}, \mathscr{M}^{\vee})$. Then M is defined on some strict neighbourhood of $]X[_{\hat{\mathscr{P}}}$ such that $]\bar{Z}[_{\hat{\mathscr{P}}}$ is smooth over K in the neighbourhood. If W is an open affinoid in $]\bar{X} - X[_{\hat{\mathscr{P}}}$, then the diagram

$$\begin{tikzcd}
DR^{\bullet}(j_{\bullet}^{log}M) \otimes_K [DR^{\bullet}(j_{\bullet}^{log}M^{\vee}) \to DR^{\bullet}(j_{\bullet}^{log}M^{\vee}|_W)] \arrow[r] \arrow[d, "Res_{\mathscr{Y}/\mathscr{X}} \otimes Res_{\mathscr{Y}/\mathscr{X}}"'] & {[DR^{\bullet}(j_{\bullet}^{log}\Gamma(j^{\dagger}O_{]\bar{X}[})) \to DR^{\bullet}(j_{\bullet}^{log}\Gamma(O_W))]} \arrow[d, "Res_{\mathscr{Y}/\mathscr{X}}"] \\
DR^{\bullet}(i^{\dagger}M)[-d] \otimes_K [DR^{\bullet}(i^{\dagger}M^{\vee}) \to DR^{\bullet}((i^{\dagger}M)^{\vee}|_{]\bar{Z}[\cap W})][-d] \arrow[r] & {[DR^{\bullet}(\Gamma(j^{\dagger}O_{]\bar{Z}[})) \to DR^{\bullet}(\Gamma(O_{]\bar{Z}[\cap W}))][-d]}
\end{tikzcd}$$

is commutative, where $Res_{\mathscr{Y}/\mathscr{X}}$ is defined in 4.3 and the horizontal arrows are natural pairings. If W runs through the set of affinoid coverings of $]\overline{X} - X[_{\hat{\mathscr{P}}}$, we get the diagram in the assertion. This completes the proof. □

By the construction of the comparison morphism between the rigid cohomology and the crystalline cohomology in Theorem 5.2.1 and by [2, Chapitre VII, Théorème 1.4.6], we have

PROPOSITION 6.2.4. *With the notation as above, assume furthermore that X is proper smooth over* $\operatorname{Spec} k$ *and that K is absolutely unramified. Let $(\mathscr{M}, \nabla, \Phi)$ be an object in $F\text{-}\mathrm{Isoc}^{\dagger}(X/K, \sigma^a)$ and let $(\mathscr{L}, \Phi)$ be the corresponding F-crystal on X/V with respect to σ^a as in Theorem* 5.1.2. *Then the diagram*

$$\begin{array}{ccc}
\mathbf{R}\Gamma_{rig}(X/K, \mathscr{M}) & \xrightarrow{\eta_{X,X}} & \mathbf{R}\operatorname{Hom}_K(\mathbf{R}\Gamma_{rig}(X/K, \mathscr{M}^{\vee}), K)[-2n] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma_{crys}(X/V, \mathscr{L}) \otimes_V K & \longrightarrow & \mathbf{R}\operatorname{Hom}_K(\mathbf{R}\Gamma_{crys}(X/V, \mathscr{L}^{\vee}) \otimes_V K, K)[-2n]
\end{array}$$

is commutative in the derived category of upper bounded complexes of K-vector spaces, where the vertical arrows are the isomorphisms in Theorem 5.2.1 *and the bottom horizontal arrow is induced by the Poincaré duality of the crystalline cohomology.*

Now we prove the Poincaré duality.

THEOREM 6.2.5. *With the notation as above, assume that k is perfect and that X is smooth over* $\operatorname{Spec} k$. *Let $(\mathscr{M}, \nabla, \Phi)$ be an object in $F\text{-}\mathrm{Isoc}^{\dagger}(X/K, \sigma^a)^0$. Then the morphism $\eta_{Z,X}$ is an isomorphism. Moreover, the induced perfect K-pairing*

$$H^l_{Z,rig}(X/K, \mathscr{M}) \otimes_K H^{2n-l}_{c,rig}(Z/K,]\bar{\imath}[^* \mathscr{M}^{\vee}) \to K(-n)$$

commutes with Frobenius structures.

PROOF. The argument of the proof is the same as in [7, Théorème 3.4]. We prove two assertions;

$(a)_d : \eta_{X,X}$ is an isomorphism for the dimension $X \leqq d$;

$(b)_d : \eta_{Z,X}$ is an isomorphism for the dimension $Z \leqq d$;

by induction on d simultaneously. The assertion $(a)_0$ is trivial.

We prove $(a)_d \Rightarrow (b)_d$. Since the rigid cohomology with supports in Z does not change if we replace Z into the reduced subscheme Z^{red} of Z, we may assume that Z is smooth over $\operatorname{Spec} k$ by Proposition 2.1.1, 2.1.2, Lemma 6.2.1, 6.2.2 and the hypothesis of induction. We can also assume the situation of the pair (X, Z) as in Lemma 6.2.3. Therefore, the assertion follows from the Gysin isomorphism (Theorem 4.1.1) and the hypothesis of induction.

We prove $(b)_d \Rightarrow (a)_{d+1}$. By [20, Theorem 1.3.1] one can find a smooth scheme X' over $\operatorname{Spec} k$ with a smooth compactification $j' : X' \to \overline{X}'$ and a generically etale proper surjective morphism $f : X' \to X$ and find a convergent unit-root F-isocrystal $\mathcal{N}$ on $\overline{X}'/K$ with respect to σ^a such that $f_{rig}^*\mathcal{M} \cong (j')^\dagger\mathcal{N}$. By the Poincaré duality of the crystalline cohomology [2, Chapitre VII, Théorème 2.1.3] the assertion $(b)_d \Rightarrow (a)_{d+1}$ follows from Lemma 6.2.1, 6.2.2 and Proposition 6.2.4 and the hypothesis of induction. □

Corollary 6.2.6. *Under the same assumption as in Theorem* 6.2.5, *let* $i_Y : Y \to X$ *be a closed immersion of codimension e of smooth schemes over* $\operatorname{Spec} k$ *such that Y includes Z. We put* $\overline{Y}$ *(resp.* $\bar{i}_Y : \overline{Y} \to \overline{X}$*) the closure of Y in* $\overline{X}$ *(resp. the closed immersion). If* $(\mathcal{M}, \nabla, \Phi)$ *is an object in* $F\text{-Isoc}^\dagger(X/K, \sigma^a)^0$, *then there is a canonical isomorphism*

$$G_{Z/Y,X} : \mathbf{R}\Gamma_{Z,rig}(Y/K,]\bar{i}_Y[^*\mathcal{M}) \to \mathbf{R}\Gamma_{Z,rig}(X/K, \mathcal{M})[2e]$$

such that the induced K-homomorphisms on the cohomology groups commute with Frobenius structures. This isomorphism is a generalization of the Gysin morphism $G_{Z/X}$ *in Section* 4. *In the case of the constant object* $j^\dagger O_{]\overline{X}[}$, $G_{Z/Y,X}$ *coincides with the Gysin isomorphism in [*6, *Théorème* 3.8*]. We also call* $G_{Z/Y,X}$ *the Gysin isomorphism.*

In the case of the constant coefficient B. Chiarellotto proved the commutativity of the Gysin isomorphism and Frobenius structures on rigid cohomologies [8, Theorem 2.4].

(6.3) We study Künneth formula of the rigid cohomology. In the case of the constant coefficient it was proved in [7]. Let X_ν (resp. $j_\nu : X_\nu \to \overline{X}_\nu$, resp. $\overline{X} \to \hat{\mathscr{P}}_\nu$, resp. Z_ν, resp. $\overline{Z}_\nu$) be a separated scheme of finite type over $\operatorname{Spec} k$ (resp. an open immersion into a proper scheme of finite type over $\operatorname{Spec} k$, resp. a closed immersion into a formal scheme of finite type over $\operatorname{Spf} V$ such that $\hat{\mathscr{P}}_\nu$ is smooth over $\operatorname{Spf} V$ around X_ν, resp. a closed k-subscheme of X_ν, resp. the closure Z_ν in $\overline{X}_\nu$) for $\nu \in \{1, 2\}$. We put $X = X_1 \times_{\operatorname{Spec} k} X_2$, $\overline{X} = \overline{X}_1 \times_{\operatorname{Spec} k} \overline{X}_2$, $\hat{\mathscr{P}} = \hat{\mathscr{P}}_1 \times_{\operatorname{Spf} V} \hat{\mathscr{P}}_2$, $Z = Z_1 \times_{\operatorname{Spec} k} Z_2$ and the closed immersion $\bar{i}_\nu : \overline{Z}_\nu \to \overline{X}_\nu$ (resp. $\bar{i} : \overline{Z} \to \overline{X}$). We also denote by $pr_\nu :]\overline{X}[_{\hat{\mathscr{P}}} \to]\overline{X}_\nu[_{\hat{\mathscr{P}}_\nu}$ the ν-th projection.

Let $(\mathcal{M}_\nu, \nabla_\nu)$ be an object in $\operatorname{Isoc}^\dagger(X_\nu/K)$ and put $(\mathcal{M}, \nabla) = pr_1^*(\mathcal{M}_1, \nabla_1) \otimes pr_2^*(\mathcal{M}_2, \nabla_2)$ to be an object in $\operatorname{Isoc}^\dagger(X/K)$. Then the natural morphisms

$$pr_1^{-1}\underline{\Gamma}^\dagger_{]Z_1[}(\mathcal{M}_1) \otimes_K pr_2^{-1}\underline{\Gamma}^\dagger_{]Z_2[}(\mathcal{M}_2) \to \underline{\Gamma}^\dagger_{]Z[}(\mathcal{M})$$

$$pr_1^{-1}\underline{\Gamma}_{]X_1[}(\mathcal{M}_1) \otimes_K pr_2^{-1}\underline{\Gamma}_{]X_2[}(\mathcal{M}_2) \to \underline{\Gamma}_{]X[}(\mathcal{M})$$

of sheaves on $]\overline{X}[_{\hat{\mathscr{P}}}$ induce functorial morphisms

$$(6.3.1)\qquad \begin{aligned} &\mathbf{R}\Gamma_{Z_1,rig}(X_1/K, \mathcal{M}_1) \otimes_K \mathbf{R}\Gamma_{Z_2,rig}(X_2/K, \mathcal{M}_2) \to \mathbf{R}\Gamma_{Z,rig}(X/K, \mathcal{M}) \\ &\mathbf{R}\Gamma_{c,rig}(X_1/K, \mathcal{M}_1) \otimes_K \mathbf{R}\Gamma_{c,rig}(X_2/K, \mathcal{M}_2) \to \mathbf{R}\Gamma_{c,rig}(X/K, \mathcal{M}). \end{aligned}$$

If φ_v is a Frobenius on $\hat{\mathscr{P}}_v$, then one can easily see that the induced homomorphisms of cohomologies from the morphisms 6.3.1 commute with the Frobenius structures for any overconvergent F-isocrystal.

One can easily prove

LEMMA 6.3.2. *With the notation as above, if $Z_2 = X_2$, then the morphisms* 6.3.1 *induce the morphisms*

$$\Delta_{rig}(X_1, Z_1, \mathscr{M}_1) \otimes_K \mathbf{R}\Gamma_{rig}(X_2/K, \mathscr{M}_2) \to \Delta_{rig}(X, Z, \mathscr{M})$$

$$\Delta_{c,rig}(X_1, Z_1, \mathscr{M}_1) \otimes_K \mathbf{R}\Gamma_{c,rig}(X_2/K, \mathscr{M}_2) \to \Delta_{c,rig}(X, Z, \mathscr{M})$$

of distinguished triangles.

LEMMA 6.3.3. *With the notation as above, assume that both X_1 and X_2 are smooth over* Spec k. *Let $f : Y_v \to X_v$ (resp. $f : Y \to X$) be a finite etale morphism for $v = 1, 2$ (resp. $f = f_1 \times_{\mathrm{Spec}\, k} f_2$).*

(1) *The adjoint map* ad : id $\to f_{vrig*}f^*_{vrig}$ *and the trace map $tr : f_{vrig*}f^*_{vrig} \to$* id *in* 2.7 *induce commutative diagrams*

$$\begin{array}{ccc}
\mathbf{R}\Gamma_{rig}(X_1/K, \mathscr{M}_1) \otimes_K \mathbf{R}\Gamma_{rig}(X_2/K, \mathscr{M}_2) & \longrightarrow & \mathbf{R}\Gamma_{rig}(X/K, \mathscr{M}) \\
\Big\downarrow{\scriptstyle \mathrm{ad}\otimes\mathrm{ad}} & & \Big\downarrow{\scriptstyle \mathrm{ad}} \\
\mathbf{R}\Gamma_{rig}(X_1/K, f_{1,rig*}f^*_{1,rig}\mathscr{M}_1) \otimes_K \mathbf{R}\Gamma_{rig}(X_2/K, f_{2,rig*}f^*_{2,rig}\mathscr{M}_2) & \longrightarrow & \mathbf{R}\Gamma_{rig}(X/K, f_{rig*}f^*_{rig}\mathscr{M}) \\
\Big\downarrow{\scriptstyle tr\otimes tr} & & \Big\downarrow{\scriptstyle tr} \\
\mathbf{R}\Gamma_{rig}(X_1/K, \mathscr{M}_1) \otimes_K \mathbf{R}\Gamma_{rig}(X_2/K, \mathscr{M}_2) & \longrightarrow & \mathbf{R}\Gamma_{rig}(X/K, \mathscr{M})
\end{array}$$

$$\begin{array}{ccc}
\mathbf{R}\Gamma_{c,rig}(X_1/K, \mathscr{M}_1) \otimes_K \mathbf{R}\Gamma_{c,rig}(X_2/K, \mathscr{M}_2) & \longrightarrow & \mathbf{R}\Gamma_{c,rig}(X/K, \mathscr{M}) \\
\Big\downarrow{\scriptstyle \mathrm{ad}\otimes\mathrm{ad}} & & \Big\downarrow{\scriptstyle \mathrm{ad}} \\
\mathbf{R}\Gamma_{c,rig}(X_1/K, f_{1,rig*}f^*_{1,rig}\mathscr{M}_1) \otimes_K \mathbf{R}\Gamma_{c,rig}(X_2/K, f_{2,rig*}f^*_{2,rig}\mathscr{M}_2) & \longrightarrow & \mathbf{R}\Gamma_{c,rig}(X/K, f_{rig*}f^*_{rig}\mathscr{M}) \\
\Big\downarrow{\scriptstyle tr\otimes tr} & & \Big\downarrow{\scriptstyle tr} \\
\mathbf{R}\Gamma_{c,rig}(X_1/K, \mathscr{M}_1) \otimes_K \mathbf{R}\Gamma_{c,rig}(X_2/K, \mathscr{M}_2) & \longrightarrow & \mathbf{R}\Gamma_{c,rig}(X/K, \mathscr{M}),
\end{array}$$

where the horizontal arrows are the morphisms 6.3.1.

(2) *For any object $\mathscr{N}_v$ in* $\mathrm{Isoc}^\dagger(Y_v/K)$, *if we put $\mathscr{N} = pr_1^*\mathscr{N}_1 \otimes pr_2^*\mathscr{N}_2$, then the diagrams*

$$\begin{array}{ccc}
\mathbf{R}\Gamma_{rig}(X_1/K, f_{1,rig*}\mathscr{N}_1) \otimes_K \mathbf{R}\Gamma_{rig}(X_2/K, f_{2,rig*}\mathscr{N}_2) & \longrightarrow & \mathbf{R}\Gamma_{rig}(X/K, f_{rig*}\mathscr{N}) \\
\Big\downarrow & & \Big\downarrow \\
\mathbf{R}\Gamma_{rig}(Y_1/K, \mathscr{N}_1) \otimes_K \mathbf{R}\Gamma_{rig}(Y_2/K, \mathscr{N}_2) & \longrightarrow & \mathbf{R}\Gamma_{rig}(Y/K, \mathscr{N})
\end{array}$$

$$\begin{array}{ccc} \mathbf{R}\Gamma_{c,rig}(X_1/K, f_{1,rig*}\mathscr{N}_1) \otimes_K \mathbf{R}\Gamma_{c,rig}(X_2/K, f_{2,rig*}\mathscr{N}_2) & \longrightarrow & \mathbf{R}\Gamma_{c,rig}(X/K, f_{rig*}\mathscr{N}) \\ \downarrow & & \downarrow \\ \mathbf{R}\Gamma_{c,rig}(Y_1/K, \mathscr{N}_1) \otimes_K \mathbf{R}\Gamma_{c,rig}(Y_2/K, \mathscr{N}_2) & \longrightarrow & \mathbf{R}\Gamma_{c,rig}(Y/K, \mathscr{N}) \end{array}$$

are commutative, where the horizontal arrows are defined in 6.3.1 *and the vertical arrows are isomorphisms which are defined in Theorem* 2.6.3.

(3) *For a finite extension K' over K, the morphisms in* 6.3.1 *commute with both $\tau^*_{K'/K}$ and $\tau_{K'/K*}$ in* 5.1.

Proof. We may assume that both X_1 and X_2 are affine by [5, Proposition 2.1.8]. Then the assertion easily follows from Proposition 2.2.1. □

Lemma 6.3.4. *With the notation as above, assume furthermore that there is an affine smooth lift of the pair (X_v, Z_v) over* Spec V *which satisfies the situation in* 4.1 *for $v = 1, 2$. If we denote by d_v (resp. d) the codimension of Z_v in X_v (resp. $d = d_1 + d_2$), then the diagram*

$$\begin{array}{ccc} \mathbf{R}\Gamma_{rig}(Z_1/K,]\bar{\imath}_1[^*\mathscr{M}_1) \otimes_K \mathbf{R}\Gamma_{rig}(Z_2/K,]\bar{\imath}_2[^*\mathscr{M}_2) & \longrightarrow & \mathbf{R}\Gamma_{rig}(Z/K,]\bar{\imath}[^*\mathscr{M}) \\ \downarrow & & \downarrow \\ \mathbf{R}\Gamma_{Z_1,rig}(X_1/K, \mathscr{M}_1)[2d_1] \otimes_K \mathbf{R}\Gamma_{Z_2,rig}(X_2/K, \mathscr{M}_2)[2d_2] & \longrightarrow & \mathbf{R}\Gamma_{Z,rig}(X/K, \mathscr{M})[2d] \end{array}$$

is commutative in the derived category of complexes of K-vector spaces, where the horizontal arrows are the morphisms in 6.3.1 *and the vertical arrows are the Gysin morphisms.*

Proof. Let $\bar{X}_v$, $\mathscr{X}_v$ and $\mathscr{P}_v$ be as in the section 4. Put $(M_v, \nabla) = \Gamma(]\bar{X}_v[_{\hat{\mathscr{P}}_v}, (\mathscr{M}_v, \nabla))$ and $M = \Gamma(]\bar{X}[_{\hat{\mathscr{P}}}, (\mathscr{M}, \nabla))$. Then $(M, \nabla) = (M_1, \nabla) \otimes_K (M_2, \nabla)$. One can easily see that the following diagram

$$\begin{array}{ccc} DR^\bullet(j^{log}_{1\bullet} M_1) \otimes_K DR^\bullet(j^{log}_{2\bullet} M_2) & \longrightarrow & DR^\bullet(j^{log}_{\bullet} M) \\ \downarrow{\scriptstyle Res_{\mathscr{Z}_1/\mathscr{X}_1} \otimes Res_{\mathscr{Z}_2/\mathscr{X}_2}} & & \downarrow{\scriptstyle Res_{\mathscr{Z}/\mathscr{X}}} \\ DR^\bullet(i_1^\dagger M_1)[-d_1] \otimes_K DR^\bullet(i_2^\dagger M_2)[-d_2] & \longrightarrow & DR^\bullet(i^\dagger M)[-d] \end{array}$$

is commutative. (See the definition of $Res_{\mathscr{Z}/\mathscr{X}}$ in 4.3.) This induces the commutativity of the diagram. □

By [2, Chapitre V, Corollaire 4.1.2] and the construction of the comparison morphism between the rigid cohomology and the crystalline cohomology in Theorem 5.2.1, we have

Proposition 6.3.5. *With the notation as above, assume furthermore that both X_1 and X_2 are proper smooth over* Spec k *and that K is absolutely* *unramified. Let* $(\mathcal{M}_v, \nabla, \Phi)$ *(resp.* $(\mathcal{M}, \nabla, \Phi)$*) be an object in* $F\text{-Isoc}^\dagger(X/K, \sigma^a)$ *(resp.* $(\mathcal{M}, \nabla, \Phi) = pr_1^*(\mathcal{M}_1, \nabla, \Phi) \otimes pr_2^*(\mathcal{M}_2, \nabla, \Phi)$*) and let* $(\mathcal{L}_v, \Phi)$ *(resp.* $(\mathcal{L}, \Phi)$*) be the corresponding F-crystal on* X/V *with respect to* σ^a *as in Theorem* 5.1.2 *(resp.* $(\mathcal{L}, \Phi) = pr_1^*(\mathcal{L}_1, \Phi) \otimes pr_2^*(\mathcal{L}_2, \Phi)$*). Then the diagram*

$$\begin{array}{ccc} \mathbf{R}\Gamma_{rig}(X_1/K, \mathcal{M}_1) \otimes_K \mathbf{R}\Gamma_{rig}(X_2/K, \mathcal{M}_2) & \longrightarrow & \mathbf{R}\Gamma_{rig}(X/K, \mathcal{M}) \\ \downarrow & & \downarrow \\ (\mathbf{R}\Gamma_{crys}(X_1/V, \mathcal{L}_1) \otimes_V K) \otimes_K (\mathbf{R}\Gamma_{crys}(X_2/V, \mathcal{L}_2) \otimes_V K) & \longrightarrow & \mathbf{R}\Gamma_{crys}(X/V, \mathcal{L}) \otimes_V K \end{array}$$

is commutative in the derived category of complexes of K-vector spaces, where the horizontal arrows are the morphisms in 6.3.1 *and the vertical arrows are the isomorphisms in Theorem* 5.2.1.

Now we prove the Künneth formulas.

Theorem 6.3.6. *With the notation as above, assume that k is perfect. Let* $(\mathcal{M}_v, \nabla_v, \Phi_v)$ *be an object in* $F\text{-Isoc}^\dagger(X/K, \sigma^a)^0$ *for* $v = 1, 2$ *and put* $(\mathcal{M}, \nabla, \Phi) = pr_1^*(\mathcal{M}_1, \nabla_1, \Phi_1) \otimes pr_2^*(\mathcal{M}_2, \nabla_2, \Phi_2)$.

(1) *If both* X_1 *and* X_2 *are smooth over* $\operatorname{Spec} k$*, then the first morphism in* 6.3.1 *is an isomorphism. Moreover, the induced K-homomorphism*

$$\bigoplus_{l_1+l_2=l} H^{l_1}_{Z_1,rig}(X_1/K, \mathcal{M}_1) \otimes_K H^{l_2}_{Z_2,rig}(X_2/K, \mathcal{M}_2) \to H^l_{Z,rig}(X/K, \mathcal{M})$$

commutes with Frobenius structures for any l.

(2) *The second morphism in* 6.3.1 *is an isomorphism. Moreover, the induced K-homomorphism*

$$\bigoplus_{l_1+l_2=l} H^{l_1}_{c,rig}(X_1/K, \mathcal{M}_1) \otimes_K H^{l_2}_{c,rig}(X_2/K, \mathcal{M}_2) \to H^l_{c,rig}(X/K, \mathcal{M})$$

commutes with Frobenius structures for any l.

Proof. (1) The argument of the proof is the same as in [7, Théorème 4.2]. We prove two assertions;

$(a)_d$: if $Z_v = X_v$ $(v = 1, 2)$, the first morphism in 6.3.1 is an isomorphism for the dimension $X \leqq d$;

$(b)_d$: the first morphism in 6.3.1 is an isomorphism for the dimension $Z \leqq d$;

by induction on d simultaneously. The assertion $(a)_0$ is trivial.

We prove $(a)_d \Rightarrow (b)_d$. Since the rigid cohomology with supports in Z (resp. Z_1, resp. Z_2) does not change if we replace Z (resp. Z_1, resp. Z_2) into the reduced subscheme Z^{red} (resp. Z_1^{red}, resp. Z_2^{red}) of Z (resp. Z_1, resp. Z_2), we may assume that Z (resp. Z_1, resp. Z_2) is smooth over $\operatorname{Spec} k$ by Proposition 2.1.1, 2.1.2, Lemma 6.3.2, 6.3.3 and the hypothesis of induction. We can also assume the situation of the pair (X_ν, Z_ν) $(\nu = 1,2)$ as in Lemma 6.3.4. Therefore, the assertion follows from the Gysin isomorphism (Theorem 4.1.1) and the hypothesis of induction.

We prove $(b)_d \Rightarrow (a)_{d+1}$. By [20, Theorem 1.3.1] one can find a smooth scheme X'_ν over $\mathrm{Spec}\, k$ with a smooth compactification $j'_\nu : X'_\nu \to \overline{X}'_\nu$ and a generically etale proper surjective morphism $f : X'_\nu \to X_\nu$ and find a convergent unit-root F-isocrystal $\mathscr{N}_\nu$ on $\overline{X}'_\nu/K$ with respect to σ^a such that $f^*_{\nu,rig}\mathscr{M} \cong (j')^\dagger_\nu \mathscr{N}_\nu$. By the Künneth formula of the crystalline cohomology [2, Chapitre V, Théorème 4.2.1], the assertion $(b)_d \Rightarrow (a)_{d+1}$ follows from Corollary 5.1.2, Theorem 5.2.1, Lemma 6.3.2, 6.3.3, Proposition 6.3.5 and the hypothesis of induction.

(2) The argument of the proof is similar as in Theorem 6.1.2 and (1). □

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