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On generalized ℓ_p-spaces

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Abstract. In [8], we characterized absolute normalized norms on $\mathbf{C}^n$ by using a continuous convex function with some appropriate conditions on a certain convex subset of $\mathbf{R}^n$. In this paper, we introduce the notion of generalized ℓ_p-spaces, that is, ℓ_ψ-spaces, and study their structures.

1. Introduction and preliminaries

The study of the geometrical structure of Banach spaces is the main line of the theory of functional analysis. In this paper, we present the systematic examples of certain Banach spaces and study the geometrical structure.

Let ℓ_0 denote the set of all infinite sequences of complex numbers with only finitely many non-zero elements. A norm $\|\cdot\|$ on ℓ_0 is called absolute if

$$\|\{x_n\}_{n=1}^\infty\| = \|\{|x_n|\}_{n=1}^\infty\|$$

for all $\{x_n\}_{n=1}^\infty \in \ell_0$, and normalized if $\|e_n\| = 1$ for all $n = 1, 2, \dots,$ where $e_n = (0, 0, \dots, 0, \overset{(n)}{1}, 0, \dots)$. As in [8], to every absolute normalized norm $\|\cdot\|$ on ℓ_0, there corresponds a unique continuous convex function ψ satisfying some appropriate conditions on the convex subset

$$\Delta_\infty = \left\{ s = \{s_n\}_{n=1}^\infty \in \ell_0 : \sum_{n=1}^\infty s_n = 1, s_n \geq 0 \ (\forall n) \right\}$$

under the equation $\psi(s) = \|s\|$. Using this, we introduce the following spaces. For a corresponding convex function ψ, we define the space ℓ_ψ by

$$\ell_\psi = \left\{ \{x_n\}_{n=1}^\infty \in \ell_\infty : \lim_{n\to\infty} \|(x_1, \dots, x_n, 0, 0, \dots)\|_\psi < \infty \right\},$$

where ℓ_∞ is the Banach space of all bounded infinite sequences of complex numbers. Then ℓ_ψ is a Banach space with the norm

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$$\|\{x_n\}_{n=1}^{\infty}\|_{\psi} = \lim_{n\to\infty} \|(x_1, \dots, x_n, 0, 0, \dots)\|_{\psi}.$$

Further, we define c_ψ by the closure of ℓ_0 in $(\ell_\psi, \|\cdot\|_\psi)$. This is a generalization of the ℓ_p-spaces. Also, this space is concerned with the infinite sequence spaces which are given by symmetric norming functions as in [3] (cf. [10]). Our norm is not necessarily symmetric. However, we believe that our theory gives many interesting examples of Banach spaces to study their geometrical structure.

In this paper, we study the norm structure of ℓ_ψ-spaces and characterize the separability and the geometrical structure using the corresponding convex function. In section 2, we consider the norm structure of ℓ_ψ-spaces using the convex functions in Δ_∞. In section 3, we introduce the notion of the absolute ideal $\mathscr{U}$ of ℓ_∞, and study the separability of $\mathscr{U}$ and ℓ_ψ-spaces. Namely, we show that if $\mathscr{U}$ is an absolute ideal of ℓ_∞, then $c_\psi = \mathscr{U}$ if and only if $\mathscr{U}$ is separable. In particular, if $c_\psi \neq \ell_\psi$, then ℓ_ψ is not separable. In section 4, we consider the strict convexity and uniform convexity of ℓ_ψ. As in [8], $(\ell_0, \|\cdot\|_\psi)$ is strictly convex if and only if ψ is strictly convex on Δ_∞. However we present an example of strictly convex functions ψ such that ℓ_ψ is not strictly convex. On the other hand, we show that ℓ_ψ is uniformly convex if and only if $(\ell_0, \|\cdot\|_\psi)$ is uniformly convex. In section 5, we consider the absolute normalized norms on ℓ_0 which are equivalent to ℓ_1- and ℓ_∞-norms, respectively.

2. ℓ_ψ-spaces

In this section, we introduce the notion of the ℓ_ψ-spaces. Let AN_∞ be the family of all absolute normalized norms on ℓ_0. The ℓ_p-norm $\|\cdot\|_p$ is a basic example:

$$\|\{x_n\}_{n=1}^{\infty}\|_p = \begin{cases} \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max_{1\le n<\infty} |x_n| & \text{if } p = \infty. \end{cases}$$

We summarize some basic properties about the absolute norms on ℓ_0. As in the proof of [8, Lemmas 2.1 and 2.2], we have

Lemma 2.1. (i) *Let* $\|\cdot\| \in AN_\infty$. *If* $x = \{x_n\}_{n=1}^{\infty}$, $y = \{y_n\}_{n=1}^{\infty} \in \ell_0$, *then* $\|x\| \le \|y\|$, *whenever* $|x_n| \le |y_n|$ *for all* n.

(ii) *For every* $\|\cdot\| \in AN_\infty$, *we have*

$$\|\cdot\|_\infty \le \|\cdot\| \le \|\cdot\|_1.$$

As in Saito, Kato and Takahashi [8], we can characterize an absolute normalized norm on ℓ_0 by using a continuous convex function on Δ_∞ (see also [2]).

Theorem 2.2. (i) *For every* $\|\cdot\| \in AN_\infty$, *we define*

$$\psi(s) = \|s\| \qquad (s \in \Delta_\infty). \tag{1}$$

Then ψ *is a continuous convex function on* Δ_∞ *satisfying the following conditions:*

$$\psi(e_n) = 1 \tag{A_0}$$

$$\psi(s) \geq (1 - s_n)\psi\left(\frac{s_1}{1-s_n}, \ldots, \frac{s_{n-1}}{1-s_n}, \overset{(n)}{0}, \frac{s_{n+1}}{1-s_n}, \ldots\right) \tag{A_n}$$

for all $n = 1, 2, \ldots$ *and every* $s = \{s_n\}_{n=1}^\infty \in \Delta_\infty$ *with* $s_n \neq 1$.

(ii) *Let* Ψ_∞ *denote the family of all continuous convex functions* ψ *on* Δ_∞ *satisfying* (A_n) *for all* $n = 0, 1, 2, \ldots$. *For any* $\psi \in \Psi_\infty$, *we define*

$$\|\{x_n\}_{n=1}^\infty\|_\psi = \begin{cases} \left(\sum_{i=1}^\infty |x_i|\right)\psi\left(\frac{|x_1|}{\sum_{i=1}^\infty |x_i|}, \ldots, \frac{|x_n|}{\sum_{i=1}^\infty |x_i|}, \ldots\right), & \textit{if } \{x_n\}_{n=1}^\infty \neq 0, \\ 0, & \textit{if } \{x_n\}_{n=1}^\infty = 0. \end{cases}$$

Then $\|\cdot\|_\psi \in AN_\infty$ *and satisfies* (1). *Therefore* AN_∞ *and* Ψ_∞ *are in a one-to-one correspondence under the equation* (1).

Proof. (i) For every $\|\cdot\| \in AN_\infty$, we define $\psi(s) = \|s\|$ $(s = \{s_n\} \in \Delta_\infty)$. Clearly, ψ is a continuous convex function on Δ_∞. By Lemma 2.1, we have for every n

$$\begin{aligned} \psi(s) &= \|(s_1, s_2, \ldots, s_n, \ldots)\| \\ &\geq \|(s_1, s_2, \ldots, s_{n-1}, \overset{(n)}{0}, s_{n+1}, \ldots)\| \\ &= (1 - s_n)\left\|\left(\frac{s_1}{1-s_n}, \frac{s_2}{1-s_n}, \ldots, \frac{s_{n-1}}{1-s_n}, \overset{(n)}{0}, \frac{s_{n+1}}{1-s_n}, \ldots\right)\right\| \\ &= (1 - s_n)\psi\left(\frac{s_1}{1-s_n}, \ldots, \frac{s_{n-1}}{1-s_n}, \overset{(n)}{0}, \frac{s_{n+1}}{1-s_n}, \ldots\right). \end{aligned}$$

Thus, we have $\psi \in \Psi_\infty$.

(ii) As in the proof of [8, Theorem 3.4], all properties of an absolute norm are clear except the triangular inequality. Let $x, y \in \ell_0$. Then there exists some $m \in \mathbf{N}$ such that $x = (x_1, \ldots, x_m, 0, 0, \ldots)$ and $y = (y_1, \ldots, y_m, 0, 0, \ldots)$. We can consider that x and y are elements in $\mathbf{C}^m$. By [8, Theorem 3.4], we have $\|x + y\|_\psi \leq \|x\|_\psi + \|y\|_\psi$. This completes the proof. □

Let ψ_p be the corresponding convex function in Ψ_∞ for $\|\cdot\|_p$. Using this characterization, we introduce the following spaces. Let $\{x_n\}_{n=1}^\infty$ be an infinite sequence of complex numbers. By Lemma 2.1, $\{\|(x_1, \ldots, x_n, 0, 0, \ldots)\|_\psi\}_{n=1}^\infty$ is an increasing sequence. Thus we have

DEFINITION 2.3. For $\psi \in \Psi_\infty$, we define the space ℓ_ψ by

$$\ell_\psi = \left\{ \{x_n\}_{n=1}^\infty \in \ell_\infty : \lim_{n\to\infty} \|(x_1, \dots, x_n, 0, 0, \dots)\|_\psi < \infty \right\}.$$

Further, we define c_ψ by the closure of ℓ_0 in $(\ell_\psi, \|\cdot\|_\psi)$.

PROPOSITION 2.4. *ℓ_ψ and c_ψ are Banach spaces with the norm*

$$\|\{x_n\}_{n=1}^\infty\|_\psi = \lim_{n\to\infty} \|(x_1, \dots, x_n, 0, 0, \dots)\|_\psi.$$

PROOF. Since c_ψ is closed in ℓ_ψ, we only show that ℓ_ψ is a Banach space. Let $\{y_k\}_{k=1}^\infty$ be any Cauchy sequence of ℓ_ψ. We put $y_k = \{x_n^k\}_{n=1}^\infty$ for every k. By Lemma 2.1, $\{x_n^k\}_{k=1}^\infty$ is a Cauchy sequence of $\mathbf{C}$ for each n. Thus there exists $x_n \in \mathbf{C}$ such that $x_n^k \to x_n$ for every n. By Lemma 2.1,

$$\begin{aligned}\|(x_1^k - x_1, \dots, x_n^k - x_n, 0, 0, \dots)\|_\psi &= \lim_{m\to\infty} \|(x_1^k - x_1^m, \dots, x_n^k - x_n^m, 0, 0, \dots)\|_\psi \\ &\le \varliminf_{m\to\infty} \|y_k - y_m\|_\psi.\end{aligned}$$

As $n \to \infty$, we have $y = \{x_n\}_{n=1}^\infty \in \ell_\psi$ and $\|y_k - y\|_\psi \to 0$. This completes the proof. □

If $1 \le p < \infty$, then $\ell_{\psi_p} = c_{\psi_p} = \ell_p$, and if $p = \infty$, then $\ell_{\psi_\infty} = \ell_\infty$ and $c_{\psi_\infty} = c_0$. Also, this space is concerned with the sequence spaces which are given by symmetric norming functions in [3] (cf. [10]). We next consider some norm properties of the ℓ_ψ-spaces.

PROPOSITION 2.5. *Let $\psi \in \Psi_\infty$. Then we have for every $x = \{x_n\}_{n=1}^\infty \in c_\psi$,*

$$\lim_{n\to\infty} \|(0, 0, \dots, 0, x_n, x_{n+1}, \dots)\|_\psi = 0.$$

PROOF. We put $x = \{x_n\}_{n=1}^\infty \in c_\psi$. Fix any $\varepsilon > 0$. Then there exists some

$$y = (y_1, \dots, y_{n_0-1}, 0, 0, \dots)$$

in ℓ_0 such that $\|x - y\|_\psi < \varepsilon$. By Lemma 2.1, we have, for every n with $n \ge n_0$,

$$\begin{aligned}\|(0, \dots, 0, x_n, x_{n+1}, \dots)\|_\psi &\le \|(0, \dots, 0, x_{n_0}, x_{n_0+1}, \dots)\|_\psi \\ &\le \|(x_1 - y_1, \dots, x_{n_0-1} - y_{n_0-1}, x_{n_0}, x_{n_0+1}, \dots)\|_\psi \\ &= \|x - y\|_\psi < \varepsilon.\end{aligned}$$
□

DEFINITION 2.6. Let $\psi \in \Psi_\infty$. Then ψ is called *regular* if $\ell_\psi = c_\psi$.

Since $\ell_{\psi_p} = c_{\psi_p} = \ell_p$ for $1 \le p < \infty$, ψ_p is regular. Next, we consider the dual spaces of c_ψ and ℓ_ψ. Let $\psi \in \Psi_\infty$ and let $\|\cdot\|_\psi^*$ be the dual norm on $(\ell_0, \|\cdot\|_\psi)$. That is, for any $y = \{y_n\}_{n=1}^\infty \in \ell_0$,

$$\|y\|_\psi^* = \sup\{|\langle \{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \rangle| : x = \{x_n\}_{n=1}^\infty \in \ell_0, \|x\|_\psi = 1\}$$
$$= \sup\left\{ \left| \sum_{n=1}^\infty x_n y_n \right| : x = \{x_n\}_{n=1}^\infty \in \ell_0, \|x\|_\psi = 1 \right\}.$$

Then it is clear that $\|\cdot\|_\psi^* \in AN_\infty$ and the corresponding convex function is given by

$$\psi^*(s) = \sup_{t \in \Delta_\infty} \frac{\sum_{n=1}^\infty s_n t_n}{\psi(t)}$$

for every $s = \{s_n\}_{n=1}^\infty \in \Delta_\infty$. We note that $\psi^* \in \Psi_\infty$ and $\|\cdot\|_\psi^* = \|\cdot\|_{\psi^*}$. Then we easily have

Proposition 2.7 (**Generalized Hölder Inequality**). *Let $\psi \in \Psi_\infty$. Then we have*

$$|\langle x, y \rangle| \le \|x\|_\psi \|y\|_{\psi^*}$$

for any $x \in \ell_\psi$ and any $y \in \ell_{\psi^*}$.

As in [10], we have

Proposition 2.8. *Let $\psi \in \Psi_\infty$. Then*

(i) $(c_\psi)^* = \ell_{\psi^*}$. *In particular, if ψ is regular, then $(\ell_\psi)^* = \ell_{\psi^*}$.*

(ii) ℓ_ψ *(resp. c_ψ) is reflexive if and only if ψ and ψ^* are regular.*

Proof. For every $y \in \ell_{\psi^*}$, we define F_y on c_ψ by

$$F_y(x) = \sum_{n=1}^\infty x_n y_n.$$

By Proposition 2.7, F_y is a bounded linear functional on c_ψ and $\|F_y\| \le \|y\|_{\psi^*}$. For every n, let $x = (x_1, \ldots, x_n, 0, 0, \ldots) \in \ell_0$. Then $|\sum_{k=1}^n x_k y_k| = |F_y(x)| \le \|F_y\| \, \|x\|_\psi$. Since ℓ_0 is dense in c_ψ, we have

$$\|(y_1, \ldots, y_n, 0, 0, \ldots)\|_{\psi^*} \le \|F_y\|$$

for every n. Thus we have $\|y\|_{\psi^*} \le \|F_y\|$ and so $\|y\|_{\psi^*} = \|F_y\|$. Thus we only prove that any $F \in (c_\psi)^*$ has the requisite form. For every $n \in \mathbf{N}$, we put $\|(x_1, \ldots, x_n)\|_n = \|(x_1, \ldots, x_n, 0, 0, \ldots)\|_\psi$ and $\|(x_1, \ldots, x_n)\|_{n^*} = \|(x_1, \ldots, x_n, 0, 0, \ldots)\|_{\psi^*}$, respectively. Since $(\mathbf{C}^n, \|\cdot\|_n)^* = (\mathbf{C}^n, \|\cdot\|_{n^*})$ for every n, we can find a sequence $\{y_n\}_{n=1}^\infty$ with $F(x) = \sum_{n=1}^\infty x_n y_n$ for any $x \in \ell_0$. For any n, we have

$$\|(y_1,\dots,y_n,0,0,\dots)\|_{\psi^*} = \sup\left\{\left|\sum_{k=1}^{n} x_k y_k\right| : \|x\|_\psi = 1\right\}$$
$$\leq \sup\{|F(x)| : \|x\|_\psi = 1\} \leq \|F\|.$$

Since $y \in \ell_{\psi^*}$, we have $F = F_y$. This implies $(c_\psi)^* = \ell_{\psi^*}$. This completes the proof. □

3. Separability

It is well known that there is a natural correspondence between all symmetric sequence spaces and all symmetric normed ideals (cf. [3, 10]). Connecting with this result, we introduce the following.

DEFINITION 3.1. Let $\mathcal{U}$ be an ideal of ℓ_∞ such that $\mathcal{U} \supset \ell_0$. Then $\mathcal{U}$ is said to be *absolute* if there exists some norm $\|\cdot\|$ on $\mathcal{U}$ such that $(\mathcal{U}, \|\cdot\|)$ is a Banach space and

$$\|e_n\| = 1 \quad (\forall n) \qquad \text{and} \qquad \|ax\| \leq \|a\|_\infty \|x\| \quad (\forall a \in \ell_\infty, \forall x \in \mathcal{U}). \tag{2}$$

PROPOSITION 3.2. *The norm $\|\cdot\|$ on $\mathcal{U}$ satisfying* (2) *is an absolute normalized norm on ℓ_0.*

PROOF. For any $x = (x_1,\dots,x_n,0,0,\dots) \in \ell_0$, we put $\rho_n = \arg x_n \in [0, 2\pi)$ for each n, where $\arg 0 = 0$. Then

$$\|(x_1,\dots,x_n,0,0,\dots)\| = \|(e^{i\rho_1}|x_1|,\dots,e^{i\rho_n}|x_n|,0,0,\dots)\|$$
$$\leq \|(e^{i\rho_1},\dots,e^{i\rho_n},0,0,\dots)\|_\infty \|(|x_1|,\dots,|x_n|,0,0,\dots)\|$$
$$= \|(|x_1|,\dots,|x_n|,0,0,\dots)\|$$

We similarly have

$$\|(x_1,\dots,x_n,0,0,\dots)\| \geq \|(|x_1|,\dots,|x_n|,0,0,\dots)\|.$$

Thus $\|\cdot\|$ is an absolute norm on ℓ_0. □

THEOREM 3.3. *Let $\mathcal{U}$ be an absolute ideal with $\|\cdot\|$. Then*
(i) *There exists some $\psi \in \Psi_\infty$ such that $c_\psi \subset \mathcal{U} \subset \ell_\psi$ and $\|\cdot\| = \|\cdot\|_\psi$.*
(ii) *$\mathcal{U}$ is separable if and only if $\mathcal{U} = c_\psi$. In particular, ℓ_ψ is separable if and only if $\ell_\psi = c_\psi$.*

PROOF. (i) By Proposition 3.2, there exists a convex function $\psi \in \Psi_\infty$ such that $\|x\| = \|x\|_\psi$ for any $x \in \ell_0$. By Definition 3.1(2), we have for any $x = (x_1,\dots,x_n,\dots) \in \mathcal{U}$

$$\|(x_1,\dots,x_n,0,0,\dots)\|_\psi = \|(x_1,\dots,x_n,0,0,\dots)\|$$
$$\le \|(1,\dots,1,0,0,\dots)\|_\infty \|(x_1,\dots,x_n,\dots)\| = \|x\|.$$

Thus $x \in \ell_\psi$ and so $\mathcal{U} \subset \ell_\psi$. Further, since $\mathcal{U}$ is a Banach space and $\ell_0 \subset \mathcal{U}$, we have $c_\psi \subset \mathcal{U}$.

(ii) Assume that $\mathcal{U} \supsetneqq c_\psi$. We take an element $b = \{b_n\}_{n=1}^\infty \in \mathcal{U}$ with $b \notin c_\psi$. Then there exists a $\delta > 0$ such that for every positive integer n,

$$\|(0,0,\dots,0,b_n,b_{n+1},\dots)\|_\psi > \delta.$$

Hence, for each n, there exists an $m(n)$ $(>n)$ such that

$$\|(0,0,\dots,0,b_n,b_{n+1},\dots,b_{m(n)-1},0,0,\dots)\|_\psi > \delta.$$

Now we take an integer n_j $(j = 1,2,\dots)$ satisfying $n_1 = 1$ and $n_j = m(n_{j-1})$ for $j \ge 2$. Define $\Lambda = \{\{\alpha_j\}_{j=1}^\infty : \alpha_j = 0 \text{ or } 1\}$. For each $\alpha = \{\alpha_j\}_{j=1}^\infty \in \Lambda$, we put

$$x_\alpha = (\alpha_1 b_1,\dots,\alpha_1 b_{n_1-1},\alpha_2 b_{n_1},\dots,\alpha_2 b_{n_2-1},\dots,\alpha_k b_{n_{k-1}},\dots,\alpha_k b_{n_k-1},\dots).$$

Since $\mathcal{U}$ is an absolute ideal of ℓ_∞, $\{x_\alpha : \alpha \in \Lambda\}$ is an uncountable subset of $\mathcal{U}$. Suppose that $\alpha \ne \alpha'$, where $\alpha = \{\alpha_j\}_{j=1}^\infty$ and $\alpha' = \{\alpha'_j\}_{j=1}^\infty$. We take any j_0 with $\alpha_{j_0} \ne \alpha'_{j_0}$ and so $|\alpha_{j_0} - \alpha'_{j_0}| = 1$. Then we have

$$\begin{aligned}
&\|x_\alpha - x_{\alpha'}\|_\psi \\
&\quad = \|(\alpha_1 b_1,\dots,\alpha_1 b_{n_1-1},\alpha_2 b_{n_1},\dots,\alpha_2 b_{n_2-1},\dots,\alpha_k b_{n_{k-1}},\dots,\alpha_k b_{n_k-1},\dots) \\
&\qquad - (\alpha'_1 b_1,\dots,\alpha'_1 b_{n_1-1},\alpha'_2 b_{n_1},\dots,\alpha'_2 b_{n_2-1},\dots,\alpha'_k b_{n_{k-1}},\dots,\alpha'_k b_{n_k-1},\dots)\|_\psi \\
&\quad \ge |\alpha_{j_0} - \alpha'_{j_0}|\, \|(0,\dots,0,b_{n_{j_0-1}},\dots,b_{n_{j_0}-1},0,0,\dots)\|_\psi \\
&\quad > |\alpha_{j_0} - \alpha'_{j_0}|\delta = \delta.
\end{aligned}$$

Therefore $\mathcal{U}$ is not separable. □

4. Strict convexity and uniform convexity

A Banach space X or its norm $\|\cdot\|$ is called *strictly convex* if, for all $x, y \in X$ such that $x \ne y$ and $\|x\| = \|y\| = 1$, $\left\|\frac{x+y}{2}\right\| < 1$. A function $\psi \in \Psi_\infty$ is called *strictly convex* if for any $s, t \in \Delta_\infty$ with $s \ne t$, one has

$$\psi\left(\frac{s+t}{2}\right) < \frac{\psi(s)+\psi(t)}{2}.$$

In [8], we characterized the strict convexity of absolute norms on $\mathbf{C}^n$. For $(\ell_0, \|\cdot\|_\psi)$ and ℓ_ψ, we similarly have

THEOREM 4.1. *Let* $\psi \in \Psi_\infty$. *Then*
(i) $(\ell_0, \|\cdot\|_\psi)$ *is strictly convex if and only if* ψ *is strictly convex on* Δ_∞.
(ii) *If* ℓ_ψ *is strictly convex, then* ψ *is strictly convex on* Δ_∞.

However, the converse of the assertion (ii) is not true in general. For example, let $\{p_n\}_{n=1}^\infty$ be the sequence defined by

$$p_n = \frac{\log 2}{\log\left(1 + \frac{1}{n^2+4n+3}\right)}.$$

Note that $1 < p_n < +\infty$ $(\forall n)$ and $p_n \to +\infty$ as $n \to +\infty$. By the induction, we define Banach spaces X_n $(n = 1, 2, \ldots)$ by

$$X_1 = (\mathbf{C} \oplus \mathbf{C})_{p_1}$$

and

$$X_n = (X_{n-1} \oplus \mathbf{C})_{p_n} \qquad (n \geq 2).$$

Let $\|\cdot\|_{X_n}$ be the norm of X_n. We define the norm $\|\cdot\|$ on ℓ_0 by

$$\|x\| = \|(x_1, x_2, \ldots, x_n)\|_{X_{n-1}} \qquad (x = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in \ell_0).$$

Then $\|\cdot\| \in AN_\infty$ and we define ϕ by the corresponding convex function, that is, $\phi(s) = \|s\|$ $(s \in \Delta_\infty)$. Since $X_1, X_2, \ldots$ are strictly convex, $(\ell_0, \|\cdot\|_\phi)$ is strictly convex. However ℓ_ϕ is not strictly convex. Indeed, we put

$$x = \left(\frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n-1}{n}, \frac{n}{n+1}, \ldots\right)$$

and

$$y = \left(-\frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n-1}{n}, \frac{n}{n+1}, \ldots\right).$$

Note that

$$\left\|\left(\frac{n+1}{n+2}, \frac{n+1}{n+2}\right)\right\|_{p_n} = \frac{n+1}{n+2} 2^{1/p_n} = \frac{n+2}{n+3} \tag{3}$$

for every positive integer n. Then we show that

$$\left\|\left(\frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \frac{n+1}{n+2}\right)\right\|_{X_n} = \frac{n+2}{n+3} \tag{4}$$

holds true for any n with $n \geq 2$. If $n = 2$, then we have by (3),

$$\left\|\left(\frac{2}{3}, \frac{2}{3}, \frac{3}{4}\right)\right\|_{X_2} = \left\|\left(\left\|\left(\frac{2}{3}, \frac{2}{3}\right)\right\|_{p_1}, \frac{3}{4}\right)\right\|_{p_2} = \left\|\left(\frac{3}{4}, \frac{3}{4}\right)\right\|_{p_2} = \frac{4}{5}.$$

Now assume that (4) holds true for n. For $n+1$, we have

$$\begin{aligned}\left\|\left(\frac{2}{3},\frac{2}{3},\frac{3}{4},\dots,\frac{n+1}{n+2},\frac{n+2}{n+3}\right)\right\|_{X_n} &= \left\|\left(\left\|\left(\frac{2}{3},\frac{2}{3},\frac{3}{4},\dots,\frac{n+1}{n+2}\right)\right\|_{X_n},\frac{n+2}{n+3}\right)\right\|_{p_{n+1}} \\ &= \left\|\left(\frac{n+2}{n+3},\frac{n+2}{n+3}\right)\right\|_{p_{n+1}} \\ &= \frac{n+3}{n+4}.\end{aligned}$$

Hence (4) holds true for every n with $n \geq 3$. Letting $n \to \infty$, $x \in \ell_\phi$ and $\|x\|_\phi = 1$, and hence $y \in \ell_\phi$ and $\|y\|_\phi = 1$. By the definition of $\|\cdot\|_\phi$, we also have for every n,

$$\frac{n}{n+1} \leq \left\|\left(0,\frac{2}{3},\frac{3}{4},\dots,\frac{n}{n+1},\frac{n+1}{n+2},0,0,\dots\right)\right\|_\phi \leq \left\|\frac{x+y}{2}\right\|_\phi \leq 1.$$

So we have $\|(x+y)/2\|_\phi = 1$. Therefore ℓ_ϕ is not strictly convex.

Problem. If ψ is strictly convex on Δ_∞, is c_ψ strictly convex?

We next consider the uniform convexity of ℓ_ψ. A Banach space X or its norm is called *uniformly convex* if, for every $\varepsilon > 0$, there exists some δ $(0 < \delta < 1)$ such that $\|x - y\| \geq \varepsilon$, $\|x\| \leq 1$ and $\|y\| \leq 1$ implies $\|(x+y)/2\| \leq 1 - \delta$.

Theorem 4.2. *Let $\psi \in \Psi_\infty$. Then ℓ_ψ is uniformly convex if and only if $(\ell_0, \|\cdot\|_\psi)$ is uniformly convex.*

Proof. The sufficiency assertion is clear since $(\ell_0, \|\cdot\|_\psi)$ is isometrically imbedded into ℓ_ψ. Assume that $(\ell_0, \|\cdot\|_\psi)$ is uniformly convex. Take arbitrary $\varepsilon > 0$. Then there exists some δ $(0 < \delta < 1)$ such that $x, y \in \ell_0$, $\|x\|_\psi \leq 1$, $\|y\|_\psi \leq 1$ and

$$\|x - y\|_\psi \geq \frac{\varepsilon}{2}$$

implies $\|(x+y)/2\|_\psi \leq 1 - \delta$. We take any $x = \{x_n\}_{n=1}^\infty$, $y = \{y_n\}_{n=1}^\infty \in \ell_\psi$ such that $\|x\|_\psi \leq 1$, $\|y\|_\psi \leq 1$ and

$$\lim_{n\to\infty} \|(x_1 - y_1, \dots, x_n - y_n, 0, 0, \dots)\|_\psi = \|x - y\|_\psi \geq \varepsilon.$$

For each n, we put

$$x^{(n)} = (x_1, x_2, \dots, x_n, 0, 0, \dots)$$

and

$$y^{(n)} = (y_1, y_2, \dots, y_n, 0, 0, \dots)$$

in ℓ_0. Then there exists some positive integer n_0 such that $\|x^{(n)} - y^{(n)}\|_\psi \geq \varepsilon/2$ for all n with $n \geq n_0$. By $\|x^{(n)}\|_\psi \leq 1$ and $\|y^{(n)}\|_\psi \leq 1$, we have for all n with $n \geq n_0$,

$$\left\| \frac{x^{(n)} + y^{(n)}}{2} \right\|_\psi \leq 1 - \delta.$$

So $\|(x+y)/2\|_\psi \leq 1 - \delta$. Thus ℓ_ψ is uniformly convex. This completes the proof. □

PROBLEM. What are the equivalent conditions for ψ that ℓ_ψ is uniformly convex?

5. Equivalent norm

In this section, we consider the norms in AN_∞ which are equivalent to ℓ_1- and ℓ_∞-norms, respectively. As in [3], we similarly have

PROPOSITION 5.1. *Let $\psi \in \Psi_\infty$. Then the following are equivalent:*
(i) $\|\cdot\|_\psi$ *is equivalent to* $\|\cdot\|_\infty$.

(ii) $$\sup_{t \in \Delta_\infty} \frac{\psi(t)}{\psi_\infty(t)} = \sup_n n\psi\Big(\overbrace{\frac{1}{n}, \dots, \frac{1}{n}}^{n}, 0, 0, \dots\Big) < \infty.$$

PROOF. (i) ⇒ (ii) If (i) holds, then there exists some $M > 0$ such that $\|x\|_\psi \leq M\|x\|_\infty$ for $x \in \ell_0$. Thus we have for all $t \in \Delta_\infty$,

$$\frac{\psi(t)}{\psi_\infty(t)} = \frac{\|t\|_\psi}{\|t\|_\infty} \leq M.$$

Thus we have (i) ⇒ (ii).

(ii) ⇒ (i) Assume that $\sup_{t \in \Delta_\infty} \frac{\psi(t)}{\psi_\infty(t)} (= M) < \infty$. Then we have, for all $x = \{x_n\}_{n=1}^\infty \in \ell_0$, $\|x\|_\infty \leq \|x\|_\psi \leq M\|x\|_\infty$. Thus $\|\cdot\|_\psi$ is equivalent to $\|\cdot\|_\infty$. We next show that

$$\sup_{t \in \Delta_\infty} \frac{\psi(t)}{\psi_\infty(t)} = \sup_n n\psi\Big(\overbrace{\frac{1}{n}, \dots, \frac{1}{n}}^{n}, 0, 0, \dots\Big) \tag{5}$$

For any $t = (t_1, t_2, \dots, t_n, 0, 0, \dots) \in \Delta_\infty$, we have

$$\frac{\psi(t)}{\psi_\infty(t)} = \frac{1}{\|t\|_\infty}\|(t_1, t_2, \ldots, t_n, 0, 0, \ldots)\|_\psi$$
$$\le \|(\overbrace{1, 1, \ldots, 1}^{n}, 0, 0, \ldots)\|_\psi$$
$$= n\psi\left(\overbrace{\frac{1}{n}, \ldots, \frac{1}{n}}^{n}, 0, 0, \ldots\right)\left(= \frac{\psi\left(\frac{1}{n}, \ldots, \frac{1}{n}, 0, 0, \ldots\right)}{\psi_\infty\left(\frac{1}{n}, \ldots, \frac{1}{n}, 0, 0, \ldots\right)}\right),$$

which implies (5). This completes the proof. □

Let $\psi \in \Psi_\infty$. Then we define Δ_ψ as the norm closure of Δ_∞ in $(\ell_\psi, \|\cdot\|_\psi)$.

Proposition 5.2. *Let* $\psi \in \Psi_\infty$. *Then*
(i) $\Delta_{\psi_1} = \{\{s_n\}_{n=1}^\infty \in \ell_\infty : \sum_{n=1}^\infty s_n = 1, s_n \ge 0\}$.
(ii) *For all* $\psi \in \Psi_\infty$, *we have* $\Delta_{\psi_1} \subset \Delta_\psi$.

Proof. (i) We put $A = \{\{s_n\}_{n=1}^\infty \in \ell_\infty : \sum_{n=1}^\infty s_n = 1, s_n \ge 0\}$. Take any $t = \{s_n\}_{n=1}^\infty \in A$ and put $t_n \in \Delta_\infty$ as

$$t_n = \frac{1}{\sum_{k=1}^n s_k}(s_1, s_2, \ldots, s_n, 0, 0, \ldots).$$

Then $\|t_n - t\|_1 \to 0$ and so $t \in \Delta_{\psi_1}$. Conversely, we take any $t = \{s_n\}_{n=1}^\infty \in \Delta_{\psi_1}$. Then there exists some sequence $\{t_k\}_{k=1}^\infty$ in Δ_∞ such that $\|t_k - t\|_1 \to 0$, where $t_k = \{s_{k,n}\}_{n=1}^\infty$. For each n, we have $|s_{k,n} - s_n| \le \|t_k - t\|_1 \to 0$. So $s_n \ge 0$ for all n. Since $\|t_k\|_1 \to \|t\|_1$ as $k \to \infty$, we have $\|t\|_1 = 1$ or $\sum_{n=1}^\infty s_n = 1$. Hence we have $t \in A$.

(ii) Take any $t \in \Delta_{\psi_1}$. Then there exists $\{t_k\}_{k=1}^\infty$ in Δ_∞ such that $\|t_k - t\|_1 \to 0$. By Lemma 2.1,

$$\|t_k - t\|_\psi \le \|t_k - t\|_1 \to 0.$$

Hence we have $t \in \Delta_\psi$. This completes the proof. □

Proposition 5.3. *Let* $\psi \in \Psi_\infty$. *Then the following assertions are equivalent:*
(i) $\|\cdot\|_\psi$ *is equivalent to* $\|\cdot\|_1$.
(ii) $\inf_{t \in \Delta_\infty} \psi(t) > 0$.
(iii) $0 \notin \Delta_\psi$.

Proof. As in the proof of Proposition 5.1, we similarly have (i) $\Leftrightarrow$ (ii). It is easy to see that $0 \in \Delta_\psi$ if and only if there exists a sequence $\{s_k\}_{k=1}^\infty$ in Δ_∞ such that $\psi(s_k) \to 0$. Therefore we have (ii) $\Leftrightarrow$ (iii). This completes the proof. □

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