Hiroshima Math. J.
31 (2001), 477–492

A new family of filtration three in the stable homotopy of spheres

Jinkun Lin

(Received October 2, 2000)
(Revised May 24, 2001)

Abstract. This paper proves the existence of a new nontrivial family of filtration three in the stable homotopy group of spheres $\pi_{2(p^n+p)(p-1)-3}S$ which is of order p and is represented by $(b_0h_n + h_1b_{n-1})$ in the $E_2^{3,2(p^n+p)(p-1)}$-term of the Adams spectral sequence, where $p \geq 5$ is a prime and $n \geq 3$.

1. Introduction

Let A be the mod p Steenrod algebra and S the sphere spectrum localized at an odd prime p. To determine the stable homotopy groups of spheres π_*S is one of the central problems in homotopy theory. One of the main tools to reach it is the Adams spectral sequence (ASS) $E_2^{s,t} = Ext_A^{s,t}(Z_p, Z_p) \Rightarrow \pi_{t-s}S$, where the $E_2^{s,t}$-term is the cohomology of A. If a family of generators x_i in $E_2^{s,*}$ converges nontrivially in the ASS, then we get a family of nontrivial homotopy elements f_i in π_*S and we say that f_i is represented by $x_i \in E_2^{s,*}$ and has filtration s in the ASS. So far, not so many families of homotopy elements in π_*S have been detected. For example, a family $\zeta_{n-1} \in \pi_{p^nq+q-3}S$ for $n \geq 2$ which has filtration 3 and is represented by $h_0b_{n-1} \in Ext_A^{3,p^nq+q}(Z_p, Z_p)$ has been detected in [2], where $q = 2(p-1)$.

From [6], $Ext_A^{1,*}(Z_p, Z_p)$ has Z_p-base consisting of a_0, h_n $(n \geq 0)$ whose internal degrees are 1, p^nq respectively and $Ext_A^{2,*}(Z_p, Z_p)$ has Z_p-base consisting of a_0^2, α_2, a_0h_n $(n > 0)$, g_n, k_n, b_n $(n \geq 0)$ and h_nh_m $(m \geq n+2, n \geq 0)$ whose internal degrees are 2, $2q+1$, p^nq+1, $(p^n+2p^{n-1})q$, $(2p^n+p^{n-1})q$, $p^{n+1}q$, p^nq+p^mq respectively.

Let M be the Moore spectrum modulo a prime $p \geq 5$ given by the cofibration

$$(1.1) \qquad S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S.$$

2000 *Mathematics Subjects Classification*. 55Q45.

Key words and phrases. Stable homotopy of spheres, Adams spectral sequence, Toda-Smith spectrum.

Let $\alpha: \Sigma^q M \to M$ be the Adams map and K be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M. \tag{1.2}$$

The spectrum which we briefly write as K is known to be the Toda-Smith spectrum $V(1)$ and the elements in the stable homotopy of spheres $\pi_* S$ are closely related to the elements in $\pi_* M$ or $\pi_* K$. The main purpose of this paper is to detect a new family of $(b_0 h_n + h_1 b_{n-1})$-element in $\pi_* S$ as stated in the following Theorem A.

Theorem A. *Let $p \geq 5$, $n \geq 3$, then,*

(1) $i_*(h_1 h_n) \in Ext_A^{2, p^n q + pq}(H^* M, Z_p)$ *is a permanent cycle in the ASS and converges to a nontrivial element* $\xi_n \in \pi_{p^n q + pq - 2} M$.

(2) *For* $\xi_n \in \pi_{p^n q + pq - 2} M$ *obtained in* (1), $j\xi_n \in \pi_{p^n q + pq - 3} S$ *is a nontrivial element of order p which is represented (up to nonzero scalar) by* $(b_0 h_n + h_1 b_{n-1}) \in Ext_A^{3, p^n q + pq}(Z_p, Z_p)$ *in the ASS.*

Theorem A(2) is an easy consequence of Theorem A(1) which will be proved by an argument processing in the Adams resolution of certain spectra related to M and using the known ζ_{n-1}-map in [2] as a geometric input. The main step is to show that there exists a map $\tilde{\tilde{\xi}}_n \in [\Sigma^{p^n q + pq - 3} M, S]$ such that

$$\alpha i \tilde{\tilde{\xi}}_n = j' \beta \eta'_n i' \qquad \text{modulo higher filtration,}$$

where $\beta \in [\Sigma^{(p+1)q} K, K]$ is the known second periodicity element and $\eta'_n i' i \in \pi_{p^n q + q - 2} K$ is an $h_0 h_n$-map induced by the known $\zeta_{n-1} \in \pi_{p^n q + q - 3} S$ so that the right hand side of the above equation has filtration 4 in the ASS.

The new family in $\pi_* S$ obtained in Theorem A(2) actually is a third periodicity family represented by $\gamma_{p^{n-2}/p^{n-2}-1} + \text{other terms} \in Ext^{3,*}_{BP_* BP}(BP_*, BP_*)$ in the Adams-Novikov spectral sequence based on the Brown-Peterson spectrum BP. Roughly speaking, we have the relation that $\alpha_1 \gamma_{p^{n-2}/p^{n-2}-1} = \beta_1 \beta_{p^{n-1}/p^{n-1}-1} \in Ext^{4,*}_{BP_* BP}(BP_*, BP_*)$ and after the Thom map $\Phi: Ext^{*,*}_{BP_* BP}(BP_*, BP_*) \to Ext_A^{*,*}(Z_p, Z_p)$ we have $\Phi(\gamma_{p^{n-2}/p^{n-2}-1}) = b_0 h_n + h_1 b_{n-1} \in Ext_A^{3,*}(Z_p, Z_p)$ since $\Phi(\beta_1 \beta_{p^{n-1}/p^{n-1}-1}) = b_0 h_0 h_n \in Ext_A^{4,*}(Z_p, Z_p)$.

After giving some preliminaries on low dimensional Ext groups in § 2, the proof of Theorem A will be given in § 3.

2. Some preliminaries on low dimensional Ext groups

In this section, we prove some results on low dimensional Ext groups which will be used in the proof of the main theorem.

Proposition 2.1. *Let* $p \geq 5$, $n \geq 3$, $h_n \in Ext_A^{1, p^n q}(Z_p, Z_p)$, $a_0 \in Ext_A^{1,1}$ ·

(Z_p, Z_p) and α_2, b_n be generators in $Ext_A^{2,*}(Z_p, Z_p)$ with internal degrees $2q+1$, $p^{n+1}q$ respectively. Then we have the following.

(1) *The product $\alpha_2 b_0 h_n \neq 0 \in Ext_A^{5, p^n q + (p+2)q+1}(Z_p, Z_p)$.*

(2) *$Ext_A^{4, p^n q + pq + 1}(Z_p, Z_p) \cong Z_p\{a_0 b_0 h_n, a_0 b_{n-1} h_1\}$,*
$Ext_A^{4, p^n q + (p+1)q}(Z_p, Z_p) \cong Z_p\{b_0 h_0 h_n\}$.

(3) *$Ext_A^{4, p^n q + (p+t)q + r}(Z_p, Z_p) = 0$ for $t = 1, 2$ and $r = 1, 2, 3$.*

(4) *$Ext_A^{4, (p+2)q + r}(Z_p, Z_p) = 0$ for $r = 2, 3$.*

Proof. From [8], p. 82, Theorem 3.2.5, there is a May spectral sequence (MSS) $\{E_r^{s,t,*}, d_r\}$ which converges to $Ext_A^{s,t}(Z_p, Z_p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{m,i} \mid m > 0, i \geq 0) \otimes P(b_{m,i} \mid m > 0, i \geq 0) \otimes P(a_n \mid n \geq 0),$$

where E is the exterior algebra, P is the polynomial algebra, and $h_{m,i} \in E_1^{1, 2(p^m-1)p^i, 2m-1}$, $b_{m,i} \in E_1^{2, 2(p^m-1)p^{i+m}, p(2m-1)}$, $a_n \in E_1^{1, 2p^n-1, 2n+1}$. Observe the second degree of the following generators $(\mathrm{mod}\, p^n q)$ for $2 \leq i < n$, $n \geq 3$:

$$|h_{1,i}| = p^i q \qquad (\mathrm{mod}\, p^n q),$$

$$|b_{1,i-1}| = p^i q \qquad (\mathrm{mod}\, p^n q),$$

$$|h_{s,i-1}| = (p^{s+i-2} + \cdots + p^{i-1})q \qquad (\mathrm{mod}\, p^n q), i \leq s+i-2 < n,$$

$$|b_{s,i-1}| = (p^{s+i-1} + \cdots + p^i)q \qquad (\mathrm{mod}\, p^n q), i \leq s+i-2 < n,$$

$$|a_{i+1}| = (p^i + \cdots + 1)q + 1 \qquad (\mathrm{mod}\, p^n q).$$

At degree $t = p^n q + (p+r)q + k$ for $r = 0, 1, 2$ and $k = 0, 1, 2, 3$, $E_1^{4,t,*}$ has no generator which has factors consisting of the above elements, because such generators will have internal degree $(c_{n-1}p^{n-1} + \cdots + c_1 p + c_0)q + d \;(\mathrm{mod}\, p^n q)$ with some $c_i \neq 0$, $2 \leq i \leq n-1$, where $0 \leq c_s < p$, $s = 0, \ldots, n-1$, $0 \leq d \leq 4$. Exclude the degree $> p^n q$, then we know that $E_1^{4,t,*}$ for $t = p^n q + (p+r)q + k$ with $r = 0, 1, 2$, $k = 0, 1, 2, 3$ has elements of the form $h_{1,n} x$ or $b_{1,n-1} x$ for some $x \in E(h_{1,0}, h_{1,1}, h_{2,0}) \otimes P(a_0, a_1, a_2, b_{1,0})$. So we have

$$E_1^{4, p^n q + (p+1)q, *} = Z_p\{h_{1,n} b_{1,0} h_{1,0}, b_{1,n-1} h_{1,1} h_{1,0}\},$$

$$E_1^{4, p^n q + (p+2)q + 1, *} = Z_p\{h_{1,n} h_{1,1} h_{1,0} a_1, h_{1,n} h_{2,0} h_{1,0} a_0, b_{1,n-1} h_{1,0} a_2, b_{1,n-1} h_{2,0} a_1\},$$

$$E_1^{4, p^n q + (p+1)q + 1, *} = Z_p\{b_{1,n-1} h_{1,1} a_1, b_{1,n-1} h_{2,0} a_0, h_{1,n} h_{1,1} h_{1,0} a_0\},$$

$$E_1^{4, p^n q + (p+2)q + 2, *} = Z_p\{b_{1,n-1} a_2 a_1, h_{1,n} h_{2,0} a_1 a_0, h_{1,n} h_{1,0} a_2 a_0, h_{1,n} h_{1,1} a_1^2\},$$

$$E_1^{4, p^n q + (p+1)q + 2, *} = Z_p\{b_{1,n-1} a_2 a_0, h_{1,n} h_{1,0} a_1 a_0, h_{1,n} h_{2,0} a_0^2\},$$

$$E_1^{4,p^nq+(p+2)q+3,*} = Z_p\{h_{1,n}a_2a_1a_0\}, \qquad E_1^{4,p^nq+(p+1)q+3,*} = Z_p\{h_{1,n}a_2a_0^2\},$$

$$E_1^{4,p^nq+pq+1,*} = Z_p\{b_{1,n-1}h_{1,1}a_0, h_{1,n}b_{1,0}a_0\},$$

$$E_1^{3,p^nq+pq+1,*} = Z_p\{h_{1,n}h_{1,1}a_0\}, \quad E_1^{3,p^nq+(p+1)q,*} = Z_p\{b_{1,n-1}h_{2,0}, h_{1,n}h_{1,1}h_{1,0}\},$$

$$E_1^{3,p^nq+pq,*} = Z_p\{h_{1,n}b_{1,0}, \ b_{1,n-1}h_{1,1}\}.$$

From [8], p. 82, Theorem 3.2.5, the formulas for the differential d_1 are $d_1(h_{1,n}) = 0$, $d_1(b_{1,n-1}) = 0$, $d_1(a_0) = 0$, $d_1(a_1) = -a_0h_{1,0}$, $d_1(h_{2,0}) = -h_{1,0}h_{1,1}$, $d_1(a_2) = -a_0h_{2,0} - a_1h_{1,1}$ and $d_r(xy) = d_r(x)y + (-1)^s x d_r(y)$ for $x \in E_r^{s,t,*}$, $y \in E_r^{s',t',*}$ $(r \geq 1)$, $xy = (-1)^{ss'+tt'}yx$ for $x, y = h_{m,i}, b_{m,i}$ or a_n. Thus, we have

$$d_1(h_{1,n}h_{1,1}h_{1,0}a_1) = 0 = d_1(h_{1,n}h_{2,0}h_{1,0}a_0),$$

$$d_1(b_{1,n-1}h_{1,0}a_2) = b_{1,n-1}h_{1,0}a_0h_{2,0} + b_{1,n-1}h_{1,0}a_1h_{1,1} \neq 0, \qquad (^*)$$

$$d_1(b_{1,n-1}h_{2,0}a_1) = b_{1,n-1}h_{2,0}a_0h_{1,0} - b_{1,n-1}h_{1,0}h_{1,1}a_1 \neq 0, \qquad (^{**})$$

and the last two elements are linearly independent. Therefore, $E_2^{4,p^nq+(p+2)q+1,*} \cong Z_p\{h_{1,n}h_{1,1}h_{1,0}a_1, h_{1,n}h_{2,0}h_{1,0}a_0\}$ and these two generators are permanent cycles in the MSS since it is known that $h_{1,n}, b_{1,n}, h_{2,0}h_{1,0}, h_{1,0}a_1, a_0 \in E_1^{*,*,*}$ are permanent cycles which converge in the MSS for all $n \geq 0$ to h_n, b_n, g_0, α_2, $a_0 \in Ext_A^{*,*}(Z_p, Z_p)$ respectively. Then, $h_{1,0}a_1b_{1,0}h_{1,n} \in E_1^{5,p^nq+(p+2)q+1,*}$ cannot be hitted by differential and it converges in the MSS nontrivially to $\alpha_2b_0h_n \in Ext_A^{5,p^nq+(p+2)q+1}(Z_p, Z_p)$, and so (1) is proved.

Note that $h_{1,n}h_{1,1}h_{1,0}a_1$, $h_{1,n}h_{1,0}h_{2,0}a_0$ converge in the MSS to $h_nh_1\alpha_2 = 0$, $h_ng_0a_0 = 0$ (Note: $\alpha_2h_1 = 0$, $g_0a_0 = 0$ by [1], Table 8.2) in Ext respectively. Combining with the linearly independent equations (*), (**), this shows that $Ext_A^{4,p^nq+(p+2)q+1}(Z_p, Z_p) = 0$. Look at the following:

$$d_1(b_{1,n-1}h_{1,1}a_1) = b_{1,n-1}h_{1,1}a_0h_{1,0} = -d_1(b_{1,n-1}h_{2,0}a_0)$$

and $b_{1,n-1}h_{1,1}a_1 + b_{1,n-1}h_{2,0}a_0 = d_1(b_{1,n-1}a_2)$. Moreover, $h_{1,n}h_{1,1}h_{1,0}a_0 \in E_1^{4,*,*}$ converges in the MSS to $h_nh_1h_0a_0 = 0$ in Ext, then we have $Ext_A^{4,p^nq+(p+1)q+1}(Z_p, Z_p) = 0$. By a straightforward calculation we have

$$d_1(b_{1,n-1}a_2a_1) = -b_{1,n-1}a_0h_{2,0}a_1 - b_{1,n-1}a_1h_{1,1}a_1 + b_{1,n-1}a_2a_0h_{1,0} \neq 0,$$

$$d_1(h_{1,n}h_{2,0}a_1a_0) = h_{1,n}h_{1,0}h_{1,1}a_1a_0 - h_{1,n}h_{2,0}a_0h_{1,0}a_0 \neq 0,$$

$$d_1(h_{1,n}h_{1,0}a_2a_0) = -h_{1,n}h_{1,0}a_1h_{1,1}a_0 - h_{1,n}h_{1,0}a_0h_{2,0}a_0 \neq 0,$$

$$d_1(h_{1,n}h_{1,1}a_1^2) = -h_{1,n}h_{1,1}a_0h_{1,0}a_1 + h_{1,n}h_{1,1}a_1a_0h_{1,0} = 2h_{1,n}h_{1,1}a_0a_1h_{1,0} \neq 0,$$

where the first three elements are linearly independent and

$$d_1(h_{1,n}(h_{2,0}a_1a_0 + h_{1,0}a_2a_0 + h_{1,1}a_1^2)) = 0.$$

However, $h_{1,n}(h_{2,0}a_1a_0 + h_{1,0}a_2a_0 + h_{1,1}a_1^2) = -d_1(h_{1,n}a_2a_1)$, which shows that all the generators in $E_1^{4,p^nq+(p+2)q+2,*}$ die, i.e. $E_2^{4,p^nq+(p+2)q+2,*} = 0$, so $Ext_A^{4,p^nq+(p+2)q+2}(Z_p, Z_p) = 0$ as desired. By a straightforward calculation we have

$$d_1(b_{1,n-1}a_2a_0) = -b_{1,n-1}a_0h_{2,0}a_0 - b_{1,n-1}a_1h_{1,1}a_0 \neq 0,$$

$$d_1(h_{1,n}h_{2,0}a_0^2) = h_{1,n}h_{1,0}h_{1,1}a_0^2 \neq 0$$

which are linearly independent. In addition, $h_{1,n}h_{1,0}a_1a_0$ is a permanent cycle which converges in the MSS to $h_n\alpha_2a_0 = 0$ in Ext (Note: $a_0\alpha_2 = 0$ by [1], Table 8.2). This shows that $Ext_A^{4,p^nq+(p+1)q+2}(Z_p, Z_p) = 0$. Moreover, we have

$$d_1(h_{1,n}a_2a_1a_0) = h_{1,n}a_0h_{2,0}a_1a_0 + h_{1,n}a_1h_{1,1}a_1a_0 + h_{1,n}a_2a_0h_{1,0}a_0 \neq 0,$$

$$d_1(h_{1,n}a_2a_0^2) = -h_{1,n}h_{2,0}a_0^3 - h_{1,n}h_{1,1}a_1a_0^2 \neq 0.$$

This shows that $Ext_A^{4,p^nq+(p+r)q+3}(Z_p, Z_p) = 0$ for $r = 1, 2$ and finishes the proof of (3).

It is easily seen that $d_r(E_1^{3,p^nq+pq+1,*}) = 0$ for all $r \geq 1$ and $d_1(E_1^{3,p^nq+(p+1)q,*}) = Z_p\{b_{1,n-1}h_{1,0}h_{1,1}\}$. Therefore, $E_2^{4,p^nq+(p+1)q,*} \cong Z_p\{h_{1,n}b_{1,0}h_{1,0}\}$, $E_2^{4,p^nq+pq+1,*} \cong Z_p\{b_{1,n-1}h_{1,1}a_0, h_{1,n}b_{1,0}a_0\}$ and $d_r(E_r^{3,p^nq+(p+1)q,*}) = 0$ for all $r \geq 2$, which proves (2). The result in (4) follows from $E_1^{4,(p+2)q+3,*} = Z_p\{h_{1,1}a_1^2a_0\}$, $E_1^{4,(p+2)q+2,*} = Z_p\{b_{1,0}a_1^2, h_{1,1}h_{1,0}a_1a_0\}$ and $d_1(h_{1,1}a_1^2a_0) \neq 0$, $d_1(b_{1,0}a_1^2) \neq 0$, $2h_{1,1}h_{1,0}a_1a_0 = d_1(h_{1,1}a_1^2)$ by a straghtforward calculation. Q.E.D.

Proposition 2.2. *Let* $p \geq 5$, $n \geq 3$, *then*
(1) $Ext_A^{4,p^nq+(p+2)q+2}(H^*K, H^*M) = 0$.
(2) $Ext_A^{r,p^nq+2q+t}(H^*K, Z_p) = 0$ *for* $r = 3, t = 0, 1$ *or* $r = 4, t = 1, 2$.

Proof. (1) Consider the exact sequence $(k = p^nq + (p+2)q, r = 2, 3)$

$$Ext_A^{4,k+r}(H^*M, Z_p) \xrightarrow{i'_*} Ext_A^{4,k+r}(H^*K, Z_p) \xrightarrow{j'_*} Ext_A^{4,k+r-q-1}(H^*M, Z_p) \xrightarrow{\alpha_*}$$

induced by (1.2), where α_* is the connecting homomorphism associated with the short exact sequence in Z_p-cohomology induced by (1.2). The first and the third groups are zero by Proposition 2.1(3) except for the third group in case $r = 2$ which has unique generator $\alpha_*i_*(b_0h_n)$ by Proposition 2.1(2),(3), since $h_0b_0h_n = j_*\alpha_*i_*(b_0h_n) \in Ext_A^{4,p^nq+(p+1)q}(Z_p, Z_p)$ (cf. Remark below). However, $\alpha_*(\alpha i)_*(b_0h_n) \neq 0 \in Ext_A^{5,p^nq+(p+2)q+2}(H^*M, Z_p)$ by the fact that $j_*\alpha_*(\alpha i)_*(b_0h_n) = \alpha_2b_0h_n \neq 0$ (cf. Proposition 2.1(1)). This shows that the above α_* is monic and im $j'_* = 0$. So the middle group is zero for $r = 2, 3$ and the result follows by the exact sequence $(k = p^nq + (p+2)q)$

$$0 = Ext_A^{4,k+3}(H^*K, Z_p) \xrightarrow{j^*} Ext_A^{4,k+2}(H^*K, H^*M) \xrightarrow{i^*} Ext_A^{4,k+2}(H^*K, Z_p) = 0$$

induced by (1.1).

(2) Look at the exact sequence

$$Ext_A^{r,p^nq+2q+t}(H^*M, Z_p) \xrightarrow{i'_*} Ext_A^{r,p^nq+2q+t}(H^*K, Z_p) \xrightarrow{j'_*} Ext_A^{r,p^nq+q+t-1}(H^*M, Z_p)$$

induced by (1.2). The right group is zero for $(r,t) = (3,0), (4,1), (4,2)$ by [1], Table 8.1 or [4], Proposition 2.1 and has unique generator $i_*(h_0b_{n-1})$ for $(r,t) = (3,1)$ which satisfies $\alpha_* i_*(h_0b_{n-1}) = i_*\alpha_2 b_{n-1} \neq 0 \in Ext_A^{4,p^nq+2q+1}(H^*M, Z_p)$ by [4], Proposition 2.1 and $Ext_A^{3,p^nq+2q}(Z_p, Z_p) = 0$ by [1], Table 8.1. The left group is zero for $(r,t) = (3,0)$ and has unique generator $i_*(\alpha_2 h_n)$, $i_*(\alpha_2 b_{n-1})$, $\alpha_*(\alpha i)_*(b_{n-1})$ for $(r,t) = (3,1)$, $(4,1)$, $(4,2)$ respectively (cf. [1], Table 8.1 and [4], Proposition 2.1(2)). However, $i'_* i_*(\alpha_2 h_n) = 0$, $i'_* i_*(\alpha_2 b_{n-1}) = 0$ and $i'_*\alpha_*(\alpha i)_*(b_{n-1}) = 0$ by $i'ij\alpha^2 i = 0 \in \pi_{2q-1}K$, then the result follows. Q.E.D.

REMARK. Let us intepret why the connecting homomorphism $p_*: Ext_A^{s,t}(Z_p, Z_p) \to Ext_A^{s+1,t+1}(Z_p, Z_p)$ is a multiplication by a_0. Let $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} \Sigma W$ be a cofibration such that h induces the zero homomorphism in Z_p-cohomology. From [8], p. 63–64, Theorem 2.3.4, the connecting homomorphism $h_*: Ext_A^{s,t}(H^*Y, Z_p) \to Ext_A^{s+1,t}(H^*W, Z_p)$ can be described as the Yoneda product with the element of $Ext_A^1(H^*W, H^*Y)$ corresponding to the short exact sequence $0 \to H^*Y \xrightarrow{g^*} H^*X \xrightarrow{f^*} H^*W \to 0$ in Z_p-cohomology. Applying this result to the cofibration (1.1), for the connecting homomorphism $p_*: Ext_A^{s,t}(Z_p, Z_p) \to Ext_A^{s+1,t+1}(Z_p, Z_p)$, $p_*(x)$ is the Yoneda product of $x \in Ext_A^{s,t} \cdot (Z_p, Z_p)$ with $a_0 \in Ext_A^{1,1}(Z_p, Z_p)$ corresponding to the short exact sequence $0 \to H^*S \xrightarrow{j^*} H^*M \xrightarrow{i^*} H^*S \to 0$. (Note: The latter also follows from the fact that the degree p map $S \to S$ is represented by $a_0 \in Ext_A^{1,1}(Z_p, Z_p)$ in the ASS.) This shows that $p_*(x) = a_0 x$ and $p_*(x) \cdot x' = p_*(xx')$. Similarly we have $p^*(x) = a_0 x$ and $j_*\alpha_* i_*(x) = h_0 x$.

Let L be the cofibre of $\alpha_1 = j\alpha i: \Sigma^{q-1}S \to S$ and K'_r be the cofibre of $\alpha^r i: \Sigma^{rq}S \to M$ given by the following cofibrations:

$$\Sigma^{q-1}S \xrightarrow{\alpha_1} S \xrightarrow{i''} L \xrightarrow{j''} \Sigma^q S, \tag{2.3}$$

$$\Sigma^{rq}S \xrightarrow{\alpha^r i} M \xrightarrow{v_r} K'_r \xrightarrow{y_r} \Sigma^{rq+1}S. \tag{2.4}$$

Then K'_2 also is the cofibre of $vj': \Sigma^{-1}K \to \Sigma^q K'$ given by the cofibration

$$\Sigma^{-1}K \xrightarrow{vj'} \Sigma^q K' \xrightarrow{\bar{\psi}} K'_2 \xrightarrow{\bar{\rho}} K, \tag{2.5}$$

where we briefly write K'_1 as K' etc., which can be seen by the following commutative diagram of 3×3 lemma in stable homotopy category (cf. [9], p. 292–293):

$$(2.6)\qquad \begin{array}{ccccc}
\Sigma^{-1}K & \xrightarrow{vj'} & \Sigma^q K' & \xrightarrow{y} & \Sigma^{2q+1}S \\
{\scriptstyle j'}\searrow & & \nearrow{\scriptstyle v} \quad {\scriptstyle \bar{\psi}}\searrow & & \nearrow{\scriptstyle y_2} \\
& \Sigma^q M & & K_2' & \\
{\scriptstyle \alpha i}\nearrow & & \searrow{\scriptstyle \alpha} \quad {\scriptstyle v_2}\nearrow & & \searrow{\scriptstyle \bar{\rho}} \\
\Sigma^{2q}S & \xrightarrow{\alpha^2 i} & M & \xrightarrow{i'} & K,
\end{array}$$

and we have the following relations:

$$(2.7)\qquad y = y_2\bar{\psi}, \qquad \bar{\psi}v = v_2\alpha, \qquad \bar{\rho}v_2 = i', \qquad j'\bar{\rho} = -\alpha i y_2.$$

Let $\overline{m}_M : \Sigma M \to M \wedge M$ be the injection and $m_M : M \wedge M \to M$ be the multiplication of M satisfying $m_M(i \wedge 1_M) = 1_M$, $(j \wedge 1_M)\overline{m}_M = (1_M \wedge j)\overline{m}_M = 1_M$ and $(i \wedge 1_M)m_M + \overline{m}_M(j \wedge 1_M) = 1_{M \wedge M}$. By the commutative diagram of 3×3 lemma

$$(2.8)\qquad \begin{array}{ccccc}
\Sigma M & \xrightarrow{v} & \Sigma K' & \xrightarrow{1_{K'} \wedge p} & \Sigma K' \\
{\scriptstyle (v \wedge 1_M)\overline{m}_M}\searrow & & \nearrow{\scriptstyle 1_{K'} \wedge j} \quad {\scriptstyle y}\searrow & & \nearrow{\scriptstyle z} \\
& K' \wedge M & & \Sigma^{q+2}S & \\
{\scriptstyle 1_{K'} \wedge i}\nearrow & & \searrow{\scriptstyle \pi} \quad {\scriptstyle jj'}\nearrow & & \searrow{\scriptstyle \alpha i} \\
K' & \xrightarrow{x} & K & \xrightarrow{j'\alpha'} & \Sigma^2 M,
\end{array}$$

we have two cofibrations

$$(2.9)\qquad K' \xrightarrow{x} K \xrightarrow{jj'} \Sigma^{q+2}S \xrightarrow{z} \Sigma K',$$

$$(2.10)\qquad \Sigma^{-1}K \xrightarrow{j'\alpha'} \Sigma M \xrightarrow{v \wedge 1_M \overline{m}_M} K' \wedge M \xrightarrow{\pi} K,$$

where $[K, \Sigma^{q+2}S] \cong Z_p\{jj'\}$, since $[M, \Sigma^{q+2}S] = 0$, $[M, \Sigma S] \cong Z_p\{j\}$ and $\alpha ijj' = -(\alpha_1 \wedge 1_M)j' = j'\alpha'$ with $\alpha' = \alpha_1 \wedge 1_K$. In addition, $(vi \wedge 1_M)m_M(\alpha i \wedge 1_M) + (v \wedge 1_M)\overline{m}_M(j\alpha i \wedge 1_M) = (v \wedge 1_M)(\alpha i \wedge 1_M) = 0$, which shows that

$$(2.11)\qquad (vi \wedge 1_M)\alpha = -(v \wedge 1_M)\overline{m}_M(\alpha_1 \wedge 1_M),$$

since $m_M(\alpha i \wedge 1_M) = \alpha = m_M(i \wedge 1_M)\alpha$.

By the commutative diagram of 3×3-lemma in the stable homotopy category

$$(2.12)\qquad \begin{array}{ccccccccc} M & & \longrightarrow & & L\wedge K & & \xrightarrow{j''\wedge 1_K} & & \Sigma^q K \\ & \searrow^{i'} & & \nearrow^{i''\wedge 1_K} & & \searrow^{\bar r} & & \nearrow^{\pi} & \\ & & K & & & & \Sigma^q K'\wedge M & & \\ & \nearrow^{\alpha'} & & \searrow^{j'} & & \nearrow^{(v\wedge 1_M)\bar m_M} & & \searrow^{\varepsilon} & \\ \Sigma^{q-1}K & & \xrightarrow{j'\alpha'} & & \Sigma^{q+1}M & & \xrightarrow{\alpha} & & \Sigma M, \end{array}$$

we have a cofibration

$$(2.13)\qquad M \xrightarrow{(i''\wedge 1_K)i'} L\wedge K \xrightarrow{\bar r} \Sigma^q K'\wedge M \xrightarrow{\varepsilon} \Sigma M$$

with the relation

$$(2.14)\qquad \varepsilon(v\wedge 1_M)\bar m_M = \alpha, \qquad \varepsilon(1_{K'}\wedge i)vj' = -2j'\alpha' \in [\Sigma^{-2}K, M].$$

Note that $\varepsilon(1_{K'}\wedge i)v \in [\Sigma^{q-1}M, M] \cong Z_p\{ij\alpha, \alpha ij\}$ so that $\varepsilon(1_{K'}\wedge i)v = \lambda_1 ij\alpha + \lambda_2\alpha ij$, $\lambda_1 ij\alpha\alpha i + \lambda_2\alpha ij\alpha i = 0$ and $\lambda_2 = -2\lambda_1$ by $2\alpha ij\alpha = ij\alpha^2 + \alpha^2 ij$. By applying d on $\varepsilon(1_{K'}\wedge i)v = \varepsilon(1_{K'}\wedge ij)(v\wedge 1_M)\bar m_M$ we have $\lambda_1 = 1$ and so $\varepsilon(1_{K'}\wedge i)vj' = -2j'\alpha'$. In addition, by the 3×3-lemma in the stable homotopy category one can easily check that there is a cofibration

$$(2.15)\qquad M \xrightarrow{vi} K' \xrightarrow{k} \Sigma L \to \Sigma S$$

with the relation that $k\cdot v = i''j$.

From [7], p. 434, there are $\bar\Delta \in [\Sigma^{-q-1}L\wedge K, K]$ and $\tilde\Delta \in [\Sigma^{-1}K, L\wedge K]$ satisfying $\bar\Delta(i''\wedge 1_K) = (j''\wedge 1_K)\tilde\Delta = i'j' \in [\Sigma^{-q-1}K, K]$ and $jj'\bar\Delta = 0$. Then, by (2.9), there is $\bar\Delta_{K'} \in [\Sigma^{-q-1}L\wedge K, K']$ such that

$$(2.16)\qquad \bar\Delta_{K'}(i''\wedge 1_K) = vj' \in [\Sigma^{-q-1}K, K'], \qquad \bar\Delta(i''\wedge 1_K) = (j''\wedge 1_K)\tilde\Delta = i'j'.$$

Proposition 2.17. *Let* $p\ge 5, n\ge 3$, *then*

(1) $Ext_A^{3,p^nq+tq+1}(H^*K, H^*M) = 0$ *for* $t = 0, 1$.

(2) $Ext_A^{3,p^nq+pq}(Z_p, H^*M) = 0$ *and* $Ext_A^{3,p^nq+(p+2)q+2}(H^*K'\wedge M, Z_p)$ *has unique generator* $(v\wedge 1_M\bar m_M)_* i^*(\overline{h_n g_0})$, *where* $\overline{h_n g_0} \in Ext_A^{3,p^nq+(p+2)q+1}(H^*M, H^*M)$ *satisfies* $j_* i^*(\overline{h_n g_0}) = h_n g_0$, *the unique generator of* $Ext_A^{3,p^nq+(p+2)q}(Z_p, Z_p)$ *stated in* [1], *Table* 8.1.

(3) $(v\wedge 1_M\bar m_M)_* : Ext_A^{5,p^nq+(p+2)q+1}(H^*M, H^*M) \to Ext_A^{5,p^nq+(p+2)q+2}(H^*K'\wedge M, H^*M)$ *is monic.*

Proof. (1) Consider the exact sequence ($k = p^nq + tq + 1$ with $t = 0, 1$)

$$Ext_A^{3,k}(H^*M, H^*M) \xrightarrow{(i')_*} Ext_A^{3,k}(H^*K, H^*M) \xrightarrow{(j')_*} Ext_A^{3,k-q-1}(H^*M, H^*M)$$

induced by (1.2). The left group is zero for $t=0$ and has unique generator $\alpha_*(\tilde{b}_{n-1})$ for $t=1$ (cf. [4], Proposition 2.3(1) and 2.4(1)), and hence im $(i')_* = 0$. The right group is zero for $t=0,1$, since $Ext_A^{3,p^nq+(t-1)q+r}(Z_p,Z_p)=0$ for $t=0,1$ and $r=-1,0$ (cf. [1], Table 8.1). Thus im $(j')_*=0$ and the result follows.

(2) By [1], Table 8.1, $Ext_A^{3,p^nq+pq+1}(Z_p,Z_p)=0$ and $Ext_A^{3,p^nq+pq}(Z_p,Z_p)$ has two generators h_nb_0, h_1b_{n-1} such that $a_0h_nb_0, a_0h_1b_{n-1}$ are linearly independent in $Ext_A^{4,*}(Z_p,Z_p)$ (cf. Proposition 2.1(2)). Then the first result follows.

For the second half, look at the exact sequence

$$Ext_A^{3,p^nq+(p+2)q+2}(H^*M\wedge M,Z_p)\xrightarrow{(v\wedge 1_M)_*}Ext_A^{3,p^nq+(p+2)q+2}(H^*K'\wedge M,Z_p)$$
$$\xrightarrow{(y\wedge 1_M)_*}Ext_A^{3,p^nq+(p+1)q+1}(H^*M,Z_p)$$

induced by (2.4). The right group is zero by $Ext_A^{3,p^nq+(p+1)q+r}(Z_p,Z_p)=0$ for $r=0,1,2$ and the left group has unique generator $(\overline{m}_M)_*i^*(\overline{h_ng_0})$ by $Ext_A^{3,p^nq+(p+2)q+r}(Z_p,Z_p)=0$ for $r=1,2$ and $Ext_A^{3,p^nq+(p+2)q}(Z_p,Z_p)\cong Z_p\{h_ng_0\}$ (cf. [1], Table 8.1). Thus the result follows.

(3) Consider the exact sequence induced by (2.10)

$$Ext_A^{4,p^nq+(p+2)q+2}(H^*K,H^*M)\xrightarrow{(j'\alpha')_*}Ext_A^{5,p^nq+(p+2)q+1}(H^*M,H^*M)$$
$$\xrightarrow{(v\wedge 1_M\overline{m}_M)_*}Ext_A^{5,p^nq+(p+2)q+2}(H^*K'\wedge M,H^*M).$$

The left group is zero by Proposition 2.2(1) and so the result follows. Q.E.D.

Proposition 2.18. *Let* $p\ge 5, n\ge 3$, *then*

(1) $[\Sigma^{-1}L\wedge K,M]=0$.

(2) $Ext_A^{1,p^nq+(p+2)q}(H^*L\wedge K,H^*M)=0$.

Proof. (1) By (2.3), it suffices to prove $[\Sigma^{-1}K,M]=0=[\Sigma^{q-1}K,M]$. Consider the exact sequence $(t=0,1)$

$$[\Sigma^{(t+1)q}M,M]\xrightarrow{(j')^*}[\Sigma^{tq-1}K,M]\xrightarrow{(i')^*}[\Sigma^{tq-1}M,M]\xrightarrow{\alpha^*}$$

induced by (1.2). The left group has unique generator α,α^2 for $t=0,1$ respectively, and hence im $(j')^*=0$. The right group has unique generator ij for $t=0$ and two generators $ij\alpha, \alpha ij$ for $t=1$. However, $ij\alpha\ne 0$, $(\lambda_1 ij\alpha+\lambda_2\alpha ij)\alpha=0$ implies $\lambda_1=\lambda_2=0$, and hence α^* is monic. Thus im $(i')^*=0$ and the result follows.

(2) Since the top cell of $L\wedge K$ has degree $2q+2$, the result follows from the fact that $Ext_A^{1,*}(Z_p,Z_p)$ has Z_p-base consisting of h_n for all $n\ge 0$ with internal degree p^nq.

3. Proof of the main theorem

We first prove Theorem A(1) which will be done by an argument processing in the Adams resolution of some spectra related to K. Let

$$\begin{array}{ccccccc} \cdots \xrightarrow{\bar a_2} & \Sigma^{-2}E_2 & \xrightarrow{\bar a_1} & \Sigma^{-1}E_1 & \xrightarrow{\bar a_0} & E_0 = S \\ & \downarrow \bar b_2 & & \downarrow \bar b_1 & & \downarrow \bar b_0 \\ & \Sigma^{-2}KG_2 & & \Sigma^{-1}KG_1 & & KG_0 = KZ_p \end{array} \tag{3.1}$$

be the minimal Adams resolution of S satisfying the following.

(1) $E_s \xrightarrow{\bar b_s} KG_s \xrightarrow{\bar c_s} E_{s+1} \xrightarrow{\bar a_s} \Sigma E_s$ are cofibrations for all $s \geq 0$ which induce short exact sequences in Z_p-cohomology.

(2) KG_s is a wedge sum of suspensions of Eilenberg-MacLane spectra of type KZ_p.

(3) $\pi_t KG_s$ are the $E_1^{s,t}$-terms, $(\bar b_s \bar c_{s-1})_* : \pi_t KG_{s-1} \to \pi_t KG_s$ are the $d_1^{s-1,t}$-differentials of the ASS and $\pi_t KG_s \cong Ext_A^{s,t}(Z_p, Z_p)$ (cf. [3], p. 180).

Then an Adams resolution of an arbitrary finite spectrum V can be obtained by smashing V on (3.1). We first prove the following lemmas.

Lemma 3.2. (1) *Let* $p \geq 3, n \geq 3$ *and* $h_n \in Ext_A^{1,p^n q}(Z_p, Z_p)$, b_0, α_2, g_0 *be the generators in* $Ext_A^{2,*}(Z_p, Z_p)$ *with internal degrees* $pq, 2q+1, (p+2)q$ *respectively, then* $d_2(h_n g_0) = \alpha_2 h_n b_0 \neq 0 \in Ext_A^{5, p^n q + (p+2)q + 1}(Z_p, Z_p)$ *(up to nonzero scalar), where* $d_2 : Ext_A^{3,*}(Z_p, Z_p) \to Ext_A^{5,*+1}(Z_p, Z_p)$ *is the differential of the ASS.*

(2) *The differential satisfies* $d_2(\bar g_0 \tilde h_n) = \alpha_* \alpha_* (\tilde b_0 \tilde h_n) \in Ext_A^{5, p^n q + (p+2)q + 2} \cdot (H^*M, H^*M)$ *up to nonzero scalar, where* $\bar g_0 \in Ext_A^{3,(p+2)q+1}(H^*M, H^*M)$, $\tilde b_0 \in Ext_A^{2,pq}(H^*M, H^*M)$ *and* $\tilde h_n \in Ext_A^{1,p^n q}(H^*M, H^*M)$ *satisfy* $i^* j_* \bar g_0 = g_0$, $i^*(\tilde b_0) = i_*(b_0)$ *and* $i^*(\tilde h_n) = i_*(h_n)$ *respectively.*

Proof. (1) Let $\beta \in [\Sigma^{(p+1)q}K, K]$ be the second periodicity element (cf. [7], p. 426). It is known that $\beta_1 = jj'\beta i' i \in \pi_{pq-2}S$ is represented by $b_0 \in Ext_A^{2,pq}(Z_p, Z_p)$ in the ASS and $\alpha_2 \beta_1 = j\alpha^2 ijj'\beta i' i = 0$, $\alpha_2 b_0 \neq 0 \in Ext_A^{4,pq+2q+1} \cdot (Z_p, Z_p)$. Then $\alpha_2 b_0$ must be hitted by the differential and the only possibility is $d_2(g_0) = \alpha_2 b_0$ up to nonzero scalar. From [8], p. 11, Theorem 1.2.14, $d_2(h_n) = a_0 b_{n-1} \in Ext_A^{3,p^n q+1}(Z_p, Z_p)$, then $d_2(h_n g_0) = d_2(h_n) g_0 + h_n d_2(g_0) = \alpha_2 h_n b_0$ up to nonzero scalar. (Note: $g_0 a_0 = 0$ by [1], Table 8.2).

(2) Since $p_*(g_0) = a_0 g_0 = 0$, we have $g_0 \in j_*\, Ext_A^{3,(p+2)q+1}(H^*M, Z_p)$. Moreover, $p^*\, Ext_A^{3,(p+2)q+1}(H^*M, Z_p) \subset i_*\, Ext_A^{4,(p+2)q+2}(Z_p, Z_p) = 0$ (cf. Proposition 2.1(4)), then there is $\bar g_0 \in Ext_A^{3,(p+2)q+1}(H^*M, H^*M)$ such that $i^* j_* \bar g_0 = g_0$. By (1), the differential satisfies $d_2(i^* j_* \bar g_0) = \alpha_2 b_0 = i^* j_* \alpha_* \alpha_* (\tilde b_0)$ and so $d_2(i^* \bar g_0) = i^* \alpha_* \alpha_* (\tilde b_0)$ modulo $i_*\, Ext_A^{4,(p+2)q+2}(Z_p, Z_p) = 0$ (cf. Proposition 2.1(4)).

Hence we have $d_2(\bar{g}_0) = \alpha_*\alpha_*(\tilde{b}_0)$ modulo $j^* \, Ext_A^{4,(p+2)q+3}(H^*M, Z_p) = 0$ by Proposition 2.1(4). Since $d_2(\tilde{h}_n) \in Ext_A^{3,p^nq+1}(H^*M, H^*M)$ which is zero by $Ext_A^{3,p^nq+r}(H^*M, Z_p) = 0$ for $r = 1, 2$ (cf. [4], Proposition 2.3(1)), we have $d_2(\bar{g}_0\tilde{h}_n) = d_2(\bar{g}_0)\tilde{h}_n = \alpha_*\alpha_*(\tilde{b}_0) \cdot \tilde{h}_n = \alpha_*\alpha_*(\tilde{b}_0\tilde{h}_n)$ (cf. Remark of Proposition 2.2) as desired. Q.E.D.

Lemma 3.3. *Let* $p \geq 5, n \geq 3$, *then there exists* $\eta'_{n,2} \in [\Sigma^{p^nq+q}K, E_2 \wedge K]$ *such that* $(\bar{b}_2 \wedge 1_K)\eta'_{n,2} = h_0h_n \wedge 1_K \in [\Sigma^{p^nq+q}K, KG_2 \wedge K]$ *and* $(1_{E_2} \wedge \alpha')\eta'_{n,2} = 0$, *where* $h_0h_n \in \pi_{p^nq+q}KG_2 \cong Ext_A^{2,p^nq+q}(Z_p, Z_p)$ *and* $\alpha' = j\alpha i \wedge 1_K \in [\Sigma^{q-1}K, K]$.

Proof. From [5], Proposition 3.4, there is a d_1-cycle $(h_0h_n)'' \in [\Sigma^{p^nq+q-1}K, KG_2 \wedge K]$ such that $(1_{KG_2} \wedge j')(h_0h_n)'' = (1_{KG_2} \wedge ijj')(h_0h_n \wedge 1_K)$ and $(\bar{c}_2 \wedge 1_K)(h_0h_n)'' = 0$. It follows that $(\bar{c}_2 \wedge 1_{L\wedge K})(1_{KG_2} \wedge \tilde{\Delta})(h_0h_n)'' = 0$ and there exists $\tilde{f}_1 \in [\Sigma^{p^nq+q-2}K, E_2 \wedge L \wedge K]$ such that $(\bar{b}_2 \wedge 1_{L\wedge K})\tilde{f}_1 = (1_{KG_2} \wedge \tilde{\Delta})(h_0h_n)'' \in [\Sigma^{p^nq+q-2}K, KG_2 \wedge L \wedge K]$. Then, by (2.16) we have

$$
\begin{aligned}
(\bar{b}_2 \wedge 1_K)(1_{E_2} \wedge j'' \wedge 1_K)\tilde{f}_1 &= (1_{KG_2} \wedge (j'' \wedge 1_K)\tilde{\Delta})(h_0h_n)'' \\
&= (1_{KG_2} \wedge i'j')(h_0h_n)'' = (1_{KG_2} \wedge i'ijj')(h_0h_n \wedge 1_K)
\end{aligned}
$$

and so

$$
\begin{aligned}
&(\bar{b}_2 \wedge 1_K)(1_{E_2} \wedge j'' \wedge 1_K)(1_{E_2} \wedge 1_L \wedge \mu)(\tilde{f}_1 \wedge 1_K)\nu \\
&\quad = (1_{KG_2} \wedge \mu)(1_{KG_2} \wedge i'ijj' \wedge 1_K)(h_0h_n \wedge 1_K \wedge 1_K)\nu = h_0h_n \wedge 1_K,
\end{aligned}
$$

where $\mu : K \wedge K \to K$ is the multiplication of K satisfying $\mu(i'i \wedge 1_K) = 1_K$ and $\nu : \Sigma^{q+2}K \to K \wedge K$ is the injection such that $(jj' \wedge 1_K)\nu = 1_K$ (cf. [7], p. 433). This shows that $\eta'_{n,2} = (1_{E_2} \wedge j'' \wedge 1_K)(1_{E_2} \wedge 1_L \wedge \mu)(\tilde{f}_1 \wedge 1_K)\nu$ is our desired map. Q.E.D.

Proof of Theorem A(1). For the map $\eta'_{n,2}$ in Lemma 3.3, we have $[(\bar{b}_2 \wedge 1_K)\eta'_{n,2}i'] = [(h_0h_n \wedge 1_K)'i'] = (\alpha_1 \wedge 1_K)_*[(h_n \wedge 1_K)i']$, then $[(\bar{b}_2 \wedge 1_{L\wedge K}) \cdot (1_{E_2} \wedge i'' \wedge 1_K)\eta'_{n,2}i'] = 0 \in Ext_A^{2,p^nq+q}(H^*L \wedge K, H^*M)$. By (2.3) and Proposition 2.17(1) we have $Ext_A^{3,p^nq+q+1}(H^*L \wedge K, H^*M) = 0$, then $(\bar{a}_0\bar{a}_1 \wedge 1_{L\wedge K}) \cdot (1_{E_2} \wedge i'' \wedge 1_K)\eta'_{n,2}i' \in [\Sigma^{p^nq+q-2}M, L \wedge K]$ has filtration ≥ 4. Moreover, the second periodicity element $\beta \in [\Sigma^{(p+1)q}K, K]$ has filtration one, then $(\bar{a}_0\bar{a}_1 \wedge 1_{L\wedge K})(1_{E_2} \wedge 1_L \wedge \beta)(1_{E_2} \wedge i'' \wedge 1_K)\eta'_{n,2}i' = (\bar{a}_0\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4 \wedge 1_{L\wedge K})f_2$ which has filtration ≥ 5 with $f_2 \in [\Sigma^{p^nq+(p+2)q+3}M, E_5 \wedge L \wedge K]$. It follows that $(1_{E_2} \wedge i'' \wedge 1_K)(1_{E_2} \wedge \beta)\eta'_{n,2}i' = (\bar{a}_2\bar{a}_3\bar{a}_4 \wedge 1_{L\wedge K})f_2 + (\bar{c}_1 \wedge 1_{L\wedge K})g$ and the d_1-cycle $g \in [\Sigma^{p^nq+(p+2)q}M, KG_1 \wedge L \wedge K]$ is zero by Proposition 2.18(2). That is, we have

$$
(3.4) \qquad (1_{E_2} \wedge (i'' \wedge 1_K)\beta)\eta'_{n,2}i' = (\bar{a}_2\bar{a}_3\bar{a}_4 \wedge 1_{L\wedge K})f_2
$$

for some $f_2 \in [\Sigma^{p^nq+(p+2)q+3}M, E_5 \wedge L \wedge K]$.

By composing $(1_{E_2} \wedge \varepsilon(1_{K'} \wedge i)\bar{\Delta}_{K'})$ on (3.4) we have $(\bar{a}_2\bar{a}_3\bar{a}_4 \wedge 1_M) \cdot (1_{E_5} \wedge \varepsilon(1_{K'} \wedge i)\bar{\Delta}_{K'})f_2 i = (1_{E_2} \wedge \varepsilon(1_{K'} \wedge i)vj'\beta)\eta'_{n,2}i'i = 0$ by (2.16), (2.14), (3.4) and Lemma 3.3.

Note that $\pi_{p^n q+(p+2)q-1}KG_2 \wedge M = 0$ and $\pi_{p^n q+(p+2)q}KG_3 \wedge M$ has unique generator $(1_{KG_3} \wedge i)h_n g_0$ (cf. [1], Table 8.1). Then we have

$$(\bar{a}_4 \wedge 1_M)(1_{E_5} \wedge \varepsilon(1_{K'} \wedge i)\bar{\Delta}_{K'})f_2 i = \lambda(\bar{c}_3 \wedge 1_M)(1_{KG_3} \wedge i)h_n g_0 \tag{3.5}$$

for some $\lambda \in Z_p$ and this shows that the differential $\lambda d_2(i_*(h_n g_0)) = 0$, since $\varepsilon \in [\Sigma^{q-1}K' \wedge M, M]$ induces the zero homomorphism in Z_p-cohomology. Note that $d_2(i_*(h_n g_0)) \neq 0$, for otherwise, we would have $d_2(h_n g_0) = p_*(g_0 b_{n-1}) = a_0 g_0 b_{n-1} = 0$ by $a_0 g_0 = 0$ in [1], Table 8.2, and $Ext_A^{4,p^n q+(p+2)q}(Z_p, Z_p) \cong Z_p\{g_0 b_{n-1}\}$ by [4], Proposition 2.1(2). This contradicts Lemma 3.2. So the scalar λ must be 0 and by applying the derivation d on (3.5) we have (cf. [10], p. 210, Theorem 2.2 and $d(\varepsilon(1_{K'} \wedge i)\bar{\Delta}_{K'}) \in [\Sigma^{-1}L \wedge K, M] = 0$ by Proposition 2.18(1)):

$$(\bar{a}_4 \wedge 1_M)(1_{E_5} \wedge \varepsilon(1_{K'} \wedge i)\bar{\Delta}_{K'})d(f_2 ij) = 0. \tag{3.6}$$

Let X be the cofibre of $\varepsilon(1_{K'} \wedge i) : \Sigma^{q-1}K' \to M$ given by the cofibration

$$\Sigma^{q-1}K' \xrightarrow{\varepsilon(1_{K'} \wedge i)} M \xrightarrow{w_2} X \xrightarrow{u_2} \Sigma^q K'. \tag{3.7}$$

It follows from (3.7) and (3.6) that

$$(\bar{a}_4 \wedge 1_{K'})(1_{E_5} \wedge \bar{\Delta}_{K'})d(f_2 ij) = (1_{E_4} \wedge u_2)f_3 \tag{3.8}$$

with $f_3 \in [\Sigma^{p^n q+(p+2)q+1}M, E_4 \wedge X]$ and by (3.8), (3.4) and (2.5) we have

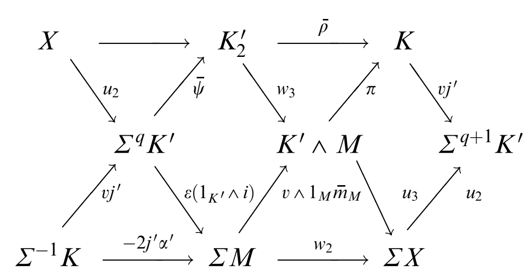
$$(\bar{a}_2\bar{a}_3 \wedge 1_{K'_2})(1_{E_4} \wedge \bar{\psi}u_2)f_3 = 0. \tag{3.9}$$

We claim that the cofibre of $\bar{\psi}u_2 : X \to K'_2$ is $K' \wedge M$ given by the cofibration

$$X \xrightarrow{\bar{\psi}u_2} K'_2 \xrightarrow{w_3} K' \wedge M \xrightarrow{u_3} \Sigma X. \tag{3.10}$$

This can be seen by (2.5), (2.10) and the following commutative diagram of 3×3 lemma in stable homotopy category.

(3.11)



Then, by (3.9) we have

(3.12) $(\bar{a}_2\bar{a}_3 \wedge 1_X)f_3 = (1_{E_2} \wedge u_3)f_4 \qquad \text{with } f_4 \in [\Sigma^{p^nq+(p+2)q}M, E_2 \wedge K' \wedge M].$

We claim that the above f_4 has filtration ≥ 4, that is

(3.13) $$f_4 = (\bar{a}_2\bar{a}_3 \wedge 1_{K' \wedge M})f_5$$

for some $f_5 \in [\Sigma^{p^nq+(p+2)q+2}M, E_4 \wedge K' \wedge M]$. This will be proved later. Then, by (3.8), (3.12), (3.13) and $u_2u_3 = -vj'\pi$ in (3.11) we have

(3.14) $$(1_{E_2} \wedge vj'\beta)d(\eta'_{n,2}i'ij) = -(\bar{a}_2\bar{a}_3 \wedge 1_{K'})(1_{E_4} \wedge vj'\pi)f_5$$

and

(3.15) $$(1_{E_2} \wedge j'\beta)d(\eta_{n,2}i'ij) = -(\bar{a}_2\bar{a}_3 \wedge 1_M)(1_{E_4} \wedge j'\pi)f_5 + (1_{E_2} \wedge \alpha i)\tilde{\xi}_{n,2}$$

with $\tilde{\xi}_{n,2} \in [\Sigma^{p^nq+pq-1}M, E_2]$. The left hand side of (3.15) has filtration 4, since $(1_{E_2} \wedge jj'\beta)\eta'_{n,2}i'i \in \pi_{p^nq+(p+1)q-2}E_2$ is represented by $b_0h_0h_n \in Ext_A^{4,p^nq+(p+1)q}(Z_p, Z_p)$ in the ASS. However, since $[(\bar{b}_4 \wedge 1_K)(1_{E_4} \wedge \pi)f_5] \in Ext_A^{4,p^nq+(p+2)q+2}(H^*K, H^*M) = 0$ (cf. Proposition 2.2), the first term of the right hand side of (3.15) has filtration ≥ 5. Then $(1_{E_2} \wedge \alpha i)\tilde{\xi}_{n,2}$ must be of filtration 4 and so $\tilde{\xi}_{n,2} \in [\Sigma^{p^nq+pq-1}M, E_2]$ should be represented by the unique generator $j^*(h_1h_n)$ of $Ext_A^{2,p^nq+pq-1}(Z_p, H^*M)$, since $Ext_A^{3,p^nq+pq}(Z_p, H^*M) = 0$ (cf. Proposition 2.17(2)). This shows that $\bar{c}_2(h_1h_n)j = 0$ and so $(\bar{c}_2 \wedge 1_M)(1_{KG_2} \wedge i)(h_1h_n) = 0$, and the theorem is proved.

Now our remaining work is to prove the claim (3.13). Note that $(\bar{b}_2 \wedge 1_{K' \wedge M})f_4 \in [\Sigma^{p^nq+(p+2)q}M, KG_2 \wedge K' \wedge M] = 0$ by the following exact sequence

$$[\Sigma^{p^nq+(p+2)q}M, KG_2 \wedge M \wedge M] \xrightarrow{(v \wedge 1_M)_*} [\Sigma^{p^nq+(p+2)q}M, KG_2 \wedge K' \wedge M]$$
$$\xrightarrow{(y \wedge 1_M)_*} [\Sigma^{p^nq+(p+1)q-1}M, KG_2 \wedge M]$$

induced by (2.4), where the first and the last groups are zero by the fact that $\pi_{p^nq+(p+t)q+r}KG_2 \cong Ext_A^{2,p^nq+(p+t)q+r}(Z_p, Z_p) = 0$ for $t = 1, 2$ and $r = -1, 0, 1$ (cf. [6]). Hence, $f_4 = (\bar{a}_2 \wedge 1_{K' \wedge M})f_6$ for some $f_6 \in [\Sigma^{p^nq+(p+2)q+1}M, E_3 \wedge K' \wedge M]$ and from (3.12) we have

(3.16) $$(\bar{a}_3 \wedge 1_X)f_3 = (1_{E_3} \wedge u_3)f_6 + (\bar{c}_2 \wedge 1_X)\tilde{l}_0$$

with $\tilde{l}_0 \in [\Sigma^{p^nq+(p+2)q}M, KG_2 \wedge X]$.

Note that $(\bar{b}_3 \wedge 1_M)(1_{E_3} \wedge y \wedge 1_M)f_6 \in [\Sigma^{p^nq+(p+1)q}M, KG_3 \wedge M] = 0$ by the fact that $\pi_{p^nq+(p+1)q+t}KG_3 = 0$ for $t = -1, 0, 1$ (cf. [1], Table 8.1), then $(\bar{b}_3 \wedge 1_{K' \wedge M})f_6 = \lambda(1_{KG_3} \wedge vi \wedge 1_M)\overline{h_ng_0} + \lambda'(1_{KG_3} \wedge v \wedge 1_M\overline{m}_M)\overline{h_ng_0}ij + \lambda''(1_{KG_3} \wedge v \wedge 1_M\overline{m}_Mij)(\overline{h_ng_0})$ for some $\lambda, \lambda', \lambda'' \in Z_p$, where $\overline{h_ng_0} \in [\Sigma^{p^nq+(p+2)q+1}M, KG_3 \wedge M]$ satisfying $(1_{KG_3} \wedge j)\overline{h_ng_0}i = h_ng_0 \in \pi_{p^nq+(p+2)q}KG_3 \cong Ext_A^{3,p^nq+(p+2)q}$.

(Z_p, Z_p). Note that $\overline{h_n g_0}$ is a d_1-cycle which represents the element $\tilde{g}_0\tilde{h}_n$ in Lemma 3.2(2), then by Lemma 3.2(2), the differential satisfies $d_2[\overline{h_n g_0}] = \alpha_*\alpha_*(\tilde{b}_0\tilde{h}_n) \in Ext_A^{5,p^nq+(p+2)q+2}(H^*M, H^*M)$ and

$$\begin{aligned} &d_2((vi \wedge 1_M)_*[\overline{h_n g_0}]) \\ &= (vi \wedge 1_M)_*\alpha_*\alpha_*(\tilde{b}_0\tilde{h}_n) = -(v \wedge 1_M\overline{m}_M)_*(\alpha_1 \wedge 1_M)_*\alpha_*(\tilde{b}_0\tilde{h}_n) \qquad (\text{by } (2.11)) \\ &= -\tfrac{1}{2}(v \wedge 1_M\overline{m}_M)_*[\alpha_2 b_0 h_n \wedge 1_M] \qquad (\text{by } 2(\alpha_1 \wedge 1_M)\alpha = \alpha_2 \wedge 1_M \text{ (cf. [7], p. 430)}) \\ &= -\tfrac{1}{2}(v \wedge 1_M\overline{m}_M)_* d_2[h_n g_0 \wedge 1_M]. \end{aligned}$$

Note that $(1_{KG_3} \wedge ij)\overline{h_n g_0}i = (1_{KG_3} \wedge i)h_n g_0 = (h_n g_0 \wedge 1_M)i$, then we have

$$h_n g_0 \wedge 1_M = (1_{KG_3} \wedge ij)\overline{h_n g_0} + \bar{\lambda} \cdot \overline{h_n g_0}ij \in [\Sigma^{p^nq+(p+2)q}M, KG_3 \wedge M]$$

for some $\bar{\lambda} \in Z_p$. Since $d(\overline{h_n g_0}) \in [\Sigma^{p^nq+(p+2)q+2}M, KG_3 \wedge M] = 0$ by $\pi_{p^nq+(p+2)q+r}KG_3 \cong Ext_A^{3,p^nq+(p+2)q+r}(Z_p, Z_p) = 0$ for $r = 1, 2, 3$ (cf. [1], Table 8.1), by applying d on the above equation we have $\bar{\lambda} = 1$. Moreover, the differential satisfies $d_2[(1_{KG_3} \wedge ij)\overline{h_n g_0} + \overline{h_n g_0}ij] \neq 0$, for otherwise, we would have $d_2[(1_{KG_3} \wedge i)h_n g_0] = 0$ and $d_2(h_n g_0) = p_*(g_0 b_{n-1}) = a_0 g_0 b_{n-1} = 0$ (cf. [4], Proposition 2.1(2) and $a_0 g_0 = 0$ in [1], Table 8.2) which contradicts Lemma 3.2. Hence, by Proposition 2.17(3) we see that $\lambda' = \lambda'' = \frac{1}{2}\lambda$, i.e.

$$(3.17) \qquad (\bar{b}_3 \wedge 1_{K' \wedge M})f_6 = \lambda(1_{KG_3} \wedge vi \wedge 1_M)\overline{h_n g_0} + \tfrac{1}{2}\lambda(1_{KG_3} \wedge v \wedge 1_M\overline{m}_M)(h_n g_0 \wedge 1_M)$$

for some $\lambda \in Z_p$ and we need to show that this λ is zero.

Observe from (3.7) that $\varepsilon(1_{K'} \wedge i)u_2 = 0$, then by (2.13), $(1_{K'} \wedge i)u_2 = \overline{ru}$ with $\bar{u} \in [X, L \wedge K]$ and $\bar{u}u_3 \in [\Sigma^{-1}K' \wedge M, L \wedge K] \cong Z_p\{(i'' \wedge 1_K)\pi(1_{K'} \wedge ij), \tilde{\Delta}\pi\}$. By (2.8), we may choose the sign of $\pi : K' \wedge M \to K$ so that $\pi(1_{K'} \wedge i)v = i'$ with positive sign and so up to sign we have $j'\pi = j'm_K(x \wedge 1_M) = m_M(j'x \wedge 1_M) = j''k \wedge 1_M$, i.e. we have

$$(3.18) \qquad \pi(1_{K'} \wedge i)v = i', \qquad j'\pi = \pm(j''k \wedge 1_M).$$

Note that, by [7], p. 434, Lemma 6.2(iii), $(1_L \wedge j')\tilde{\Delta}\pi = -(i'' \wedge 1_M)ijj'\pi = \pm(i'' \wedge 1_M)ij(j''k \wedge 1_M) = \pm(1_L \wedge j')(i'' \wedge 1_K)\pi(1_{K'} \wedge ij)$ and so $\tilde{\Delta}\pi \pm (i'' \wedge 1_K)\pi(1_{K'} \wedge ij) = (1_L \wedge i')(k \wedge 1_M)$, since $[\Sigma^{-1}K' \wedge M, L \wedge M] \cong Z_p\{k \wedge 1_M\}$. Hence we have $\bar{u}u_3 = \lambda_1(i'' \wedge 1_K)\pi(1_{K'} \wedge ij) + \lambda_2(1_L \wedge i')(k \wedge 1_M)$, and by $\bar{\Delta} = x(1_{K'} \wedge j)\bar{r}$ and $\overline{ru} = (1_{K'} \wedge i)u_2$ we have

$$\begin{aligned} 0 = \bar{\Delta}\bar{u}u_3 &= \lambda_1 i'j'\pi(1_{K'} \wedge ij) - \lambda_2 i'ij(j''k \wedge 1_M) \\ &\qquad (\text{by } \bar{\Delta}(1_L \wedge i') = -i'ij(j'' \wedge 1_M), \text{cf. [7], p. 434}) \\ &= (\pm\lambda_1 - \lambda_2)i'ij(j''k \wedge 1_M). \end{aligned}$$

This shows that $\lambda_2 = \pm\lambda_1$ and

$$\bar{u}u_3 = \lambda_1(i'' \wedge 1_K)\pi(1_{K'} \wedge ij) \pm \lambda_1(1_L \wedge i')(k \wedge 1_M). \tag{3.19}$$

By (3.16), (3.19) and (3.17), modulo d_1-boundary $-(\bar{b}_3\bar{c}_2 \wedge 1_{L\wedge K})(1_{KG_2} \wedge \bar{u})\tilde{l}_0$ we have

$$\begin{aligned}
0 &= (\bar{b}_3 \wedge 1_{L\wedge K})(1_{E_3} \wedge \bar{u}u_3)f_6 \\
&= \lambda\lambda_1(1_{KG_3} \wedge (i'' \wedge 1_K)\pi(1_{K'} \wedge ij)(vi \wedge 1_M))(\overline{h_n g_0}) \\
&\quad + \tfrac{1}{2}\lambda\lambda_1(1_{KG_3} \wedge (i'' \wedge 1_K)\pi(1_{K'} \wedge ij)(v \wedge 1_M\overline{m}_M))(h_n g_0 \wedge 1_M) \\
&\quad \pm \tfrac{1}{2}\lambda\lambda_1(1_{KG_3} \wedge (1_L \wedge i')(k \wedge 1_M)(v \wedge 1_M\overline{m}_M))(h_n g_0 \wedge 1_M) \\
&= \lambda\lambda_1(1_{KG_3} \wedge (i'' \wedge 1_K)i'ij)(\overline{h_n g_0}) + \tfrac{1}{2}\lambda\lambda_1(1_{KG_3} \wedge (i'' \wedge 1_K)i')(h_n g_0 \wedge 1_M) \\
&\quad \pm \tfrac{1}{2}\lambda\lambda_1(1_{KG_3} \wedge (i'' \wedge 1_K)i')(h_n g_0 \wedge 1_M) \qquad \text{(by (3.18), (2.15))}
\end{aligned}$$

and so we have $\lambda_0\lambda\lambda_1(1_{KG_3} \wedge (i'' \wedge 1_K)i'ij)(\overline{h_n g_0})i = 0$ for $\lambda_0 = 1$ or 2. However, $(i'' \wedge 1_K)_*(i'i)_*(h_n g_0) \neq 0 \in Ext_A^{3,p^nq+(p+2)q}(H^*L \wedge K, Z_p)$, since $Ext_A^{2,p^nq+(p+1)q} \cdot (H^*K, Z_p) = 0$ (cf. [6]). This shows that the scalar λ in (3.17) is zero and proves the claim (3.13). Q.E.D.

Proof of Theorem A(2). From Theorem A(1), there is $\xi_{n,2} \in \pi_{p^nq+pq}E_2 \wedge M$ such that $(\bar{b}_2 \wedge 1_M)\xi_{n,2} = (1_{KG_2} \wedge i)(h_1h_n)$ and so $\bar{b}_2(1_{E_2} \wedge j)\xi_{n,2} = 0$ and we have $(1_{E_2} \wedge j)\xi_{n,2} = \bar{a}_2 f'$ for some $f' \in \pi_{p^nq+pq}E_3$. It follows that $\bar{a}_2(1_{E_3} \wedge p)f' = 0$ and $(1_{E_3} \wedge p)f' = \lambda'\bar{c}_2(h_1h_n)$ for some $\lambda' \in Z_p$, since $\pi_{p^nq+pq}KG_2 \cong Ext_A^{2,p^nq+pq}(Z_p, Z_p) \cong Z_p\{h_1h_n\}$. We claim that the scalar $\lambda' \neq 0$, which can be shown as follows. If $\lambda' = 0$, then $(1_{E_3} \wedge p)f' = 0$, $f' = (1_{E_3} \wedge j)f''$ for some $f'' \in \pi_{p^nq+pq+1}E_3 \wedge M$ and so $\xi_{n,2} = (\bar{a}_2 \wedge 1_M)f'' + (1_{E_2} \wedge i)f'''$ with $f''' \in \pi_{p^nq+pq}E_2$ which must have filtration 2 and it is represented by $h_1h_n \in Ext_A^{2,p^nq+pq}(Z_p, Z_p)$. This contradicts the following nontrivial differential: $d_2(h_1h_n) = d_2(h_1)h_n - h_1d_2(h_n) = a_0b_0h_n - h_1a_0b_{n-1} = a_0(b_0h_n + h_1b_{n-1}) > \neq 0 \in Ext_A^{3,p^nq+pq+1}(Z_p, Z_p)$ (cf. [8], p. 11, Theorem 1.2.14 and Proposition 2.1). This shows that $(1_{E_3} \wedge p)f' = \lambda'\bar{c}_2(h_1h_n)$ with $\lambda' \neq 0$, or equivalently, the differential satisfies $d_2(\lambda'h_1h_n) = p_*[\bar{b}_3 f'] = \lambda' a_0(b_0h_n + h_1b_{n-1})$ and so $[\bar{b}_3 f'] = \lambda'(b_0h_n + h_1b_{n-1})$, since $Ext_A^{3,p^nq+pq}(Z_p, Z_p) \cong Z_p\{b_0h_n, h_1b_{n-1}\}$ by [1], Table 8.1. This shows the theorem. Q.E.D.

Acknowledgement

The author would like to thank the referee for his corrections of proofs of some propositions in the original manuscript. He would also like to thank Professors Qibing Zheng and Xiangjun Wang for their help on calculation in the proof of Proposition 2.1.

References

[1] T. Aikawa, 3-Dimensional cohomology of the mod p Steenrod algebra, Math. Scand. **47** (1980), 91–115.
[2] R. Cohen, Odd primary families in stable homotopy theory, Memoirs of Amer. Math. Soc. No. 242 (1981).
[3] R. Cohen and P. Goerss, Secondary cohomology operations that detect homotopy classes, Topology **23** (1984), 177–194.
[4] Jinkun Lin and Qibing Zheng, A new family of filtration seven in the stable homotopy of spheres, Hiroshima Math. J. **28** (1998), 183–205.
[5] Jinkun Lin, Some new families in the stable homotopy of spheres revisited, To appear in Acta Math. Sinica.
[6] A. Liulevicius, The factorizations of cyclic reduced powers by secondary cohomology operations, Memoirs of Amer. Math. Soc. No. 42 (1962).
[7] S. Oka, Multiplicative structure of finite ring spectra and stable homotopy of spheres. Algebraic Topology (Aarhus), Lect. Notes in Math. v. 1051, Springer-Verlag (1984).
[8] D. C. Ravenel, Complex cobordism and stable homotopy groups of spheres, Academic Press, Inc, 1986.
[9] E. Thomas and R. Zahler, Generalized higher order cohomology operations and stable homotopy groups of spheres, Advances in Math. **20** (1976), 287–328.
[10] H. Toda, Algebra of stable homotopy of Z_p-spaces and applications, J. Math. Kyoto Univ. **11** (1971), 197–251.
[11] H. Toda, On spectra realizing exterior part of the Steenrod algebra, Topology **10** (1971), 53–65.

Department of Mathematics, Nankai University
Tianjin, 300071, *People's Republic of China*
email: jklin@nankai.edu.cn