

A note on Yamabe-type equations on the Heisenberg group

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ABSTRACT. We obtain some existence and nonexistence results for the Yamabe-type equation (1.1) on the Heisenberg group, improving recent theorems by Lu and Wei [LW].

1. Introduction

After the fundamental works by Jerison and Lee on the Yamabe problem for CR manifolds, the last few years have shown a growing interest on semi-linear equations on the Heisenberg group (see [GL], [Bi], [C], [BC], [BRS], [LW], [BCC1-2], [B1-2], [LU1-2], [U1-2-3], [CC], [CU], [BP]). The Yamabe problem on a CR manifold leads to the following semilinear equation

$$\Delta_{\mathbf{H}^n} u + a(\xi)u + K(\xi)u^p = 0, \quad \xi = (z, t) \in \mathbf{H}^n, \quad (1.1)$$

where $\Delta_{\mathbf{H}^n}$ denotes the Kohn Laplacian on the Heisenberg group \mathbf{H}^n and $p = 1 + 2/n$ (see [JL1-2-3]). For this reason, for any $p \geq 1$, (1.1) is called by several authors a Yamabe-type equation. Throughout this paper a, K will be supposed to be locally Hölder continuous on \mathbf{H}^n .

In this note, by using some recent results in [U2], we obtain some existence and nonexistence results for (1.1) which slightly improve analogous theorems by Lu and Wei in [LW] and by Brandolini, Rigoli and Setti in [BRS]. We give a contribution in the same direction as in the papers [N], [Na], [LN], [CN], in which the classical Laplace-Yamabe equation is studied.

In order to state our results we introduce some notation. In §2, which contains the proofs of our theorems, we give the other definitions and notation we need. Let us denote by $Q = 2n + 2$ the homogeneous dimension of \mathbf{H}^n , by d the intrinsic distance and by X the vector field $(z, 2t)$ generating the group of dilations on \mathbf{H}^n (see [GL]). A function $K : \mathbf{H}^n \rightarrow \mathbf{R}$ will be said to belong to $L^{q_1, q_2}(\mathbf{H}^n)$ if there exist $q_1 < q$ and $q_2 > q$ such that $K \in L^{q_1}(\mathbf{H}^n) \cap L^{q_2}(\mathbf{H}^n)$. Our first result is the following nonexistence theorem that improves Theorem 4.1 in [LW].

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THEOREM 1.1. *Let $K \in C^1(\mathbf{H}^n)$, $a \equiv 0$, $p \geq 1$ and assume that the function $L := (Q - \frac{1}{2}(Q - 2)(p + 1))K + XK$ never changes sign in \mathbf{H}^n . If*

$$K \in L^{]Q/2[}(\mathbf{H}^n) \cap L^{]Q[}(\mathbf{H}^n), \tag{1.2}$$

then (1.1) has no positive bounded solutions u such that

$$\liminf_{d(\xi) \rightarrow \infty} u(\xi) = 0. \tag{1.3}$$

Lu and Wei in [LW], Theorem 4.1, instead of (1.2) require the stronger condition

$$K(\xi) = O\left(\frac{1}{d(\xi)^\ell}\right), \quad \text{as } d(\xi) \rightarrow \infty, \quad \text{for some } \ell > 2. \tag{1.4}$$

The main step in the proof of Theorem 1.1 is the following asymptotic behavior of a solution to (1.1) that easily follows from the Liouville theorem for $\Delta_{\mathbf{H}^n}$ and from our results in [U2].

THEOREM 1.2. *Let $K \in L^{]Q/2[}(\mathbf{H}^n)$, $p \geq 1$ and let u be a bounded solution to*

$$\Delta_{\mathbf{H}^n} u + K(\xi)|u|^{p-1}u = 0, \quad \xi \in \mathbf{H}^n.$$

Then we have:

- (i) *There exists $c \in \mathbf{R}$ such that $u(\xi) \rightarrow c$, as $d(\xi) \rightarrow \infty$.*
- (ii) *If (1.3) holds, then*

$$u(\xi) = O\left(\frac{1}{d(\xi)^s}\right), \quad \text{as } d(\xi) \rightarrow \infty, \quad \text{for every } s < Q - 2. \tag{1.5}$$

- (iii) *If (1.3) holds and $p > 1$, then $u(\xi) = O\left(\frac{1}{d(\xi)^{Q-2}}\right)$ at ∞ .*

We explicitly remark that this result improves Theorem 3.1 of [LW] (when $p > 1$).

Next theorem, an existence result, is suggested by the Naito methods in [Na].

THEOREM 1.3. *Let $n \geq 2$, $p > 1$. Assume that there exist locally Hölder continuous functions α, β such that*

$$|K(z, t)| \leq \beta(|z|), \quad \int_0^\infty r\beta(r)dr < +\infty, \tag{1.6}$$

$$|a(z, t)| \leq \alpha(|z|), \quad \int_0^\infty r\alpha(r)dr < 2n - 2. \tag{1.7}$$

Then for every $\lambda > 0$ small enough, there exists a bounded positive solution u to (1.1) such that $u(z, t) \rightarrow \lambda$, as $|z| \rightarrow \infty$, uniformly with respect to t .

REMARK 1.4. *Let $m \geq 3$ and $\tilde{z} = (z_{j_1}, \dots, z_{j_m})$. Then Theorem 1.3 holds true even replacing $\alpha(|z|)$, $\beta(|z|)$ with $\alpha(|\tilde{z}|)$, $\beta(|\tilde{z}|)$, if we replace also the assertion $u(z, t) \rightarrow \lambda$ as $|z| \rightarrow \infty$ with $u(z, t) \rightarrow \lambda$ as $|\tilde{z}| \rightarrow \infty$.*

COROLLARY 1.5. *Let $n \geq 2$, $a \equiv 0$ and $p > 1$. If (1.6) holds and*

$$K \in L^{Q/2}(\mathbf{H}^n), \tag{1.8}$$

then for every $\lambda > 0$ small enough, there exists a positive solution u to (1.1) such that

$$u(\xi) \rightarrow \lambda, \quad \text{as } d(\xi) \rightarrow \infty.$$

This result improves (in the case of \mathbf{H}^n and for $n \geq 2$) Theorems 2.1–2.2 by Lu and Wei [LW] which require the stronger condition (1.4) instead of (1.6) and (1.8).

We finally make a little remark on a theorem in [BRS].

THEOREM 1.6. *Let $n \geq 2$, $p > 1$, $a \equiv 0$ and suppose that there exists $\beta \in C([0, \infty[)$ such that*

$$0 < -K(z, t) \leq \beta(|z|), \tag{1.9}$$

$$\int_0^\infty r\beta(r)dr < +\infty. \tag{1.10}$$

Then there exists a unique, positive, maximal solution U to equation (1.1) satisfying $U(\xi) \rightarrow +\infty$ as $d(\xi) \rightarrow \infty$.

The above theorem improves Theorem 5 of [BRS] (in the case $n \geq 2$) where the inequality $-K(z, t) \leq \psi(z, t)\beta(d(z, t))$ involving the degenerate term $\psi(z, t) = |z|^2/d(z, t)^2$ is required.

REMARK 1.7. *In the case a is not identically zero, Theorem 1.6 is still true if we add the hypothesis $0 \leq a \leq \psi\alpha(d)$, for a continuous $\alpha(r) \leq n^2/r^2$; it is also sufficient to assume $K \leq 0$, $K < 0$ at infinity, instead of $K < 0$ everywhere (as in [BRS]).*

2. Notation and proofs

The Heisenberg group \mathbf{H}^n , whose points will be denoted by $\xi = (z, t) = (x, y, t)$, is the Lie group $(\mathbf{R}^{2n+1}, \circ)$ with composition law defined by

$$\xi \circ \xi' = (z + z', t + t' + 2(\langle x', y \rangle - \langle x, y' \rangle)),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbf{R}^n . The Kohn Laplacian on \mathbf{H}^n is the operator

$$\Delta_{\mathbf{H}^n} = \sum_{j=1}^n (X_j^2 + Y_j^2),$$

where

$$X_j = \partial_{x_j} + 2y_j \partial_t, \quad Y_j = \partial_{y_j} - 2x_j \partial_t, \quad j \in \{1, \dots, n\},$$

generate the real Lie algebra of left-invariant vector fields on \mathbf{H}^n . The operator $\Delta_{\mathbf{H}^n}$ is the prototype of the so-called sublaplacians on stratified nilpotent Lie groups. We set

$$\nabla_{\mathbf{H}^n} = (X_1, \dots, X_n, Y_1, \dots, Y_n).$$

A natural group of dilations on \mathbf{H}^n is given by

$$\delta_\lambda(\xi) = (\lambda z, \lambda^2 t), \quad \lambda > 0.$$

The Jacobian determinant of δ_λ is λ^Q where

$$Q = 2n + 2$$

is the *homogeneous dimension* of \mathbf{H}^n . The operators $\nabla_{\mathbf{H}^n}$ and $\Delta_{\mathbf{H}^n}$ are invariant w.r.t. the left translations τ_ξ of \mathbf{H}^n and homogeneous w.r.t. the dilations δ_λ of degree one and of degree two, respectively.

A remarkable analogy between the Kohn Laplacian and the classical Laplace operator is that a fundamental solution of $-\Delta_{\mathbf{H}^n}$ with pole at zero is given by [F]

$$\Gamma(\xi) = \frac{c_Q}{d(\xi)^{Q-2}},$$

where c_Q is a suitable positive constant and

$$d(\xi) = (|z|^4 + t^2)^{1/4}.$$

Moreover, if we define $d(\xi, \xi') = d_{\xi'}(\xi) = d(\xi'^{-1} \circ \xi)$, then d is a distance on \mathbf{H}^n (see [Cy] for a complete proof of this statement). We shall denote by $B_d(\xi, r)$ the d -ball of center ξ and radius r . By the left translation invariance of the distance d , we have $\tau_\xi(B_d(0, r)) = B_d(\xi, r)$. Moreover, since d is homogeneous of degree one with respect to the dilations δ_λ , we also have $\delta_\lambda(B_d(0, r)) = B_d(0, \lambda r)$ and $|B_d(\xi, r)| = r^Q |B_d(0, 1)|$. Here $|\cdot|$ denotes the Lebesgue measure on \mathbf{R}^{2n+1} . We also recall that the Lebesgue measure is a Haar measure on \mathbf{H}^n .

PROOF OF THEOREM 1.2. Let $w = \Gamma * (-\Delta_{\mathbf{H}^n} u) := \int_{\mathbf{H}^n} \Gamma(\cdot, \xi') (-\Delta_{\mathbf{H}^n} u)(\xi') d\xi'$. Here we have set $\Gamma(\xi, \xi') = \Gamma_{\xi'}(\xi) = \Gamma(\xi'^{-1} \circ \xi)$. There exist $q_1 < \frac{Q}{2} < q_2$ such that $K \in L^{q_1} \cap L^{q_2}$. Denoting q'_j the conjugate exponent of q_j ($j = 1, 2$), we have

$$\begin{aligned}
 |w(\xi)| &\leq \int_{\mathbf{H}^n} \Gamma_\xi |K| |u|^p \leq c \int_{\mathbf{H}^n} \Gamma_\xi |K| \\
 &\leq \|\Gamma_\xi\|_{L^{q'_2}(B_d(\xi, R))} \|K\|_{L^{q_2}(B_d(\xi, R))} + \|\Gamma_\xi\|_{L^{q'_1}(\mathbf{H}^n \setminus B_d(\xi, R))} \|K\|_{q_1} \\
 &= \|\Gamma\|_{L^{q'_2}(B_d(0, R))} \|K\|_{L^{q_2}(B_d(\xi, R))} + c \|\Gamma\|_{L^{q'_1}(\mathbf{H}^n \setminus B_d(0, R))} =: F(\xi, R) + G(R)
 \end{aligned}$$

with $G(R) \rightarrow 0$ as $R \rightarrow +\infty$ and $F(\xi, R) \rightarrow 0$ as $d(\xi) \rightarrow \infty$ (for every fixed $R > 0$), since $\Gamma \in L^{q'_2}_{\text{loc}}$, $K \in L^{q_1} \cap L^{q_2}$ and $\Gamma \in L^{q'_1}(\infty)$. Hence

$$w(\xi) \rightarrow 0, \quad \text{as } d(\xi) \rightarrow \infty. \tag{2.1}$$

By the classical Liouville theorem for $\Delta_{\mathbf{H}^n}$ (see [BCC1]), being $u - w$ a bounded $\Delta_{\mathbf{H}^n}$ -harmonic function on \mathbf{H}^n , we get $u = w + c$ for an absolute constant $c \in \mathbf{R}$. This proves (i). From (1.3) and (2.1) we obtain $c = 0$, so that $u = w$. In particular $u(\xi) \rightarrow 0$ as $d(\xi) \rightarrow \infty$. Since $K|u|^{p-1} \in L^{Q/2}$, using [U2, Theorem 1.3] we finally obtain (1.5).

Moreover if $p > 1$, setting $s = (Q - 2)/p$, from (1.5) we get (for large $d(\xi)$)

$$\begin{aligned}
 |u(\xi)| = |w(\xi)| &\leq \left(\int_{d(\xi, \eta) > d(\xi)/2} + \int_{d(\xi, \eta) \leq d(\xi)/2} \right) (\Gamma(\xi, \eta) |K(\eta)| |u(\eta)|^p d\eta) \\
 &\leq \frac{c}{d(\xi)^{Q-2}} \|K|u|^p\|_1 + \frac{c}{d(\xi)^{sp}} \int_{\mathbf{H}^n} \Gamma_\xi |K| \leq \frac{c}{d(\xi)^{Q-2}}. \quad \square
 \end{aligned}$$

PROOF OF THEOREM 1.1. By Theorem 1.2, we have that a solution u satisfies

$$u = \Gamma * (Ku^p) = O(d^{-s}) \quad \text{at } \infty, \quad \forall s < Q - 2. \tag{2.2}$$

Moreover $f := Ku^p \in L^1 \cap L^Q$ and

$$\nabla_{\mathbf{H}^n} u = (\nabla_{\mathbf{H}^n} \Gamma) * f \in L^q \quad \forall q \in \left] \frac{Q}{Q-1}, \infty \right[, \tag{2.3}$$

$$\partial_t u \in L^q \quad \forall q \in]1, Q] \tag{2.4}$$

(see [FS]). From (1.2), (2.2) and (2.3) we get (for a $q_2 > Q$ and large $d(\xi)$)

$$\begin{aligned}
 |\nabla_{\mathbf{H}^n} u(\xi)| &\leq c \left(\int_{d(\xi, \eta) > d(\xi)/2} + \int_{d(\xi, \eta) \leq d(\xi)/2} \right) \left(\frac{|f(\eta)|}{d(\xi, \eta)^{Q-1}} d\eta \right) \\
 &\leq \frac{c}{d(\xi)^{Q-1}} \|f\|_1 + \frac{c}{d(\xi)^{sp}} \|K\|_{q_2} \left(\int_{d(\xi, \eta) \leq d(\xi)/2} \frac{d\eta}{d(\xi, \eta)^{(Q-1)q'_2}} \right)^{1/q'_2} \\
 &\leq \frac{c}{d(\xi)^{Q-1}} + \frac{c}{d(\xi)^{sp+(Q/q_2)-1}}.
 \end{aligned}$$

Hence

$$|\nabla_{\mathbf{H}^n} u| = O(d^{-1-\varepsilon}) \quad \text{at } \infty, \quad \text{for some } \varepsilon > 0. \quad (2.5)$$

We now use the Rellich-Pohozaev integral identity proved in [GL, Theorem 2.1]. We denote by σ the surface measure, by $\nu = (\nabla d)/|\nabla d|$ the outer unit normal to $B := B_d(0, R)$ and by A the matrix

$$\begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \\ 2y & -2x & 4|z|^2 \end{pmatrix}$$

which allows us to write $\Delta_{\mathbf{H}^n}$ in the divergence form $\Delta_{\mathbf{H}^n} = \operatorname{div}(A\nabla)$, ($\nabla =$ gradient operator in \mathbf{R}^{2n+1}). We obtain

$$\begin{aligned} & 2 \int_{\partial B} \langle A\nabla u, \nu \rangle Xu d\sigma - \int_{\partial B} |\nabla_{\mathbf{H}^n} u|^2 \langle X, \nu \rangle d\sigma \\ &= (2 - Q) \int_B |\nabla_{\mathbf{H}^n} u|^2 - 2 \int_B Ku^p Xu \\ &= (2 - Q) \int_B (\operatorname{div}(uA\nabla u) - u\Delta_{\mathbf{H}^n} u) - 2 \int_B \left(\left\langle X, \nabla \left(\frac{Ku^{p+1}}{p+1} \right) \right\rangle - \frac{u^{p+1}}{p+1} XK \right) \\ &= (2 - Q) \int_{\partial B} u \langle A\nabla u, \nu \rangle d\sigma + (2 - Q) \int_B Ku^{p+1} \\ &\quad - \frac{2}{p+1} \int_B (\operatorname{div}(Ku^{p+1} X) - Ku^{p+1} \operatorname{div} X - u^{p+1} XK) \\ &= (2 - Q) \int_{\partial B} u \langle A\nabla u, \nu \rangle d\sigma - \frac{2}{p+1} \int_{\partial B} Ku^{p+1} \langle X, \nu \rangle d\sigma + \frac{2}{p+1} \int_B u^{p+1} L. \end{aligned}$$

Hence, setting

$$G := 2 \langle A\nabla u, \nu \rangle Xu - \langle X, \nu \rangle |\nabla_{\mathbf{H}^n} u|^2 + \frac{2}{p+1} \langle X, \nu \rangle Ku^{p+1} + (Q - 2)u \langle A\nabla u, \nu \rangle,$$

we have

$$\frac{2}{p+1} \int_B u^{p+1} L = \int_{\partial B} G d\sigma. \quad (2.6)$$

Since

$$|\langle A\nabla u, \nu \rangle| = \left| \left\langle A\nabla u, \frac{\nabla d}{|\nabla d|} \right\rangle \right| = \frac{|\langle \nabla_{\mathbf{H}^n} u, \nabla_{\mathbf{H}^n} d \rangle|}{|\nabla d|} \leq \frac{|\nabla_{\mathbf{H}^n} u|}{|\nabla d|},$$

$$|\langle X, \nu \rangle| = \frac{|Xd|}{|\nabla d|} = \frac{d}{|\nabla d|},$$

$$|Xu| = |\langle z, \nabla_{\mathbf{H}^n} u \rangle + 2t\partial_t u| \leq d|\nabla_{\mathbf{H}^n} u| + 2d^2|\partial_t u|,$$

we get

$$|G| \leq c \frac{d}{|\nabla d|} g, \tag{2.7}$$

where

$$g := |\nabla_{\mathbf{H}^n} u|^2 + d|\nabla_{\mathbf{H}^n} u| |\partial_t u| + \frac{u|\nabla_{\mathbf{H}^n} u|}{d} + |K|u^{p+1} \in L^1(\infty),$$

by means of (1.2), (2.2), (2.3), (2.4) and (2.5). Using coarea formula we obtain (for a large $M > 0$)

$$\int_M^\infty \left(\int_{\partial B_d(0, R)} \frac{g}{|\nabla d|} d\sigma \right) dR = \int_{\mathbf{H}^n \setminus B_d(0, M)} g < +\infty.$$

Hence there exists a divergent sequence of radii $(R_k)_{k \in \mathbb{N}}$ such that

$$\int_{\partial B_d(0, R_k)} \frac{g}{|\nabla d|} d\sigma = o\left(\frac{1}{R_k}\right), \quad \text{as } k \rightarrow +\infty.$$

Therefore, from (2.6) and (2.7) we deduce that

$$\left| \int_{B_d(0, R_k)} u^{p+1} L \right| \leq c \int_{\partial B_d(0, R_k)} |G| d\sigma \rightarrow 0, \quad \text{as } k \rightarrow +\infty.$$

Since L never changes sign in \mathbf{H}^n and $u > 0$, we infer

$$L \equiv 0 \quad \text{in } \mathbf{H}^n. \tag{2.8}$$

We now want to prove that $K \equiv 0$ in \mathbf{H}^n . We argue by contradiction and suppose that there exists $B = B_d(\eta_0, r)$ such that $K|_B \geq \sigma > 0$ (the case $K|_B \leq -\sigma < 0$ can be treated in the same way). We fix $\xi_0 \in B \setminus \{0\}$ and we set for brevity $\lambda = Q - \frac{1}{2}(Q - 2)(p + 1)$. If we consider the function $\varphi \in C^1([0, +\infty[)$ defined by

$$\varphi(\varrho) = K(\delta_\varrho \xi_0),$$

we have

$$\varrho \varphi'(\varrho) = \varrho \langle \nabla K(\delta_\varrho \xi_0), (z_0, 2\varrho t_0) \rangle = (XK)(\delta_\varrho \xi_0).$$

Hence (2.8) yields

$$0 = L(\delta_\varrho \xi_0) = \lambda \varphi(\varrho) + \varrho \varphi'(\varrho) \quad \forall \varrho \geq 0. \tag{2.9}$$

We now distinguish two cases. If $\lambda \leq 0$, then

$$\varphi'(\varrho) = -\frac{\lambda \varphi(\varrho)}{\varrho}, \quad \varphi(1) = K(\xi_0) \geq \sigma > 0$$

imply $\varphi(\varrho) \geq \sigma$ for every $\varrho \geq 1$. Hence $K(\delta_\varrho \xi) \geq \sigma$ for every $\xi \in B \setminus \{0\}$ and for every $\varrho \geq 1$. This contradicts (1.2).

If $\lambda > 0$, then (2.9) gives $\varphi(0) = 0$. Moreover $\varphi(1) = K(\xi_0) \geq \sigma > 0$. Hence there exists $\varrho_0 \in]0, 1[$ such that $\varphi(\varrho_0) > 0, \varphi'(\varrho_0) \geq 0$. This yields

$$\lambda\varphi(\varrho_0) + \varrho_0\varphi'(\varrho_0) > 0,$$

contradicting (2.9).

Therefore $K \equiv 0$ in \mathbf{H}^n and u is a bounded $\Delta_{\mathbf{H}^n}$ -harmonic function on \mathbf{H}^n satisfying (1.3). By means of the Liouville theorem for $\Delta_{\mathbf{H}^n}$ we conclude that $u \equiv 0$ in \mathbf{H}^n . □

PROOF OF THEOREM 1.3. We shall find super- and sub-solutions to equation (1.1) depending only on $|z|$. We denote for brevity $\alpha_1 = \alpha, \alpha_2 = \beta, A_i = \int_0^\infty r\alpha_i(r)dr$ ($i = 1, 2$), $A = A_1, B = A_2$, and we fix

$$\lambda \in \left] 0, \min \left\{ \left(\frac{2n-2-A}{B} \right)^{1/(p-1)}, \frac{p-1}{2n-2} B \left(\frac{2n-2-A}{pB} \right)^{p/(p-1)} \right\} \right[\quad (2.10)$$

(it is not restrictive to assume $B \neq 0$). Then we set

$$\lambda_1 = \lambda_2 = \frac{\lambda}{2}, \quad c_1 = \lambda, \quad c_2 = \lambda^p, \quad \ell_i = \lambda_i - \frac{c_i A_i}{2n-2} \quad (i = 1, 2), \quad \ell = \ell_1 + \ell_2.$$

Following an idea of Naito [Na], we define ($i = 1, 2$)

$$\begin{cases} w_i(z, t) = W_i(|z|) = \lambda_i - c_i |z|^{2-2n} \int_0^{|z|} s^{2n-3} \int_s^\infty r\alpha_i(r) dr ds, & \text{if } z \neq 0 \\ w_i(0, t) = W_i(0) = \ell_i. \end{cases}$$

Then we have

$$w_i \in C^2(\mathbf{H}^n), \quad \Delta_{\mathbf{H}^n} w_i(z, t) = c_i \alpha_i(|z|) \quad \text{in } \mathbf{H}^n, \quad (2.11)$$

$$W_i(r) \nearrow \lambda_i, \quad \text{as } r \rightarrow +\infty. \quad (2.12)$$

We now set $w = w_1 + w_2, W = W_1 + W_2$. From (2.10) it follows that

$$W(0) = \ell = \ell_1 + \ell_2 = \lambda - \frac{\lambda A + \lambda^p B}{2n-2} = \lambda \frac{2n-2-A}{2n-2} - \frac{\lambda^p B}{2n-2} > 0.$$

Hence, by (2.12), we have

$$0 < \ell = W(0) \leq W(r) \nearrow \lambda, \quad \text{as } r \rightarrow +\infty \quad (2.13)$$

and from (2.11), (1.7) and (1.6) we finally get

$$\begin{aligned} \Delta_{\mathbf{H}^n} w(z, t) &= \lambda \alpha(|z|) + \lambda^p \beta(|z|) \geq \alpha(|z|) W(|z|) + \beta(|z|) W(|z|)^p \\ &\geq -a(z, t) w(z, t) - K(z, t) w(z, t)^p. \end{aligned}$$

Therefore w is a positive sub-solution to equation (1.1).

We now look for a super-solution. Because of (2.10), there exists $\ell^* > 0$ such that

$$\ell^*(2n - 2 - A) - \ell^{*p} B = (2n - 2)\lambda. \tag{2.14}$$

We set

$$\ell_1^* = \ell_2^* = \frac{\ell^*}{2}, \quad c_1^* = \ell^*, \quad c_2^* = \ell^{*p}$$

and we define ($i = 1, 2$)

$$\begin{cases} v_i(z, t) = V_i(|z|) = \ell_i^* - c_i^* |z|^{2-2n} \int_0^{|z|} s^{2n-3} \int_0^s r \alpha_i(r) dr ds, & \text{if } z \neq 0 \\ v_i(0, t) = V_i(0) = \ell_i^*. \end{cases}$$

Then we have

$$v_i \in C^2(\mathbf{H}^n), \quad \Delta_{\mathbf{H}^n} v_i(z, t) = -c_i^* \alpha_i(|z|) \quad \text{in } \mathbf{H}^n, \tag{2.15}$$

$$V_i(r) \searrow \ell_i^* - \frac{c_i^* A_i}{2n - 2}, \quad \text{as } r \rightarrow +\infty. \tag{2.16}$$

We now set $v = v_1 + v_2$, $V = V_1 + V_2$. From (2.14) and (2.16) it follows that

$$\ell^* = V(0) \geq V(r) \searrow \lambda > 0, \quad \text{as } r \rightarrow +\infty \tag{2.17}$$

and from (2.15), (1.7) and (1.6) we finally get

$$\begin{aligned} \Delta_{\mathbf{H}^n} v(z, t) &= -\ell^* \alpha(|z|) - \ell^{*p} \beta(|z|) \leq -\alpha(|z|) V(|z|) - \beta(|z|) V(|z|)^p \\ &\leq -a(z, t) v(z, t) - K(z, t) v(z, t)^p. \end{aligned}$$

Therefore v is a super-solution to equation (1.1). Moreover, by (2.13) and (2.17) we have

$$0 < \ell \leq w \leq \lambda \leq v \leq \ell^* \quad \text{in } \mathbf{H}^n.$$

Starting from our super- and sub-solution, by standard monotone methods we find a solution u to (1.1) such that $w \leq u \leq v$. Moreover, by (2.13) and (2.17), $u(z, t) \rightarrow \lambda$ as $|z| \rightarrow \infty$, uniformly with respect to t . \square

PROOF OF COROLLARY 1.5. Since $a \equiv 0$ and (1.6) holds, from Theorem 1.3 we get that there exists a bounded positive solution u to equation (1.1) such that $u(z, t) \rightarrow \lambda$ as $|z| \rightarrow \infty$, uniformly with respect to t . Since also (1.8) holds, from Theorem 1.2-(i) we obtain $u(\xi) \rightarrow \lambda$, as $d(\xi) \rightarrow \infty$. \square

PROOF OF THEOREM 1.6. It is not restrictive to suppose β locally Hölder continuous. By means of [Na], there exists a positive solution $\varphi \in C^2(\mathbf{R}^{2n})$ to

$$\Delta\varphi(z) - \beta(|z|)\varphi(z)^p = 0, \quad z \in \mathbf{R}^{2n}.$$

Setting $w(z, t) = \varphi(z)$ and recalling that $a \geq 0$, from (1.9) we get

$$\begin{aligned} \Delta_{\mathbf{H}^n} w(z, t) &= \Delta\varphi(z) = \beta(|z|)\varphi(z)^p \geq -K(z, t)w(z, t)^p \\ &\geq -a(z, t)w(z, t) - K(z, t)w(z, t)^p. \end{aligned}$$

Hence w is a positive sub-solution to (1.1) and we can complete the proof by applying [BRS, Theorem 6]. \square

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