HIROSHIMA MATH. J.
31 (2001), 345–366

Investigation of the nonlocal initial boundary value problems for some hyperbolic equations

David GORDEZIANI and Gia AVALISHVILI

(Received March 15, 2000)
(Revised October 27, 2000)

ABSTRACT. In the present article we are interested in the analysis of nonlocal initial boundary value problems for some medium oscillation equations. More precisely, we investigate different types of nonlocal problems for one-dimensional oscillation equations and prove existence and uniqueness theorems. In some cases algorithms for direct construction of the solution are given. We also consider nonlocal problem for multidimensional hyperbolic equation and prove the uniqueness theorem for the formulated initial boundary value problem applying the theory of characteristics under rather general assumptions.

1. Introduction

While applying mathematical modelling to various phenomena of physics, biology and ecology there often arise problems with non-classical boundary conditions, which connect the values of unknown function on the boundary and inside of the given domain. Boundary conditions of such type are called nonlocal boundary conditions. Nonlocal initial boundary value problems are important from the point of view of their practical application to modelling and investigating of pollution processes in rivers, seas, which are caused by sewage. It is possible by nonlocal boundary conditions to simulate decreasing of pollution under influence of natural factors of filtration and settling that causes self-purification of the medium.

One of the first works, where nonlocal conditions were considered, is [1]. The nonlocal problem was investigated, applying the method of separation of variables and the corresponding eigenvalues and eigenfunctions were considered. First, the systematic investigation of a certain class of spatial nonlocal problems was carried out by A. Bitsadze and A. Samarskii in [2]. Further, in the works [3, 4] resolution methods for such type problems in the case of rather

2000 *Mathematics Subject Classification.* 35L20, 45B05.
Key words and phrases. Hyperbolic equations, Caracteristic Surfaces, Nonlocal boundary conditions.

general elliptic equations were suggested. In [5] for the equations of shell and elasticity theories boundary conditions similar to Bitsadze-Samarskii ones were considered. Under rather strict conditions the uniqueness of the solution of the nonlocal problem for the three-dimensional models of the elasticity theory is proved. The stated nonlocal problems were effectively solved in the case of circular plates for the Kirchhoff model. Later, in [6–9] generalizations of Bitsadze-Samarskii conditions were suggested. Particularly, in [6] discrete spatial nonlocal problems were studied for rather general elliptic and parabolic differential equations. Suggested iteration procedures allow not only to prove existence of the solution of formulated problems, but also to construct algorithms for numerical resolution.

Note that theoretical study of nonlocal problems is connected with great difficulties. Too many things are expected to be done in this direction, though a lot of interesting works are already devoted to these questions ([1–17]). Complications in investigation of problems of these type are essentially caused by the fact, that it is usually impossible to apply the classical methods of functional analysis, the energetic method, the method of singular integral equations. This is the reason for existing only separate results for nonlocal initial boundary value problems.

It must be emphasized that in the papers devoted to nonlocal problems the cases of elliptic and parabolic equations have been mainly considered. In the present work we study nonlocal problems for hyperbolic equations. In § 2 we state the theorem of uniqueness for rather general discrete spatial nonlocal problem for hyperbolic equation. In § 3 and § 4 we study in details one-dimensional problems of the mechanics of solids with different nonlocal boundary conditions. More precisely, in § 3 we consider the string oscillation equation with the classical initial and discrete nonlocal boundary conditions, which are the generalizations of Bitsadze-Samarskii conditions. In the same section we discuss the problem with the integral nonlocal conditions. There are proved the theorems of existence and uniqueness of the solution, which in some cases can be constructed directly using algorithms given ibidem. In § 4 we consider the telegraph equation. As in the case of string oscillation equation we study nonlocal problems with discrete and integral nonlocal boundary conditions.

2. Nonlocal problem for multidimensional medium oscillation equation

Let us consider the bounded domain $\Omega \subset \mathbf{R}^n$, $n \geq 1$, $x = (x_1, \dots, x_n)$, and Γ be the boundary of Ω. Let $\Omega_i(t)$ $(i = 1, \dots, m)$ be the subsets of Ω. Assume that boundaries $\Gamma_i(t)$ of $\Omega_i(t)$ are diffeomorphic images of Γ, i.e. $I_i(\cdot, t) : \Gamma \to \Gamma_i(t)$ are diffeomorphisms, and $I_i(x, t)$ are continuous functions,

Γ, Γ_i are sufficiently smooth surfaces and that the distance between them is positive $(i = \overline{1,m})$.

Definition. Let $A : C^2(\Sigma \times [a,b]) \to X(\Sigma \times [a,b])$ be an operator, where X is a functional space defined on $\Sigma \times [a,b]$, $\Sigma \subset \mathbf{R}^l$, $l \in \mathbf{N}$. We say, that for the operator A the condition of localization by $t \in [a,b]$ is valid if there exists a class of operators $A_{\alpha,\beta} : C^2(\Sigma \times [\alpha,\beta]) \to X(\Sigma \times [\alpha,\beta])$, such that the following is true: if $[\alpha,\beta] \subset [\gamma,\delta] \subset [a,b]$, $v \in C^2(\Sigma \times [\alpha,\beta])$, $w \in C^2(\Sigma \times [\gamma,\delta])$ and $v(x,t) \equiv w(x,t)$, $t \in [\alpha,\beta]$, then $(A_{\alpha,\beta}v)(x,t) \equiv (A_{\gamma,\delta}w)(x,t)$, for $\alpha \le t \le \beta$, and $A_{a,b} = A$.

Let L be a strongly elliptic operator

$$L \equiv \sum_{i,k=1}^{n} a_{ik}(x,t)\frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial}{\partial x_i} + b(x,t)\frac{\partial}{\partial t} + c(x,t),$$

$$(x,t) \in \Omega \times (0,T),$$

$$\sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k \ge \gamma(\xi_1^2 + \cdots + \xi_n^2), \qquad \gamma = const > 0, \qquad \forall \xi = (\xi_1,\ldots,\xi_n) \in \mathbf{R}^n,$$

where a_{ik}, b_i, b, c are prescribed functions.

Consider the nonlocal problem for the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - Lu = f(x,t), \qquad (x,t) \in Q_T = \Omega \times (0,T), \tag{2.1}$$

with the classical initial conditions

$$\begin{aligned} u(x,0) &= u_0(x),\\ u_t(x,0) &= u_1(x), \end{aligned} \qquad x \in \Omega, \tag{2.2}$$

and the nonlocal boundary conditions

$$u(x,t) = \sum_{i=1}^{m} p_i(x,t)u(I_i(x,t),t) + g(x,t), \qquad (x,t) \in S_T = \Gamma \times [0,T], \tag{2.3}$$

where p_i, g, u_0, u_1 are prescribed continuous functions and $u(x,t)$ is an unknown function, which is the classical solution of equation (2.1) satisfying conditions (2.2) and (2.3) at the same time. The following uniqueness theorem is valid.

Theorem 2.1. *If a_{ik} are continuously differentiable functions and coefficients b_i, b, c are continuous $(i,k = \overline{1,n})$, then the nonlocal problem* (2.1)–(2.3) *has no more than one solution.*

Proof. Assume that there exist two $u(x,t)$ and $v(x,t)$ solutions of the problem. Then obviously their difference $w(x,t) = u(x,t) - v(x,t)$ is the solution of the homogeneous equation (2.1) under homogeneous initial and nonlocal conditions. Note that

$$\rho_i(t) = \operatorname{dist}(\Gamma_i(t), \Gamma) = \inf_{x, y \in \Gamma} \rho(I_i(x,t), y)$$

continuously depends on t, since $I_i(x,t)$ is a continuous function and hence it is uniformly continuous.

Taking into account that distance between Γ and $\Gamma_i(t)$ is positive, we get $\rho_i(t) > 0$ for all $t \in [0,T]$ and consequently, there exists such a $\delta > 0$, that $\rho_i(t) > \delta$, $t \in [0,T]$ $(i = \overline{1,m})$. Therefore, for any point $(x,t) \in \Omega_i(t)$, the ball of a radius δ centered at x is placed entirely in a "horizontal" cross-section $\Omega \times \{t\}$.

Strong ellipticity of the operator L allows to inscribe as well as to overdraw cones respectively inside and outside of a characteristic conoid, defined by the operator L. Tangents of angles between the axis and elements of the cones are denoted by α and β $(\alpha \le \beta)$ and we call them spreads of the cones.

Note that since $w(x,0) = w_t(x,0) = 0$, $x \in \Omega$, then $w(x,t)$ equals to zero in any point (x,t) for which the base of the characteristic conoid, passing through this point, lies in Ω [17]. Let us now consider an interval $0 \le t \le t^*$, where $t^* = \delta/\beta$. Then for any point $(\bar{x},\bar{t})$, which belong to the curvilinear cylinder $\Omega_i(t)$ $(i = 1,\dots,m)$ base of the cone with a top in $(\bar{x},\bar{t})$, axis parallel to the axis t and with a spread β lies in Ω, as $\bar{t}\beta \le t^*\beta = (\delta/\beta)\cdot\beta = \delta$. Therefore, $w(\bar{x},\bar{t}) = 0$, i.e. in any point of the curvilinear cylinders $\Omega_i(t)$ $(i = \overline{1,m})$, $w(x,t) = 0$, for $0 \le t \le t^*$. Taking into account that $w(x,t)$ satisfies the homogeneous nonlocal boundary conditions, we obtain

$$w(x,t) = 0, \qquad (x,t) \in S_{t^*},$$

and therefore $w(x,t)$ is the solution of the homogeneous equation (2.1) under homogeneous initial and boundary conditions. Since the classical problem has a unique solution, then

$$w(x,t) \equiv 0, \qquad 0 \le t \le t^*.$$

Now take for an initial moment of time t^*, i.e. change the variable $\tau = t - t^*$. The function $w^*(x,\tau) = w(x,\tau + t^*)$ satisfies the following problem

$$w^*_{\tau\tau} = Lw^*, \qquad (x,\tau) \in Q_{T-t^*}, \tag{2.4}$$

$$w^*(x,0) = w^*_\tau(x,0) = 0, \qquad x \in \Omega, \tag{2.5}$$

$$w^*(x,\tau) = \sum_{i=1}^{m} p_i(x,\tau + t^*) w^*(I_i(x,\tau + t^*),\tau), \qquad (x,\tau) \in S_{T-t^*}. \tag{2.6}$$

Repeating the proceeding reasoning, we get $w^*(x,\tau) \equiv 0$, $0 \le \tau \le t^*$, and consequently $w(x,t) \equiv 0$, $0 \le t \le 2t^*$. Similarly $w(x,t) \equiv 0$ for $t \in [0, nt^*]$, $n \in \mathbf{N}$ up to the moment T. So $w(x,t) \equiv 0$, $(x,t) \in Q_T$, which means, that $u(x,t) \equiv v(x,t)$ and the solution is unique. $\square$

It should be noted that in the similar way we can prove a uniqueness theorem in more general case.

Theorem 2.2. *The nonlocal problem for the equation* (2.1) *with the initial conditions* (2.2) *and the following nonlocal boundary conditions*

$$u(x,t) = \sum_{i=1}^{m} [A^i u(I_i(x,t),t)](x,t) + g(x,t), \qquad (x,t) \in S_T = \Gamma \times [0,T], \tag{2.7}$$

has no more than one regular solution, where $A^i : C^2(S_T) \to C^2(S_T)$ are linear operators, and for each A^i $(i = 1, \dots, m)$ the condition of localization by t is satisfied.

Remark. If the boundaries of Ω and $\Omega_i(t)$ $(i = 1, \dots, m)$ and the given functions are smooth enough, then not only the uniqueness theorem is true for the problem (2.1)–(2.3), but also the theorem of existence is valid. In particular, we can find so large N, that if all the functions mentioned in Theorem 2.1 are N-times continuously differentiable and compatibility conditions are valid, then the nonlocal problem (2.1)–(2.3) has a unique solution.

3. Nonlocal problems for the string oscillation equation

In the following two sections we study nonlocal problems for one-dimensional hyperbolic equations. Unlike multidimensional case we formulate the theorems of existence and uniqueness of the solutions for more general problems with non-linear nonlocal boundary conditions and in some cases give algorithms for direct construction of the solutions.

Let us consider the nonlocal problem for the string oscillation equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < l, \; 0 < t < T, \tag{3.1}$$

with the classical initial conditions

$$\begin{aligned} u(x,0) &= \varphi(x), \\ u_t(x,0) &= \psi(x), \end{aligned} \qquad 0 \le x \le l, \tag{3.2}$$

and the nonlocal boundary conditions

$$
(3.3)\qquad \begin{aligned}
&\alpha(t)u(0,t)+\beta(t)\frac{\partial u}{\partial x}(0,t)=\sum_{i=1}^{m}[A^i u(\xi_i(t),t)](t)+f(t),\\
&\gamma(t)u(l,t)+\theta(t)\frac{\partial u}{\partial x}(l,t)=\sum_{j=1}^{p}[B^j u(\eta_j(t),t)](t)+g(t),
\end{aligned}\qquad 0\le t\le T,
$$

where α, β, γ, θ, f, g are prescribed functions, which satisfy the compatibility conditions, A^i, B^j $(i=1,\dots,m; j=1,\dots,p)$ is the system of generally nonlinear operators $C^2([0,T])\to C^2([0,T])$, satisfying conditions of localization, $\xi_i(t), \eta_j(t)$ $(i=\overline{1,m}; j=\overline{1,p})$ are sliding points of the string $(0,l)$. We say, that $u(x,t)$ is classical solution of the problem (3.1)–(3.3) if it is twice continuously differentiable on $\bar{D}=\{0\le x\le l, 0\le t\le T\}$, satisfies equation (3.1) and conditions (3.2), (3.3). The following theorem is true.

Theorem 3.1. *Assume that the following conditions are valid:*

i) $f,g,\alpha,\beta,\gamma,\theta\in C^2([0,T])$, $\varphi\in C^2([0,l])$, $\psi\in C^1([0,l])$, $\alpha(t)\beta(t)\ne 0$, $\gamma(t)\theta(t)\ne 0$, $0\le t\le T$;

ii) $\xi_i,\eta_j\in C^2([0,T])$, $0<\xi_i(t)$, $\eta_j(t)<l$, *when* $t\in[0,T]$, $i=1,\dots,m$; $j=1,\dots,p$;

iii) *each of the functions* $\beta(t)$, $\theta(t)$ *either is not equal to zero for any* $t\in[0,T]$, *or is equal to zero identically.*

Then the nonlocal problem (3.1)–(3.3) *has a unique classical solution* $u(x,t)$.

Proof. Note that if the solution of the problem (3.1)–(3.3) is found, then we get some functions on the ends of the string

$$
(3.4)\qquad \begin{aligned} u(0,t)&=\mu_1(t),\\ u(l,t)&=\mu_2(t),\end{aligned}\qquad 0\le t\le T,
$$

and then $u(x,t)$ is the solution of the Cauchy-Dirichlet problem for the equation (3.1) with the initial and boundary conditions (3.2), (3.4), which has a unique solution

$$
(3.5)\qquad u(x,t)=F(x,t)+\sum_{n=0}^{\infty}\bar{\mu}_1(t-2nl-x)-\sum_{n=1}^{\infty}\bar{\mu}_1(t-2nl+x)
+\sum_{n=0}^{\infty}\bar{\mu}_2(t-(2n+1)l+x)-\sum_{n=0}^{\infty}\bar{\mu}_2(t-(2n+1)l-x),
$$

where $\lambda=\pi/(l+1)$;

$$
F(x,t)=\frac{\Phi(x+t)+\Phi(x-t)}{2}+\frac{1}{2}\int_{x-t}^{x+t}\Psi(\alpha)d\alpha
-\frac{\varphi''(0)}{\lambda^2\sin\lambda l}\sin(\lambda(l-x))\cos\lambda t-\frac{\varphi''(l)}{\lambda^2\sin\lambda l}\sin\lambda x\cos\lambda t;
$$

$\Phi(x)$ and $\Psi(x)$ represent continuations of the functions $\varphi(x) + \varphi''(0) \cdot \sin(\lambda(l-x))/(\lambda^2 \sin \lambda l) + \varphi''(l) \sin \lambda x/(\lambda^2 \sin \lambda l)$ and $\psi(x)$ respectively on the whole axis retaining smoothness in such a way, that

$$\Phi(x) + \Phi(-x) = 2\Phi(0), \qquad \Psi(x) + \Psi(-x) = 2\Psi(0),$$
$$\Phi(l-x) + \Phi(l+x) = 2\Phi(l), \qquad \Psi(l-x) + \Psi(l+x) = 2\Psi(l);$$

$$(3.6)\quad \bar{\mu}_1(t) = \begin{cases} \mu_1(t) - (\varphi(0) + \varphi''(0)/\lambda^2) - \psi(0)t + \varphi''(0)\cos(\lambda t)/\lambda^2, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

and for $\bar{\mu}_2(t)$ we have the corresponding expression, where 0 is replaced by l. Thus, any classical solution of the problem (3.1)–(3.3) can be represented by the form (3.5). If we find twice continuously differentiable functions $\mu_1(t)$, $\mu_2(t)$, then the problem is solved. Consequently, due to this fact under the solution of the problem (3.1)–(3.3) we sometimes mean the couple $\{\mu_1, \mu_2\}$.

Taking into account nonlocal conditions (3.3), we get that the problem (3.1)–(3.3) will be solved, if we find the couple $\{\mu_1, \mu_2\}$, which satisfies the equations

$$(3.7)\quad \begin{aligned} &\alpha(t)\mu_1(t) + \beta(t)\left(F_x(0,t) - \bar{\mu}_1'(t) - 2\sum_{n=1}^{\infty} \bar{\mu}_1'(t-2nl) + 2\sum_{n=0}^{\infty} \bar{\mu}_2'(t-(2n+1)l)\right) \\ &\qquad = \sum_{i=1}^{m} [A^i u(\xi_i(t), t)](t) + f(t), \\ &\gamma(t)\mu_2(t) + \theta(t)\left(F_x(l,t) + \bar{\mu}_2'(t) + 2\sum_{n=1}^{\infty} \bar{\mu}_2'(t-2nl) - 2\sum_{n=0}^{\infty} \bar{\mu}_1'(t-(2n+1)l)\right) \\ &\qquad = \sum_{j=1}^{p} [B^j u(\eta_j(t), t)](t) + g(t). \end{aligned} \qquad 0 \leq t \leq T,$$

It should be mentioned, that from the above reasonings it follows that solution of the problem is completely reduced to finding the pair $\{\mu_1, \mu_2\}$, i.e. existence and uniqueness of the solution $u(x,t)$ and of the pair $\{\mu_1, \mu_2\}$ are equivalent.

Since the functions $\xi_i(t)$ and $\eta_j(t)$ $(i = 1, \ldots, m; j = 1, \ldots, p)$ are continuous on $[0,T]$ and for all $t \in [0,T]$ they belong to the interval $(0,l)$, then there exist

$$\varepsilon_1 = \min_{\substack{0 \le t \le T \\ 1 \le i \le m}} \xi_i(t), \qquad \tilde{\varepsilon}_1 = \max_{\substack{0 \le t \le T \\ 1 \le i \le m}} \xi_i(t),$$

$$\varepsilon_2 = \min_{\substack{0 \le t \le T \\ 1 \le j \le p}} \eta_j(t), \qquad \tilde{\varepsilon}_2 = \max_{\substack{0 \le t \le T \\ 1 \le j \le p}} \eta_j(t),$$

where each of the numbers ε_1, ε_2, $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$ belongs to $(0,l)$. Obviously, all the curves ξ_i, η_j are located in the stripe $[t^*, l - t^*] \times [0,T]$, where $t^* = \min\{\varepsilon_1, \varepsilon_2, l - \tilde{\varepsilon}_1, l - \tilde{\varepsilon}_2\}$.

By (3.7), taking into account the definition of $\bar{\mu}_1(t)$, $\bar{\mu}_2(t)$, we get that if the pair $\{\mu_1, \mu_2\}$ is the solution of the problem, then it has to satisfy the following equalities

$$\left.\begin{aligned} \alpha(t)\mu_1(t) - \beta(t)\mu_1'(t) &= \sum_{i=1}^{m} [A_{0,t^*}^i F(\xi_i(t),t)](t) + \tilde{f}(t), \\ \gamma(t)\mu_2(t) + \theta(t)\mu_2'(t) &= \sum_{j=1}^{p} [B_{0,t^*}^j F(\eta_j(t),t)](t) + \tilde{g}(t), \end{aligned}\right. \qquad 0 \le t \le t^*,$$

where $\tilde{f}(t)$, $\tilde{g}(t)$ are expressed through prescribed functions. Obviously $\tilde{f}(t)$, $\tilde{g}(t) \in C^1([0,t^*])$. Consequently, for $\mu_1(t)$ and $\mu_2(t)$ we get ordinary differential equations of the first order. Assume that first of the conditions of the point iii) in the Theorem 3.1 is true, i.e. $\beta(t) \ne 0$, when $t \in [0,T]$. Then taking into account compatibility condition $\mu_1(0) = \varphi(0)$, for $0 \le t \le t^*$ we obtain

$$\mu_1(t) = \exp\left(\int_0^t \frac{\alpha(\tau)}{\beta(\tau)}\, d\tau\right)\Bigg(\varphi(0) - \int_0^t \exp\left(-\int_0^\tau \frac{\alpha(s)}{\beta(s)}\, ds\right)$$
$$\left(\sum_{i=1}^{m} [A_{0,t^*}^i F(\xi_i(t),t)](\tau) + \tilde{f}(\tau)\right) \frac{1}{\beta(\tau)}\, d\tau\Bigg).$$

In the second case, $\mu_1(t)$ can be directly expressed by the functions in the right-hand part of the equation. Here, corresponding functions $\tilde{f}(t)$ or $\tilde{g}(t)$ will be twice continuously differentiable. In both cases, as we see, $\mu_1(t)$ is equal to twice continuously differentiable function, for $0 \le t \le t^*$. Therefore, in the time interval $[0,t^*]$ we can define the unknown pair of functions $\{\mu_1, \mu_2\}$ and, using the formula (3.5), we get the solution of the problem (3.1)–(3.3) on $[0,t^*]$.

Now, take for the initial moment t^*. Introducing a new time variable $\tau = t - t^*$, the nonlocal problem for the function $v(x,\tau) = u(x, \tau + t^*)$ considered in $[0,l] \times [0,t^*]$ takes the following form

$$v_{\tau\tau} = v_{xx}, \qquad 0 < x < l, \ 0 < \tau < t^*, \tag{3.8}$$

$$\begin{aligned}
&\alpha(\tau + t^*)v(0,\tau) + \beta(\tau + t^*)\frac{\partial v}{\partial x}(0,\tau) \\
&\quad = \sum_{i=1}^{m}[A^i_{t^*,2t^*}v(\xi_i(t), t - t^*)](\tau + t^*) + f(\tau + t^*), \\
&\gamma(\tau + t^*)v(l,\tau) + \theta(\tau + t^*)\frac{\partial v}{\partial x}(l,\tau) \\
&\quad = \sum_{j=1}^{p}[B^j_{t^*,2t^*}v(\eta_j(t), t - t^*)](\tau + t^*) + g(\tau + t^*),
\end{aligned} \qquad 0 \le \tau \le t^*, \tag{3.9}$$

where the initial conditions are

$$\begin{aligned} v(x,0) &= u(x,t^*), \\ v_\tau(x,0) &= u_t(x,t^*), \end{aligned} \qquad 0 \le x \le l. \tag{3.10}$$

Note that in the right-hand parts of (3.9) we mean that $t^* \le t \le 2t^*$ to make it possible to act with corresponding operators. Finally we get the function defined on $[t^*, 2t^*]$ with the argument $t = \tau + t^*$, where $0 \le \tau \le t^*$. As in the previous case we can find the solution of the nonlocal problem (3.8)–(3.10) on $[0, t^*]$ and it will be an expansion of $u(x,t)$ on the time interval $[t^*, 2t^*]$. Let's show now that obtained $u(x,t)$ is the solution of the problem (3.1)–(3.3), when $0 \le t \le 2t^*$. Obviously, it's sufficient to check twice continuously differentiability of $u(x,t)$ at the moment $t = t^*$. Since $u(x,t)$ is the solution of (3.1)–(3.3) on $[0, t^*]$, then it is twice continuously differentiable by x, when $t = t^*$, and

$$\lim_{t - t^* \to 0^-} u(x,t) = u(x,t^*) = v(x,0) = \lim_{t - t^* \to 0^+} u(x,t),$$

and consequently $u(x,t)$ is continuous in the point t^*. Analogously,

$$\begin{aligned}
\lim_{\delta \to 0^-}\frac{u(x,t^* + \delta) - u(x,t^*)}{\delta} &= u_t(x,t^*) \\
&= v_\tau(x,0) = \lim_{\delta \to 0^+}\frac{u(x,t^* + \delta) - u(x,t^*)}{\delta},
\end{aligned}$$

$$\lim_{t - t^* \to 0^-} u_t(x,t) = u_t(x,t^*) = v_\tau(x,0) = \lim_{t - t^* \to 0^+} u_t(x,t).$$

Therefore, $u_t(x,t)$ exists and is continuous for $t = t^*$.

In the same way we can check that $u_{tt}(x,t)$ is continuous for $t = t^*$. Taking into account the conditions (3.3), we can say that they are true since the operators A^i and B^j satisfy conditions of localization $(i = \overline{1,m}, j = \overline{1,n})$.

Consequently $u(x,t)$ is the solution of the nonlocal problem (3.1)–(3.3), when $0 \le t \le 2t^*$.

Applying the same method, we find $u(x,t)$ on the intervals $[0, nt^*]$ $(n = 2, 3, \dots)$ until the moment T. Therefore we can find $u(x,t)$ for the whole time interval $[0, T]$, i.e. the solution of the problem (3.1)–(3.3) exists, is unique, and expressed through the given functions and their integrals. $\square$

Remark. Let us consider particular case of nonlocal conditions (3.3):

$$
\begin{aligned}
u(0,t) &= \sum_{i=1}^{m} \alpha_i u(\xi_i, t) + f(t), \\
u(l,t) &= \sum_{j=1}^{p} \beta_j u(\eta_j, t) + g(t),
\end{aligned}
\qquad 0 \le t \le T, \tag{3.11}
$$

where α_i, β_j are prescribed numbers, ξ_i, η_j $(i = \overline{1,m}, j = \overline{1,p})$ are points of the string $(0, l)$. Then, the corresponding operators, which are in the right-hand parts of the nonlocal conditions (3.11), satisfy the conditions of localization and therefore according to the Theorem 3.1 nonlocal problem for the string oscillation equation with the initial conditions (3.2) and the nonlocal boundary conditions (3.11) has a unique solution, which can be found directly. The posed problem can be interpreted as the problem of controllability by the boundary conditions, where the boundary meanings of unknown function are required to differ from the linear combination of its meanings in certain points by a value given beforehand. This type of problems arises in building constructions and generators.

It should be mentioned that the nonlocal conditions (3.3) generally are not linear. Therefore, instead of (3.11) we can consider, for example, the following conditions

$$
\begin{aligned}
u(0,t) &= \sum_{i=1}^{m} \alpha_i u^{p_i}(\xi_i, t) + f(t), \\
u(l,t) &= \sum_{j=1}^{p} \beta_j u^{q_j}(\eta_j, t) + g(t),
\end{aligned}
\qquad 0 \le t \le T,
$$

where p_i, q_j $(i = \overline{1,m}, j = \overline{1,p})$ are non-negative integers. In this case the corresponding operators satisfy conditions of localization and we get nonlinear nonlocal initial boundary value problem which, according to the Theorem 3.1, has a unique classical solution.

Consider now the nonlocal problem for the equation (3.1) with the conditions (3.2) and the following integral nonlocal boundary conditions

$$\begin{aligned}&\alpha(t)u(0,t)+\beta(t)\frac{\partial u}{\partial x}(0,t)=\sum_{i=1}^{m}\int_{\xi_i^1(t)}^{\xi_i^2(t)}p_i(t,x)u(x,t)dx+f(t),\\ &\gamma(t)u(l,t)+\theta(t)\frac{\partial u}{\partial x}(l,t)=\sum_{j=1}^{p}\int_{\eta_j^1(t)}^{\eta_j^2(t)}q_j(t,x)u(x,t)dx+g(t),\end{aligned}\qquad 0\le t\le T, \tag{3.12}$$

where $\xi_i^1(t)\le\xi_i^2(t),\eta_j^1(t)\le\eta_j^2(t)$ $(i=\overline{1,m},j=\overline{1,p})$ are sliding points of the string $[0,l]$; α, β, γ, θ, p_i, q_j, f, g are prescribed sufficiently smooth functions.

Throughout the paper we shall use $C^{2,1}([0,T]\times[0,l])$ to denote the set of continuously differentiable functions twice continuously differentiable with respect to t.

The following statement is true.

Theorem 3.2. *Assume that the conditions* i), iii) *of the Theorem* 3.1 *are valid, functions* $\xi_i^1(t)$, $\xi_i^2(t)$, $\eta_j^1(t)$, $\eta_j^2(t)$ *are twice continuously differentiable,* $p_i,q_j\in C^{2,1}([0,T]\times[0,l])$ *and* $0\le\xi_i^1(t)\le\xi_i^2(t)\le l$, $0\le\eta_j^1(t)\le\eta_j^2(t)\le l$, *for* $t\in[0,T]$. *Then nonlocal problem* (3.1), (3.2), (3.12) *has a unique classical solution.*

Proof. Conducting the same reasoning as in the case of the Theorem 3.1, we get that the stated problem is equivalent to the one connected with determination of a pair $\{\mu_1,\mu_2\}$, satisfying the equations

$$\begin{aligned}&\alpha(t)\mu_1(t)+\beta(t)\Bigg(F_x(0,t)-\bar{\mu}_1'(t)-2\sum_{n=1}^{\infty}\bar{\mu}_1'(t-2nl)\\ &\qquad+2\sum_{n=0}^{\infty}\bar{\mu}_2'(t-(2n+1)l)\Bigg)\\ &=\sum_{i=1}^{m}\int_{\xi_i^1(t)}^{\xi_i^2(t)}p_i(t,x)u(x,t)dx+f(t),\\ &\gamma(t)\mu_2(t)+\theta(t)\Bigg(F_x(l,t)+\bar{\mu}_2'(t)+2\sum_{n=1}^{\infty}\bar{\mu}_2'(t-2nl)\\ &\qquad-2\sum_{n=0}^{\infty}\bar{\mu}_1'(t-(2n+1)l)\Bigg)\\ &=\sum_{j=1}^{p}\int_{\eta_j^1(t)}^{\eta_j^2(t)}q_j(t,x)u(x,t)dx+g(t),\end{aligned}\tag{3.13}$$

where $0\le t\le T$. In order to solve this integral nonlocal problem we use the same method which was applied in the case of discrete nonlocal problem, i.e.

we break down time interval $[0, T]$ into subintervals, solve the problem separately on each of them and finally unite obtained solutions into one, which is the solution of the problem (3.1), (3.2), (3.12).

Let's break $[0, T]$ into equal intervals with the length $t^* < l$. Consider the first interval $[0, t^*]$. On the basis of the definition of $\bar{\mu}_1(t)$ and $\bar{\mu}_2(t)$ equations (3.13), where $u(x, t)$ is substituted by its expression according to the formula (3.5), on the interval $[0, t^*]$ take the following form

$$
\begin{aligned}
\alpha(t)\mu_1(t) - \beta(t)\mu_1'(t) &= \sum_{i=1}^{m} \int_{\xi_i^1(t)}^{\xi_i^2(t)} p_i(t,x)\bar{\mu}_1(t-x)dx \\
&\quad + \sum_{i=1}^{m} \int_{\xi_i^1(t)}^{\xi_i^2(t)} p_i(t,x)\bar{\mu}_2(t-l+x)dx \\
&\quad - \beta(t)F_x(0,t) + \hat{f}(t), \\
\gamma(t)\mu_2(t) + \theta(t)\mu_2'(t) &= \sum_{j=1}^{p} \int_{\eta_j^1(t)}^{\eta_j^2(t)} q_j(t,x)\bar{\mu}_1(t-x)dx \\
&\quad + \sum_{j=1}^{p} \int_{\eta_j^1(t)}^{\eta_j^2(t)} q_j(t,x)\bar{\mu}_2(t-l+x)dx \\
&\quad - \theta(t)F_x(l,t) + \hat{g}(t),
\end{aligned}
\qquad 0 \le t \le t^*, \tag{3.14}
$$

where $\hat{f}$, $\hat{g}$ are expressed through given functions. Obviously, $\hat{f}, \hat{g} \in C^2([0, t^*])$. Changing variables in the integrals in the right-hand parts of (3.14), for $0 \le t \le t^*$ we get

$$
\begin{aligned}
\alpha(t)\mu_1(t) - \beta(t)\mu_1'(t) &= \sum_{i=1}^{m} \int_{t-\xi_i^2(t)}^{t-\xi_i^1(t)} p_i(t,t-\tau)\bar{\mu}_1(\tau)d\tau \\
&\quad + \sum_{i=1}^{m} \int_{t-l+\xi_i^1(t)}^{t-l+\xi_i^2(t)} p_i(t,\tau+l-t)\bar{\mu}_2(\tau)d\tau \\
&\quad - \beta(t)F_x(0,t) + \hat{f}(t),
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
\gamma(t)\mu_2(t) + \theta(t)\mu_2'(t) &= \sum_{j=1}^{p} \int_{t-\eta_j^2(t)}^{t-\eta_j^1(t)} q_j(t,t-\tau)\bar{\mu}_1(\tau)d\tau \\
&\quad + \sum_{j=1}^{p} \int_{t-l+\eta_j^1(t)}^{t-l+\eta_j^2(t)} q_j(t,\tau+l-t)\bar{\mu}_2(\tau)d\tau - \theta(t)F_x(l,t) \\
&\quad + \hat{g}(t).
\end{aligned} \tag{3.16}
$$

Let us introduce notations $\tilde{\xi}_i^1(t) = \chi(t - \xi_i^1(t))(t - \xi_i^1(t))$, $\tilde{\xi}_i^2(t) = \chi(t - \xi_i^2(t))(t - \xi_i^2(t))$, $\tilde{\xi}_i^{1^*}(t) = \chi(t - l + \xi_i^1(t))(t - l + \xi_i^1(t))$, $\tilde{\xi}_i^{2^*}(t) = \chi(t - l + \xi_i^2(t))(t - l + \xi_i^2(t))$ and in the same way $\tilde{\eta}_j^1(t)$, $\tilde{\eta}_j^2(t)$, $\tilde{\eta}_j^{1^*}(t)$, $\tilde{\eta}_j^{2^*}(t)$, where

$$\chi(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

is Heaviside's function. We denote by $h_1(t)$ and $h_2(t)$ additional functions, except μ_1, μ_2, taking part in the definition of $\bar{\mu}_1$ and $\bar{\mu}_2$. Taking these notations into account, we rewrite (3.15), (3.16) as follows:

$$\begin{aligned}(3.17)\qquad \alpha(t)\mu_1(t) - \beta(t)\mu_1'(t) = {} & \sum_{i=1}^{m} \int_{\tilde{\xi}_i^2(t)}^{\tilde{\xi}_i^1(t)} p_i(t, t - \tau)(\mu_1(\tau) + h_1(\tau))d\tau \\ & + \sum_{i=1}^{m} \int_{\tilde{\xi}_i^{1^*}(t)}^{\tilde{\xi}_i^{2^*}(t)} p_i(t, \tau + l - t)(\mu_2(\tau) + h_2(\tau))d\tau \\ & - \beta(t)F_x(0, t) + \hat{f}(t),\end{aligned}$$

$$\begin{aligned}(3.18)\qquad \gamma(t)\mu_2(t) + \theta(t)\mu_2'(t) = {} & \sum_{j=1}^{p} \int_{\tilde{\eta}_j^2(t)}^{\tilde{\eta}_j^1(t)} q_j(t, t - \tau)(\mu_1(\tau) + h_1(\tau))d\tau \\ & + \sum_{j=1}^{p} \int_{\tilde{\eta}_j^{1^*}(t)}^{\tilde{\eta}_j^{2^*}(t)} q_j(t, \tau + l - t)(\mu_2(\tau) + h_2(\tau))d\tau \\ & - \theta(t)F_x(l, t) + \hat{g}(t),\end{aligned}$$

where $0 \leq t \leq t^*$. Therefore, for the functions $\mu_1(t)$ and $\mu_2(t)$ we get the system of integro-differential equations.

According to the condition iii) of the theorem, $\beta(t)$ is either equal or unequal to zero everywhere. Due to this reason in the first case instead of (3.17) we have a special type integral equation, and in the second one, taking an integral from 0 to t, for each $0 \leq t \leq t^*$, we obtain an integral equation too.

Similarly we conclude that the equation (3.18) can be reduced to a special type integral equation. We consider only the case where the functions $\beta(t)$, $\theta(t)$ are equal to zero everywhere since all the rest cases can be considered without any significant changes. In this case $\alpha(t), \gamma(t) \neq 0$, for $0 \leq t \leq t^*$. Dividing the above expressions by $\alpha(t)$, $\gamma(t)$ respectively, for $0 \leq t \leq t^*$ we get

$$\mu_1(t) = \sum_{i=1}^{m} \int_{\tilde{\xi}_i^2(t)}^{\tilde{\xi}_i^1(t)} p_i^*(t,\tau)(\mu_1(\tau) + h_1(\tau))d\tau + \sum_{i=1}^{m} \int_{\tilde{\xi}_i^{1^*}(t)}^{\tilde{\xi}_i^{2^*}(t)} p_i^*(t, 2t - l - \tau)(\mu_2(\tau) + h_2(\tau))d\tau + \frac{\hat{f}(t)}{\alpha(t)}, \tag{3.19}$$

$$\mu_2(t) = \sum_{j=1}^{p} \int_{\tilde{\eta}_j^2(t)}^{\tilde{\eta}_j^1(t)} q_j^*(t,\tau)(\mu_1(\tau) + h_1(\tau)d\tau + \sum_{j=1}^{p} \int_{\tilde{\eta}_j^{1^*}(t)}^{\tilde{\eta}_j^{2^*}(t)} q_j^*(t, 2t - l - \tau)(\mu_2(\tau) + h_2(\tau)d\tau + \frac{\hat{g}(t)}{\gamma(t)}, \tag{3.20}$$

where $p_i^*(t,\tau) = \dfrac{p_i(t, t-\tau)}{\alpha(t)}$, $q_j^*(t,\tau) = \dfrac{q_j(t, t-\tau)}{\gamma(t)}$. Note that all the functions in the system (3.19), (3.20) are continuous. Now we prove that the operator $K : C([0,t^*]) \times C([0,t^*]) \to C([0,t^*]) \times C([0,t^*])$ is compact, $K = (K_1, K_2)$,

$$\begin{aligned} K_1\begin{pmatrix} v \\ w \end{pmatrix} &= \sum_{i=1}^{m} \int_{\tilde{\xi}_i^2(t)}^{\tilde{\xi}_i^1(t)} p_i^*(t,\tau)v(\tau)d\tau + \sum_{i=1}^{m} \int_{\tilde{\xi}_i^{1^*}(t)}^{\tilde{\xi}_i^{2^*}(t)} p_i^*(t, 2t-l-\tau)w(\tau)d\tau, \\ K_2\begin{pmatrix} v \\ w \end{pmatrix} &= \sum_{j=1}^{p} \int_{\tilde{\eta}_j^2(t)}^{\tilde{\eta}_j^1(t)} q_j^*(t,\tau)v(\tau)d\tau + \sum_{j=1}^{p} \int_{\tilde{\eta}_j^{1^*}(t)}^{\tilde{\eta}_j^{2^*}(t)} q_j^*(t, 2t-l-\tau)w(\tau)d\tau. \end{aligned} \tag{3.21}$$

Indeed, let $A \subset C([0,t^*]) \times C([0,t^*])$ be a bounded set, i.e.

$$\forall \begin{pmatrix} v \\ w \end{pmatrix} \in A, \qquad \left\| \begin{matrix} v \\ w \end{matrix} \right\|_{C([0,t^*]) \times C([0,t^*])} = \|v\| + \|w\| \le c, \tag{3.22}$$

where by $\|\cdot\|$ we denote a norm in $C([0,t^*])$.

To show, that K is a compact operator, it is sufficient to prove, that both components $K_1, K_2 : C([0,t^*]) \times C([0,t^*]) \to C([0,t^*])$ of the operator K are compact. By the Ascoli-Arzela Theorem, the operator K_1 is compact whenever $K_1(A)$ is uniformly bounded and equicontinuous. From (3.21) we have

$$\left\| K_1\begin{pmatrix} v \\ w \end{pmatrix} \right\| \le C_1 Tm(\|v\| + \|w\|), \qquad C_1 = \max_{\substack{[0,t^*]\times[0,l] \\ 1 \le i \le m}} \{|p_i(t,x)/\alpha(t)|\}.$$

Therefore, as A is bounded, $K_1(A)$ and analogously $K_2(A)$ are uniformly bounded. Furthermore, since all the functions ξ_i^1, ξ_i^2 are twice continuously differentiable, it is easy to see that $\tilde{\xi}_i^1$, $\tilde{\xi}_i^2$, $\tilde{\xi}_i^{1^*}$, $\tilde{\xi}_i^{2^*}$ are Lipschitz continuous on $[0,t^*]$ and consequently

$$(3.23)\qquad \left|K_1\begin{pmatrix} v \\ w \end{pmatrix}(t_2) - K_1\begin{pmatrix} v \\ w \end{pmatrix}(t_1)\right|$$
$$\leq \sum_{i=1}^{m}[C_1\|v\|(|\tilde{\xi}_i^1(t_2) - \tilde{\xi}_i^1(t_1)| + |\tilde{\xi}_i^2(t_2) - \tilde{\xi}_i^2(t_1)|)$$
$$+ C_1\|w\|(|\tilde{\xi}_i^{1^*}(t_2) - \tilde{\xi}_i^{1^*}(t_1)| + |\tilde{\xi}_i^{2^*}(t_2) - \tilde{\xi}_i^{2^*}(t_1)|)]$$
$$\leq \sum_{i=1}^{m}[C_1\|v\|(\tilde{C}_i^1 + \tilde{C}_i^2) + C_1\|w\|(\tilde{C}_i^{1^*} + \tilde{C}_i^{2^*})]|t_2 - t_1|,$$

where $\tilde{C}_i^1$, $\tilde{C}_i^2$, $\tilde{C}_i^{1^*}$, $\tilde{C}_i^{2^*}$ are Lipschitz constants corresponding to the functions $\tilde{\xi}_i^1$, $\tilde{\xi}_i^2$, $\tilde{\xi}_i^{1^*}$, $\tilde{\xi}_i^{2^*}$. (3.22) and (3.23) imply that $K_1(A)$ is equicontinuous set of functions and thus K_1 is a compact operator. By the similar discussion we can deduce that the operator K_2 is compact too.

So, the operator K is compact and therefore the alternative of Fredholm is true for the system (3.19), (3.20), i.e. the system has a unique solution only if the homogeneous system has only the trivial one. Though, using the method of mathematical induction we can prove the validity of the following inequalities for any $n \in \mathbf{N}$:

$$(3.24)\quad \max\{|\mu_1(t)|, |\mu_2(t)|\} \leq 2^{n-1}\bar{c}^n\frac{t^n}{n!}\|\mu_1\| + 2^{n-1}\bar{c}^n\frac{t^n}{n!}\|\mu_2\|, \qquad 0 \leq t \leq t^*,$$

where $\bar{c} = \max\{mC_1, pC_2\}$, $C_2 = \max\limits_{[0,t^*]\times[0,l];\,1\leq j\leq p}\{|q_j(t,x)/\gamma(t)|\}$. From the (3.24), tending $n \to \infty$, we get that the homogeneous system has only the trivial solution. Therefore, system (3.19), (3.20) has a unique solution.

To show that the pair $\{\mu_1, \mu_2\}$ is a solution of the nonlocal problem, it is sufficient to prove that $\mu_1, \mu_2 \in C^2([0,t^*])$. According to the continuity of $\mu_1(t)$ and $\mu_2(t)$ it is obvious that $\bar{\mu}_1(t)$ and $\bar{\mu}_2(t)$ are also continuous. Then from (3.14) where $\beta(t) \equiv \theta(t) \equiv 0$ we get that the right-hand parts of the equations are continuously differentiable, since $\xi_1(t), \xi_2(t), \eta_1(t), \eta_2(t) \in C^2([0,t^*])$, and consequently $\mu_1, \mu_2 \in C^1([0,t^*])$. It is not difficult to check that $\bar{\mu}_1, \bar{\mu}_2 \in C^1([0,t^*])$. Repeating the above reasoning, we similarly obtain that $\mu_1, \mu_2 \in C^2([0,t^*])$. Substituting this pair into the formula (3.5), we get the solution $u(x,t)$ of the problem (3.1), (3.2), (3.12) for $t \in [0,t^*]$.

Taking for the initial moment of time t^*, i.e. changing time variable t by $\tau = t - t^*$, for the function $v(x,\tau) = u(x,\tau + t^*)$ we get the following nonlocal problem

$$\frac{\partial^2 v}{\partial \tau^2} = \frac{\partial^2 v}{\partial x^2}, \qquad 0 < x < l,\ \ 0 < \tau < T - t^*,$$

$$\left.\begin{aligned}
&\alpha(\tau+t^*)v(0,\tau)+\beta(\tau+t^*)\frac{\partial v}{\partial x}(0,\tau)\\
&\qquad=\sum_{i=1}^{m}\int_{\xi_i^1(\tau+t^*)}^{\xi_i^2(\tau+t^*)}p_i(\tau+t^*,x)v(x,\tau)dx+f(\tau+t^*),\\
&\gamma(\tau+t^*)v(l,\tau)+\theta(\tau+t^*)\frac{\partial v}{\partial x}(l,\tau)\\
&\qquad=\sum_{j=1}^{p}\int_{\eta_j^1(\tau+t^*)}^{\eta_j^2(\tau+t^*)}q_j(\tau+t^*,x)v(x,\tau)dx+g(\tau+t^*),
\end{aligned}\right\}\quad 0\le\tau\le T-t^*,$$

with the initial conditions

$$\left.\begin{aligned}
v(x,0)&=u(x,t^*),\\
v_\tau(x,0)&=u_t(x,t^*),
\end{aligned}\right\}\quad 0\le x\le l,$$

where the function $u(x,t)$ is already defined on the interval $[0,t^*]$. Analogous to the above we can find the function $v(x,\tau)$ $(0\le\tau\le t^*)$, which is the expansion of $u(x,t)$ on the set $[0,l]\times[t^*,2t^*]$. Repeating the reasoning conducted in the proof of the Theorem 3.1 it is easy to check that the function $u(x,t)$ obtained by such a way is the solution of the formulated problem on $[0,2t^*]$. Analogously the function $u(x,t)$ can be determined for $t\in[0,nt^*]$, $n\in\mathbf{N}$ until the moment of time T and consequently the nonlocal problem (3.1), (3.2), (3.12) has a unique solution. □

4. Nonlocal problems for the telegraph equation

As in the case of string oscillation equation in this section we consider nonlocal problems with discrete and integral nonlocal conditions for the telegraph equation. However, in contrast to the case of string oscillation, here the main method of solution constructing is the application of a special type potential, which allows to reduce posed nonlocal problems to integral equations. Here we also use corresponding notations of the § 3.

Let us consider the nonlocal problem for the telegraph equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+c^2u,\qquad 0<x<l,\ 0<t<T,\tag{4.1}$$

with the initial conditions

$$u(x,0)=\varphi(x),\qquad u_t(x,0)=\psi(x),\qquad 0\le x\le l,\tag{4.2}$$

and the nonlocal boundary conditions

$$
\begin{aligned}
&\alpha(t)u(0,t)+\beta(t)\frac{\partial u}{\partial x}(0,t)=\sum_{i=1}^{m}[A^i u(\xi_i(t),t)](t)+f(t),\\
&\gamma(t)u(l,t)+\theta(t)\frac{\partial u}{\partial x}(l,t)=\sum_{j=1}^{p}[B^j u(\eta_j(t),t)](t)+g(t),
\end{aligned}
\qquad 0\le t\le T, \tag{4.3}
$$

where $0\neq c=const$ is a real or an imaginary number, and $u(x,t)$ is an unknown function, twice continuously differentiable on $[0,l]\times[0,T]$, satisfying the equation (4.1) and conditions (4.2), (4.3).

The following theorem is true.

Theorem 4.1. *If all the conditions of the Theorem* 3.1 *are valid, then the nonlocal problem* (4.1)–(4.3) *has a unique solution.*

Proof. Note that if we find the solution $u(x,t)$ of the problem (4.1)–(4.3), then it takes certain meanings on the boundary and, consequently, it is the solution of the telegraph equation with classical Dirichlet conditions on the boundary. In this case we can show that

$$
\begin{aligned}
u(x,t)&=G(x,t)+\frac{\partial}{\partial x}\left[\int_0^{t-x}v_1(\tau)I(c^2((t-\tau)^2-x^2))d\tau\right.\\
&\qquad\left.+\int_0^{t-l+x}v_2(\tau)I(c^2((t-\tau)^2-(l-x)^2))d\tau\right],\\
G(x,t)&=\frac{\partial}{\partial t}\left[\frac{1}{2}\int_{x-t}^{x+t}I(c^2(t^2-(\alpha-x)^2))\tilde{\varphi}(\alpha)d\alpha\right]\\
&\qquad+\frac{1}{2}\int_{x-t}^{x+t}I(c^2(t^2-(\alpha-x)^2))\tilde{\psi}(\alpha)d\alpha,
\end{aligned}
\tag{4.4}
$$

where $I(z)=\sum_{s=0}^{\infty}\frac{1}{(s!)^2}\left(\frac{z}{4}\right)^s$, $\tilde{\varphi}$, $\tilde{\psi}$ are continuations of the functions φ, ψ on the whole axis retaining smoothness, $v_1,v_2\in C^2([0,T])$, $v_1(\tau)=v_2(\tau)=0$, for $\tau\le 0$. Therefore solution of the problem (4.1)–(4.3) is uniquely defined by the functions v_1,v_2 and due to this fact resolution of the posed problem is reduced to determination of functions $v_1(t)$ and $v_2(t)$. For definiteness we consider the case when $\beta(t)\neq 0$, $\theta(t)\neq 0$ for $0\le t\le T$. All other cases can be treated similarly.

Substituting formula (4.4) into the first condition (4.3), we get an equation for $v_1(t)$ and $v_2(t)$

$$
\begin{aligned}
&\alpha(t)u(0,t)+\beta(t)\frac{\partial u}{\partial x}(0,t)\\
&\quad=\sum_{i=1}^{m}A^{i}\Bigg[G(\xi_i(t),t)-v_1(t-\xi_i(t))\\
&\qquad-2\int_0^{t-\xi_i(t)}v_1(\tau)c^2\xi_i(t)I'(c^2((t-\tau)^2-\xi_i^2(t)))d\tau+v_2(t-l+\xi_i(t))\\
&\qquad+2\int_0^{t-l+\xi_i(t)}v_2(\tau)c^2(l-\xi_i(t))I'(c^2(t-\tau)^2-(l-\xi_i(t)^2))d\tau\Bigg](t)+f(t).
\end{aligned}
$$

We may obtain the similar equation from the second boundary condition (4.3). As in the proof of the Theorem 3.1 we consider the same time interval $[0,t^*]$ for which we have

$$
\begin{aligned}
&-\alpha(t)v_1(t)+\beta(t)v_1'(t)-\beta(t)\int_0^t v_1(\tau)I_1(t-\tau)d\tau\\
&\qquad=\sum_{i=1}^{m}[A^i_{0,t^*}(G(\xi_i(t),t))](t)-\alpha(t)G(0,t)-\beta(t)G_x(0,t)+f(t),\\
&\gamma(t)v_2(t)+\theta(t)v_2'(t)-\theta(t)\int_0^t v_2(\tau)I_1(t-\tau)d\tau\\
&\qquad=\sum_{j=1}^{p}[B^j_{0,t^*}(G(\eta_j(t),t))](t)-\gamma(t)G(l,t)-\theta(t)G_x(l,t)+g(t),
\end{aligned}
\tag{4.5}
$$

where $I_1(t-\tau)=2c^2I'(c^2(t-\tau)^2)$, and denoting the right-hand parts of the equations (4.5) by $f^*(t)$ and $g^*(t)$ respectively, we obtain

$$
\begin{aligned}
v_1(t)&=\int_0^t\int_0^s v_1(\tau)I_1(s-\tau)d\tau ds+\int_0^t\frac{\alpha(\tau)}{\beta(\tau)}v_1(\tau)d\tau+\int_0^t\frac{f^*(\tau)}{\beta(\tau)}\,d\tau,\\
v_2(t)&=\int_0^t\int_0^s v_2(\tau)I_1(s-\tau)d\tau ds-\int_0^t\frac{\gamma(\tau)}{\theta(\tau)}v_2(\tau)d\tau+\int_0^t\frac{g^*(\tau)}{\theta(\tau)}\,d\tau,
\end{aligned}
\tag{4.6}
$$

for $0\le t\le t^*$.

Both equations (4.6) are integral equations of the same type. Due to this fact we consider the first for $v_1(t)$. Let's introduce the following operator

$$
Kv(t)=\int_0^t\int_0^s v(\tau)I_1(s-\tau)d\tau ds+\int_0^t\frac{\alpha(\tau)}{\beta(\tau)}v(\tau)d\tau,\qquad \forall v\in C([0,t^*]).
$$

Then the first equation (4.6) takes the following form

$$
v_1=Kv_1+f^{**}, \tag{4.7}
$$

where $f^{**}(t) = \int_0^t \frac{f^*(\tau)}{\beta(\tau)}\, d\tau$ and obviously $f^{**} \in C^2([0, t^*])$. Let's prove now that K is a compact operator from $C([0, t^*])$ to $C([0, t^*])$. Let us consider the bounded set $A \subset C([0, t^*])$ and prove that the closure of $K(A)$ is compact in $C([0, t^*])$. By the Ascoli-Arzela Theorem, we have to check uniform boundedness and equicontinuity of $K(A)$. Indeed, if we denote a norm in $C([0, t^*])$ by $\|\cdot\|$, we get

$$\|Kv\| \le T^2 C_1 \|v\| + TC_2 \|v\|, \qquad \forall v \in A,$$

where $C_1 = \max_{[0, t^*]} |I_1(t)|$, $C_2 = \max_{[0, t^*]} |\alpha(t)/\beta(t)|$, and since $\|v\|$ is bounded, $K(A)$ is uniformly bounded. Also

$$\begin{aligned} |Kv(t_1) - Kv(t_2)| &= \left| \int_{t_1}^{t_2} \int_0^s v(\tau) I_1(s-\tau)\, d\tau ds + \int_{t_1}^{t_2} \frac{\alpha(\tau)}{\beta(\tau)} v(\tau)\, d\tau \right| \\ &\le |t_2 - t_1| (TC_1 \|v\| + C_2 \|v\|) \end{aligned}$$

and, consequently, $K(A)$ is equicontinuous. This implies that K is compact. Taking the latter into account for equation (4.7) the Fredholm theorems are true. Hence, if we prove, that the homogeneous equation has only a trivial solution, the equation (4.7) has a unique solution. Let's consider the homogeneous equation

$$v = Kv$$

or

$$v(t) = \int_0^t \int_0^s v(\tau) I_1(s-\tau)\, d\tau ds + \int_0^t \frac{\alpha(\tau)}{\beta(\tau)} v(\tau)\, d\tau.$$

Let's show that $v(t)$ satisfies the following estimate

$$|v(t)| \le \|v\| \sum_{k=0}^{n} C_n^k C_1^k C_2^{n-k} \frac{t^{n+k}}{(n+k)!}. \tag{4.8}$$

Indeed, (4.8) is obvious for $n = 1$. Assume that the above estimate is true for n and let it show for $n+1$,

$$\begin{aligned} |v(t)| &\le C_1 \int_0^t \int_0^s |v(\tau)|\, d\tau ds + C_2 \int_0^t |v(\tau)|\, d\tau \\ &= \|v\| \sum_{k=0}^{n+1} C_{n+1}^k C_1^k C_2^{n+1-k} \frac{t^{n+1+k}}{(n+1+k)!}. \end{aligned}$$

Therefore the inequality (4.8) is true for any $n \in \mathbf{N}$. (4.8) implies, that

$$|v(t)| \le \|v\|(C_1 + C_2)^n \sum_{k=0}^{n} \frac{t^{n+k}}{(n+k)!} \le \|v\|(C_1 + C_2)^n \frac{e^{t^*}}{n!} \to 0$$

as $n \to \infty$ for $t \in [0, t^*]$ and therefore $v(t) \equiv 0$ on $[0, t^*]$.

So, the first equation (4.6) has a unique continuous solution. However, as in the proof of the Theorem 3.2 the form of the equation provides twice continuously differentiability of v_1 on $[0, t^*]$. Similarly we can check existence and uniqueness of the function $v_2(t)$. Thus, we conclude that the pair $\{v_1, v_2\}$ is uniquely defined and, consequently, the stated problem (4.1)–(4.3) has a unique solution on $[0, t^*]$.

Now, consider the time interval $[t^*, 2t^*]$. Here we also obtain equations similar to (4.6), where f^* and g^* are changed by combinations of the functions $v_1(t)$ and $v_2(t)$, already defined on $[0, t^*]$, since $t^* \le \xi_i(t)$, $\eta_j(t) \le l - t^*$ and consequently for $t^* \le t \le 2t^*$,

$$2t^* - l \le t - \xi_i(t) \le t^*, \qquad 2t^* - l \le t - l + \eta_j(t) \le t^*,$$
$$2t^* - l \le t - l + \xi_i(t) \le t^*, \qquad 2t^* - l \le t - \eta_j(t) \le t^*.$$

Repeating above reasoning for these equations we determine $v_1(t)$ and $v_2(t)$ for $t \in [t^*, 2t^*]$. It is not difficult to show that the obtained functions $v_1(t)$ and $v_2(t)$ are twice continuously differentiable on $[0, 2t^*]$. Therefore $u(x, t)$ solution of the posed problem is uniquely found on $[0, 2t^*]$. Similarly, we can define $u(x, t)$ on $[0, nt^*]$, $n \in \mathbf{N}$ until the moment T. $\square$

REMARK. It should be pointed out that the operators A^i, B^j in nonlocal boundary conditions in general are neither linear nor continuous. However, if we do not consider the operators as continuous, nonlocal problem might be incorrect. In the case of continuity of the operators it is easy to show that solution $u(x, t)$ continuously depends on initial data. Consequently, in any case, a solution of the nonlocal problem (4.1)–(4.3) exists and is unique, but depending on continuity of the operators the posed problem will either be correct or incorrect. Similar reasoning is true for the problem (3.1)–(3.3).

Let us consider now integral nonlocal problem for the telegraph equation (4.1), with the initial conditions (4.2) and the nonlocal boundary conditions

$$\begin{aligned} \alpha(t)u(0,t) + \beta(t)\frac{\partial u}{\partial x}(0,t) &= \sum_{i=1}^{m} \int_{\xi_i^1(t)}^{\xi_i^2(t)} p_i(t,x)u(x,t)dx + f(t), \\ \gamma(t)u(l,t) + \theta(t)\frac{\partial u}{\partial x}(l,t) &= \sum_{j=1}^{p} \int_{\eta_j^1(t)}^{\eta_j^2(t)} q_j(t,x)u(x,t)dx + g(t), \end{aligned} \qquad 0 \le t \le T, \tag{4.9}$$

where the given functions satisfy all conditions required in the Theorem 3.2. Under these conditions the following theorem is correct.

Theorem 4.2. *The nonlocal problem* (4.1), (4.2), (4.9) *has a unique solution* $u(x,t)$, *which is twice continuously differentiable on* $[0,l]\times[0,T]$, *satisfies the equation* (4.1) *and conditions* (4.2), (4.9).

Proof. As for the proceeding theorem, even in this case solution of the problem (4.1), (4.2), (4.9) is equivalent to determination of a pair of twice continuously differentiable functions $\{v_1, v_2\}$. In order to simplify the following discussions we assume, that $\beta(t)=\theta(t)=0$ for $t\in[0,T]$. Then, using the properties of v_1, v_2, from (4.9) we obtain the following equations

$$\begin{aligned}
&-\alpha(t)v_1(t)\\
&\quad=\sum_{i=1}^{m}\int_{\xi_i^1(t)}^{\xi_i^2(t)}p_i(t,x)\biggl(G(x,t)-v_1(t-x)-\int_0^{t-x}v_1(\tau)I_2(t,\tau,x)d\tau\\
&\qquad+v_2(t-l+x)+\int_0^{t-l+x}v_2(\tau)I_2(t,\tau,l-x)d\tau\biggr)dx\\
&\qquad-\alpha(t)G(0,t)+f(t),\\
&\gamma(t)v_2(t)\\
&\quad=\sum_{j=1}^{p}\int_{\eta_j^1(t)}^{\eta_j^2(t)}g_j(t,x)\biggl(G(x,t)-v_1(t-x)-\int_0^{t-x}v_1(\tau)I_2(t,\tau,x)d\tau\\
&\qquad+v_2(t-l+x)+\int_0^{t-l+x}v_2(\tau)I_2(t,\tau,l-x)d\tau\biggr)dx\\
&\qquad-\gamma(t)G(l,t)+g(t),
\end{aligned}\tag{4.10}$$

for $0\le t\le t^*$, where t^* is an arbitrary positive real number $t^*<l$, $I_2(t,\tau,x)=2c^2xI'(c^2((t-\tau)^2-x^2))$ and consequently we get a system of integral equations for which we can show that there exists a unique twice continuously differentiable solution and therefore, $v_1(t)$ and $v_2(t)$ are defined on $[0,t^*]$.

As in the proof conducted for the previous theorem, considering time interval $[t^*,2t^*]$ we get equations similar to those of (4.10) from which we define the functions v_1, v_2 on $[t^*,2t^*]$, twice continuously differentiable continuations of $v_1(t)$, $v_2(t)$. Consequently, the pair $\{v_1,v_2\}$ is found on $[0,2t^*]$. Similarly, v_1 and v_2 can be defined for all $t\in[0,T]$. Therefore, the integral nonlocal problem (4.1), (4.2), (4.9) has a unique solution. □

References

[1] E. Hilb, Zur Theorie der Entwicklungen willkürlicher Funktionen nach Eigenfunktionen, Math. Z. **58** (1918), 1–9.

[2] A. V. Bitsadze – A. A. Samarskii, On some simplest generalizations of linear elliptic problems, Dokl. Akad. Nauk SSSR **185** (1969), 739–740.

[3] D. G. Gordeziani, On a method of resolution of Bitsadze-Samarskii boundary value problem, Abstracts of reports of Inst. Appl. Math. Tbilisi State Univ. **2** (1970), 38–40.

[4] D. G. Gordeziani – T. Z. Djioev, On solvability of a boundary value problem for a nonlinear elliptic equation, Bull. Acad. Sci. Georgian SSR **68** (1972), 289–292.

[5] D. G. Gordeziani, On methods of resolution of a class of nonlocal boundary value problems, Tbilisi University Press, Tbilisi, 1981.

[6] D. Gordeziani – N. Gordeziani – G. Avalishvili, Nonlocal boundary value problems for some partial differential equations, Bull. Georgian Acad. Sci. **157** (1998), 3, 365–368.

[7] M. P. Sapagovas – R. I. Chegis, On some boundary value problems with nonlocal conditions, Diff. Equat. **23** (1987), 1268–1274.

[8] V. A. Il'in – E. I. Moiseev, Two-dimensional nonlocal boundary value problems for Poisson's operator in differential and difference variants, Mat. Mod. **2** (1990), 139–159.

[9] D. V. Kapanadze, On the Bitsadze-Samarskii nonlocal boundary value problem, Diff. Equat. **23** (1987), 543–545.

[10] Z. G. Mansourati – L. L. Campbell, Non-classical diffusion equations related to birth-death processes with two boundaries, Quart. Appl. Math. **54** (1996), 423–443.

[11] E. Obolashvili, Nonlocal problems for some partial differential equations, Appl. Anal. **45** (1992), 269–280.

[12] C. V. Pao, Reaction diffusion equations with nonlocal boundary and nonlocal initial conditions, J. Math. Anal. Appl. **195** (1995), 702–718.

[13] B. P. Paneyakh, On some nonlocal boundary value problems for linear differential operators, Mat. Zam. **35** (1984), 425–433.

[14] A. Fridman, Monotonic decay of solutions of parabolic equations with nonlocal boundary conditions, Quart. Appl. Math. **44** (1986), 401–407.

[15] A. L. Skubachevskii, Nonlocal elliptic problems and multidimensional diffusion processes, J. Math. Phys. **3** (1995), 327–360.

[16] A. Bouziani, Strong solution for a mixed problem with nonlocal condition for certain pluriparabolic equations, Hiroshima Math. J. **27** (1997), 373–390.

[17] F. Frankl, The Cauchy problem for linear and nonlinear second order hyperbolic partial differential equations, Math. Sb. **2** (1937), 5, 793–814.

I. Vekua Institute of Applied Mathematics,
Tbilisi State University,
Tbilisi 380043, *GEORGIA*
e-mail: gord@viam.hepi.edu.ge

Faculty of Mechanics and Mathematics,
Tbilisi State University,
Tbilisi 380043, *GEORGIA*
e-mail: gavalish@viam.hepi.edu.ge