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# A white noise approach to an evolutional equation in biology

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ABSTRACT. In this paper we shall discuss a white noise differential equation that comes from some biological phenomena. Having applied the so-called $S$-transform to the white noise functionals the given equation turns into an equation for a $U$-functional. There the advantage of the white noise calculus is heavily used. Thus the solution is obtained in an explicit form in terms of white noise, and we see that the solution is a generalized functional which is in the space of Cochran-Kuo-Sengupta. Some characteristic properties of the solution are shown, for instance, positivity of the solution. Further, its mean lies in between 0 and 1. The expression of the solution shows the most significant property that there is an *asymmetry* in time for the phenomenon in question.

## 1. Introduction

A biological organism is composed of one cell or many cells. The surface of a cell is covered with a plasma membrane and the membrane is the border between the inside and the outside of a cell. Disproportion of sodium, potassium, calcium and chlorine ions exist between the inside and the outside of a cell and this fact brings about the difference in electric potential. Ion channels are macromolecules that open and close in a random fashion on membrane and play the role of gatekeepers which control the flux of their ions coming in and out of the cell.

F. Oosawa et al. [10] has introduced a differential equation which is proposed to describe the probabilistic behavior of ion channels in a fluctuating electric field. They claim that the open—close fluctuation in an assembly of channels has an asymmetry with respect to time reversal. In their theory, the probability which is the ratio of the number of channels in open state to the total number of channels is denoted by $p(t)$ and it is given by the solution to the equation.

$$\frac{dp(t)}{dt} = -k_{+o}\exp\{-\beta E(t)\}p(t) + k_{-o}\exp\{+\beta E(t)\}(1-p(t)), \tag{1.1}$$

where $k_{+o} > 0$, $k_{-o} > 0$, $\beta = (1/2)\delta\mu/kT$ ($\delta\mu$: the free energy difference between

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an opened state and a closed state, $k$: the Boltzman constant, $T$: the absolute temperature, $\langle E(t)\rangle = 0$, and $E(t)$ is Gaussian).

We are interested in their approach and understand the equation (1.1) to be expressed in terms of fluctuation. Namely, we regard the $E(t)$ in the above equation as an operator describing the fluctuation expressed by creating operators of the field.

Thus, the purpose of this paper is to give a mathematical interpretation to the solution of (1.1) from the viewpoint of white noise analysis and to show asymmetry of the phenomenon for ion channels with respect to time reversal, by using the explicit expression of $p(t)$.

We get a solution of the equation in the space $[E]^*_{\bar{B}_j}$ which is recently obtained by Cochran-Kuo-Sengupta [1]. Our method gives an investigation of the equation introduced by F. Oosawa et al. and also gives an example which is actually useful to study the theory of the space $[E]^*_{\bar{B}_j}$.

The paper is organized as follows. In § 2 we summarize basic concepts of white noise calculus, including the space of *white noise distributions* in the sense of Cochran-Kuo-Sengupta [1] as well as the creation operator and the annihilation operator acting on the space. The space is provided rich enough so as the solution can live within the space constructed. In § 3 we obtain the solution of the equation (1.1) which can be transformed to the equation of the *U-functional*. The solution is, in fact, a generalized white noise functional which is in the space of Cochran-Kuo-Sengupta. In § 4 we prove the positivity of the solution, which is requested as a probability. This fact should be clarified since the solution itself is a generalized white noise function and it is impossible to show positivity in the ordinary sense. The positivity of a generalized white noise functional is rephrased as the positive definiteness of its $S$-transform, and actually this property is proved. In the last section which is the main section we prove an asymmetry of the equation with respect to time reversal.

## 2. Preliminaries

We now prepare some background of white noise calculus to discuss the equation (1.1). Basic notation is introduced following Hida [2], [4], Cochran-Kuo-Sengupta [1] and Kuo [6].

Let $L^2(\mathbf{R})$ be the Hilbert space consisting of real-valued square integrable functions on $\mathbf{R}$ with norm $|\cdot|_0$. We start with the real Gel'fand triple:

$$E = \mathscr{S}(\mathbf{R}) \subset L^2(\mathbf{R}) \subset E^* = \mathscr{S}'(\mathbf{R}),$$

where $\mathscr{S}(\mathbf{R})$ is the Schwartz space consisting of rapidly decreasing $C^\infty$-

functions on $\mathbf{R}$ and $\mathscr{S}'(\mathbf{R})$ is its dual space, i.e., the space of tempered distributions.

Let $A = 1 + t^2 - \frac{d^2}{dt^2}$ be densely defined self-adjoint operator on $L^2(\mathbf{R})$ such that there exists an orthonormal basis $\{e_j;\, j = 0, 1, 2, \dots\} \subset E$ for $L^2(\mathbf{R})$ satisfying $Ae_j = (2j+2)e_j$.

Define the norm $|\cdot|_p$ by $|f|_p = |A^p f|_0$ for $f \in L^2(\mathbf{R})$ and $p \in \mathbf{R}$. For any $p \in \mathbf{R}$, also define $E_p$ by $\{f \in L^2(\mathbf{R});\, |f|_p < \infty\}$ for $p \geq 0$ and the completion of $L^2(\mathbf{R})$ with respect to the norm $|\cdot|_p$ for $p < 0$. Then the space $E_p$ is a Hilbert space with norm $|\cdot|_p$ for each $p \in \mathbf{R}$, and we get $E = \bigcap_p E_p$ and $E^* = \bigcup_p E_p$ with the projective limit topology and the inductive limit topology, respectively.

Take $C(\xi) = e^{-(1/2)|\xi|_0^2}$, $\xi \in E$. Since the functional $C(\xi)$ satisfies conditions (1) continuous, (2) positive definite, (3) $C(0) = 1$, we can appeal to the Bochner–Minlos Theorem to guarantee the existence of a probability measure $\mu$ on $E^*$ such that the characteristic functional is given by

$$\int_{E^*} e^{i\langle x, \xi \rangle} d\mu(x) = C(\xi), \qquad \xi \in E.$$

The measure $\mu$ is called the standard Gaussian measure or the white noise measure defined on $E^*$ and $(E^*, \mu)$ is called a *white noise probability space.*

The Hilbert space $(L^2) = L^2(E^*, \mu)$ of complex-valued $\mu$-square-integrable functionals defined on $E^*$ admits the well-known Wiener-Itô decomposition:

$$(L^2) = \bigoplus_{n=0}^{\infty} H_n,$$

where $H_n$ is the space of multiple Wiener integrals of degree $n \in \mathbf{N}$ and $H_0 = \mathbf{C}$. Let $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$ denote the $n$-fold symmetric tensor product of the complexification $L^2_{\mathbf{C}}(\mathbf{R})$ of $L^2(\mathbf{R})$. It is known that a multiple Wiener integral of degree n has a representation in terms of a kernel $f \in L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$, and hence it is denoted by $\mathbf{I}_n(f)$ [3]. For $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (L^2)$, the $(L^2)$-norm $\|\varphi\|_0$ is equal to

$$\|\varphi\|_0 = \sqrt{\sum_{n=0}^{\infty} n! |f_n|_0^2},$$

where $|\cdot|_0$ denotes the $L^2_{\mathbf{C}}(\mathbf{R})^{\otimes n}$-norm.

Set $\exp_0(x) = x$ and define

$$\exp_{j+1}(x) = \exp[\exp_j(x)], \qquad j = 0, 1, 2, \dots.$$

inductively. Denote by $B_j(n)$ the $n$-th coeffícent of the power series expansion

$$\exp_j(x) = \sum_{n=0}^{\infty} \frac{B_j(n)}{n!} x^n.$$

For each positive integer $j \geq 1$, $\bar{B}_j(n) = B_j(n)/\exp_j(0)$ $(n \geq 0)$ are called the *j-th order Bell numbers.*

For $p \geq 0$ and $j \in \mathbf{N} \cup \{0\}$, we define a norm $\|\cdot\|_{p,\bar{B}_j}$ by

$$\|\varphi\|_{p,\bar{B}_j} = \sqrt{\sum_{n=0}^{\infty} n!\bar{B}_j(n)|f_n|_p^2},$$

for $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (L^2)$. Let $[E_p]_{\bar{B}_j} = \{\varphi \in (L^2); \|\varphi\|_{p,\bar{B}_j} < \infty\}$. Then the projective limit space, denoted by $[E]_{\bar{B}_j}$, of the spaces $[E_p]_{\bar{B}_j}$, $p \geq 0$, is a nuclear space (for proof see [1]). Let $[E_p]_{\bar{B}_j}^*$ be the dual space of $[E_p]_{\bar{B}_j}$. The space $[E_p]_{\bar{B}_j}^*$ is defined in a usual manner and it is in agreement with the Hilbert space obtained by the completion of $(L^2)$ with respect to the norm

$$\|\varphi\|_{-p,\bar{B}_j^{-1}} = \sqrt{\sum_{n=0}^{\infty} n!\bar{B}_j(n)^{-1}|f_n|_{-p}^2},$$

for each $p \geq 0$. The dual space of $[E]_{\bar{B}_j}$ is denoted by $[E]_{\bar{B}_j}^*$, which is one of the spaces of *white noise distributions* in the sense of Cochran-Kuo-Sengupta. We denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the canonical bilinear form on $[E]_{\bar{B}_j}^* \times [E]_{\bar{B}_j}$. Then we have

$$\langle\langle \Phi, \varphi \rangle\rangle = \sum_{n=0}^{\infty} n!\langle F_n, f_n \rangle$$

for any $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(F_n) \in [E]_{\bar{B}_j}^*$ and $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in [E]_{\bar{B}_j}$, where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form that links $(E_{\mathbf{C}}^{\otimes n})^*$ and $E_{\mathbf{C}}^{\otimes n}$.

Since $\exp\langle\cdot,\xi\rangle$ and $\exp(i\langle\cdot,\xi\rangle)$ are in $[E]_{\bar{B}_j}$, the $S$-transform $S[\Phi]$ and the $T$-transform $T[\Phi]$ of $\Phi \in [E]_{\bar{B}_j}^*$ are, by definition, of the forms

$$S[\Phi](\xi) = \exp\left(-\frac{1}{2}\langle\xi,\xi\rangle\right)\langle\langle\Phi, \exp\langle\cdot,\xi\rangle\rangle\rangle,$$

and

$$T[\Phi](\xi) = \langle\langle\Phi, e^{i\langle\cdot,\xi\rangle}\rangle\rangle, \qquad \xi \in E_{\mathbf{C}},$$

respectively. By the expansion

$$e^{i\langle x,\xi\rangle} = e^{-(1/2)|\xi|_0^2} \sum_{n=0}^{\infty} \frac{i^n}{n!} :\langle x,\xi\rangle^n:,$$

where $:\langle x,\xi\rangle^n:$ is the wick ordering of $\langle x,\xi\rangle^n$ (See [6] for example.), we can calculate the norm $\|e^{i\langle\cdot,\xi\rangle}\|_{p,\bar{B}_j}$ for any $p > 0$ and $j \geq 1$ as follows:

$$\|e^{i\langle\cdot,\xi\rangle}\|_{p,\bar{B}_j}^2 = e^{-|\xi|_0^2} \sum_{n=0}^{\infty} \frac{\bar{B}_j}{n!}|\xi|_p^{2n} = e^{-|\xi|_0^2} \frac{\exp_j(|\xi|_p^2)}{\exp_j(0)} < \infty.$$

Thus, for each $j \geq 1$, we get that $e^{i\langle\cdot,\xi\rangle} \in [E]_{\bar{B}_j}$ and therefore we can estimate $|T[\Phi](\xi)|$ for $\Phi \in [E]^*_{\bar{B}_j}$ and $p > 0$ as follows:

$$|T[\Phi](\xi)| = |\langle\!\langle \Phi, e^{i\langle\cdot,\xi\rangle} \rangle\!\rangle| \leq \|\Phi\|_{-p,\bar{B}_j^{-1}} \|e^{i\langle\cdot,\xi\rangle}\|_{p,\bar{B}_j}.$$

This estimation implies that, for each $\Phi \in [E]^*_{\bar{B}_j}$, the map $T[\Phi] : E_{\mathbf{C}} \to \mathbf{C}$ is continuous. From $S[\Phi](\xi) = \exp\left(-\frac{1}{2}\langle\xi,\xi\rangle\right) T[\Phi](-i\xi)$, the $S$-transform $S[\Phi](\xi)$ is also continuous in $\xi$.

For any $\varphi \in [E]_{\bar{B}_j}$, define the *Gâteaux derivative* $D_y\varphi$ in the direction $y \in E^*$ by

$$D_y\varphi(x) = \lim_{\lambda \to 0} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda},$$

where the limit is taken in the topology of $[E]_{\bar{B}_j}$. For $y \in E^*$ and $f \in E^{\hat{\otimes} n}$, we introduce a notation $\star$ by

$$y \star f(u_1, \ldots, u_{n-1}) = \int y(u_n) \cdot f(u_1, \ldots, u_n) du_n.$$

Then, for $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in [E]_{\bar{B}_j}$, $D_y\varphi$ is expressed in the form

$$D_y\varphi = \sum_{n=1}^{\infty} n\mathbf{I}_{n-1}(y \star f_n),$$

provided that the convergence of the sum is guaranteed. We denote the adjoint-operator of $D_y$ by $D_y^*$.

The *white noise differential operator* $\partial_t$ is defined to be the operator $D_{\delta_t}$ acting on $[E]_{\bar{B}_j}$. Then $\partial_t$ is a continuous linear operator from $[E]_{\bar{B}_j}$ to itself, and its adjoint operator $\partial_t^*$ is also a continuous linear operator from $[E]^*_{\bar{B}_j}$ into itself.

It is noted that $\partial_t$ is an annihilation operator, while the adjoint $\partial_t^*$ is a creation operator.

## 3. A white noise version of the equation that describes an evolutional phenomenon in biology

In this section we find a solution of the equation in the space $[E]^*_{\bar{B}_j}$ which is recently obtained by Cochran-Kuo-Sengupta [1].

We consider the equation (1.1) with the following setting. Since F. Oosawa et al. [10] assumed that $E(t)$ is a centered stationary Gaussian process, we can regard that $E(t)$ is expressed in the form, $F(t,u) \in L^2(\mathbf{R}) \cap C^\infty(\mathbf{R})$ and $E(t)1 = \int_{-\infty}^{t} F(t,u)\dot{\mathbf{B}}(u)du$, with the nondeterministic property.

Let $G^t(u) = F(t,u)1_{[t_0,t]}(u)$ for each $t > 0$. Set $E(t) = D^*_{G^t}$ $(= \int_{t_0}^t F(t,u)\partial_u^* du)$ and denote the set of all $\Phi \in [E]^*_{\bar{B}_j}$ such that $\sum_{n=0}^{\infty} \frac{\alpha^n}{n!}(D^*_{G^t})^n \Phi$ exists in $[E]^*_{\bar{B}_j}$ by $Dom(e^{\alpha E(t)})$. Then we can define an operator $\exp(\alpha E(t))$, $\alpha \in \mathbf{R}$ by $\exp(\alpha E(t)) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}(D^*_{G^t})^n$.

Replacing $p(t)$ in (1.1) with a white noise functional $\Phi(t,\cdot)$ we may discuss the following equation:

$$\frac{\partial}{\partial t}\Phi(t,x) = -k_{+o}\exp\{-\beta E(t)\}\Phi(t,x) + k_{-o}\exp\{\beta E(t)\}(1 - \Phi(t,x)). \tag{3.1}$$

The operator $E(t)$ is continuous linear from $[E]^*_{\bar{B}_j}$ into itself.

A generalized white noise functional $\Phi(t,x)$ is called to be a solution of (3.1) if $\Phi(t,x)$ satisfies the following conditions:

(1) for each $\xi$, $U(t,\xi) = S[\Phi(t,\cdot)](\xi)$ is differentiable in $t$.

(2) for each $\xi$ and $t$, $U(t,\xi)$ satisfies

$$\begin{aligned}\frac{\partial}{\partial t}U(t,\xi) = (-k_{+o}\exp\{-\beta\tilde{E}(t,\xi)\} - k_{-o}\exp\{\beta\tilde{E}(t,\xi)\})U(t,\xi)\\ + k_{-o}\exp\{\beta\tilde{E}(t,\xi)\},\end{aligned} \tag{3.2}$$

where $\tilde{E}(t,\xi) = \int_{t_0}^t F(t,u)\xi(u)du$, which is the $S$-transform of $E(t)1$.

The method in this section gives an investigation of the equation introduced by F. Oosawa [10] and also gives an example which is actually useful to study the theory of the space $[E]^*_{\bar{B}_j}$.

Note that $S[\exp\{\alpha E(t)\}\Phi](\xi) = \exp\{\alpha\tilde{E}(t,\xi)\}S\Phi(\xi)$ for $\Phi \in [E]^*_{\bar{B}_j}$.

We can solve (3.2) which is a linear ordinary differential equation of the first order and get the solution of the form.

$$\begin{aligned}U(t,\xi) = \exp\left\{\int_{t_0}^t (-k_{+o}\exp\{-\beta\tilde{E}(s,\xi)\} - k_{-o}\exp\{\beta\tilde{E}(s,\xi)\})ds\right\}\\ \cdot\left\{C_1 + k_{-o}\int_{t_0}^t \exp\left(\beta\tilde{E}(s,\xi) + \int_{t_0}^s [k_{+o}\exp\{-\beta\tilde{E}(u,\xi)\}\right.\right.\\ \left.\left. + k_{-o}\exp\{\beta\tilde{E}(u,\xi)\}]du\right)ds\right\},\end{aligned} \tag{3.3}$$

where $C_1$ is the initial value of $U(t,\xi)$ satisfying $0 < C_1 < 1$.

It can be easily checked that $U(t,\xi)$ satisfies the following two properties: for each $t$ $(> t_0)$

i) $U(t, z\xi + \eta)$, $z \in \mathbf{C}$ is entire function in $z$ for any $\xi, \eta \in E_{\mathbf{C}}$.

ii) there exist $K_1 > 0$ and $K_2 > 0$ such that

$$|U(t,\xi)| \le K_1 \exp[\exp(K_2|\xi|_0^2)], \qquad \xi \in E_{\mathbf{C}}.$$

In fact, we can take the constants $K_2 = 1$ and

$$K_1 = \{C_1 + k_{-o}(t - t_0)\}$$
$$\cdot \exp\left(\frac{1}{2}\{2(k_{+o} + k_{-o})(t - t_0) + 1\}^2 \exp\{\beta^2 \sup_{t_0 \le s \le t} \int_{t_0}^t F(s, u)^2 du\}\right).$$

These properties show that for each $t \ge t_0$, $U(t, \cdot)$ is a $U$-functional of the element in $[E]^*_{\bar{B}_2}$, i.e., $U(t, \cdot) \in S[[E]^*_{\bar{B}_2}]$. (See [1].)

For $\Phi, \Psi \in [E]^*_{\bar{B}_j}$, the Wick product, denoted by $\Phi \diamond \Psi$, can be defined by

$$S[\Phi \diamond \Psi](\xi) = S[\Phi](\xi) S[\Psi](\xi), \qquad \xi \in E_{\mathbf{C}},$$

since the product in right hand side is again $U$-functional by the characteristic theorem. Similarly $S[\Phi](\xi)^n$ defined a white noise distribution which are denoted by $\Phi^{\diamond n}$:

$$S[\Phi^{\diamond n}](\xi) = (S[\Phi](\xi))^n.$$

With the notation established above we prove the following theorem.

THEOREM 3.1. *For each* $t > t_0$ *the equation* (3.1) *has a unique solution in* $[E]^*_{\bar{B}_2}$ *given by*

$$\Phi(t, x) = \exp^\diamond\left\{\int_{t_0}^t (-k_{+o} \exp\{-\beta E(s)1\} - k_{-o} \exp\{\beta E(s)1\}) ds\right\}$$
$$\diamond \left\{C_1 + k_{-o} \cdot \int_{t_0}^t \exp^\diamond\left(\beta E(s)1 + \int_{t_0}^s [k_{+o} \exp\{-\beta E(u)1\}\right.\right.$$
$$\left.\left. + k_{-o} \exp\{\beta E(u)1\}] du\right) ds\right\}, \tag{3.4}$$

*where* $\exp^\diamond[\Psi] = \sum_{n=0}^\infty \frac{1}{n!} \Psi^{\diamond n}$ for $\Psi \in [E]^*_{\bar{B}_2}$.

REMARK. By (3.2) we can check the above conditions i) and ii) for $\frac{\partial}{\partial t} U(t, \xi)$. In fact, $\left|\frac{\partial}{\partial t} U(t, \xi)\right| \le K_1 \exp\{\exp\{K_2 |\xi|_0^2\}\}$ for each $t > t_0$ with

$$K_1 = (C_1 k_{+o} + k_{-o}\{1 + C_1 + (k_{+o} + k_{-o})(t - t_0)\})$$
$$\cdot \exp\left(\{(k_{+o} + k_{-o})(t - t_0) + 1\}^2 \cdot \exp\left\{\beta^2 \sup_{t_0 \le s \le t} \int_{t_0}^t F(s, u)^2 du\right\}\right)$$

and $K_2 = 1$. Therefore $\frac{\partial}{\partial t} \Phi(t, x)$ is a member of $[E]^*_{\bar{B}_2}$.

## 4. Characteristic properties of $\Phi(t,x)$

In this section, as a characteristic property, we prove the positivity of the solution $\Phi(t,x)$ of (3.1), which is requested as a probability. This fact should be clarified since the solution itself is a generalized white noise function so that it is impossible to show positivity in the ordinary sense. Probability is a number between 0 and 1, while the $\Phi(t,x)$ is a generalized function. We expect that $\Phi$ is a generalization of the probability $p(t)$. To give a plausible interpretation to the $\Phi(t,x)$, we show that its mean is in between 0 and 1.

DEFINITION 4.1. *A generalized function $\Phi$ in $[E]_{\bar{B}_j}^*$ is called positive if $\langle\langle \Phi, \varphi\rangle\rangle \geq 0$ for all nonnegative test function $\varphi$ in $[E]_{\bar{B}_j}$.* (cf. [6])

LEMMA 4.2. *Let $\Phi$ by a generalized white noise function $\Phi$ in $[E]_{\bar{B}_j}^*$. Then the following are equivalent:*

(a) *$\Phi$ is positive.*

(b) *$T\Phi$ is positive definite on $[E]_{\bar{B}_j}$.*

PROOF. The proof is almost same to Theorem 15.3 in [6].

(a)$\to$(b): Let $\xi_k \in E$, $z_k \in \mathbf{C}$, $k = 1, \dots, n$. Then we have

$$\sum_{l,k=1}^{n} z_l T\Phi(\xi_l - \xi_k)\overline{z_k} = \left\langle\!\!\left\langle \Phi, \left|\sum_{l=1}^{n} z_l e^{i\langle \cdot, \xi_l\rangle}\right|^2 \right\rangle\!\!\right\rangle.$$

Observe that

$$\left|\sum_{l=1}^{n} z_l e^{i\langle \cdot, \xi_l\rangle}\right|^2 = \sum_{l,k=1}^{n} z_l \overline{z_k} e^{i\langle \cdot, \xi_l - \xi_k\rangle}$$

is a nonnegative test function in $[E]_{\bar{B}_j}$. Hence by the positivity of $\Phi$,

$$\sum_{l,k=1}^{n} z_l T\Phi(\xi_l - \xi_k)\overline{z_k} \geq 0.$$

This shows that $T\Phi$ is positive definite.

(b)$\to$(a): Suppose $T\Phi$ is positive definite on $E$. From the argument in §2, we see that $T\Phi$ is continuous on $E$. Hence by the Minlos Theorem there exists a finite measure $\nu$ on $E^*$ such that

$$T\Phi(\xi) = \int_{E^*} e^{i\langle x, \xi\rangle} d\nu(x), \qquad \forall \xi \in E,$$

or equivalently,

$$\langle\!\langle \Phi, e^{i\langle \cdot, \xi\rangle} \rangle\!\rangle = \int_{E^*} e^{i\langle x, \xi\rangle} dv(x), \qquad \forall \xi \in E. \tag{4.1}$$

We need to show that $v$ is a Hida measure inducing $\Phi$, i.e., $[E]_{\bar{B}_j} \subset L^1(v)$ and

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \int_{E^*} \varphi(x) dv(x), \qquad \forall \varphi \in [E]_{\bar{B}_j}. \tag{4.2}$$

Let $V$ be the subspace of $[E]_{\bar{B}_j}$ spanned by the set $\{e^{i\langle \cdot, \xi\rangle}; \xi \in E\}$. It follows from equation (4.1) that

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \int_{E^*} \varphi(x) dv(x), \qquad \forall \varphi \in V. \tag{4.3}$$

Note that if $\varphi, \psi \in V$, then $\varphi\psi \in V$. In paticular, if $\varphi \in V$, then $|\varphi|^2 \in V$ and by equation (4.3) we have

$$\int_{E^*} |\varphi(x)|^2 dv(x) = \langle\!\langle \Phi, |\varphi|^2 \rangle\!\rangle < \infty.$$

Hence $\varphi \in L^2(v)$. This shows that $V \subset L^2(v)$. Since $V$ is dense in $[E]_{\bar{B}_j}$, by using the same method in [[6]; Theorem 15.3], we can prove that $[E]_{\bar{B}_j} \subset L^2(v)$ (so $[E]_{\bar{B}_j} \subset L^1(v)$) and equation (4.2) holds. This implies the positivity of $\Phi$.

LEMMA 4.3. *If $U(\xi)$ and $V(\xi)$ are positive definite functions in $S[(E)^*]$, then $U(\xi) \cdot V(\xi)$ is also a positive definite function in $S[(E)^*]$.*

PROOF. See [7] or [8] for example.

THEOREM 4.4. *For each $t$ $(>t_0)$ the solution $\Phi(t,x)$ of* (3.1) *is positive.*

PROOF. By Lemma 4.2., it is sufficient to prove that $T[\Phi(t,\cdot)](\xi)$ is positive definite.

Then we can calculate $T[\Phi(t,\cdot)](\xi)$ as follows:

$$\begin{aligned}\sum_{k,l=1}^{N} \alpha_k \bar{\alpha}_l T[\Phi(t,\cdot)](\xi_k - \xi_l) &= \sum_{k,l=1}^{N} \alpha_k \bar{\alpha}_l C(\xi_k - \xi_l) S[\Phi(t,\cdot)](i(\xi_k - \xi_l)) \\ &= \sum_{k,l=1}^{N} \alpha_k \bar{\alpha}_l C(\xi_k - \xi_l) U(t, i(\xi_k - \xi_l))\end{aligned}$$

Since a functional $C(\xi)$ is positive definite, by Lemma 4.3., it is sufficient to prove $U(t, i\xi)$ is positive definite for each $t \geq t_0$.

$$\sum_{k,l=1}^{N} \alpha_k \bar{\alpha}_l U(t, i(\xi_k - \xi_l))$$

$$= \sum_{k,l=1}^{N} \alpha_k \bar{\alpha}_l \left[ \sum_{\nu_1=0}^{\infty} \frac{(-k_{+o})^{\nu_1}}{\nu_1!} \left\{ \int_{t_0}^{t} e^{-q(s,k)+q(s,l)} ds \right\}^{\nu_1} \right]$$

$$\cdot \left[ \sum_{\nu_2=0}^{\infty} \frac{(-k_{-o})^{\nu_2}}{\nu_2!} \left\{ \int_{t_0}^{t} e^{q(s,k)-q(s,l)} ds \right\}^{\nu_2} \right]$$

$$\cdot \left[ C_1 + k_{-o} \int_{t_0}^{t} \left\{ e^{q(s,k)-q(s,l)} \sum_{\nu_3=0}^{\infty} \frac{k_{+o}^{\nu_3}}{\nu_3!} \left( \int_{t_0}^{s} e^{-q(r,k)+q(r,l)} dr \right)^{\nu_3} \right. \right.$$

$$\left. \left. \cdot \sum_{\nu_4=0}^{\infty} \frac{k_{-o}^{\nu_4}}{\nu_4!} \left( \int_{t_0}^{s} e^{q(r,k)-q(r,l)} dr \right)^{\nu_4} \right\} ds \right]$$

$$= C_1 \sum_{\nu_1=0}^{\infty} \frac{(-k_{+o})^{\nu_1}}{\nu_1!} \sum_{\nu_2=0}^{\infty} \frac{(-k_{-o})^{\nu_2}}{\nu_2!} \int_{t_0}^{t} \cdots (\nu_1 + \nu_2) \cdots$$

$$\int_{t_0}^{t} \left| \sum_{k=1}^{N} \alpha_k e^{-\sum_{a_1=1}^{\nu_1} q(s_{a_1},k) + \sum_{a_2=1}^{\nu_2} q(s'_{a_2},k)} \right|^2 d\mathbf{u} d\mathbf{u}'$$

$$+ k_{-o} \sum_{\nu_1,\dots,\nu_4=0}^{\infty} \frac{(-1)^{\nu_1+\nu_2} k_{+o}^{\nu_1+\nu_3} k_{-o}^{\nu_2+\nu_4}}{\nu_1! \cdots \nu_4!}$$

$$\cdot \int_{t_0}^{t} \cdots (\gamma) \cdots \int_{t_0}^{t} \prod_{i=1}^{\nu_3} 1_{[t_0,s]}(s''_i) \times \prod_{i=1}^{\nu_3} 1_{[t_0,s]}(s'''_j)$$

$$\cdot \left| \sum_{k=1}^{N} \alpha_k e^{q(s.k) - \sum_{a_1=1}^{\nu_1} q(s_{a_1},k) + \sum_{a_2=1}^{\nu_2} q(s'_{a_2},k) - \sum_{a_3=1}^{\nu_3} q(s''_{a_3},k) + \sum_{a_4=1}^{\nu_4} q(s'''_{a_4},k)} \right|^2 ds d\mathbf{u} d\mathbf{u}' d\mathbf{v} d\mathbf{v}'.$$

where $d\mathbf{u} = ds_1 \cdots ds_{\nu_1}$, $d\mathbf{u}' = ds'_1 \cdots ds'_{\nu_2}$, $d\mathbf{v} = ds''_1 \cdots ds''_{\nu_3}$, $d\mathbf{v}' = ds'''_1 \cdots ds'''_{\nu_4}$, and $\gamma = \nu_1 + \nu_2 + \nu_3 + \nu_4 + 1$, and $q(x,y) = i\beta \int_{t_0}^{x} F(x,u)\xi_y(u)du$.

Therefore we have $\sum_{k,l=1}^{N} \alpha_k \bar{\alpha}_l T[\Phi(t,\cdot)](\xi_k - \xi_l) \geq 0$. Thus the assertion is proved. $\square$

THEOREM 4.5. *Let $0 < C_1 < 1$ and let $\Phi(t,x)$ be the solution of* (3.1). *Then, it holds that $0 < E[\Phi(t,\cdot)] < 1$ for each $t \geq t_0$.*

PROOF. Since we have $E[\Phi(t,\cdot)] = U(t,\xi)|_{\xi=0}$ for each $t > t_0$ by the definition of generalized expectation, from (3.3) the expectation $E[\Phi(t,\cdot)]$ is given by

$$E[\Phi(t,\cdot)] = \exp\{-(k_{+o}+k_{-o})(t-t_0)\} \cdot \left[C_1 + \frac{k_{-o}}{k_{+o}+k_{-o}}(\exp\{(k_{+o}+k_{-o})(t-t_0)\}-1)\right]. \tag{4.1}$$

By the condition it is obvious that $E[\Phi(t,\cdot)] > 0$. The expectation $E[\Phi(t,\cdot)]$ is equal to

$$\left(C_1 - \frac{k_{-o}}{k_{+o}+k_{-o}}\right)\exp\{-(k_{+o}+k_{-o})(t-t_0)\} + \frac{k_{-o}}{k_{+o}+k_{-o}}.$$

Since $0 < \exp\{-(k_{+o}+k_{-o})(t-t_0)\} < 1$, $0 < \dfrac{k_{-o}}{k_{+o}+k_{-o}} < 1$ and the condition $0 < C_1 < 1$, we obtain $E[\Phi(t,\cdot)] < 1$. $\square$

## 5. Asymmetry in time

In this section we discuss the asymmetry with respect to time reversal for the solution $\Phi(t,x)$, as in (3.4), of the equation (3.1) with $F(t,u) = e^{-\alpha(t-u)}$, $\alpha > 0, t, u \in \mathbf{R}$.

If we justify that $E[p(t)\dot{p}(t+h)] = E[\dot{p}(t)p(t+h)]$ for any $t$ and $h$, this implies an asymmetry in time of $p(t)$. Since we now regard $p(t)$ as a generalized white noise functional $\Phi(t) \equiv \Phi(t,x)$, we introduce an asymmetry, the $[E]^*_{\bar{B}_2}$-asymmetry, in time for $\Phi(t,x)$ by

$$(\Phi(t), \dot{\Phi}(t+h))_{0,\bar{B}_2^{-1}} \neq (\Phi(t+h), \dot{\Phi}(t))_{0,\bar{B}_2^{-1}}. \tag{5.1}$$

As we can assume that $\Phi(t)$ is stationary, (5.1) is equal to

$$(\Phi'(t), \Phi(t+h))_{0,\bar{B}_2^{-1}} \neq (\Phi'(t), \Phi(t-h))_{0,\bar{B}_2^{-1}} \tag{5.2}$$

The $S$-transform $U(t,\xi)$ of $\Phi(t,x)$ is given as in (3.3). Set

$$P(t,\xi) = \exp\left\{\int_{t_0}^{t}(-k_{+o}\exp\{-\beta\tilde{E}(s,\xi)\} - k_{-o}\exp\{\beta\tilde{E}(s,\xi)\})ds\right\},$$

and set also

$$Q(t,\xi) = \int_{t_0}^{t}\exp\left(\beta\tilde{E}(s,\xi) + \int_{t_0}^{s}[k_{+o}\exp\{-\beta\tilde{E}(u,\xi)\} + k_{-o}\exp\{\beta\tilde{E}(u,\xi)\}]du\right)ds.$$

Then, $P(t,\xi)$ and $Q(t,\xi)$ have the following expansions:

$$P(t,\xi) = \sum_{n=0}^{\infty}\langle \xi^{\otimes n}, f_n(t)\rangle, \qquad Q(t,\xi) = \sum_{n=0}^{\infty}\langle \xi^{\otimes n}, g_n(t)\rangle,$$

where $f_n(t)$ and $g_n(t)$ are given by

$$f_n(t) = \frac{1}{n!}(-\beta)^n e^{-(k_{+o}+k_{-o})(t-t_0)}$$

$$\sum_{\substack{0\le k_1\le\cdots\le k_n\le n\\ k_1+\cdots+k_n=n}} \mathscr{A}_{k_1,\dots,k_n}\prod_{v=1}^{n}((-1)^{k_v}k_{+o}+k_{-o})L_{k_1}(t)\,\hat{\otimes}\cdots\hat{\otimes}\,L_{k_n}(t), \quad n\ge 1$$

$f_0(t) = e^{-(k_{+o}+k_{-o})(t-t_0)}$, and

$$g_n(t) = \int_{t_0}^{t}((-1)^n e^{2(k_{+o}+k_{-o})(s-t_0)}f_n(s) + f_n^*(s))ds, \qquad n = 0,1,2,\dots$$

with

$$\mathscr{A}_{k_1,\dots,k_n} = \frac{n!}{(1!)^{\lambda_1}\cdots(n!)^{\lambda_n}\lambda_1!\cdots\lambda_n!},$$

(for $1\le j\le n$, $\lambda_j$ denotes the frequancy of appearance of $j$ in $k_i$'s, $i = 1,2,\dots,n$),

$$f_n^*(s) = \frac{1}{n!}\beta^n e^{(k_{+o}+k_{-o})(s-t_0)}\sum_{\substack{0\le k_1\le\cdots\le k_n\le n\\ k_1+\cdots+k_n=n}}\mathscr{A}_{k_1,\dots,k_n}\prod_{v=1}^{n}((-1)^{k_v}k_{+o}+k_{-o})$$

$$\cdot\sum_{j=1}^{n}\frac{1}{(-1)^{k_j}k_{+o}+k_{-o}}L_{k_1}(s)\hat{\otimes}\cdots\hat{\otimes}L'_{k_j}(s)\hat{\otimes}\cdots\hat{\otimes}L_{k_n}(s), \qquad n\ge 1,$$

$$f_0^*(s) = 0,$$

and

$$L_n(t)(v_1,\dots,v_n) = \int_{t_0}^{t}F(s,v_1)\cdots F(s,v_n)1_{[t_0,s]}(v_1)\cdots 1_{[t_0,s]}(v_n)ds.$$

Since $U(t,\xi) = P(t,\xi)(C_1 + k_{-o}Q(t,\xi))$, we have the chaos expansion of $\Phi(t,x)$:

$$\Phi(t,x) = \sum_{\ell=0}^{\infty}\left\langle :x^{\otimes\ell}:, C_1 f_\ell(t) + k_{-o}\sum_{m+n=\ell}f_m(t)\,\hat{\otimes}\,g_n(t)\right\rangle, \tag{5.3}$$

where $:x^{\otimes\ell}:$ is the Wick tensor of $x^{\otimes\ell}$ (see [6]) and $f_m(t)\,\hat{\otimes}\,g_n(t)$ means the symmetrization of $f_m(t)\otimes g_n(t)$. Therefore from (5.3) the chaos expansion of $\frac{\partial}{\partial t}\Phi(t,x)$ is given by

$$\frac{\partial}{\partial t}\Phi(t,x)$$

$$= \sum_{\ell=0}^{\infty}\left\langle :x^{\otimes \ell}:, C_1 f_\ell'(t) + k_{-o} \sum_{m+n=\ell} \{f_m'(t) \hat{\otimes} g_n(t) + f_m(t) \hat{\otimes} g_n'(t)\} \right\rangle. \quad (5.4)$$

It is sufficient to prove (5.2) to imply an asymmetry in time in $\Phi(t,x)$, where $(\cdot,\cdot)_{0,\bar{B}_2^{-1}}$ is the inner product of $[E_0]^*_{\bar{B}_2}$. From (5.2), we may prove $\|\Phi'(t)\|_{0,\bar{B}_2^{-1}} \neq 0$. By (5.4) the norm $\|\Phi'(t)\|_{0,\bar{B}_2^{-1}}$ is given by

$$\|\Phi'(t)\|_{0,\bar{B}_2^{-1}}$$

$$= \sqrt{\sum_{\ell=0}^{\infty} \ell! \bar{B}_2(\ell)^{-1} \left| C_1 f_\ell'(t) + k_{-o} \sum_{m+n=\ell} \{f_m'(t) \hat{\otimes} g_n(t) + f_m(t) \hat{\otimes} g_n'(t)\} \right|_0^2}.$$

Using $f_0'(t) = -(k_{+o} + k_{-o}) f_0(t)$ and $\int_{t_0}^t f_0(s)ds = \frac{1}{k_{+o}+k_{-o}}(1 - f_0(t))$, we have

$$C_1 f_0'(t) + k_{-o} f_0'(t) \otimes g_0(t) + k_{-o} f_0(t) \otimes g_0'(t)$$

$$= -f_0(t)[C_1(k_{+o} + k_{-o}) - k_{-o}],$$

because we employed $g_0(t) = (\exp\{(k_{+o}+k_{-o})(t-t_0)\} - 1)/(k_{+o}+k_{-o})$.

Thus, we have the following.

THEOREM 5.1. *Let $\Phi(t,x)$ be a solution of* (3.1) *as in* (3.4) *and let* $C_1 \neq \lim_{t\to\infty} E[\Phi(t)] \left(= \frac{k_{-o}}{k_{+o}+k_{-o}}\right)$. *Then, for any $t \in \mathbf{R}$ and $h > 0$, $\Phi(t,x)$ has the $[E]^*_{\bar{B}_2}$-asymmetry in time.*


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