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Classification of tilings of the 2-dimensional sphere by congruent triangles

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ABSTRACT. We give a new classification of tilings of the 2-dimensional sphere by congruent triangles accompanied with a complete proof. This accomplishes the old classification by Davies, who only gave an outline of the proof, regrettably with some redundant tilings. We clarify Davies' obscure points, give a complete list, and show that there exist ten sporadic and also ten series of such tilings, including some unfamiliar twisted ones. We also give their figures, development maps in a way easy to understand their mutual relations. In Appendix, we give curious examples of tilings on non-compact spaces of constant positive curvature with boundary possessing a special 5-valent vertex that never appear in the tiling of the usual sphere.

0. Introduction

In this paper, we give a complete classification of tilings of the 2-dimensional sphere consisting of one congruent triangle. We consider this problem as a purely combinatorial problem, not assuming a transitive group action on the set of tiles.

Concerning this problem, Sommerville [9] gave a partial classification, particularly he classified tilings by isosceles triangles. But for the scalene case, he only treated a restricted case, i.e., under the condition of "regularity", meaning that the corners at each vertex have the same angle.

Later, Davies [4] gave a classification without the assumption of "regularity". But Davies only gave a rough outline of the proof of the classification in [4], and detailed examinations are left to the readers. It seems to the authors that to fill this blank space and reconstruct the complete proof is by no means an easy (rather a quite hard) problem. In addition, Davies' final classification contains some duplicates (though his list covers all tilings without any lack). For example, the tiling F_{48} in our notation (see Table in page 465) appears twice in his list, and the last example in [4; p. 50] is redundant because TI_{24} is identical to I_{24}. Also other duplicates exist such as $MTG_{12}^{II} = TG_{12}$ $(\alpha = 2/3, \beta = \gamma = 1/3)$, etc. (For details, see the end of §2 of

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this paper. Precise definition of these tilings will be given in §2.) Moreover, the numbers of faces and angles of triangles of some tilings are not explicitly stated in [4]. And we regret to say that there remain some obscure points in Davies' classification. Hence, it is desirable to give a complete classification, not containing redundant tilings, and also to give a complete proof. To settle these unsatisfactory points and to clarify the whole situation is the main purpose of the present paper.

After the complete classification (Theorem 1, Table), we know that there exist ten sporadic and also ten series of tilings by congruent triangles. There appear some unfamiliar twisted tilings, and we give their figures, development maps and state mutual relations of these tilings in detail. Note that among these tilings three of them are continuously deformable, and the numbers of faces are multiples of 4, except one series of tilings. We also give a characterization of triangles that can tile the whole sphere monohedrally (Corollary 3), and give a list of tilings with a given number of faces (Corollary 2).

In our previous paper [10], we gave a classification of tilings of the 2-dimensional sphere consisting of congruent "right" triangles. The principle of classification in this paper is almost same as that of [10]. But for the sake of completeness, we give here a complete proof, not depending on the results of [10]. We remark that Azevedo Breda's classification of a special type of monohedral tilings [1] can be also verified directly from our classification. (Actually, tilings containing only even valent vertices are necessarily monohedral f-tilings in the sense of [1], as a result of our classification. But, it seems that the classification in [1] unfortunately lacks the tiling I_{16n} $(n \geq 3)$ in our notation.)

Now we explain the contents of this paper. In §1, after some preliminaries on notations and terminologies, we state our main result (Theorem 1, Table). In addition, we state two corollaries obtained immediately from this classification. In §2, we give a detailed explanation of tilings in Table, give their figures, development maps and state their mutual relations. We remark that five regular polyhedrons are all related to each other through some tilings in Table (Figure 21). In addition, we summarize special isomorphisms between some tilings with small number of faces. (It seems to the authors that the lack of such consideration is the principal defect in Davies' "classification".)

The rest sections are devoted to the proof of Theorem 1. After treating a preliminary case (equilateral triangles and the case where the number of faces takes the smallest value four) in §3, we give a classification by isosceles triangles in §4. This result was already proved by Sommerville [9]. But we give here a complete proof because Sommerville [9; p. 90] stated only a brief outline of the proof. The remaining scalene case is the most complicated. We carry out the classification through §5∼§8. In §5, we first classify the type of

Table

	V	E	$\alpha,\ \beta,\ \gamma$	*type of vertices* [*number*]
$\bullet F_4$	4	6	$\alpha+\beta+\gamma=2$, $\frac{1}{2}<\alpha,\beta,\gamma<1$	$\alpha+\beta+\gamma$ [4]
F_{12}^{I}	8	18	$\alpha=\frac{2}{3},\ \beta=\gamma=\frac{1}{3}$	3α [4], 6β [4]
F_{12}^{II}	8	18	$\alpha=\frac{2}{3},\ \beta=\gamma=\frac{1}{3}$	$\begin{cases}3\alpha\ [2],\ 2\alpha+2\beta\ [2]\\ \alpha+4\beta\ [2],\ 6\beta\ [2]\end{cases}$
F_{12}^{III}	8	18	$\alpha=\frac{2}{3},\ \beta=\gamma=\frac{1}{3}$	$\begin{cases}3\alpha\ [1],\ 2\alpha+2\beta\ [3]\\ \alpha+4\beta\ [3],\ 6\beta\ [1]\end{cases}$
F_{24}	14	36	$\alpha=\frac{2}{3},\ \beta=\gamma=\frac{1}{4}$	3α [8], 8β [6]
F_{48}	26	72	$\alpha=\frac{1}{2},\ \beta=\frac{1}{3},\ \gamma=\frac{1}{4}$	$\begin{cases}4\alpha\ [12],\ 6\beta\ [8]\\ 8\gamma\ [6]\end{cases}$
TF_{48}	26	72	$\alpha=\frac{1}{2},\ \beta=\frac{1}{3},\ \gamma=\frac{1}{4}$	$\begin{cases}4\alpha\ [8],\ 6\beta\ [8]\\ 8\gamma\ [2],\ 2\alpha+4\gamma\ [8]\end{cases}$
F_{60}^{I}	32	90	$\alpha=\frac{2}{3},\ \beta=\gamma=\frac{1}{5}$	3α [20], 10β [12]
F_{60}^{II}	32	90	$\alpha=\frac{2}{5},\ \beta=\gamma=\frac{1}{3}$	5α [12], 6β [20]
F_{120}	62	180	$\alpha=\frac{1}{2},\ \beta=\frac{1}{3},\ \gamma=\frac{1}{5}$	$\begin{cases}4\alpha\ [30],\ 6\beta\ [20]\\ 10\gamma\ [12]\end{cases}$
$\bullet G_{4n}\ \ (n\geq 2)$	$2n+2$	$6n$	$\alpha+\beta=1,\gamma=\frac{1}{n}$, $\frac{1}{2n}<\alpha,\beta<\frac{2n-1}{2n}$	$\begin{cases}2\alpha+2\beta\ [2n]\\ 2n\gamma\ [2]\end{cases}$
$G_{4n+2}\ \ (n\geq 1)$	$2n+3$	$6n+3$	$\alpha=\beta=\frac{1}{2},\ \gamma=\frac{2}{2n+1}$	$\begin{cases}4\alpha\ [2n+1]\\ (2n+1)\gamma\ [2]\end{cases}$
$TG_{8n}\ \ (n\geq 2)$	$4n+2$	$12n$	$\alpha=\beta=\frac{1}{2},\ \gamma=\frac{1}{2n}$	$\begin{cases}4\alpha\ [4n-2]\\ 2\alpha+2n\gamma\ [4]\end{cases}$
$\bullet TG_{8n+4}\ \ (n\geq 1)$	$4n+4$	$12n+6$	$\alpha+\beta=1,\ \gamma=\frac{1}{2n+1}$, $\frac{1}{4n+2}<\alpha,\beta<\frac{4n+1}{4n+2}$	$\begin{cases}\alpha+\beta+(2n+1)\gamma\ [4]\\ 2\alpha+2\beta\ [4n]\end{cases}$
$MTG_{8n+4}^{I}\ \ (n\geq 1)$	$4n+4$	$12n+6$	$\alpha=\frac{n+1}{2n+1},\ \beta=\frac{n}{2n+1}$, $\gamma=\frac{1}{2n+1}$	$\begin{cases}\alpha+\beta+(2n+1)\gamma\ [2]\\ \alpha+3\beta+\gamma\ [2]\\ 2\alpha+2\beta\ [4n-2]\\ 2\alpha+2n\gamma\ [2]\end{cases}$
$MTG_{8n+4}^{II}\ \ (n\geq 2)$	$4n+4$	$12n+6$	$\alpha=\frac{n+1}{2n+1},\ \beta=\frac{n}{2n+1}$, $\gamma=\frac{1}{2n+1}$	$\begin{cases}\alpha+3\beta+\gamma\ [4]\\ 2\alpha+2\beta\ [4n-4]\\ 2\alpha+2n\gamma\ [4]\end{cases}$
$H_{4n}\ \ (n\geq 3)$	$2n+2$	$6n$	$\alpha=\beta=\frac{n-1}{2n},\ \gamma=\frac{2}{n}$	$4\alpha+\gamma$ [$2n$], $n\gamma$ [2]
$TH_{8n+4}\ \ (n\geq 3)$	$4n+4$	$12n+6$	$\alpha=\beta=\frac{n}{2n+1},\ \gamma=\frac{2}{2n+1}$	$\begin{cases}4\alpha+\gamma\ [4n]\\ 2\alpha+(n+1)\gamma\ [4]\end{cases}$
$I_{8n}\ \ (n\geq 3)$	$4n+2$	$12n$	$\alpha=\frac{1}{2},\ \beta=\frac{n-1}{2n},\ \gamma=\frac{1}{n}$	$\begin{cases}4\alpha\ [2n]\\ 4\beta+2\gamma\ [2n]\\ 2n\gamma\ [2]\end{cases}$
$TI_{16n+8}\ \ (n\geq 2)$	$8n+6$	$24n+12$	$\alpha=\frac{1}{2},\ \beta=\frac{n}{2n+1},\ \gamma=\frac{1}{2n+1}$	$\begin{cases}4\alpha\ [4n+2]\\ 4\beta+2\gamma\ [4n]\\ 2\beta+(2n+2)\gamma\ [4]\end{cases}$

The subscript of each tiling indicates the number of faces.
The mark $\bullet$ indicates that the tiling is continuously deformable.

vertices appearing in the tiling. And by using this result, in § 6 and § 7, we classify monohedral tilings by scalene triangles containing an odd-valent vertex. In the final section (§ 8), we classify tilings containing only even-valent vertices. We remark that the result in § 5 (Proposition 11) is already stated in Davies' paper [4; p. 44]. But to prove this result, some combinatorial considerations are necessary in addition to Davies' explanation.

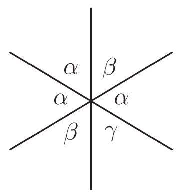
In Appendix, we give some examples of curious tilings containing a special 5-valent vertex. To understand this curiosity, we must add some explanation on the results of § 5. We may say that the determination of the type of vertices carried out in § 5 is a local problem, because it concerns only partial tilings around one point of the sphere. On the contrary, patching these local data to construct a whole tiling is a global problem. From this viewpoint, a special 5-valent vertex $3\alpha + \beta + \gamma = 2$ $(\alpha > \beta > \gamma)$ appeared in the local classification (Proposition 11) possesses a quite delicate feature because this vertex never appear in the actual tiling as a result of the classification. (The proof of this fact requires a relatively long argument, essentially given in the proof of Lemmas 13 and 14.) In Appendix, we give examples of tilings on non-compact spaces of constant positive curvature with boundary which contain this special 5-valent vertex in its interior. And so, we may say that the existence or non-existence of this 5-valent vertex $3\alpha + \beta + \gamma = 2$ is a quite delicate "global" result, depending on the topology of the sphere. By considering a special case of this tiling, we can construct a dihedral tiling on the sphere consisting of $10n$ congruent triangles and two regular n-gons ($n = 3\sim 7$). The existence of such curious examples indicates the difficulty in classifying dihedral tilings of the 2-dimensional sphere, though it is a quite interesting problem which deserve to be investigated as a next problem.

1. Main results

In this section, we state our main theorem (Theorem 1) which gives a classification of monohedral triangular tilings of the sphere and also state the results immediately obtained from this theorem (Corollaries 2 and 3). We first fix our notation.

We consider a tiling of the 2-dimensional sphere with radius $= 1$, consisting of one congruent spherical triangle (or its reflection). We assume that no vertex of any triangle lies on the interior of an edge of any other triangle. We call such a tiling *monohedral*. (For the usual terminology on tilings, see the excellent book [8].) We say that two tilings are *identical* if they are mapped to each other by a rotation or a reflection of $\mathbf{R}^3$. We denote by $\alpha\pi$, $\beta\pi$, $\gamma\pi$ the angles of the triangle in the tiling. In this paper, we always assume that these angles satisfy the inequalities $0 < \alpha\pi, \beta\pi, \gamma\pi < \pi$.

If a vertex in the tiling is surrounded by angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$ with multiplicities k, l, m, respectively, we say that the type of this vertex is $k\alpha + l\beta + m\gamma = 2$. (Clearly, this vertex is $(k + l + m)$-valent.)



$$3\alpha + 2\beta + \gamma = 2$$

Fig. 1

For simplicity, we often drop the symbol π in expressing the angles as in the above figure. Note that in the expression $k\alpha + l\beta + m\gamma = 2$, we ignore the order of angles appearing around the vertex. In general, there appear several types of vertices in one tiling.

We denote by V, E, F the number of vertices, edges and faces of the tiling, respectively. If one triangle tiles the whole sphere, we can easily see that the inequality $F \geq 4$ holds by a combinatorial reason.

Now, under these notations and assumptions, we state our main theorem of this paper. The meaning of the symbols in the left column of Table in Introduction and the explicit construction of each tiling will be explained in the next section.

Theorem 1. *Monohedral tilings of the* 2*-dimensional sphere by a triangle with angles* $\alpha\pi$, $\beta\pi$, $\gamma\pi$ $(0 < \alpha, \beta, \gamma < 1)$ *are exhausted by Table in Introduction. None of these tilings are isomorphic to each other except the trivial case given by the exchange of angles in* $\alpha + \beta + \gamma = 2$ (F_4), $\alpha + \beta = 1$ (G_{4n}, TG_{8n+4}).

Remark. Almost all tilings can be distinguished to each other by the data given in Table. Only one exception is the case of TG_{12} $(\alpha = 2/3, \beta = \gamma = 1/3)$ and MTG_{12}^{I}. These two tilings have completely the same data in Table, and we must consider an additional combinatorial property to distinguish them. For details, see the explicit construction and development maps given in the next section.

Next, we give another classification, from the viewpoint of the number of faces.

Corollary 2. *The number of faces of monohedral tilings of the* 2*-dimensional sphere by a triangle is a multiple of* 4*, except for* G_{4n+2} $(n \geq 1)$. *In the case* $F \equiv 0 \pmod 4$*, such tilings are exhausted by the following list.* (*The right* [ ] *indicates the number of tilings*):

$F=4$	: $\bullet F_4$,	[1]
$F=8$	: $\bullet G_8$,	[1]
$F=12$	: $F_{12}^{I}, F_{12}^{II}, F_{12}^{III}, \bullet G_{12}, \bullet TG_{12}, MTG_{12}^{I}, H_{12}$,	[7]
$F=16$	: $\bullet G_{16}, TG_{16}, H_{16}$,	[3]
$F=20$	: $\bullet G_{20}, \bullet TG_{20}, MTG_{20}^{I}, MTG_{20}^{II}, H_{20}$,	[5]
$F=24$	: $F_{24}, \bullet G_{24}, TG_{24}, H_{24}, I_{24}$	[5]
$F=48$	: $F_{48}, TF_{48}, \bullet G_{48}, TG_{48}, H_{48}, I_{48}$,	[6]
$F=60$	: $F_{60}^{I}, F_{60}^{II}, \bullet G_{60}, \bullet TG_{60}, MTG_{60}^{I}, MTG_{60}^{II}, H_{60}, TH_{60}$,	[8]
$F=120$	: $F_{120}, \bullet G_{120}, TG_{120}, H_{120}, I_{120}, TI_{120}$,	[6]
$F=8n+4$ $(n \geq 3, n \neq 7)$	: $\bullet G_{8n+4}, \bullet TG_{8n+4}, MTG_{8n+4}^{I}, MTG_{8n+4}^{II}, H_{8n+4}, TH_{8n+4}$,	[6]
$F=16n$ $(n \geq 2, n \neq 3)$	: $\bullet G_{16n}, TG_{16n}, H_{16n}, I_{16n}$,	[4]
$F=16n+8$ $(n \geq 2, n \neq 7)$	: $\bullet G_{16n+8}, TG_{16n+8}, H_{16n+8}, I_{16n+8}, TI_{16n+8}$.	[5]

(As in Table, the mark $\bullet$ indicates that it is continuously deformable. In this list, we count it as "one" species.)

As another corollary, we have

Corollary 3. *Only the following triangles can tile the whole sphere monohedrally*:

$$\alpha+\beta+\gamma=2 \qquad \left(\frac{1}{2}<\gamma \leq \beta \leq \alpha<1\right) \qquad [F=4],$$

$$\alpha=\frac{2}{3}, \ \beta=\gamma=\frac{1}{4} \qquad [F=24],$$

$$\alpha=\frac{2}{3}, \ \beta=\gamma=\frac{1}{5} \qquad [F=60],$$

$$\alpha=\frac{2}{5}, \ \beta=\gamma=\frac{1}{3} \qquad [F=60],$$

$$\alpha=\frac{1}{2}, \ \beta=\frac{1}{3}, \ \gamma=\frac{1}{4} \qquad [F=48],$$

$$\alpha=\frac{1}{2}, \ \beta=\frac{1}{3}, \ \gamma=\frac{1}{5} \qquad [F=120],$$

$$\alpha = \beta = \frac{1}{2},\ \ \gamma = \frac{2}{n} \qquad (n \geq 3) \qquad [F = 2n],$$

$$\alpha = \beta = \frac{n-1}{2n},\ \ \gamma = \frac{2}{n} \qquad (n \geq 4) \qquad [F = 4n],$$

$$\alpha = \frac{1}{2},\ \ \beta = \frac{n-1}{2n},\ \ \gamma = \frac{1}{n} \qquad (n \geq 3) \qquad [F = 8n],$$

$$\alpha + \beta = 1,\ \ \gamma = \frac{1}{n} \qquad \left(\frac{1}{2} < \alpha < \frac{2n-1}{2n},\ \ n \geq 2\right) \qquad [F = 4n].$$

Note that the first and the last triangles in this corollary are both deformable, and the remaining ones are rigid. These two corollaries are the immediate consequences of Theorem 1, and we leave the examination of these facts to the readers.

Before giving the explicit construction of each tiling and the proof of Theorem 1, we review some fundamental properties of spherical triangles which we use in this paper.

Proposition 4 (cf. [12; p. 62]). *The angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$ of a spherical triangle satisfy the following inequalities*:

$$\alpha + \beta + \gamma > 1,$$

$$\alpha + \beta < 1 + \gamma, \qquad \beta + \gamma < 1 + \alpha, \qquad \gamma + \alpha < 1 + \beta.$$

Conversely, if α, β, γ satisfy these conditions, then up to a motion, there exists uniquely a spherical triangle with angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$.

The area of this triangle is given by the formula

$$S = \pi(\alpha + \beta + \gamma - 1).$$

Hence, if this triangle tiles the whole sphere, the number of faces is equal to $4/(\alpha + \beta + \gamma - 1)$, which implies

$$\alpha + \beta + \gamma = 1 + \frac{4}{F}.$$

We use this equality frequently in this paper.

We denote by a, b, c the lengths of edges opposite to the angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$, respectively.

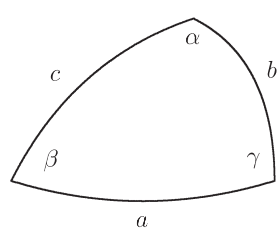


Fig. 2

Then, the following cosine rule holds:

$$\cos a = \frac{\cos \alpha\pi + \cos \beta\pi \cos \gamma\pi}{\sin \beta\pi \sin \gamma\pi},$$

$$\cos b = \frac{\cos \beta\pi + \cos \gamma\pi \cos \alpha\pi}{\sin \gamma\pi \sin \alpha\pi},$$

$$\cos c = \frac{\cos \gamma\pi + \cos \alpha\pi \cos \beta\pi}{\sin \alpha\pi \sin \beta\pi}.$$

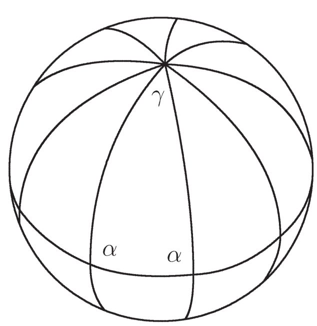
These equalities indicate that the lengths a, b, c are uniquely determined by three angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$. By using these formulas, we can easily show that the inequality $a > b$ holds if and only if $\alpha > \beta$, etc. We also use these formulas in constructing explicit "models" of tilings on some spherical material.

2. Explicit construction of tilings

In this section, we explain the explicit construction of tilings in Table, give their figures, development maps, and state their mutual relations. Note that the construction of these tilings is already explained in Davies' paper [4] (in a somewhat different fashion, but unfortunately with some duplicates).

(1) G_{2n} $(n \geq 3)$ and TG_{4n} $(n \geq 3)$.

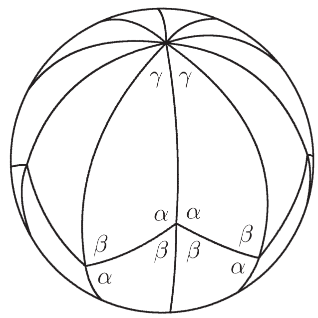
Tilings G_{4n} $(\alpha = \beta = 1/2, \gamma = 1/n, n \geq 2)$ and G_{4n+2} $(n \geq 1)$ are simply expressed in the following figure, which we usually see as a figure of the globe:



G_{18}

Fig. 3

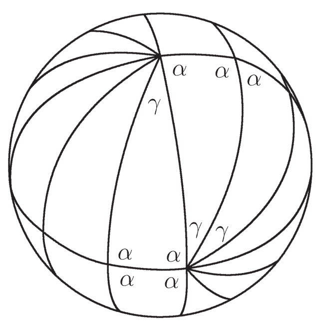
In case $F = 4n$, we can deform this tiling under the conditions $\alpha + \beta = 1$ and $1/2n < \alpha, \beta < (2n-1)/2n$ as in the following way:



G_{20}

Fig. 4

(But in case $F = 4n + 2$, on the contrary, the above tiling G_{4n+2} is rigid. The inequality $1/2n < \alpha, \beta < (2n - 1)/2n$ follows from the condition $1 + \frac{4}{F} - \alpha = \beta + \gamma < 1 + \alpha$, etc.) In addition, if $F = 4n$, we can "twist" the tiling G_{4n} in the following way. First, in case $F \equiv 0 \pmod 8$, we consider the tiling G_{8n} with $\alpha = \beta = 1/2$. Then we can rotate a hemisphere along one longitude and obtain the following tiling TG_{8n}.



TG_{8n}

Fig. 5

Next, in case $F \equiv 4 \pmod 8$, we prepare two copies of the following hemisphere of G_{8n+4} $(\alpha + \beta = 1, \gamma = 1/(2n + 1))$.

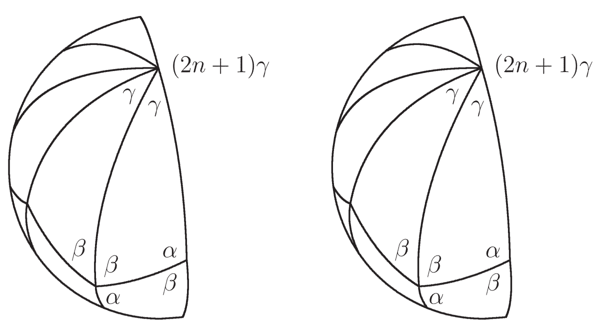
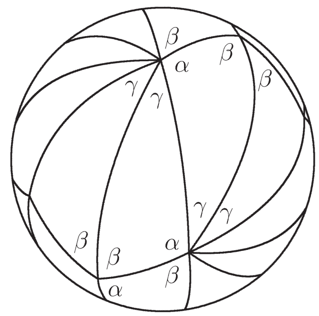


Fig. 6

And join these two copies as follows. This tiling is TG_{8n+4}.



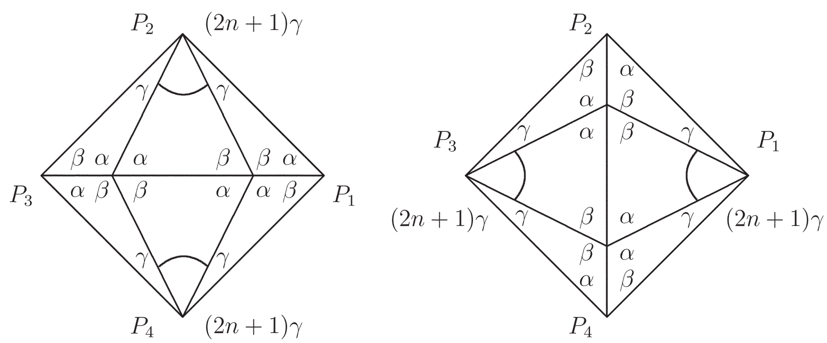
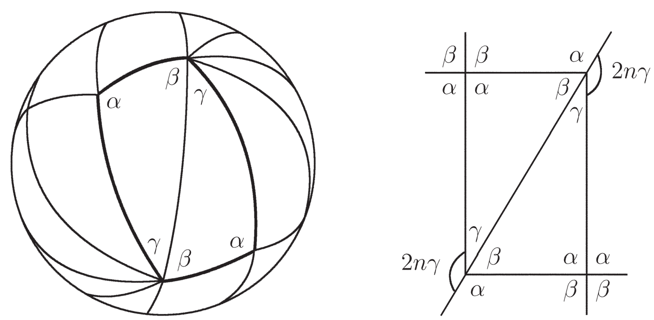


Fig. 7

Note that two P_i's ($i = 1{\sim}4$) in the above development map express the same points in the sphere. It should be remarked that this tiling cannot be obtained by a rotation of G_{8n+4} along a great circle $P_1P_2P_3P_4$, unless $\alpha = \beta = 1/2$. Clearly, this tiling is continuously deformable.

(2) MTG^{I}_{8n+4} $(n \geq 1)$ and MTG^{II}_{8n+4} $(n \geq 2)$.

We consider the special case of TG_{8n+4} where $\alpha = (n+1)/(2n+1)$ and $\beta = n/(2n+1)$. In this case, there appear two "rectangles" at the opposite side of the sphere:



TG_{8n+4}

Fig. 8

(Here, we say that a quadrangle is a "rectangle" if the opposite edges have the same lengths.) Then, since $\alpha = \beta + \gamma$, we may draw a reversed diagonal line in each rectangle instead of the usual one.

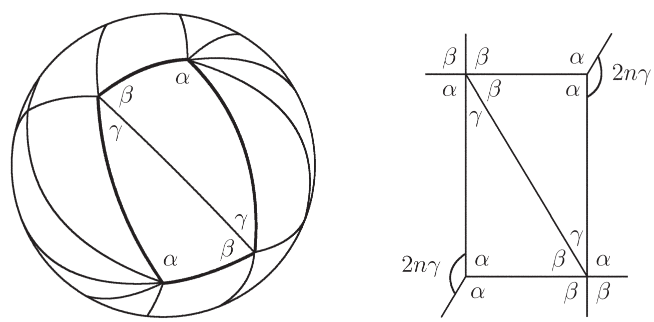


Fig. 9

If we reverse one diagonal line in TG_{8n+4}, we obtain the tiling MTG^{I}_{8n+4}. And if we reverse both diagonal lines in two rectangles, we obtain the tiling MTG^{II}_{8n+4}. The development maps of these tilings are a little complicated, which are expressed in the following form:

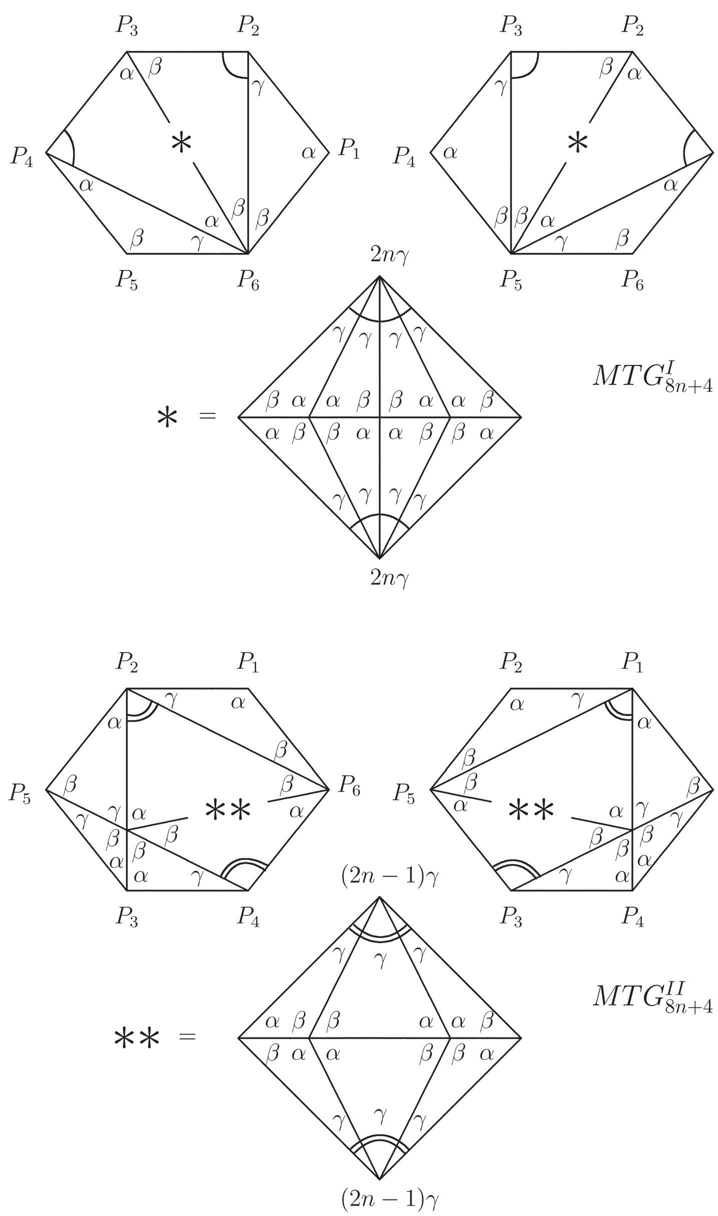


Fig. 10

(As above, two P_i's ($i = 1 \sim 6$) express the same points in the sphere.)

(3) H_{4n} ($n \geq 3$), TH_{8n+4} ($n \geq 3$), I_{8n} ($n \geq 3$) and TI_{16n+8} ($n \geq 2$).

Next, we consider the tiling G_{4n} with $\alpha = (n-1)/n$ and $\beta = \gamma = 1/n$. Then, since $\beta = \gamma$ in this case, we can construct a tiling consisting of congruent rhombuses (=quadrangles whose four edges have the same lengths) by deleting suitable edges of G_{4n}.

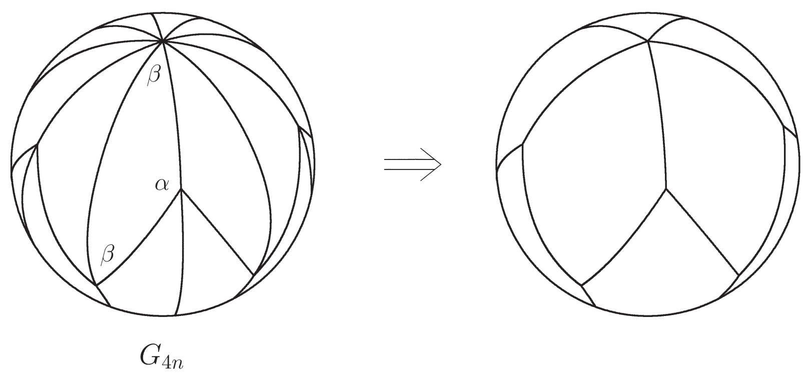


Fig. 11

Then, in this new tiling, by drawing a reversed diagonal line in each rhombus, we obtain the tiling H_{4n}:

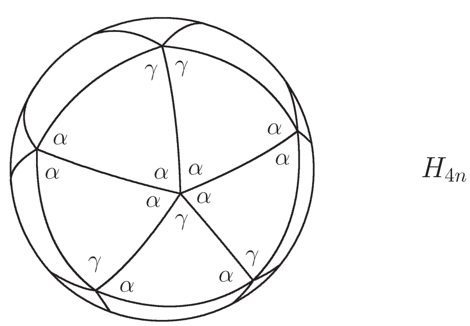
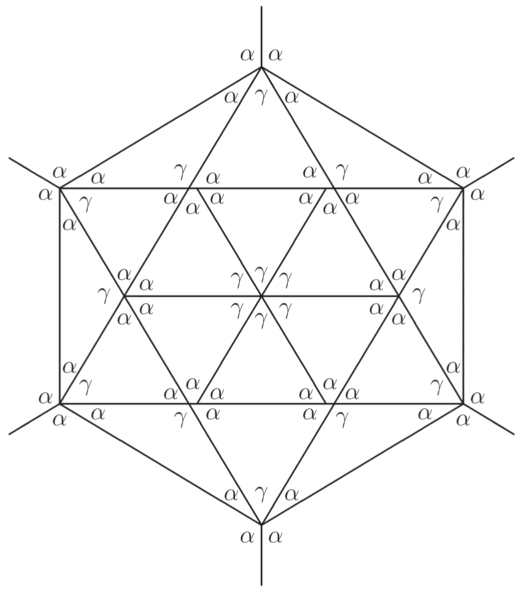


Fig. 12

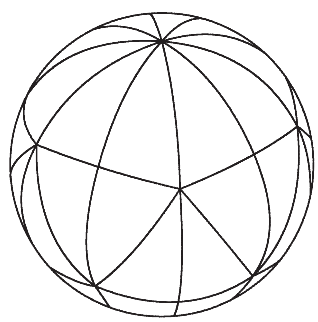
This tiling consists of isosceles triangles with $\alpha = \beta = (n-1)/2n$ and $\gamma = 2/n$. The development map of this tiling is given as follows:



H_{24}

Fig. 13

In addition, if we draw two diagonal lines in each rhombus, we obtain the tiling I_{8n}:



I_{8n}

Fig. 14

Next, starting from the tiling TG_{8n+4} with $\alpha = 2n/(2n+1)$, $\beta = \gamma = 1/(2n+1)$, we can repeat the same procedure as above. Then, we obtain new tilings TH_{8n+4} and TI_{16n+8} as twisted versions of H_{8n+4} and I_{16n+8}, respectively.

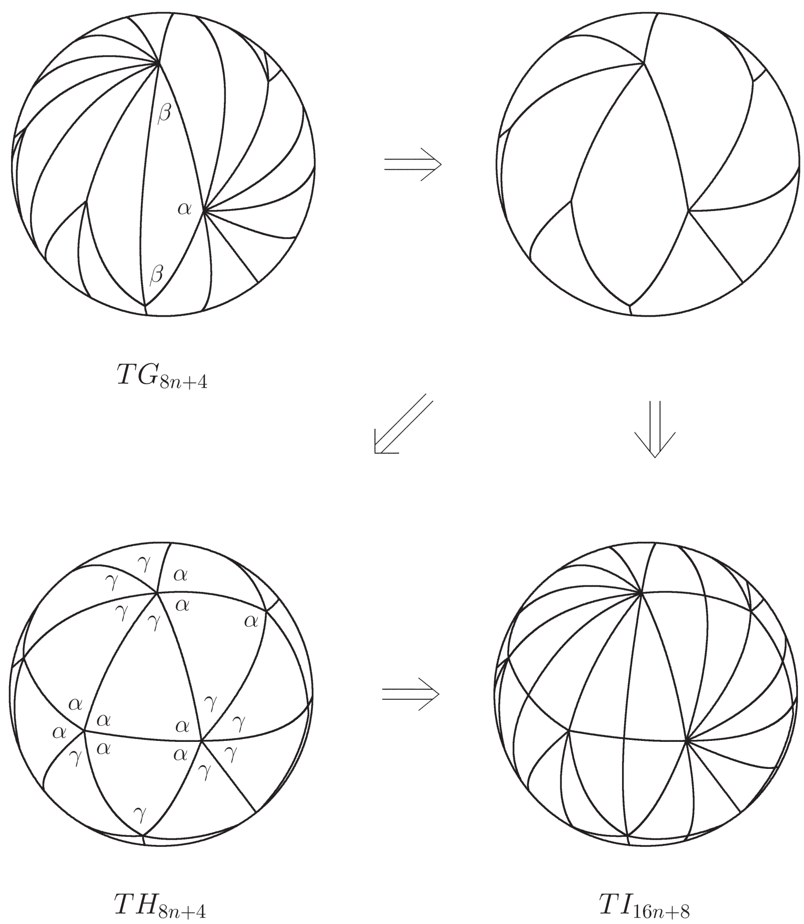


Fig. 15

The development map of TH_{8n+4} is expressed as follows:

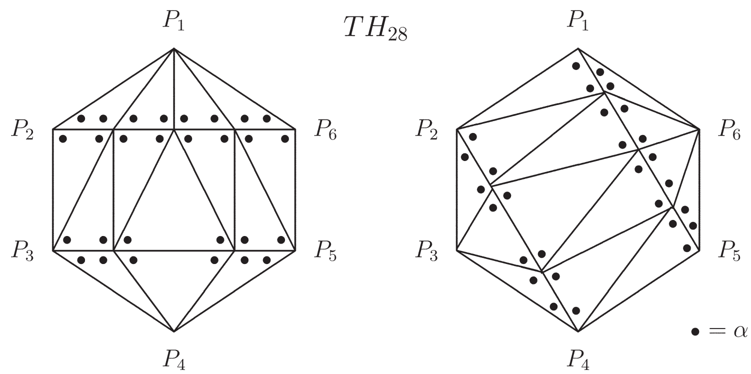


Fig. 16

(As before, two P_i's ($i = 1 \sim 6$) express the same points in the sphere. But we remark that the surrounding hexagon $P_1P_2P_3P_4P_5P_6$ in TH_{8n+4} does not constitute a great circle.)

(4) For the remaining ten sporadic tilings $F_4 \sim F_{120}$, we give here their figures and (or) development maps.

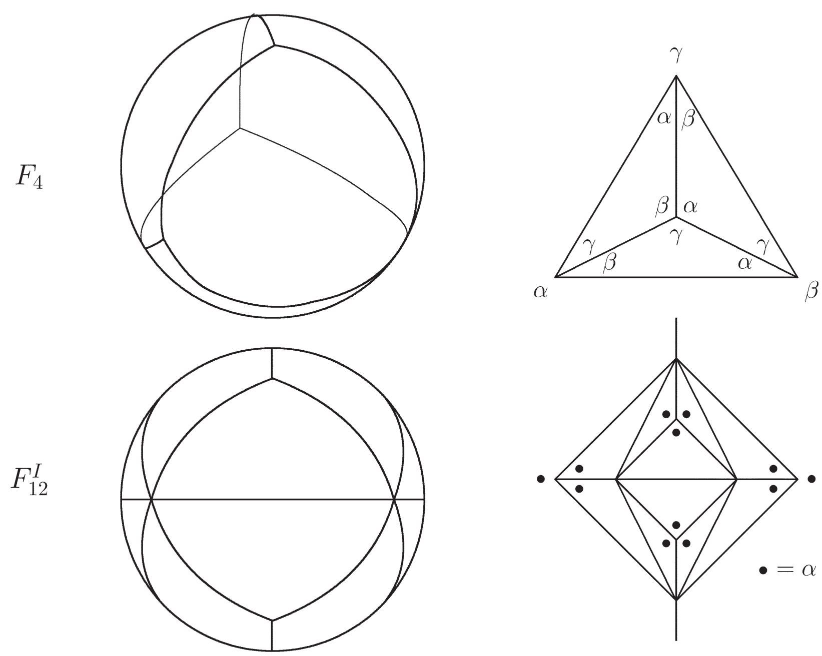
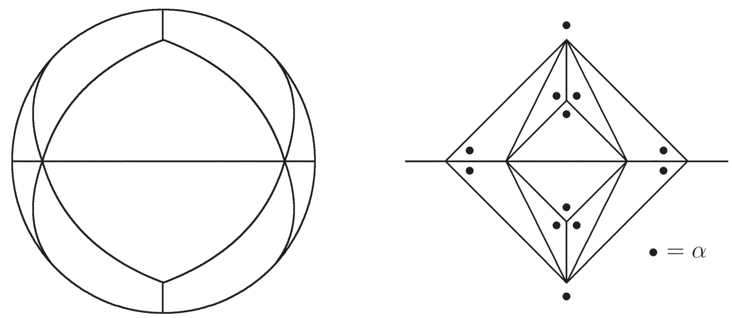
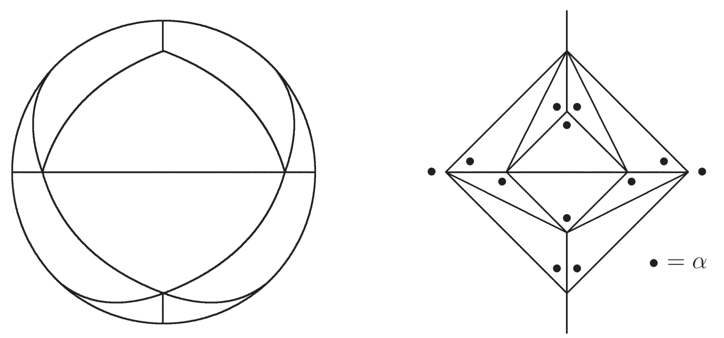


Fig. 17

F_{12}^{II}



F_{12}^{III}



F_{24}

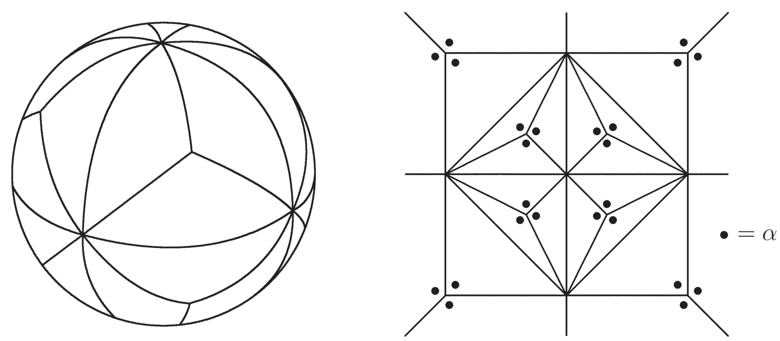


Fig. 17 (continued)

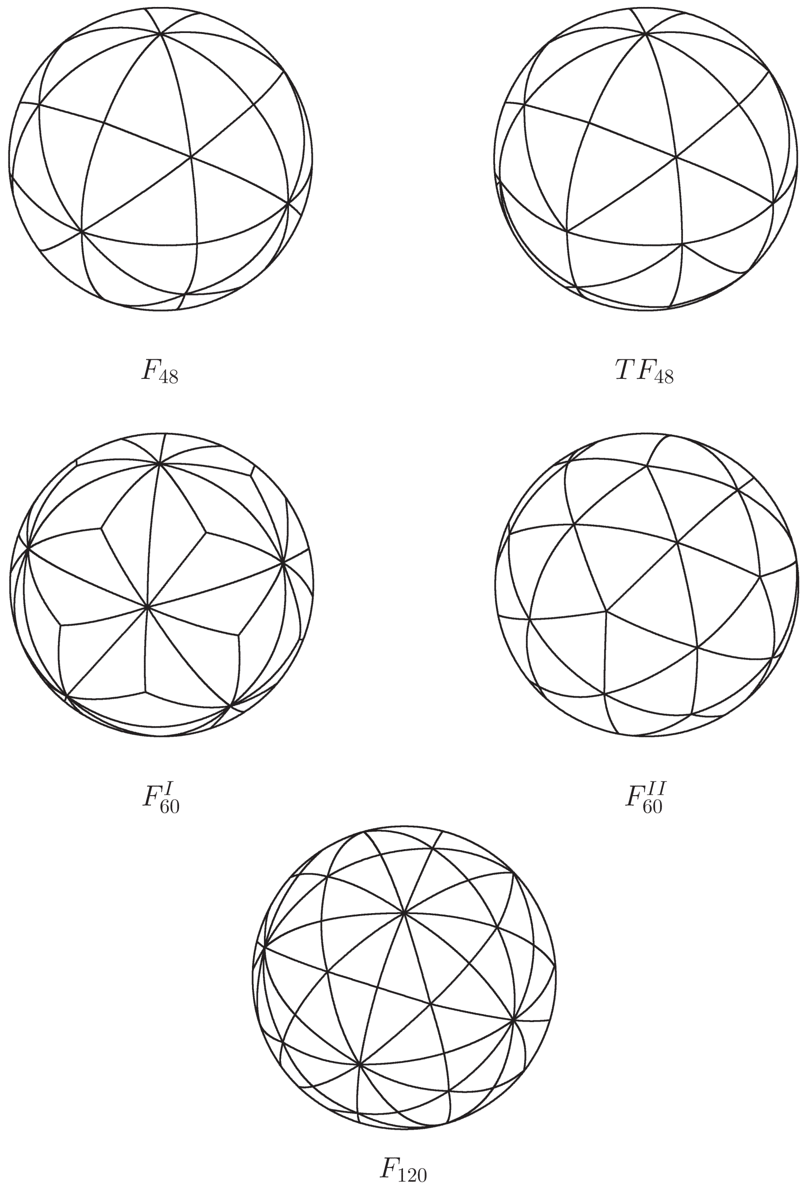


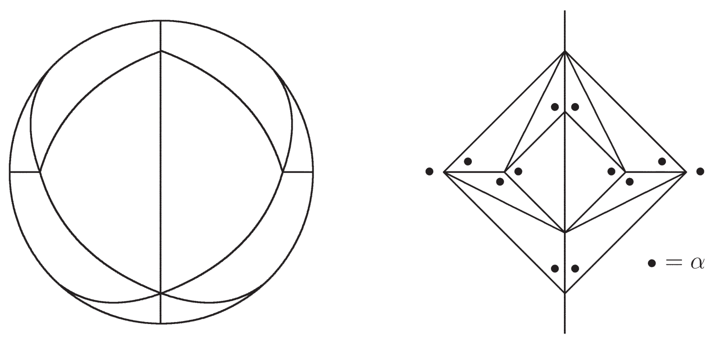
Fig. 17 (continued)

Explicit construction of these tilings can be read from the explanation below. Note that the condition $1/2 < \alpha, \beta, \gamma$ in F_4 follows from the condition $2 - \gamma = \alpha + \beta < 1 + \gamma$, etc.

Next, we summarize typical features and relations of these tilings. We obtain the most familiar triangular monohedral tilings by projecting the regular tetrahedron, octahedron and icosahedron to their circumspheres. These tilings are F_4 $(\alpha = \beta = \gamma = 2/3)$, G_8 $(\alpha = \beta = \gamma = 1/2)$ and H_{20}, respectively.

Also by projecting the regular hexahedron to the circumsphere, we obtain a tiling consisting of six squares on the sphere. We draw one diagonal line in each square. Then, we obtain monohedral tilings consisting of isosceles triangles with $\alpha = 2/3, \beta = \gamma = 1/3$. Combinatorially, there exist just seven tilings of this type: F_{12}^{I}, F_{12}^{II}, F_{12}^{III}, G_{12} $(\alpha = 2/3, \beta = \gamma = 1/3)$, TG_{12} $(\alpha = 2/3, \beta = \gamma = 1/3)$, MTG_{12}^{I} and H_{12}. Three of them are sporadic and we already gave their development maps (Figure 17). We give here the remaining ones:

G_{12}



TG_{12}

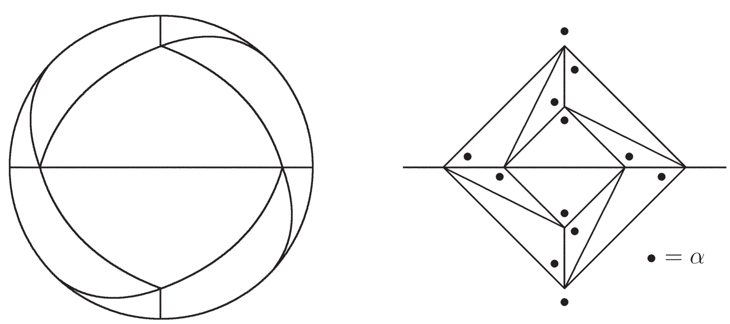
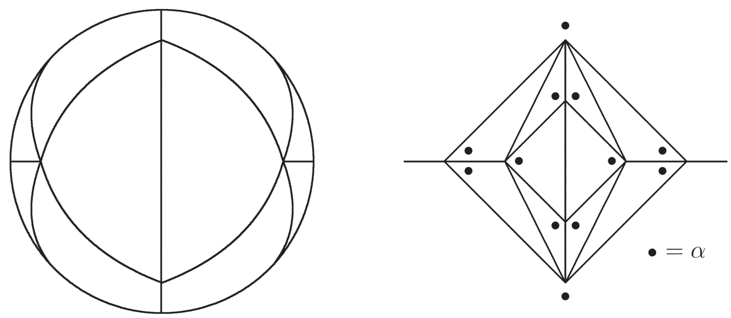


Fig. 18

MTG_{12}^{I}



H_{12}

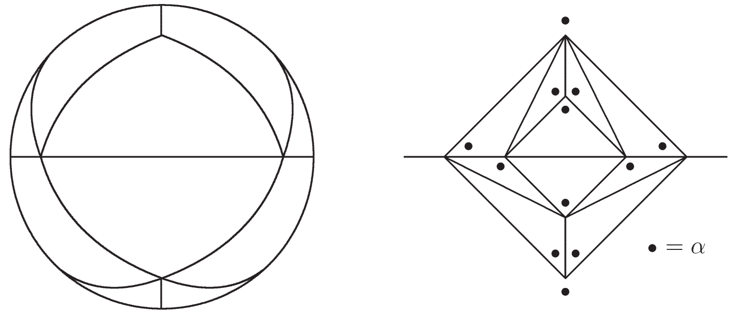


Fig. 18 (continued)

These four tilings are the special cases of the series of tilings listed in the lower half of Table. We can easily check these identifications by comparing the development maps of these tilings (or by making models on some spherical materials).

Note that four 5-valent vertices in TG_{12} $(\alpha = 2/3, \beta = \gamma = 1/3)$ lie in one great circle. But the tiling MTG_{12}^{I} does not possess this property. Hence $TG_{12} \neq MTG_{12}^{I}$, though they have completely the same data in Table as stated before. The tiling F_{12}^{I} can be also obtained by dividing each equilateral triangle in F_4 $(\alpha = \beta = \gamma = 2/3)$ into three congruent isosceles triangles.

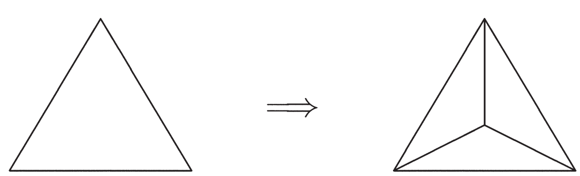
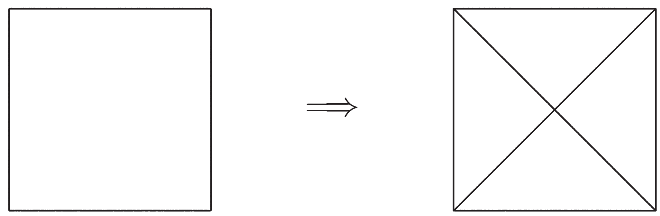
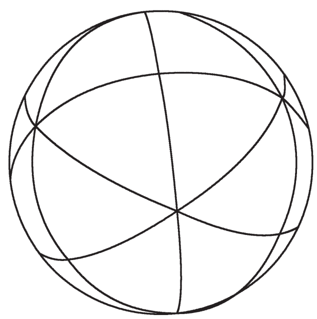


Fig. 19

In a similar way, by dividing each square of regular hexahedron into four isosceles triangles, we obtain the tiling I_{24}.





I_{24}

Fig. 20

In this way, we obtain a new tiling from the old one by dividing equilateral or isosceles triangles (or regular polygons in regular polyhedrons). We give here a diagram consisting of all relations of this type. (Davies [4] also stated a similar result. But unfortunately, his list is not complete.) In this diagram the symbol "$A \to B$" implies that the tiling B is obtained by dividing A in a suitable way.

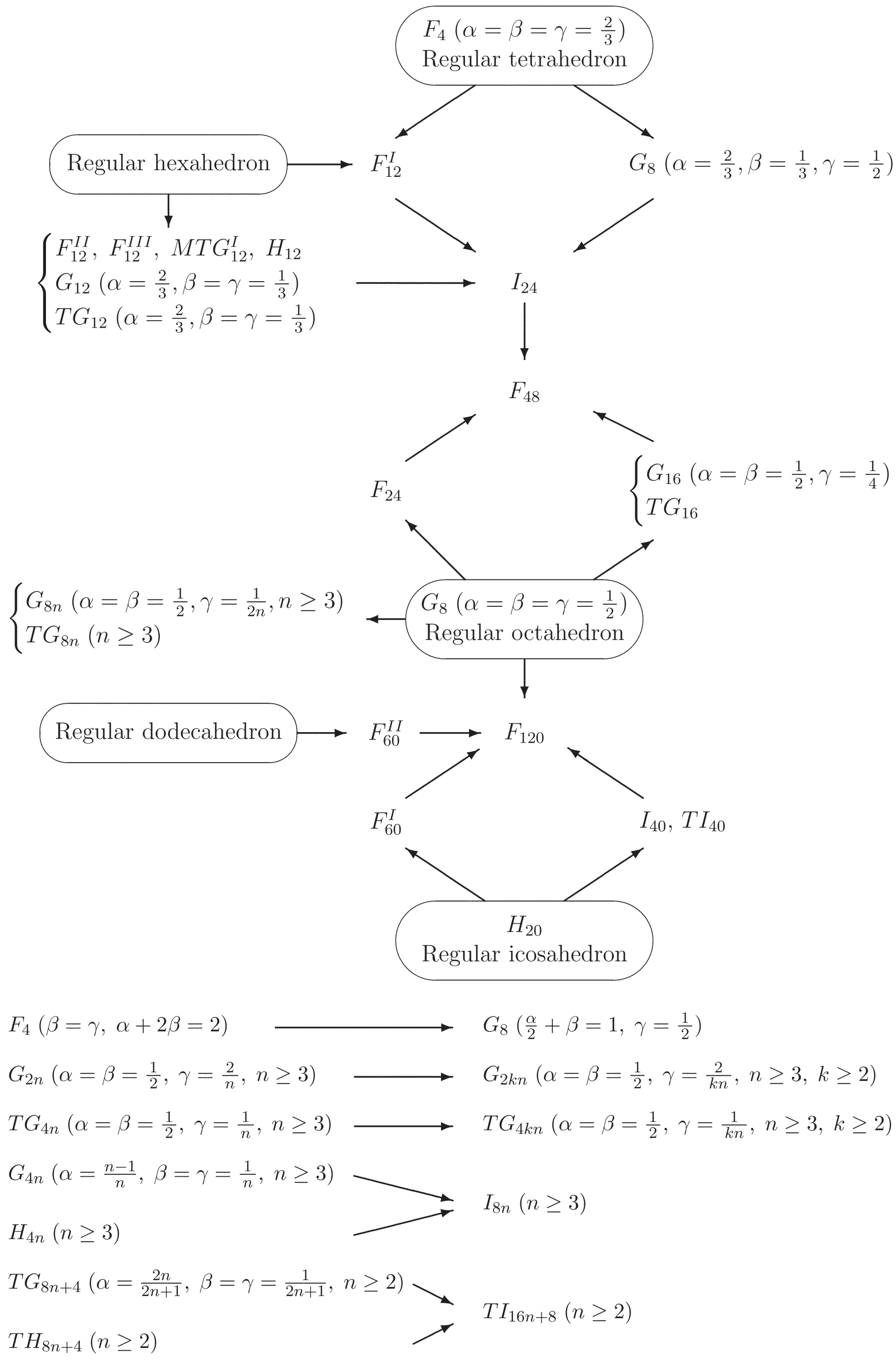

F_4 $(\alpha=\beta=\gamma=\frac{2}{3})$
Regular tetrahedron
Regular hexahedron
F_{12}^{I}
G_8 $(\alpha=\frac{2}{3}, \beta=\frac{1}{3}, \gamma=\frac{1}{2})$
F_{12}^{II}, F_{12}^{III}, MTG_{12}^{I}, H_{12}
G_{12} $(\alpha=\frac{2}{3}, \beta=\gamma=\frac{1}{3})$
TG_{12} $(\alpha=\frac{2}{3}, \beta=\gamma=\frac{1}{3})$
I_{24}
F_{48}
F_{24}
G_{16} $(\alpha=\beta=\frac{1}{2}, \gamma=\frac{1}{4})$
TG_{16}
G_{8n} $(\alpha=\beta=\frac{1}{2}, \gamma=\frac{1}{2n}, n\geq 3)$
TG_{8n} $(n\geq 3)$
G_8 $(\alpha=\beta=\gamma=\frac{1}{2})$
Regular octahedron
Regular dodecahedron
F_{60}^{II}
F_{120}
F_{60}^{I}
I_{40}, TI_{40}
H_{20}
Regular icosahedron
F_4 $(\beta=\gamma,\ \alpha+2\beta=2)$
G_8 $(\frac{\alpha}{2}+\beta=1,\ \gamma=\frac{1}{2})$
G_{2n} $(\alpha=\beta=\frac{1}{2},\ \gamma=\frac{2}{n},\ n\geq 3)$
G_{2kn} $(\alpha=\beta=\frac{1}{2},\ \gamma=\frac{2}{kn},\ n\geq 3,\ k\geq 2)$
TG_{4n} $(\alpha=\beta=\frac{1}{2},\ \gamma=\frac{1}{n},\ n\geq 3)$
TG_{4kn} $(\alpha=\beta=\frac{1}{2},\ \gamma=\frac{1}{kn},\ n\geq 3,\ k\geq 2)$
G_{4n} $(\alpha=\frac{n-1}{n},\ \beta=\gamma=\frac{1}{n},\ n\geq 3)$
I_{8n} $(n\geq 3)$
H_{4n} $(n\geq 3)$
TG_{8n+4} $(\alpha=\frac{2n}{2n+1},\ \beta=\gamma=\frac{1}{2n+1},\ n\geq 2)$
TI_{16n+8} $(n\geq 2)$
TH_{8n+4} $(n\geq 2)$

Fig. 21

(We consider $TH_{20} = H_{20}$, as explained below.) We remark that five regular polyhedrons are all related to each other through some tilings, as indicated in the above diagram.

The remaining sporadic tiling TF_{48} is obtained by rotating a hemisphere in F_{48} along the equator through the angle $\pi/4$. The tiling F_{12}^{II} contains one great circle, and if we rotate the hemisphere along this circle through the angle π, we obtain the tiling H_{12}.

For the ten series of tilings listed in the lower half part of Table, we can also consider the case where the number n takes a small value outside of the indicated range, such as TG_8, MTG_{12}^{II}, etc. But these tilings are identical to other tilings listed in Table. We give here the list of such identical tilings. (It seems to the authors that the lack of such consideration is the principal defect of Davies' classification in [4].)

- $TG_8 = G_8$ $(\alpha = \beta = \gamma = 1/2)$ (= Regular octahedron),
- $MTG_{12}^{II} = TH_{12} = TG_{12}$ $(\alpha = 2/3, \beta = \gamma = 1/3)$,
- $I_{16} = TG_{16}$,
- $TH_{20} = H_{20}$ (= Regular icosahedron),
- $TI_{24} = I_{24}$.

For example, by writing a great circle passing through four 5-valent vertices in the development maps of MTG_{12}^{II} and TH_{12}, we can easily see that these two tilings are identical to TG_{12} $(\alpha = 2/3, \beta = \gamma = 1/3)$. Remark that from the construction, TI_{40} is not identical to I_{40}, though $TH_{20} = H_{20}$. In a similar way, we have $TI_{24} = I_{24}$ in spite of $TH_{12} \neq H_{12}$.

3. Preliminary case

The rest of this paper is devoted to the proof of Theorem 1. We divide the proof into several cases according as the type of triangles and the number of faces. In this section, we treat the preliminary case: tilings with $F = 4$ and tilings by equilateral triangles. Classification of tilings by isosceles and scalene triangles will be given in §4 and §5∼§8, respectively.

Now we first show the following (maybe well known) proposition. We give here a proof for the sake of completeness.

Proposition 5. (1) *Monohedral tilings with* $F = 4$ *are given by* F_4 *with* $\alpha + \beta + \gamma = 2$ *and* $1/2 < \alpha, \beta, \gamma < 1$.

(2) *Monohedral tilings by equilateral triangles are*

$$F_4 \ \left(\alpha = \beta = \gamma = \frac{2}{3}\right), \qquad G_8 \ \left(\alpha = \beta = \gamma = \frac{1}{2}\right), \qquad H_{20}.$$

These are the spherical projections of regular tetrahedron, octahedron and icosahedron to their circumspheres.

Proof. We first prove (2). Assume that the sphere is tiled by the following triangle:

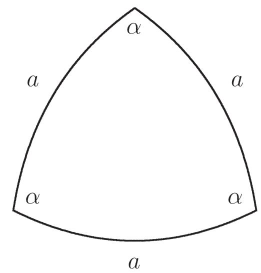


Fig. 22

Let $k\alpha = 2$ be the type of vertices of this tiling. From the condition $\alpha < 1$, we have $k \geq 3$. And from the equality

$$\alpha + \beta + \gamma = 3\alpha = 1 + \frac{4}{F},$$

we have $(k-6)(F+4) + 24 = 0$, which implies $k = 3, 4, 5$. For each case, we have $F = 4, 8, 20$, respectively, and it is clear that the corresponding spherical tilings are just equal to the projections of regular tetrahedron, octahedron and icosahedron to their circumspheres. In Table, they correspond to F_4 $(\alpha = \beta = \gamma = 2/3)$, G_8 $(\alpha = \beta = \gamma = 1/2)$ and H_{20}.

Next, we prove (1). Assume $F = 4$, and we first consider the scalene case. We may assume that $\alpha > \beta > \gamma$. Then, by using the cosine rule, we can show that $a > b > c$. In particular, the lengths of three edges differ to each other. Hence, two edges must contact in one of the following way:

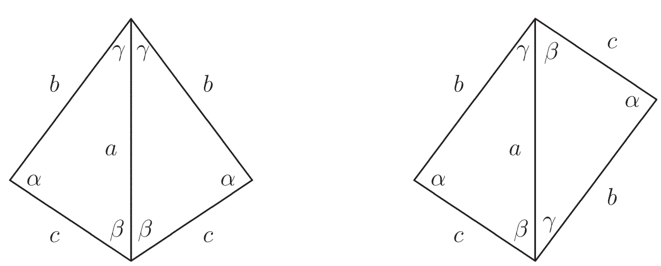


Fig. 23

Using this fact, we can show that the development map of this tiling is uniquely determined in the following way:

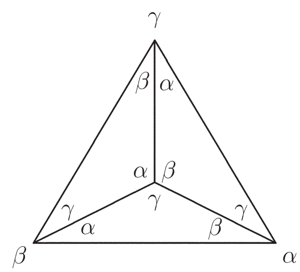


Fig. 24

From Proposition 4 and the condition $\alpha > \beta > \gamma$, we have $\alpha + \beta + \gamma = 2$ and $1/2 < \gamma < \beta < \alpha < 1$. Conversely, from Proposition 4 again, it follows that if α, β, γ satisfy these conditions, there exists a spherical triangle with angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$, which tiles the whole sphere with $F = 4$.

In the case of isosceles triangles, we may assume $\alpha \neq \beta = \gamma$. Then, from the cosine rule, we have $a \neq b = c$, and the sphere must be tiled by the following rhombus:

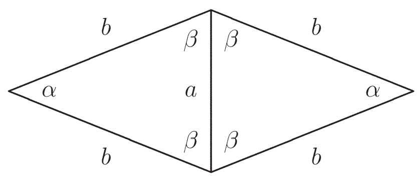


Fig. 25

Hence the tiling is of the form

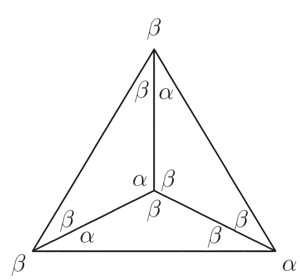


Fig. 26

This figure just coincides with Figure 24 if we drop the condition $\alpha > \beta > \gamma$ and instead, put $\beta = \gamma$. Hence, by including the equilateral case F_4 ($\alpha = \beta = \gamma = 2/3$) stated above, the conditions on the angles are simply summarized in the form

$$\alpha + \beta + \gamma = 2 \qquad \text{and} \qquad \frac{1}{2} < \alpha, \beta, \gamma < 1$$

if we drop the assumption on the order of the size of α, β, γ. Thus, we obtain all tilings with $F = 4$, which just coincides with F_4 in Table. q.e.d.

4. Case of isosceles triangles

In this section, we classify monohedral tilings by isosceles triangles with $F > 4$. There are two types of isosceles triangles:

(I) $\alpha > \beta = \gamma$,
(II) $\alpha = \beta > \gamma$.

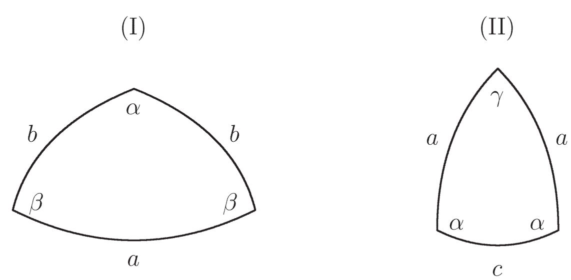


Fig. 27

A classification by these triangles is given by the following proposition. This result was already obtained by Sommerville [9; p. 91∼92], and a similar proof was outlined there (but with some misprints). We give here the detailed proof of this proposition for the sake of completeness. Note that among these tilings, G_{4n} and TG_{8n+4} are deformable to tilings by scalene triangles.

Proposition 6 (cf. [9]). *Monohedral tilings of the* 2*-dimensional sphere by isosceles triangles with* $F > 4$ *are exhausted by*:

(I) $G_6, F_{12}^{I}, F_{12}^{II}, F_{12}^{III}, MTG_{12}^{I}, H_{12}, H_{16}, F_{24}, I_{24}, F_{60}^{I}, F_{60}^{II}$,

$$G_{4n} \ \left(\alpha = \frac{n-1}{n}, \beta = \gamma = \frac{1}{n}, n \geq 3\right),$$

$$TG_{8n+4} \ \left(\alpha = \frac{2n}{2n+1}, \beta = \gamma = \frac{1}{2n+1}, n \geq 1\right).$$

(II) G_{4n} $\left(\alpha = \beta = \dfrac{1}{2}, \gamma = \dfrac{1}{n}, n \geq 3\right)$,

G_{4n+2} $(n \geq 2)$, TG_{8n} $(n \geq 2)$,

TG_{8n+4} $\left(\alpha = \beta = \dfrac{1}{2}, \gamma = \dfrac{1}{2n+1}, n \geq 1\right)$,

H_{4n} $(n \geq 6)$, TH_{8n+4} $(n \geq 3)$.

(Remark that for the tilings G_6, H_{12} and H_{16} in (I), the condition $\gamma > \alpha = \beta$ holds instead of $\alpha > \beta = \gamma$. See Table.)

We first consider the case (I). In this case, by using the cosine rule, we can easily show that $a > b = c$, and it is clear that the sphere is tiled by the rhombus with angles $\alpha\pi$, $2\beta\pi$, $\alpha\pi$, $2\beta\pi$:

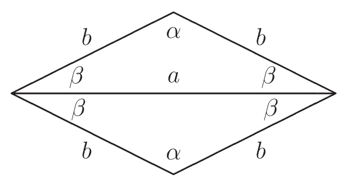


Fig. 28

Hence, the number of faces must be even, and we put $F = 2F'$. (F' expresses the number of rhombuses.) In the following, we put $B = 2\beta$. Then the whole sphere is tiled by the rhombus with angles $\alpha\pi$, $B\pi$, $\alpha\pi$, $B\pi$, and these angles satisfy the conditions:

$$\alpha + B = 1 + \frac{2}{F'}, \qquad 0 < B < 2\alpha < 2.$$

(Note that the condition $B = \beta + \beta < 1 + \alpha$ in Proposition 4 is automatically satisfied under these conditions.) In particular, from these conditions, we have

$$\frac{F' + 2}{3F'} < \alpha < 1.$$

In addition, the inequality $F' \geq 3$ holds because $F > 4$.

We first determine the type of vertices appearing in the tiling.

Lemma 7. *The type of vertices in the tiling by the rhombus with angles $\alpha\pi$, $B\pi$, $\alpha\pi$, $B\pi$ $(2\alpha > B, F' \geq 3)$ is one of the following*:

$$
\begin{aligned}
&3\alpha = 2, \\
&4\alpha = 2, \\
&5\alpha = 2, \\
&2\alpha + B = 2, \\
&3\alpha + B = 2, \\
&lB = 2 \qquad (l \geq 2), \\
&\alpha + mB = 2 \qquad (m \geq 2).
\end{aligned}
$$

Remark. Actually, among the above list, a vertex of type $3\alpha + B = 2$ does not appear in the tiling. We know this result after the classification given in this section.

Proof. We assume that there exists a vertex of type $k\alpha + lB = 2$ in the tiling by rhombuses. Then, we have $k \neq l$. In fact, if $k = l$, we have

$$k\alpha + kB = 2 = k\left(1 + \frac{2}{F'}\right),$$

which implies $(k-2)(F'+2) + 4 = 0$. From this equality, we have $k = 1$ and $F' = 2$, which contradicts the assumption $F' \geq 3$. Hence, from two equalities $k\alpha + lB = 2$ and $\alpha + B = 1 + 2/F'$, we have

$$\alpha = \frac{(2-l)F' - 2l}{F'(k-l)}, \qquad B = \frac{(k-2)F' + 2k}{F'(k-l)}.$$

Now we consider two cases. If $k > l$, then from the condition

$$\frac{F'+2}{3F'} < \alpha = \frac{(2-l)F' - 2l}{F'(k-l)},$$

we have $(k + 2l - 6)F' + 2k + 4l < 0$. From this inequality, we have $k + 2l \leq 5$. Combining with the conditions $k > l$ and $\alpha < 1$, it follows that (k, l) must be equal to one of the following:

$$(3,0), \qquad (4,0), \qquad (5,0), \qquad (2,1), \qquad (3,1).$$

If $k < l$, then from the condition $\alpha < 1$, we have $(k-2)F' + 2l < 0$, which implies $k = 0, 1$. Combining these results, it follows that the vertices must be one of the following type: $3\alpha = 2$, $4\alpha = 2$, $5\alpha = 2$, $2\alpha + B = 2$, $3\alpha + B = 2$, $lB = 2$, $\alpha + mB = 2$. Since $B < 2$, we have $l \geq 2$. In addition, the integer m is greater than 1 because $k = 1$ in this case. q.e.d.

Now, we prove Proposition 6 (I). We divide the proof into four cases according as the value of α.

(i) The case $\alpha = 2/3$.

If there exists a vertex of type $2\alpha + B = 2$, then we have $\alpha = B = 2/3$ and $F' = 6$. In this case, possible types of vertices are exhausted by $3\alpha = 2$, $2\alpha + B = 2$, $\alpha + 2B = 2$ and $3B = 2$. But, since $\alpha = B$, these types are the same, and it is easy to see that this tiling is equal to the projection of the regular hexahedron to the sphere. Hence, triangular tilings by isosceles triangles are obtained by drawing six diagonal lines in each face. Thus, the classification is purely reduced to the combinatorial examination, and the result is given as follows:

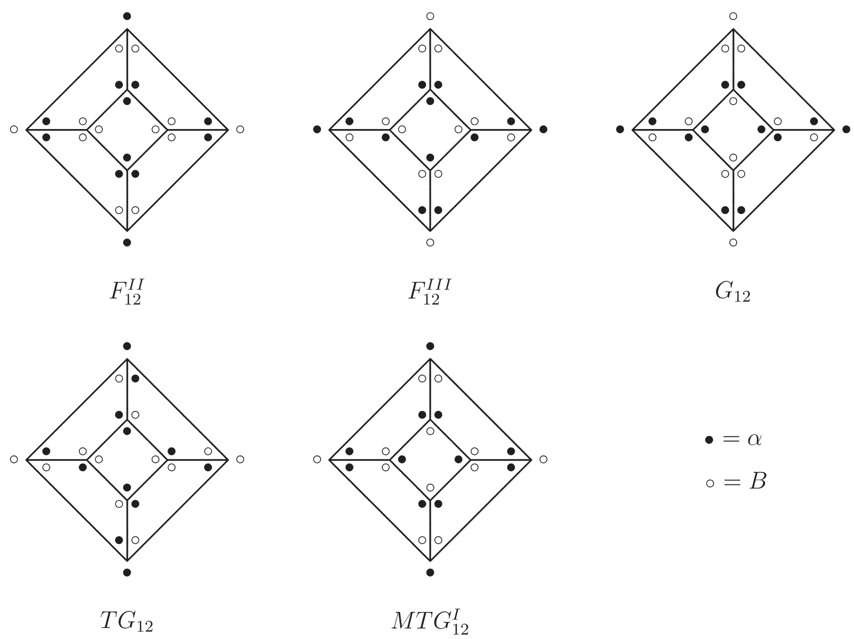


Fig. 29

Note that the vertex of type $2\alpha + B = 2$ must appear from the assumption. By drawing diagonal lines which divide the angle B in these five figures, we obtain the tilings F_{12}^{II}, F_{12}^{III}, G_{12} $(\alpha = 2/3, \beta = \gamma = 1/3)$, TG_{12} $(\alpha = 2/3, \beta = \gamma = 1/3)$ and MTG_{12}^{I}, respectively.

Next, we consider the case where a vertex of type $2\alpha + B = 2$ does not appear. Since $\alpha = 2/3$, a vertex of type $3\alpha + B = 2$ also does not exist. If a vertex of type $\alpha + mB = 2$ exists, then we have $B = 4/3m$. And from the condition $\alpha + B = 1 + 2/F'$, we have

$$m = \frac{4F'}{F' + 6} < 4,$$

and hence $m = 2, 3$. If $m = 3$, we have $B = 4/9$, and in this case, possible types of vertices are exhausted by $3\alpha = 2$ and $\alpha + 3B = 2$. Starting from the vertex $\alpha + 3B = 2$, we draw a development map of this tiling. But, as the following figure shows, this case cannot occur.

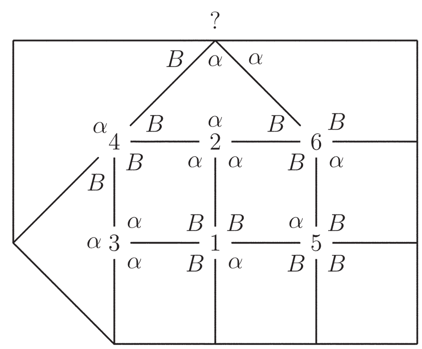


Fig. 30

(The numbers in the figure indicate the order of drawing.) Therefore, we have $m = 2$. In this case we have $\alpha = B = 2/3$ again, and the tiling is obtained by projecting the regular hexahedron to the sphere, as above. But, since a vertex of type $2\alpha + B = 2$ does not exist, possible vertices are of type $3\alpha = 2$, $\alpha + 2B = 2$ and $3B = 2$, and the vertex $\alpha + 2B = 2$ must exist from the assumption. In this situation, we can draw uniquely the following development map:

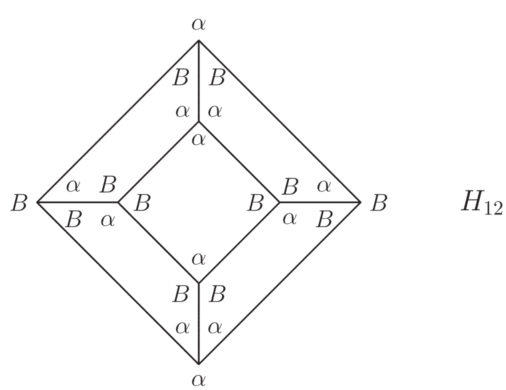


Fig. 31

By drawing diagonal lines which divide the angle B, we obtain the tiling H_{12}.

Next, we consider the case where neither vertex of type $2\alpha + B = 2$ nor $\alpha + mB = 2$ appears in the tiling. In this case, possible types of vertices are exhausted by $3\alpha = 2$ and $lB = 2$, and hence we have $B = 2/l$. From the equality $\alpha + B = 1 + 2/F'$, we have

$$l = \frac{6F'}{F' + 6} < 6,$$

and hence $l = 2, 3, 4, 5$. If $l = 2$, then we have $\alpha = 2/3$, $B = 1$, and we can uniquely draw the development map with $F' = 3$.

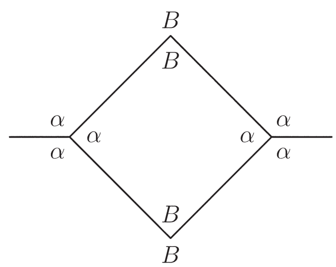
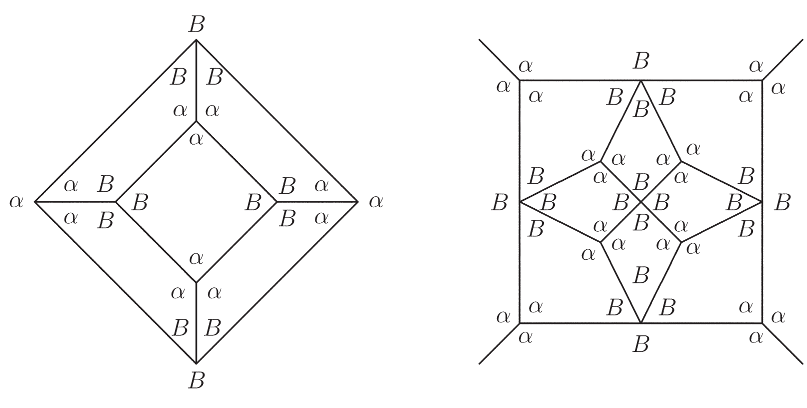


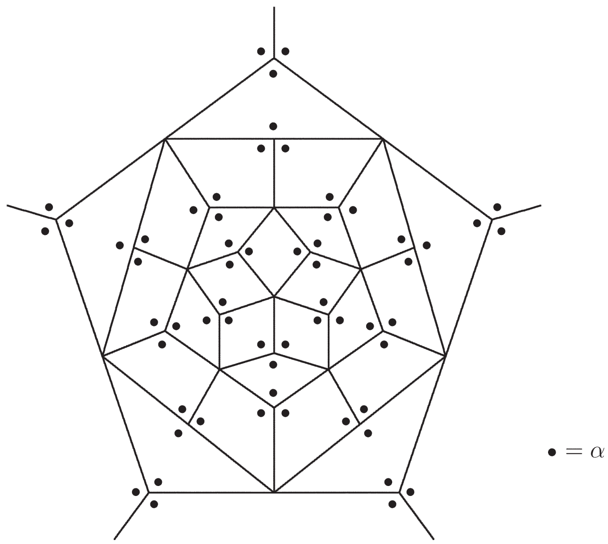
Fig. 32

Clearly, this tiling corresponds to G_6 after a suitable exchange of angles. In the same way, we can uniquely draw the development maps for the cases $l = 3$, 4 and 5 as follows:



F_{12}^{I} F_{24}

Fig. 33



F^I_{60}

Fig. 33 (continued)

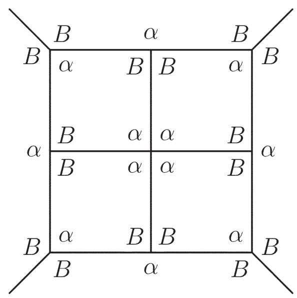
These tilings correspond to F^I_{12}, F_{24} and F^I_{60}, respectively.

(ii) The case $\alpha = 1/2$.

If a vertex of type $2\alpha + B = 2$ exists, then we have $B = 1$, and this contradicts the assumption $2\alpha > B$. If a vertex of type $3\alpha + B = 2$ exists, we have $B = 1/2$. From the equality $\alpha + B = 1 + 2/F'$, we have $2/F' = 0$, which is also a contradiction. Hence, neither vertices of type $2\alpha + B = 2$ nor $3\alpha + B = 2$ appears in the tiling. If a vertex of type $\alpha + mB = 2$ exists, we have $B = 3/2m$. Then, from the equality $\alpha + B = 1 + 2/F'$, we have

$$m = \frac{3F'}{F' + 4} < 3,$$

which implies $m = 2$. Hence, we have $\alpha = 1/2$, $B = 3/4$ and $F' = 8$. In this situation, possible types of vertices are exhausted by $4\alpha = 2$ and $\alpha + 2B = 2$, and the development map can be uniquely drawn as follows:



H_{16}

Fig. 34

Then, it is easy to see that this tiling corresponds to H_{16} after a suitable exchange of angles.

If a vertex of type $\alpha + mB = 2$ does not appear, possible types of vertices are exhausted by $4\alpha = 2$ and $lB = 2$. In this case, we have $B = 2/l$, and from the equality $\alpha + B = 1 + 2/F'$, we have

$$l = \frac{4F'}{F' + 4} < 4,$$

which implies $l = 2, 3$. If $l = 2$, we have $B = 1$, and this contradicts the assumption $2\alpha > B$. If $l = 3$, then possible vertices are of type $4\alpha = 2$ and $3B = 2$, and we can uniquely draw the following development map:

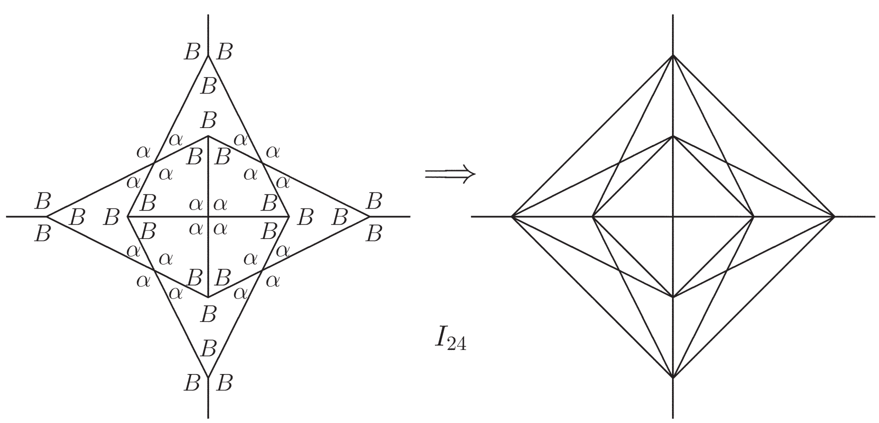


Fig. 35

By dividing the angle B, we obtain the tiling I_{24}.

(iii) The case $\alpha = 2/5$.

If a vertex of type $2\alpha + B = 2$ exists, we have $B = 6/5$, which contradicts the assumption $2\alpha > B$. By the same reason, we can show the non-existence of a vertex of type $3\alpha + B = 2$. If a vertex of type $\alpha + mB = 2$ exists, then we have $B = 8/5m$. From the equality $\alpha + B = 1 + 2/F'$, we have

$$m = \frac{8F'}{3F' + 10} < \frac{8}{3},$$

which implies $m = 2$. Then, we have $B = 4/5$, which also contradicts the assumption $2\alpha > B$. Hence, a vertex of type $\alpha + mB = 2$ does not exist. Therefore, possible vertices are exhausted by $5\alpha = 2$ and $lB = 2$. From the equalities $B = 2/l$ and $\alpha + B = 1 + 2/F'$, we have

$$l = \frac{10F'}{3F' + 10} < \frac{10}{3}.$$

On the other hand, from the assumption $2\alpha > B$, we have $2l > 5$, and hence $l = 3$. Then, possible vertices are of type $5\alpha = 2$ and $3B = 2$, and we can uniquely draw the following development map:

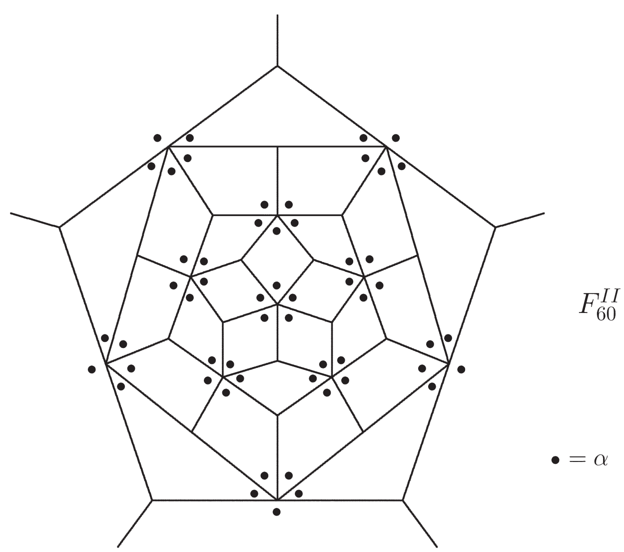


Fig. 36

This tiling corresponds to F_{60}^{II}.

(iv) The case $\alpha \neq \frac{2}{3}, \frac{1}{2}, \frac{2}{5}$.

In this case, we first prepare the following lemma.

Lemma 8. *Assume* $\alpha \neq \frac{2}{3}, \frac{1}{2}, \frac{2}{5}$. *Then any monohedral tiling of the* 2-*dimensional sphere by rhombuses with angles* $\alpha\pi$, $B\pi$, $\alpha\pi$, $B\pi$ $(2\alpha > B, F' \geq 3)$

contains a vertex of type $2\alpha + B = 2$*, but does not contain a vertex of type* $3\alpha + B = 2$.

Proof. First, we assume that a vertex of type $3\alpha + B = 2$ exists. Then, a vertex of type $2\alpha + B = 2$ cannot exist because $\alpha > 0$. If there exists a vertex of type $\alpha + mB = 2$, we have

$$\alpha = \frac{2m-2}{3m-1}, \qquad B = \frac{4}{3m-1}.$$

Then, from the condition $\alpha + B = 1 + 2/F'$, we have $(3-m)F' = 2(3m-1) > 0$, which implies $m = 2$. But in this case, we have $\alpha = 2/5$, which contradicts the assumption $\alpha \neq 2/5$. Therefore, a vertex of type $\alpha + mB = 2$ cannot exist. Hence, possible types of vertices are exhausted by $3\alpha + B = 2$ and $lB = 2$. If a vertex $lB = 2$ does not appear in the tiling, then all vertices are of type $3\alpha + B = 2$. But this case cannot occur because the number of angles α and B appearing in the tiling must coincide. Hence, a vertex of type $lB = 2$ exists, and we have

$$\alpha = \frac{2l-2}{3l}, \qquad B = \frac{2}{l}.$$

From the equality $\alpha + B = 1 + 2/F'$, we have $(4-l)F' = 6l > 0$. On the other hand, from the assumption $2\alpha > B$, we have $2l > 5$, and hence $l = 3$. Therefore, the types of vertices are $3\alpha + B = 2$ and $3B = 2$. Then, starting from the vertex $3\alpha + B = 2$, we draw the development map as follows:

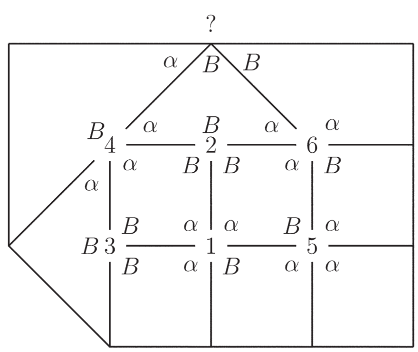


Fig. 37

As this figure shows, a contradiction occurs and this tiling cannot exist. Hence, a vertex of type $3\alpha + B = 2$ does not exist.

Next, assume that a vertex of type $2\alpha + B = 2$ does not appear in the tiling. Then, from Lemma 7, possible types of vertices are exhausted by

$lB = 2$ and $\alpha + mB = 2$ $(l, m \geq 2)$. Clearly, a vertex of type $\alpha + mB = 2$ must appear in the tiling. Then, starting from this vertex $\alpha + mB = 2$, we draw a development map. But, as the following figure shows, this case cannot occur.

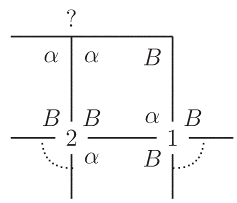


Fig. 38

Therefore, a vertex of type $2\alpha + B = 2$ must exist in the tiling. q.e.d.

Now, under this preparation, we complete the proof of Proposition 6 (I). We first consider the case where a vertex of type $\alpha + mB = 2$ exists. Combining with the equality $2\alpha + B = 2$, we have

$$\alpha = \frac{2m-2}{2m-1}, \qquad B = \frac{2}{2m-1}.$$

Then, from the equality $\alpha + B = 1 + 2/F'$, we have $F' = 4m - 2$. From the assumption $\alpha \neq 2/3$, we have $m \neq 2$, and hence $m \geq 3$. In this situation, possible types of vertices are exhausted by

$$2\alpha + B = 2, \qquad \alpha + mB = 2, \qquad (2m-1)B = 2.$$

Then, starting from the vertex $\alpha + mB = 2$, we can essentially uniquely draw the development map as follows:

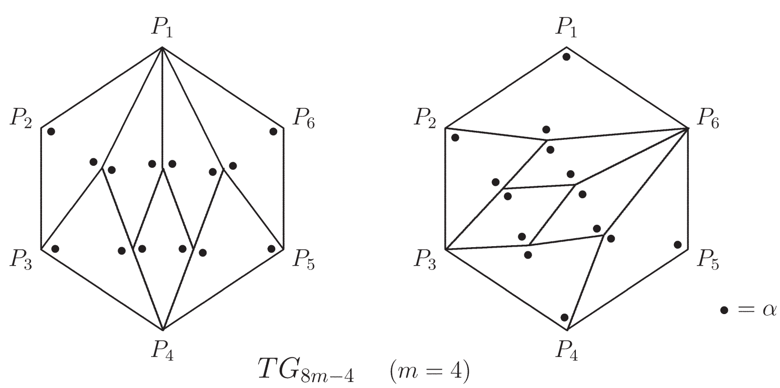
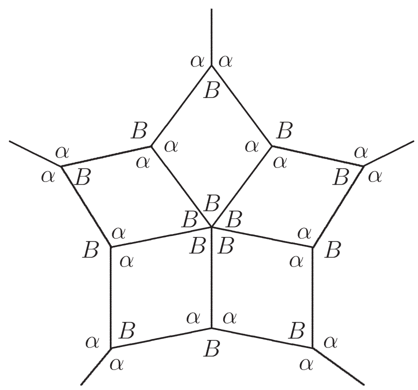


Fig. 39

(We start to draw this figure from the point P_1. As before, join two figures such that two P_i's express the same point in the sphere.) By dividing the angle B, we can easily see that this tiling corresponds to TG_{8n+4} with $\alpha = 2n/(2n+1)$ and $\beta = \gamma = 1/(2n+1)$, where we put $n = m - 1 \geq 2$.

Next, assume that a vertex of type $\alpha + mB = 2$ does not appear in the tiling. In this case, possible types of vertices are $2\alpha + B = 2$ and $lB = 2$. If a vertex $lB = 2$ does not appear in the tiling, we are lead to a contradiction by the same reason as stated in the proof of Lemma 8. Hence, a vertex of type $lB = 2$ actually exists, and we have $\alpha = (l-1)/l$, $B = 2/l$. From the equality $\alpha + B = 1 + 2/F'$, we have $F' = 2l$. Since $\alpha \neq 1/2, 2/3$, we have $l \neq 2, 3$, and hence $l \geq 4$. In this case, we can uniquely draw the development map as follows:



G_{4l} $(l = 5)$

Fig. 40

By dividing the angle B, we obtain the tiling G_{4n} with $\alpha = (n-1)/n$ and $\beta = \gamma = 1/n$, where we put $n = l \geq 4$.

Combining (i)$\sim$(iv), we complete the proof of Proposition 6 (I).

Next, we consider the case (II): $\alpha = \beta > \gamma$. In this case, we can show that $a = b > c$, by using the cosine rule. Hence, the sphere is tiled by the rhombus with angles $\gamma\pi$, $2\alpha\pi$, $\gamma\pi$, $2\alpha\pi$.

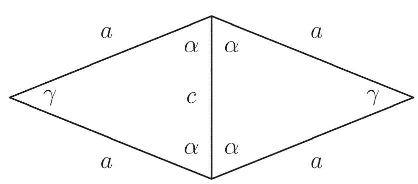


Fig. 41

We put $A = 2\alpha$ and $F = 2F'$, in this case. Then, from Proposition 4, these angles satisfy the following conditions:

$$A + \gamma = 1 + \frac{2}{F'}, \qquad 0 < 2\gamma < A < 1 + \gamma < 2.$$

In particular, the angle γ satisfies the inequality

$$\frac{1}{F'} < \gamma < \frac{F' + 2}{3F'}.$$

Of course, we have $F' \geq 3$ from the assumption $F > 4$. Now, we prove the following lemma.

LEMMA 9. *The type of vertices appearing in the tiling by the rhombus with angles $\gamma\pi$, $A\pi$, $\gamma\pi$, $A\pi$ $(A > 2\gamma, F' \geq 3)$ is one of the following:*

$$\begin{aligned} &2A = 2, \\ &2A + \gamma = 2, \\ &l\gamma = 2 \qquad &(l \geq 3), \\ &A + m\gamma = 2 \qquad &(m \geq 2). \end{aligned}$$

PROOF. Assume that there exists a vertex of type $kA + l\gamma = 2$. Then, as in the case of Lemma 7, we can show that $k \neq l$. Then, combined with the equality $A + \gamma = 1 + 2/F'$, we have

$$A = \frac{(2 - l)F' - 2l}{F'(k - l)}, \qquad \gamma = \frac{(k - 2)F' + 2k}{F'(k - l)}.$$

If $k > l$, then from the condition $\gamma < (F' + 2)/(3F')$, we have $(2k + l - 6)F' + 4k + 2l < 0$, which implies $2k + l \leq 5$. Hence, by using the condition $A < 2$, we have $(k, l) = (2, 0), (2, 1)$. If $k < l$, then from the condition $1/F' < \gamma$, we have $(k - 2)F' + k + l < 0$, which implies $k = 0, 1$. If $k = 0$, then we have $l \geq 3$ because $\gamma < 1$. In case $k = 1$, then from the assumption $k > l$, we have $l \geq 2$. q.e.d.

Using this lemma, we prove Proposition 6 (II). We divide the proof into two cases according as the value of A.

(i) The case $A = 1$.

Since $\gamma > 0$, a vertex of type $2A + \gamma = 2$ cannot exist. If a vertex of type $A + m\gamma = 2$ appears in the tiling, we have $\gamma = 1/m$. Since $2\gamma < A$, we have $m \geq 3$. In addition, from the equality $A + \gamma = 1 + 2/F'$, we have $F' = 2m$. Then, possible types of vertices are exhausted by $2A = 2$, $A + m\gamma = 2$ and

$2m\gamma = 2$. Starting from the vertex $A + m\gamma = 2$, we can uniquely draw the development map as follows:

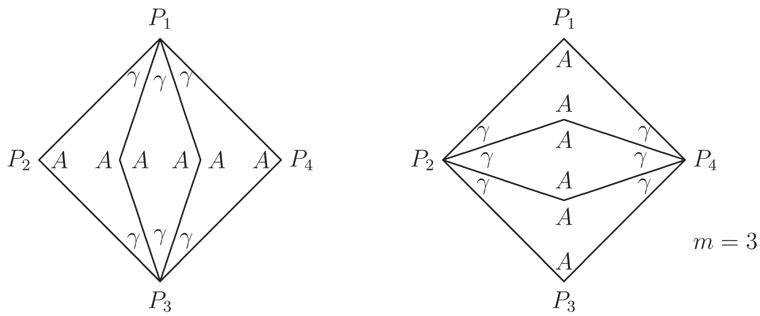


Fig. 42

(As before, we join two figures such that two P_i's express the same point in the sphere.) By dividing the angle A, we obtain the tiling TG_{8n} or TG_{8n+4} with $\alpha = \beta = 1/2$ and $\gamma = 1/(2n+1)$, where we put $m = 2n$ $(n \geq 2)$ or $m = 2n+1$ $(n \geq 1)$, respectively.

If a vertex of type $A + m\gamma = 2$ does not exist, then possible types of vertices are exhausted by $2A = 2$ and $l\gamma = 2$, and both vertices must actually appear. Then, we have $\gamma = 2/l$, and from the equality $A + \gamma = 1 + 2/F'$, we have $F' = l$. In this case, from the condition $2\gamma < A$, we have $l \geq 5$, and we can uniquely draw the following development map:

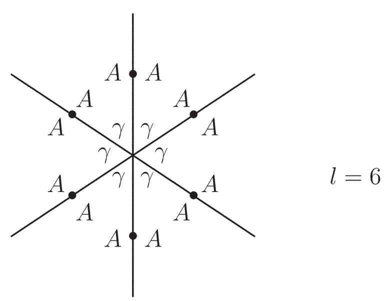


Fig. 43

By dividing the angle A as above, we obtain the tiling G_{4n} with $\alpha = \beta = 1/2$ and $\gamma = 1/n$ or G_{4n+2}, where we put $l = 2n$ $(n \geq 3)$ or $l = 2n+1$ $(n \geq 2)$, respectively.

(ii) The case $A \neq 1$.

Assume that a vertex of type $2A + \gamma = 2$ does not exist in the tiling. Then, possible types of vertices are exhausted by $l\gamma = 2$ and $A + m\gamma = 2$.

Since $m \geq 2$, both vertices must appear in the tiling by the same reason as stated in the proof of Lemma 8. But, as the following figure shows, this tiling cannot exist.

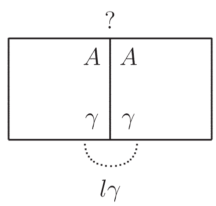


Fig. 44

Hence, a vertex of type $2A + \gamma = 2$ must appear in the tiling. If a vertex of type $A + m\gamma = 2$ exists in the tiling, we have

$$A = \frac{2m-2}{2m-1}, \qquad \gamma = \frac{2}{2m-1}$$

from these two equalities. Then, from the equality $A + \gamma = 1 + 2/F'$, we have $F' = 4m - 2$. In addition, from the condition $2\gamma < A$, we have $m \geq 4$. In this situation, possible types of vertices are exhausted by $2A + \gamma = 2$, $A + m\gamma = 2$ and $(2m-1)\gamma = 2$. Then, as in the case of Figure 39, we can essentially uniquely draw the following development map by starting from the vertex $A + m\gamma = 2$:

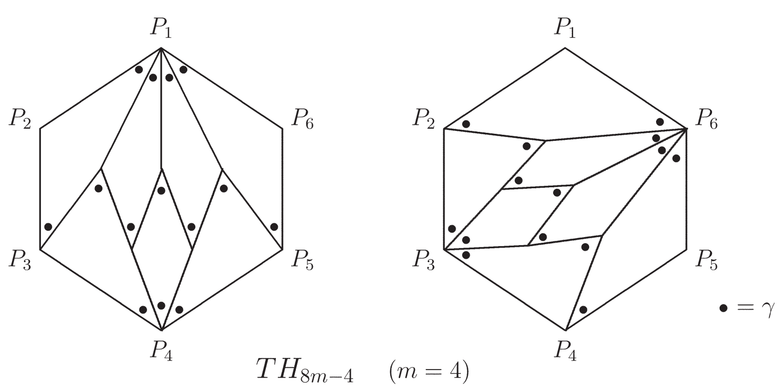


Fig. 45

By dividing the angle A, we obtain the tiling TH_{8n+4}, where we put $n = m - 1 \geq 3$.

If a vertex of type $A + m\gamma = 2$ does not exist, then possible types of vertices are exhausted by $2A + \gamma = 2$ and $l\gamma = 2$. From the same reason as above, both vertices must actually appear. Then, from these two equalities, we have $A = (l-1)/l$ and $\gamma = 2/l$. And from the equality $A + \gamma = 1 + 2/F'$, we have $F' = 2l$. In this situation, we have $l \geq 6$ on account of the condition $2\gamma < A$, and we can uniquely draw the following development map:

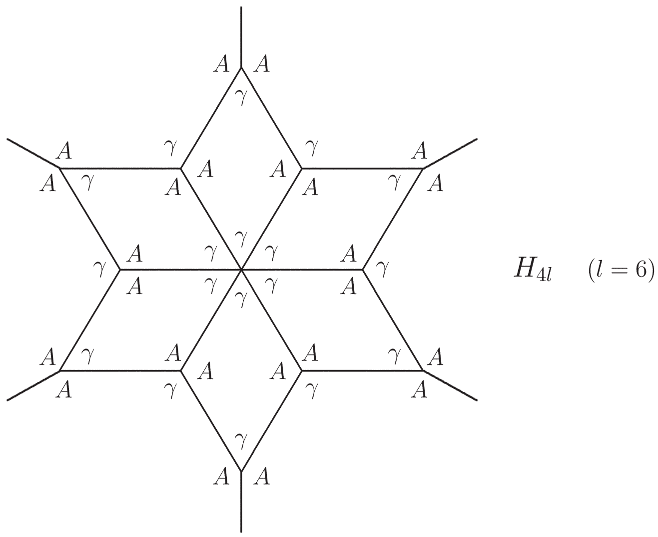


Fig. 46

By dividing the angle A, we obtain the tiling H_{4n}, where we put $n = l \geq 6$.

Combining these results, we complete the proof of Proposition 6 (II).

5. Case of scalene triangles I: Determination of the type of vertices

Next, we consider monohedral tilings by scalene triangles with $F > 4$. In this section, in order to classify these tilings, we first determine the type of vertices appearing in the tiling as a preliminary step. In the subsequent sections, we carried out the classification based on the results of this section. In the following, we always assume $0 < \gamma < \beta < \alpha < 1$, unless otherwise stated.

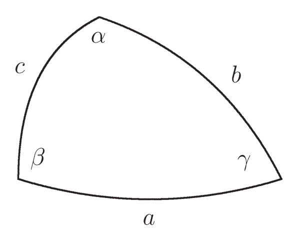


Fig. 47

In this case, by using the cosine rule, we can show that $a > b > c$. Hence, for example, the edge a must be touched to the other triangle in one of the following way:

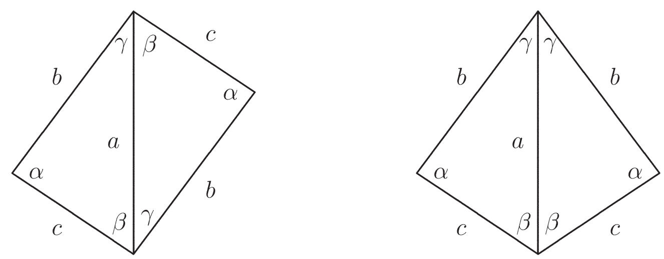


Fig. 48

From Proposition 4, these angles α, β, γ satisfy the following conditions:

$$\alpha + \beta + \gamma = 1 + \frac{4}{F}, \qquad \frac{2}{F} < \gamma < \beta < \alpha < 1.$$

In addition, α satisfies the inequality $\alpha > 1/3$ because $3\alpha > \alpha + \beta + \gamma > 1$.

Now, we determine the type of vertices appearing in the tiling. For this purpose, we first consider the following purely combinatorial figure: Let P be a point in the 2-dimensional plane, surrounded by n triangles ($n \geq 3$). We assume that these triangles satisfy the following conditions:

(i) Three vertices of each triangle are marked by three symbols λ, μ and ν in some order.

(ii) The end points of two contact edges of two triangles are marked by symbols in one of the following way (allowing the exchange of symbols λ, μ, ν):

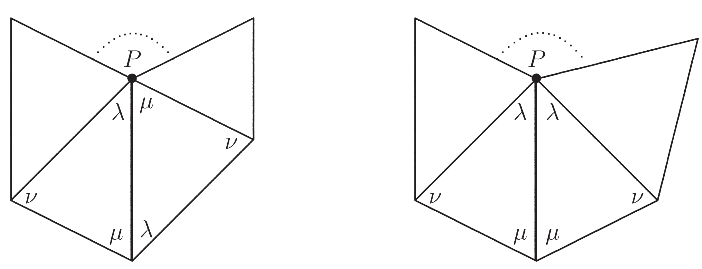


Fig. 49

For example, we are considering a figure such as

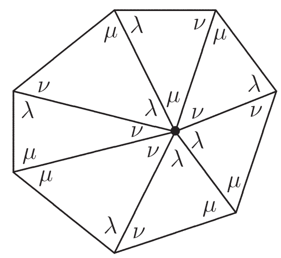


Fig. 50

where • indicates the point P. If P is surrounded by k λ's, l μ's and m v's, then we say that the type of this figure is $k\lambda + l\mu + mv$. (We ignore the order of symbols around P in this expression.) Under these conditions, we classify possible types of this combinatorial figure. It should be remarked that the symbols λ, μ, v do not express the angles of the triangle. But if the sphere is tiled by one scalene triangle with angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$, then we clearly obtain a combinatorial figure satisfying the above two conditions (i), (ii) at each vertex by replacing three angles by the symbols λ, μ, v. Thus, the above figure is an abstract combinatorial object of the surroundings of a point of the tiling. Now we prove the following lemma.

Lemma 10. *The type of this figure is expressed in the form* $(2p+1)\lambda + (2q+1)\mu + (2r+1)v$ *or* $2p\lambda + 2q\mu + 2rv$.

Proof. If P is surrounded by three triangles, then it is easy to see that the figure is essentially in the following form:

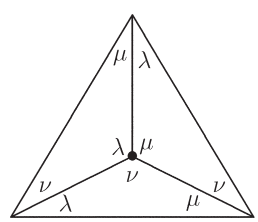


Fig. 51

Hence, in this case, the type of this figure is $\lambda + \mu + v$. In the 4-valent case, after some examinations, we can show that the type of this figure is 4λ or $2\lambda + 2\mu$, by exchanging the symbols λ, μ, v suitably.

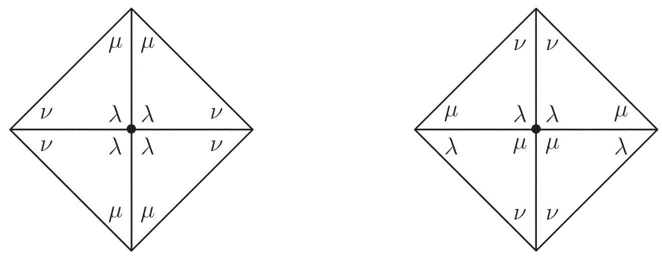


Fig. 52

Hence, the lemma holds for these two cases.

Now, we consider the k-valent case $(k \geq 5)$. If there exist two adjacent triangles possessing the same symbols at P as in the following figure, we may delete these two and join the remaining two edges, because both edges possess the symbols λ and ν at their end points.

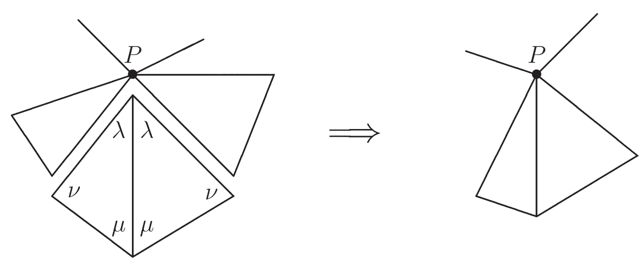


Fig. 53

Clearly, the resulting figure is $(k-2)$-valent. We repeat this procedure for several times. Then, finally, we obtain a 3- or 4-valent figure, or a figure whose adjacent symbols at P always differ. In the 3- or 4-valent case, the initial vertex is conversely obtained by adding symbols 2λ (or 2μ, 2ν) to $\lambda + \mu + \nu$, 4λ, $2\lambda + 2\mu$ (or their exchanged ones). Hence, it must be of the form $(2p+1)\lambda + (2q+1)\mu + (2r+1)\nu$ or $2p\lambda + 2q\mu + 2r\nu$. In the latter case, the same symbol does not appear adjacently around P, and hence, it must be of the form:

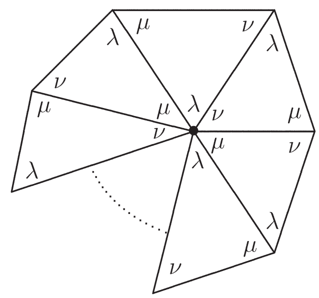


Fig. 54

In particular, this figure is of type $k(\lambda+\mu+\nu)$ for some $k \geq 1$. Hence, the initial figure is obtained by adding 2λ, 2μ or 2ν to $k(\lambda+\mu+\nu)$ for several times, and thus we arrive at the same conclusion as above. This completes the proof of the lemma. q.e.d.

Using this lemma, we prove the following proposition, giving a classification of the type of vertices for the scalene case.

Proposition 11. *The type of vertices appearing in the monohedral tiling of the 2-dimensional sphere by scalene triangles with angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$ $(2/F < \gamma < \beta < \alpha < 1$ and $F > 4)$ must be one of the following*:

$$\alpha + (2q+1)\beta + (2r+1)\gamma = 2 \qquad q, r \geq 0, \ (q, r) \neq (0, 0),$$

$$3\alpha + \beta + \gamma = 2,$$

$$4\alpha = 2,$$

$$2\alpha + 2\beta = 2,$$

$$2\alpha + 2s\gamma = 2 \qquad s \geq 1,$$

$$2t\beta + 2u\gamma = 2 \qquad t, u \geq 1,$$

$$2v\beta = 2 \qquad v \geq 2,$$

$$2w\gamma = 2 \qquad w \geq 2.$$

Proof. We first consider the odd valent case. Assume that there exists an odd valent vertex in the tiling. Then, from Lemma 10, its type is expressed as $(2p+1)\alpha + (2q+1)\beta + (2r+1)\gamma = 2$ $(p, q, r \geq 0)$. For some time, we drop the assumption $\gamma < \beta < \alpha$, but instead, assume that $p \geq q \geq r \geq 0$. Then, from the conditions $\alpha + \beta + \gamma > 1$ and $(2r+1)\gamma = 2 - (2p+1)\alpha - (2q+1)\beta$, we have

$$\begin{aligned}2r+1 &< (2r+1)(\alpha+\beta+\gamma)\\ &= (2r+1)\alpha+(2r+1)\beta+(2r+1)\gamma\\ &= 2-2(p-r)\alpha-2(q-r)\beta.\end{aligned}$$

Hence, we have $0 \le 2(p-r)\alpha+2(q-r)\beta < 1-2r$, which implies $r=0$. Hence, reviving the condition $\gamma<\beta<\alpha$ again, we have $\min\{p,q,r\}=0$ and we know that possible vertices are of type $(2p+1)\alpha+(2q+1)\beta+\gamma=2$, $(2p+1)\alpha+\beta+(2r+1)\gamma=2$ and $\alpha+(2q+1)\beta+(2r+1)\gamma=2$ $(p,q,r\ge 0)$. But a vertex of type $\alpha+\beta+\gamma=2$ cannot exist because $\alpha+\beta+\gamma=1+4/F$ <2 from the assumption.

Now, assume that a vertex of type $(2p+1)\alpha+(2q+1)\beta+\gamma=2$ $(p\ge 1)$ exists. Then, combining with the condition $\alpha+\beta+\gamma=1+4/F$, we have

$$\begin{aligned}(p-q)\beta+p\gamma &= \frac{(2p-1)F+8p+4}{2F},\\ (q-p)\alpha+q\gamma &= \frac{(2q-1)F+8q+4}{2F},\\ p\alpha+q\beta &= \frac{F-4}{2F}.\end{aligned}$$

Now, assume $q\ge 1$. Then, from the second equality, we have

$$(2q-p)\alpha > (q-p)\alpha+q\gamma = \frac{(2q-1)F+8q+4}{2F} > 0,$$

which implies $2q>p$. Next, by subtracting the third equality from the first, we have

$$(p-2q)\beta-p(\alpha-\gamma) = \frac{(p-1)F+4p+4}{F} > 0,$$

and hence, $(p-2q)\beta > p(\alpha-\gamma) > 0$. From this inequality, we have $p>2q$, but this contradicts the above inequality $2q>p$. Therefore, we have $q=0$. Then, from the first and the third equalities, we have

$$\alpha = \frac{F-4}{2pF}, \qquad \beta+\gamma = \frac{(2p-1)F+8p+4}{2pF}.$$

Substituting these values to the inequality $2\alpha>\beta+\gamma$, we have $(2p-3)F+8p+12<0$, which implies $p=1$. Hence, the vertex must be of type $3\alpha+\beta+\gamma=2$.

Next, we consider the vertex of type $(2p+1)\alpha+\beta+(2r+1)\gamma=2$ $(p,r\ge 1)$. If this type exists, then combined with the equality $\alpha+\beta+\gamma=1+4/F$, we have

$$p\beta + (p - r)\gamma = \frac{(2p-1)F + 8p + 4}{2F},$$
$$p\alpha + r\gamma = \frac{F-4}{2F},$$
$$(r-p)\alpha + r\beta = \frac{(2r-1)F + 8r + 4}{2F}.$$

Then, from the third equality, we have

$$(2r-p)\alpha > (r-p)\alpha + r\beta = \frac{(2r-1)F + 8r + 4}{2F} > 0,$$

which implies $2r > p$. Next, taking a difference of the first and the second equality, we have

$$(p-2r)\gamma - p(\alpha - \beta) = \frac{(p-1)F + 4p + 4}{F} > 0.$$

In particular, we have $(p-2r)\gamma > p(\alpha - \beta) > 0$, and hence $p > 2r$. This contradicts the above inequality $2r > p$, and therefore this case does not occur.

Combining these results, it follows that odd valent vertices are of type $3\alpha + \beta + \gamma = 2$ or $\alpha + (2q+1)\beta + (2r+1)\gamma = 2$ $(q, r \geq 0, (q, r) \neq (0, 0))$.

Next, we consider the even valent case. From Lemma 10, it is of type $2p\alpha + 2q\beta + 2r\gamma = 2$ $(p, q, r \geq 0)$. Then, by the same argument as in the odd valent case, we can prove $\min\{p, q, r\} = 0$, and hence, the type is $2p\alpha + 2q\beta = 2$, $2p\alpha + 2r\gamma = 2$ or $2q\beta + 2r\gamma = 2$. We first treat the case $2p\alpha + 2q\beta = 2$ $(p \geq 1)$. If $q = 0$, then from the condition $1/3 < \alpha < 1$, we have $p = 2$. In case $q \geq 1$, combined with the equality $\alpha + \beta + \gamma = 1 + 4/F$, we have

$$\beta = \frac{1 - p\alpha}{q}, \qquad \gamma = \frac{(p-q)\alpha F + (q-1)F + 4q}{qF}.$$

Since $\beta > \gamma$, we obtain the inequality

$$(q-2)F + 4q < (q-2p)\alpha F. \tag{$*$}$$

Now assume $q > 2p$. Then from this inequality, we have

$$\frac{(q-2)F + 4q}{(q-2p)F} < \alpha < 1,$$

from which we have $(p-1)F + 2q < 0$. But this contradicts the assumption $p, q \geq 1$, and hence, we have $q \leq 2p$. Then, from the above inequality $(*)$ again, we have $(q-2)F + 4q < 0$, which implies $q = 1$. Thus, we have

$$\beta = 1 - p\alpha, \qquad \gamma = \frac{(p-1)\alpha F + 4}{F},$$

and hence, from the condition $\gamma < \beta < \alpha$, we have

$$\frac{1}{p+1} < \alpha < \frac{F-4}{(2p-1)F}.$$

From this inequality, we have $(p-2)F + 4p + 4 < 0$, and we arrive at the conclusion $p = 1$. Therefore, the vertex is of type $2\alpha + 2\beta = 2$.

Next, we consider the case $2p\alpha + 2r\gamma = 2$ $(p, r \geq 1)$. In this case, we have

$$2 = 2p\alpha + 2r\gamma > 2p\alpha > \frac{2p}{3}$$

because $\alpha > 1/3$. Hence, we have $p = 1$ or 2. Now assume $p = 2$. Then we have $4\alpha + 2r\gamma = 2$. Combined with the equality $\alpha + \beta + \gamma = 1 + 4/F$, we have

$$\beta = \frac{(2-r)\alpha F + (r-1)F + 4r}{rF}, \qquad \gamma = \frac{1-2\alpha}{r}.$$

Then, from the condition $\alpha > \beta$, we have $2(r-1)\alpha F > (r-1)F + 4r > 0$. From this inequality, we have $r \geq 2$ and

$$\alpha > \frac{(r-1)F + 4r}{2(r-1)F}.$$

On the other hand, from the condition $\gamma > 2/F$, we have $\alpha < (F - 2r)/(2F)$. Hence we have

$$\frac{(r-1)F + 4r}{2(r-1)F} < \frac{F-2r}{2F},$$

from which the inequality $r^2 + r < 0$ follows. This is a contradiction, and therefore the case $p = 2$ does not occur. Hence we have $p = 1$, and the vertex is of type $2\alpha + 2r\gamma = 2$ $(r \geq 1)$.

For the third case $2q\beta + 2r\gamma = 2$, we can easily check that the pair (q, r) must satisfy the condition $p, q \geq 1$ or $p \geq 2$, $q = 0$ or $p = 0$, $q \geq 2$.

Combining these results, we complete the proof of Proposition 11. q.e.d.

Remark. (1) As for two vertices $\alpha + (2q+1)\beta + (2r+1)\gamma = 2$ and $2t\beta + 2u\gamma = 2$ in the above list, there may appear several types of vertices in one tiling because they contain two independent parameters. On the contrary, as for the remaining vertices, the type of vertices is uniquely determined if they exist. In fact, for example, if vertices of type $2\alpha + 2s\gamma = 2$ and $2\alpha + 2s'\gamma = 2$ exist, we have clearly $s = s'$ and the type is uniquely determined.

(2) As a result of the classification that will be carried out in §6∼§8, we know that the following types of vertices in Proposition 11 do not actually

appear in the tilings by scalene triangles with $F > 4$:

$$\alpha + (2q+1)\beta + (2r+1)\gamma = 2 \qquad q, r \geq 1,$$

$$3\alpha + \beta + \gamma = 2,$$

$$2\beta + 2u\gamma = 2 \qquad u = 1, 2,$$

$$4\beta + 2u\gamma = 2 \qquad u \geq 2,$$

$$2t\beta + 2u\gamma = 2 \qquad t \geq 3, u \geq 1,$$

$$4\gamma = 2.$$

Among these, the non-existence of a vertex of type $3\alpha + \beta + \gamma = 2$ is a quite delicate result, depending on the topology of the sphere. For details, see Appendix. But, if we consider a partial tiling around only one point of the sphere, the above vertices can all exist. For example, a vertex of type $\alpha + (2q+1)\beta + (2r+1)\gamma = 2$ $(q, r \geq 1)$ appears in the following way:

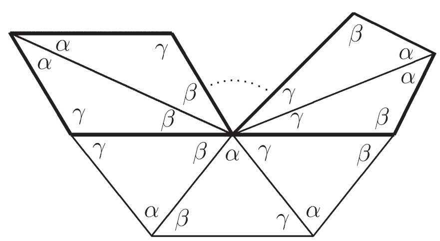


Fig. 55

(As for the vertex $3\alpha + \beta + \gamma = 2$, see Appendix.)

(3) Davies [4; p. 44] already obtained the same result as this proposition (but without giving a detailed proof). There, he wrote that other types of vertices are "prohibited by the conditions $\alpha > \beta > \gamma$ and $\alpha + \beta + \gamma > 1$" (in our notation). But actually, we need additional conditions to prove Proposition 11. In fact, for example, consider the case $\alpha = 3/5$, $\beta = 1/2$ and $\gamma = 3/10$. Then by putting $F = 10$, these angles satisfy the conditions $\alpha + \beta + \gamma = 1 + 4/F$, $2/F < \gamma < \beta < \alpha < 1$ and $2\alpha + \beta + \gamma = 2$. Hence, there may exist a vertex of type $2\alpha + \beta + \gamma = 2$ in the tiling, and in order to exclude this type of vertex, we need the combinatorial argument as in Lemma 10.

6. Case of scalene triangles II: Odd tilings (1)

Now, under these preliminaries, we classify monohedral tilings of the 2-dimensional sphere by scalene triangles with $F > 4$. We carry out the classification by considering two cases. In §6 and §7, we treat the case where a tiling contains an odd valent vertex, and in § 8, we classify the remaining tilings containing only even valent vertices. In the following, we say that a tiling is *odd* if it contains an odd valent vertex, and *even* if all vertices are even valent. The purpose of § 6 and § 7 is to prove the following proposition.

PROPOSITION 12. *Monohedral odd tilings of the* 2*-dimensional sphere by scalene triangles with* $F > 4$ *are given by the following*:

(i) TG_{8n+4} $\left(\alpha+\beta=1, \gamma=\dfrac{1}{2n+1}, \dfrac{1}{2}<\alpha<\dfrac{2n}{2n+1}, n \geq 1\right)$,

(ii) TG_{8n+4} $\left(\alpha+\beta=1, \gamma=\dfrac{1}{2n+1}, \dfrac{2n}{2n+1}<\alpha<\dfrac{4n+1}{4n+2}, n \geq 1\right)$,

(iii) MTG^{I}_{8n+4} $(n \geq 2)$,

(iv) MTG^{II}_{8n+4} $(n \geq 2)$.

We remark that for the tiling (ii) in the above list, we impose the condition $\beta<\gamma<\alpha$ instead of $\gamma<\beta<\alpha$ in order to make the list in a consistent form. Clearly, the boundary case $\alpha = 2n/(2n+1)$ between (i) and (ii) corresponds to the isosceles triangle with $\beta=\gamma$ (see Proposition 6 (I)). But to avoid confusion, we always assume $\gamma<\beta<\alpha$ in the following arguments as before.

In this section, to prove this proposition, we prepare four lemmas which exclude several types of vertices in Proposition 11 for the odd case. And by using these results, we prove Proposition 12 in the next section.

LEMMA 13. *Under the same conditions as in Proposition* 12, *the tiling must contain a vertex of type* $\alpha+(2q+1)\beta+(2r+1)\gamma=2$ *for some* $(q,r) \neq (0,0)$.

PROOF. Assume that there exists an odd tiling not containing a vertex of type $\alpha+(2q+1)\beta+(2r+1)\gamma=2$. Then, by Proposition 11, this tiling must contain a vertex of type $3\alpha+\beta+\gamma=2$. Combined with the equality $\alpha+\beta+\gamma=1+4/F$, we have

$$\alpha=\frac{F-4}{2F}, \qquad \beta+\gamma=\frac{F+12}{2F}.$$

In addition, from the condition $2\alpha>\beta+\gamma$, we have $F>20$. Since $\alpha = (F-4)/2F < 1/2$, a vertex of type $4\alpha=2$ cannot exist in the tiling. A vertex of type $2\alpha+2\beta=2$ also cannot exist because $\beta<\alpha<1/2$.

Therefore, from Proposition 11, possible types of vertices are exhausted by

$$\begin{aligned}
&3\alpha+\beta+\gamma=2, \\
&2\alpha+2s\gamma=2 \qquad s \geq 1, \\
&2t\beta+2u\gamma=2 \qquad t,u \geq 0.
\end{aligned}$$

In this situation, we draw the development map, starting from the vertex $3\alpha+\beta+\gamma=2$. We remark that if a vertex contains the angles α and β, then it must be of type $3\alpha+\beta+\gamma=2$, and it is easy to see that three angles 3α must appear adjacently around this vertex.

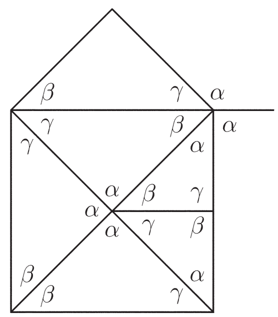


Fig. 56

Then, this figure shows that there exists a vertex containing the angles $\beta+2\gamma$. Hence, this tiling must contain a vertex of type $2t\beta+2u\gamma=2$ with $t,u\geq 1$.

Assume $t=u$. Then, we have $1=t\beta+t\gamma=t(F+12)/2F$. From this equality, we have $(t-2)F+12t=0$, which implies $t=1$ and $F=12$. This contradicts $F>20$, and therefore, we have $t\neq u$. Then, from two equalities $\beta+\gamma=(F+12)/2F$ and $t\beta+u\gamma=1$, we have

$$\beta=\frac{(2-u)F-12u}{2(t-u)F},\qquad \gamma=\frac{(t-2)F+12t}{2(t-u)F}.$$

Now, we divide the situation into the following two cases (i) and (ii).

(i) The case there exists a vertex of type $2t\beta+2u\gamma=2$ satisfying $t>u\geq 1$.

In this case, from the condition $\beta>\gamma$, we have $(t+u-4)F+12t+12u<0$. Hence, we have $t+u\leq 3$, which implies $t=2$ and $u=1$. Therefore, there exists a vertex of type $4\beta+2\gamma=2$, and we have

$$\alpha=\frac{F-4}{2F},\qquad \beta=\frac{F-12}{2F},\qquad \gamma=\frac{12}{F}.$$

In addition, from the condition $\beta>\gamma$, we have $F>36$.

Now, we determine remaining possible types of vertices $2t\beta+2u\gamma=2$ with $t,u\geq 0$ and $(t,u)\neq(2,1)$. We express it as $2k\beta+2l\gamma=2$ to avoid confusion. Substituting the above values of β and γ into $k\beta+l\gamma=1$, we have $(k-2)(F-12)+24(l-1)=0$. If $l=0$, then we have $(k-2)(F-12)=24$. But this is impossible because $F-12>24$. If $l=1$, then we have $k=2$ from the same reason. But, this contradicts the assumption $(k, l) \neq (2, 1)$. Hence, we have $l \geq 2$. In this case, we have clearly $k = 1$ or 0 from the above equality. Hence, remaining possible types of vertices $2k\beta + 2l\gamma = 2$ are exhausted by $2\beta + 2l\gamma = 2$ $(l \geq 2)$ and $2m\gamma = 2$ $(m \geq 2)$. If a vertex of type $2\beta + 2l\gamma = 2$ actually exists, then substituting the above values of β, γ, we have $F = 24l - 12$. If a vertex of type $2m\gamma = 2$ actually exists, we have $F = 12m$. From the condition $F > 36$, l and m must satisfy the inequalities $l \geq 3$ and $m \geq 4$. Now, we consider the following three cases.

(i-a) The case $F \not\equiv 0 \pmod{12}$. In this case, from the above argument, it follows that possible types of vertices are exhausted by

$$3\alpha + \beta + \gamma = 2,$$

$$2\alpha + 2s\gamma = 2,$$

$$4\beta + 2\gamma = 2.$$

In addition, in case a vertex of type $2\alpha + 2s\gamma = 2$ actually appears, we have $F = 24s - 4$, and hence $s \geq 2$. Now, starting from the vertex $3\alpha + \beta + \gamma = 2$, we draw the development map as follows:

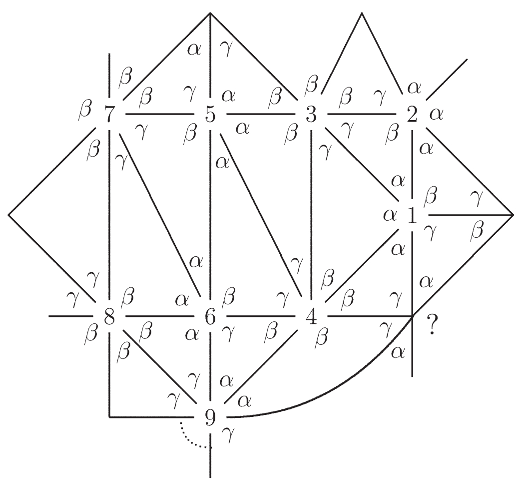


Fig. 57

The numbers in the figure indicate the order of drawing, as before. As stated before, we can show that three α's in $3\alpha + \beta + \gamma = 2$, two α's in $2\alpha + 2s\gamma = 2$ and four β's in $4\beta + 2\gamma = 2$ must appear adjacently around a vertex. Then, after drawing the development map surrounding the initial vertex as above, a vertex containing the angles $2\alpha + 2\gamma$ finally appears. This vertex must be of type $2\alpha + 2s\gamma = 2$. But two α's cannot be adjacent because $s \geq 2$. This is a contradiction, and hence this tiling cannot exist. (It should be remarked that the condition $s \geq 2$ in $2\alpha + 2s\gamma = 2$ is essential. In fact, if $s = 1$, we can uniquely complete the development map without any contradiction, and obtain the tiling MTG_{20}^{II}. In this case, we have $\alpha = 2/5$, $\beta = 1/5$, $\gamma = 3/5$, and of course, these angles do not satisfy the assumption $\gamma < \beta < \alpha$.)

(i-b) The case $F \equiv 0 \pmod{24}$. In this case, we put $F = 24p$ $(p \geq 2)$. Then, we can easily show that the possible types of vertices are exhausted by $3\alpha + \beta + \gamma = 2$, $4\beta + 2\gamma = 2$ and $4p\gamma = 2$. Then, as the following development map shows, this tiling also cannot exist.

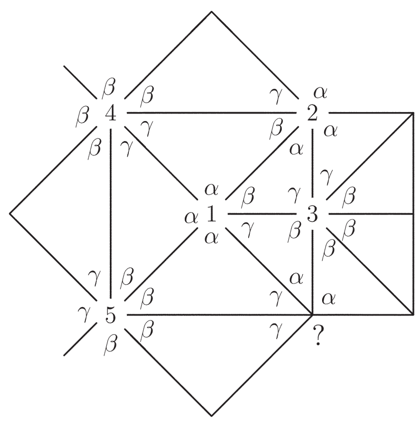


Fig. 58

(i-c) $F \equiv 12 \pmod{24}$. In this case, we put $F = 24p - 12$ $(p \geq 3)$. Then, by the same arguments as above, we can show that possible types of vertices are exhausted by $3\alpha + \beta + \gamma = 2$, $4\beta + 2\gamma = 2$, $2\beta + 2p\gamma = 2$ and $(4p - 2)\gamma = 2$. In this situation, if the tiling does not contain a vertex of type $2\beta + 2p\gamma = 2$, then we can show the non-existence of this tiling completely in the same way as the above case (i-b). Hence, we may assume that there exists a vertex of type $2\beta + 2p\gamma = 2$ $(p \geq 3)$. Then, starting from this vertex, we draw the following development map.

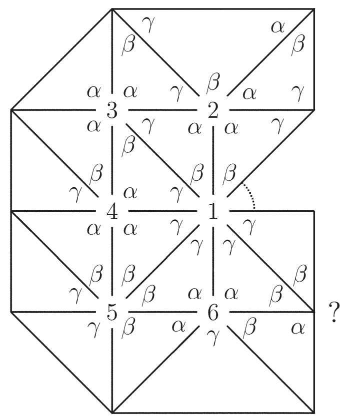


Fig. 59

Then, a vertex containing the angles $\alpha + 2\beta$ appears, and this is a contradiction. Hence, we conclude that this case also does not occur.

Combining these results, it follows that a tiling cannot exist in the case (i).

(ii) The case where all vertices of type $2t\beta + 2u\gamma = 2$ with $t, u \geq 1$ satisfy the inequality $u > t \geq 1$.

In this case, from the condition

$$\gamma = \frac{(t-2)F + 12t}{2(t-u)F} > \frac{2}{F},$$

we have $(t-2)F + 8t + 4u < 0$. Hence, we have $t = 1$, $u \geq 2$, and

$$\beta = \frac{(u-2)F + 12u}{2(u-1)F}, \qquad \gamma = \frac{F - 12}{2(u-1)F}.$$

In particular, the coefficient of β in $2t\beta + 2u\gamma$ with $t, u \geq 1$ contains only one parameter, and hence the vertex of this type is uniquely limited to the case $2\beta + 2u\gamma = 2$. Therefore, possible vertices are exhausted by $3\alpha + \beta + \gamma = 2$, $2\alpha + 2s\gamma = 2$, $2\beta + 2u\gamma = 2$, $2l\beta = 2$ and $2m\gamma = 2$. Now, starting from the vertex $3\alpha + \beta + \gamma = 2$, we draw the development map as follows:

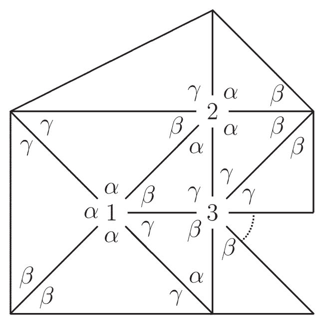


Fig. 60

(Note that $u \geq 2$.) Then, a vertex containing 3β appears, and hence a vertex of type $2l\beta = 2$ must exist in the tiling. From the condition $1/2 > \alpha > \beta = 1/l$, we have $l \geq 3$, and from the equality

$$\beta = \frac{(u-2)F + 12u}{2(u-1)F} = \frac{1}{l},$$

we have $\{(l-2)(u-2) - 2\}F + 12lu = 0$. Hence, we have $(l-2)(u-2) < 2$, from which we have $u = 2$ or $l = u = 3$ because $l \geq 3$ and $u \geq 2$.

If $u = 2$, then we have

$$F = 12l, \qquad \alpha = \frac{3l-1}{6l}, \qquad \beta = \frac{1}{l}, \qquad \gamma = \frac{l-1}{2l},$$

and a vertex of type $2\alpha + 2s\gamma = 2$ cannot exist in the tiling. In the case $l = u = 3$, we have $F = 108$, $\alpha = 13/27$, $\beta = 1/3$, $\gamma = 2/9$, and a vertex of type $2\alpha + 2s\gamma = 2$ also cannot exist. Hence, a vertex containing α is necessarily of type $3\alpha + \beta + \gamma = 2$. In this situation, we continue to draw the development map in Figure 60 as follows:

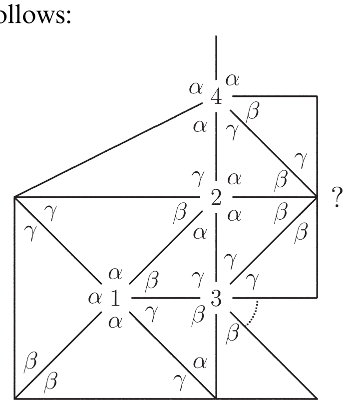


Fig. 61

Then, there appears a vertex containing $3\beta+\gamma$, which is a contradiction. Hence, a tiling of this type does not exist.

In conclusion, this tiling cannot exist for both cases (i) and (ii), and therefore, a monohedral odd tiling must contain a vertex of type $\alpha+(2q+1)\beta+(2r+1)\gamma=2$. q.e.d.

Next, we prove the following lemma.

LEMMA 14. *Under the same conditions as in Proposition* 12, *a vertex of type* $3\alpha+\beta+\gamma=2$ *does not exist in the tiling.*

PROOF. Assume there exists a vertex of type $3\alpha+\beta+\gamma=2$ in the tiling. By Lemma 13, there is a vertex of type $\alpha+(2q+1)\beta+(2r+1)\gamma=2$ $((q,r)\neq(0,0))$. Hence, combined with the equality $\alpha+\beta+\gamma=1+4/F$, we have

$$\alpha=q\beta+r\gamma=\frac{F-4}{2F},\qquad \beta+\gamma=\frac{F+12}{2F}.$$

If $q=r$, then from these equalities, we have $(q-1)F+12q+4=0$, which implies $q=0$ and $F=4$. But this contradicts the assumption $F>4$. Hence, we have $q\neq r$, and

$$\beta=\frac{(1-r)F-12r-4}{2(q-r)F},\qquad \gamma=\frac{(q-1)F+12q+4}{2(q-r)F}.$$

If $q>r$, then from the inequality $\beta>\gamma$, we have $(q+r-2)F+12q+12r+8<0$, and hence, $q+r\leq 1$. This implies $q=1$ and $r=0$. Then, we have $\alpha=\beta=(F-4)/2F$, which contradicts the assumption $\alpha>\beta$. Therefore, we have $q<r$. Then, from the condition $\gamma>2/F$, we obtain the inequality $(q-1)F+8q+4r+4<0$, and hence $q=0$. Since the vertex $\alpha+\beta+(2r+1)\gamma=2$ contains only one parameter r, other types of odd valent vertex cannot exist in the tiling. Hence, we conclude that there are just two types of odd valent vertices: $\alpha+\beta+(2r+1)\gamma=2$ and $3\alpha+\beta+\gamma=2$.

Now, under this situation, we have

$$\alpha=\frac{F-4}{2F},\qquad \beta=\frac{(r-1)F+12r+4}{2rF},\qquad \gamma=\frac{F-4}{2rF}.$$

From the condition $\alpha>\gamma$, we have $r\geq 2$. And since $\beta<\alpha<1/2$, vertices of type $4\alpha=2$ and $2\alpha+2\beta=2$ cannot exist in the tiling. Hence, possible types of vertices are exhausted by

$$\alpha+\beta+(2r+1)\gamma=2 \qquad r\geq 2,$$
$$3\alpha+\beta+\gamma=2,$$
$$2\alpha+2s\gamma=2 \qquad s\geq 1,$$
$$2t\beta+2u\gamma=2 \qquad t,u\geq 0.$$

Then, starting from the vertex $\alpha+\beta+(2r+1)\gamma=2$, we draw the development map as follows. Note that $2r+1$ γ's must appear adjacently around this vertex.

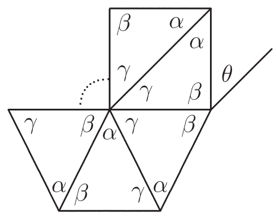


Fig. 62

Then, since a vertex containing the angles $\alpha+2\beta$ cannot exist, we have $\theta=\beta$ in this figure. In addition, a vertex containing the angles 3α must be of type $3\alpha+\beta+\gamma=2$, and we can uniquely continue to draw the development map as follows:

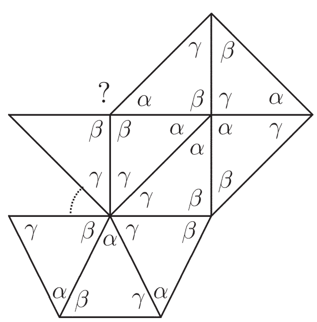


Fig. 63

But finally, we arrive at a contradiction as in the above figure. Therefore, a vertex of type $3\alpha+\beta+\gamma=2$ cannot exist in the tiling. q.e.d.

Remark. To prove the non-existence of a vertex $3\alpha+\beta+\gamma=2$, we developed complicated arguments in Lemmas 13 and 14, especially in Figure

57. If we ignore the topology of the sphere, but instead, consider a non-compact space of constant positive curvature with boundary, we can construct a new type of tilings on this space containing a vertex $3\alpha+\beta+\gamma=2$ in its interior. For details, see Appendix.

Next, we prove the following lemma.

LEMMA 15. *Under the same conditions as in Proposition* 12, *a vertex of type* $4\alpha=2$ *does not exist in the tiling.*

PROOF. Assume $\alpha=1/2$, and we show that a contradiction necessarily occurs, after a similar argument as above. By Lemma 13, there is a vertex of type $\alpha+(2q+1)\beta+(2r+1)\gamma=2$ $((q,r)\neq(0,0))$. Then, combining with the equality $\alpha+\beta+\gamma=1+4/F$, we have

$$\beta+\gamma=\frac{F+8}{2F}, \qquad q\beta+r\gamma=\frac{F-4}{2F}.$$

If $q=r$, then from these equalities, we have $(q-1)F+8q+4=0$, which implies $q=0$, $F=4$. This contradicts the assumption $F>4$, and hence $q\neq r$. Then, we have

$$\beta=\frac{(1-r)F-8r-4}{2(q-r)F}, \qquad \gamma=\frac{(q-1)F+8q+4}{2(q-r)F}.$$

Now, we consider two cases.

(i) The case where there exists a vertex of type $\alpha+(2q+1)\beta+(2r+1)\gamma=2$ satisfying $q>r\geq 0$.

In this case, from the condition $\beta>\gamma$, we have $(q+r-2)F+8q+8r+8<0$, which implies $q+r\leq 1$. Therefore, we have $q=1$, $r=0$, i.e., there exists a vertex of type $\alpha+3\beta+\gamma=2$. In addition, we have $\beta=(F-4)/2F$, $\gamma=6/F$, and a vertex of type $2\alpha+2\beta=2$ does not exist because $\beta<\alpha=1/2$.

Now, we show that an odd valent vertex other than $\alpha+3\beta+\gamma=2$ does not exist in the tiling. Assume there exists a vertex of type $\alpha+(2k+1)\beta+(2l+1)\gamma=2$ with $(k,l)\neq(1,0)$. Then, we have clearly $l\neq 0$ because $\alpha+3\beta+\gamma=2$. Substituting the above values of α, β and γ into $\alpha+(2k+1)\beta+(2l+1)\gamma=2$, we have $(k-1)(F-4)+12l=0$. Since $F>4$ and $l>0$, we have $k=0$ and $F=12l+4$. Hence, the type of this vertex is $\alpha+\beta+(2l+1)\gamma=2$, and we have

$$\alpha=\frac{1}{2}, \qquad \beta=\frac{3l}{6l+2}, \qquad \gamma=\frac{3}{6l+2}.$$

In addition, from the condition $\beta>\gamma$, we have $l\geq 2$. It is clear from the above argument that odd valent vertices are limited to $\alpha+3\beta+\gamma=2$ and $\alpha+\beta+(2l+1)\gamma=2$.

In this situation, we can easily see that vertices of type $2\alpha + 2s\gamma = 2$ and $2t\beta + 2u\gamma = 2$ cannot exist in the tiling. Therefore, possible types of vertices are exhausted by $\alpha + 3\beta + \gamma = 2$, $\alpha + \beta + (2l+1)\gamma = 2$ $(l \geq 2)$ and $4\alpha = 2$. Starting from the vertex $\alpha + 3\beta + \gamma = 2$, we draw the development map. Then, as the following figure shows, a contradiction occurs, and this tiling does not exist.

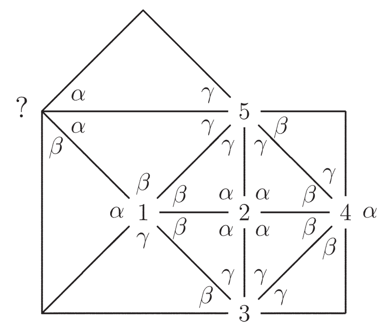


Fig. 64

Therefore, it follows that an odd valent vertex other than $\alpha + 3\beta + \gamma = 2$ does not exist in the tiling.

Then, from the above arguments, possible types of vertices are now exhausted by

$$\alpha + 3\beta + \gamma = 2,$$

$$4\alpha = 2,$$

$$2\alpha + 2s\gamma = 2 \qquad s \geq 1,$$

$$2t\beta + 2u\gamma = 2 \qquad t, u \geq 0.$$

Starting from the vertex $\alpha + 3\beta + \gamma = 2$, we draw the following development map.

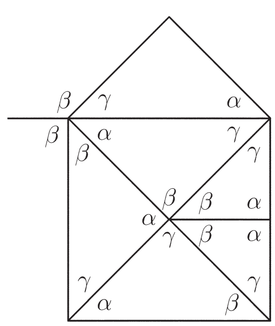


Fig. 65

Then, there appears a vertex containing the angles $\alpha + 2\gamma$. Hence, a vertex of type $2\alpha + 2s\gamma = 2$ actually exists in the tiling. Substituting $\alpha = 1/2$ and $\gamma = 6/F$ to this equality, we have $F = 12s$, and

$$\beta = \frac{F-4}{2F} = \frac{3s-1}{6s}, \qquad \gamma = \frac{1}{2s}.$$

From the condition $\alpha > \gamma$, we have $s \geq 2$. If a vertex of type $2t\beta + 2u\gamma = 2$ exists, then we have $(3s-1)(t-2) = 2 - 3u$, and from this equality, we can easily show that $t = 0$ and $u = 2s$. Therefore, possible types of vertices are restricted to

$$\begin{aligned} \alpha + 3\beta + \gamma &= 2, \\ 4\alpha &= 2, \\ 2\alpha + 2s\gamma &= 2, \\ 4s\gamma &= 2 \end{aligned}$$

$(s \geq 2)$. We continue to draw the development map in Figure 65 as follows. (Remind that in the vertex of type $2\alpha + 2s\gamma = 2$, $2s$ γ's must appear adjacently.)

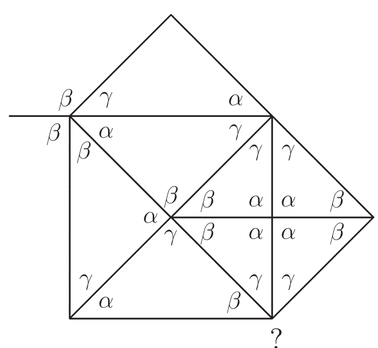


Fig. 66

Then, a vertex containing $\beta + 2\gamma$ appears. But this is a contradiction, and hence we conclude that a tiling of this type does not exist.

(ii) The case where all vertices of type $\alpha + (2q+1)\beta + (2r+1)\gamma = 2$ satisfy $r > q \geq 0$.

From the condition

$$\gamma = \frac{(q-1)F + 8q + 4}{2(q-r)F} > \frac{2}{F},$$

we have $(q-1)F+4q+4r+4<0$. Hence, we have $q=0$ and $r \geq 1$. And in particular, a vertex of type $\alpha+(2q+1)\beta+(2r+1)\gamma=2$ is uniquely limited to the case $\alpha+\beta+(2r+1)\gamma=2$. It is clear that remaining possible vertices are $4\alpha=2$, $2\alpha+2s\gamma=2$, $2t\beta+2u\gamma=2$. By starting from the vertex $\alpha+\beta+(2r+1)\gamma=2$, we draw the development map as follows:

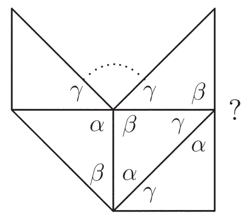


Fig. 67

Then, as this figure shows, a contradiction occurs at the right point because $2r+1$ γ's must appear adjacently in the vertex $\alpha+\beta+(2r+1)\gamma=2$. Hence, a tiling of this type also cannot exist.

Therefore, in any case we arrive at a contradiction, and hence we have $\alpha \neq 1/2$. In particular, a vertex of type $4\alpha=2$ does not appear in the tiling. q.e.d.

Remark. In our previous paper [10], we already proved Lemma 15 essentially. In fact, we showed in [10] that the vertices appearing in the tilings by congruent "right" scalene triangles are always even valent, as a result of the classification. We give here a new proof of this fact for the sake of completeness.

Finally, we prepare the following lemma, which is a refinement of Lemma 13. We impose the same condition as in Proposition 12.

Lemma 16. *Let $\alpha+(2q+1)\beta+(2r+1)\gamma=2$ $((q,r)\neq(0,0))$ be a type of a vertex appearing in the tiling. Then, we have $q=0$ or $r=0$.*

Proof. The vertex of type $\alpha+(2q+1)\beta+(2r+1)\gamma=2$ with $(q,r)\neq(0,0)$ can be expressed in the following form:

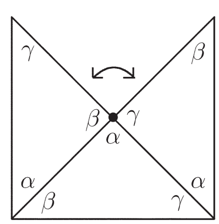


Fig. 68

And it is easy to see that the remaining angle denoted by the arrow must be filled by several copies of the following two figures:

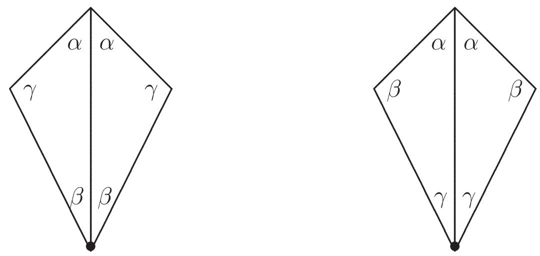


Fig. 69

By Proposition 11, Lemmas 14 and 15, we know that a vertex containing the angle 2α must be of type $2\alpha + 2\beta = 2$ or $2\alpha + 2s\gamma = 2$ $(s \geq 1)$. Consequently, the arrowed angle in Figure 68 is filled by the following figures:

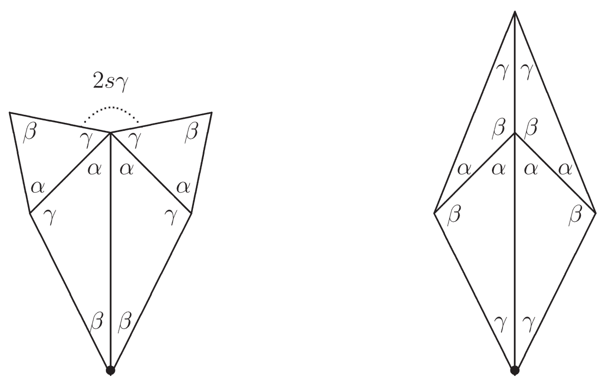


Fig. 70

In this situation, if $q, r \geq 1$, then these two figures must contact at some place, and a vertex containing the angles $2\alpha + \beta + \gamma$ appears as in the following figure.

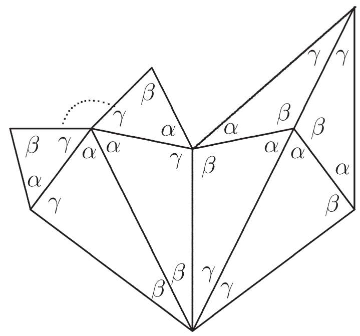


Fig. 71

But, this is a contradiction, and therefore we have $q = 0$ or $r = 0$. q.e.d.

7. Case of scalene triangles III: Odd tilings (2)

Under these preliminaries, we prove Proposition 12 in this section. We divide the proof into two cases (i) and (ii).

(i) The case there exists a vertex of type $\alpha + \beta + (2r+1)\gamma = 2$ $(r \geq 1)$.

In this case, from the arguments in the proof of Lemma 16, this vertex is expressed in the following form:

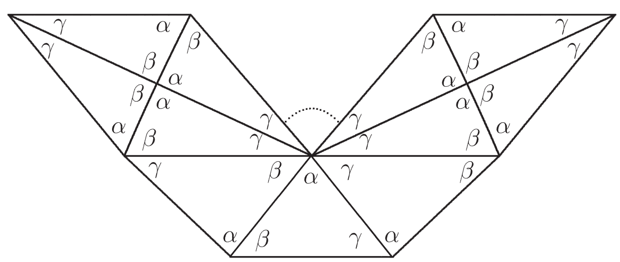


Fig. 72

In particular, a vertex of type $2\alpha + 2\beta = 2$ actually exists, and hence, combined with the equalities $\alpha + \beta + (2r+1)\gamma = 2$, $\alpha + \beta + \gamma = 1 + 4/F$, we have

$$\alpha + \beta = 1, \qquad \gamma = \frac{1}{2r+1}, \qquad F = 8r + 4.$$

In addition, from the condition $\alpha > \beta > \gamma$, we have $1/2 < \alpha < (2r)/(2r+1)$. In this situation, possible types of vertices are exhausted by

$$\begin{aligned}
&\alpha+\beta+(2r+1)\gamma=2 \qquad r\geq 1,\\
&\alpha+(2q+1)\beta+\gamma=2 \qquad q\geq 1,\\
&2\alpha+2\beta=2,\\
&2\alpha+2s\gamma=2 \qquad s\geq 2,\\
&2t\beta+2u\gamma=2 \qquad t,u\geq 0.
\end{aligned}$$

(The inequality $s \geq 2$ follows immediately from the condition $\beta > \gamma$.) Then, starting from the vertex $\alpha+\beta+(2r+1)\gamma=2$, we can uniquely draw the development map as follows:

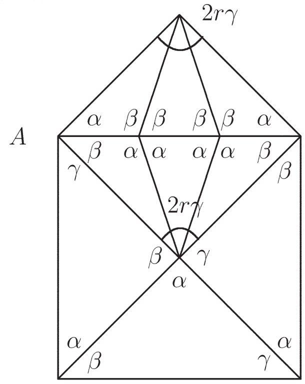


Fig. 73

The vertex A in this figure cannot be of type $\alpha+(2q+1)\beta+\gamma=2$ because $(2q+1)$ β's must appear adjacently. Hence this vertex is of type $\alpha+\beta+(2r+1)\gamma=2$. In this way, we can uniquely continue to draw the development map, and finally, we obtain the following figure:

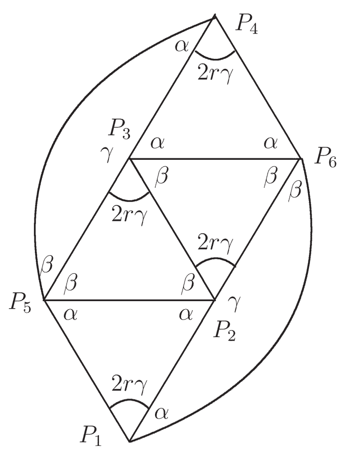


Fig. 74

The number of triangles appearing in this figure is $8r+2$, and we must divide the quadrangle situated outside of this figure into two triangles. This quadrangle can be represented in the following form:

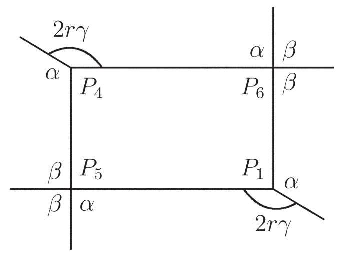


Fig. 75

One way to draw a diagonal line is given by the following:

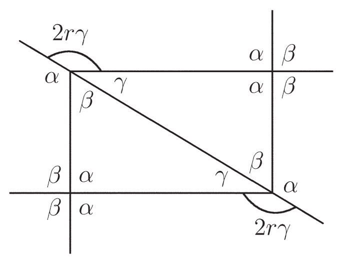


Fig. 76

And it is easy to see that the resulting tiling can be represented as follows:

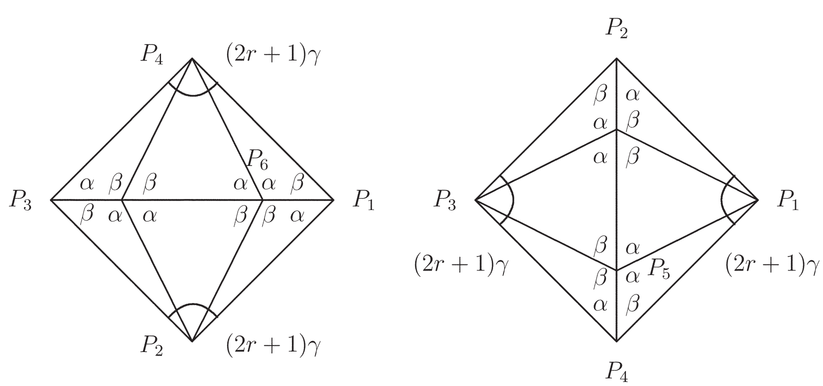


Fig. 77

This tiling is TG_{8r+4} with

$$\alpha+\beta=1, \qquad \gamma=\frac{1}{2r+1}, \qquad \frac{1}{2}<\alpha<\frac{2r}{2r+1}, \qquad r\geq 1.$$

If α, β, γ satisfy the relation $\alpha=\beta+\gamma$, then we have

$$\alpha=\frac{r+1}{2r+1}, \qquad \beta=\frac{r}{2r+1}, \qquad \gamma=\frac{1}{2r+1},$$

and in this special case we can draw the opposite diagonal line in the quadrangle in Figure 75.

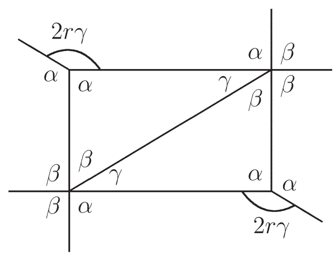


Fig. 78

In this case, from the condition $\beta>\gamma$, we have $r\geq 2$. The resulting tiling can be expressed in the following form:

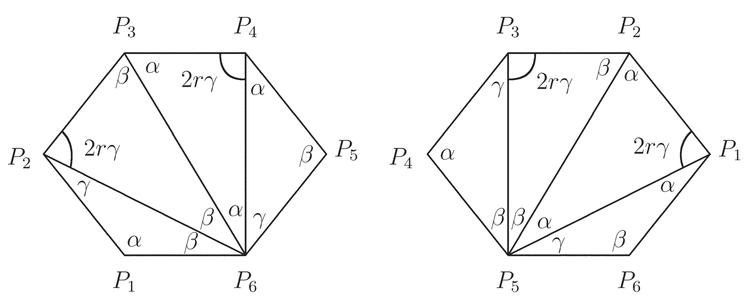


Fig. 79

This figure coincides with Figure 10 (above), and hence this tiling is MTG^{I}_{8r+4} $(r\geq 2)$.

(ii) The case where a vertex of type $\alpha+\beta+(2r+1)\gamma=2$ $(r\geq 1)$ does not exist.

In this case, by Lemmas 13 and 16, a vertex of type $\alpha+(2q+1)\beta+\gamma=2$ $(q\geq 1)$ necessarily exists, and it is expressed in the following form:

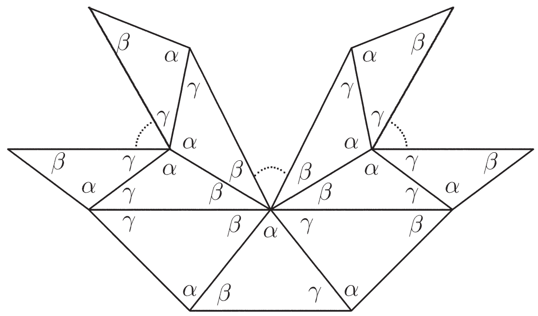


Fig. 80

In this situation, possible types of vertices are given by

$$\begin{aligned} \alpha + (2q+1)\beta + \gamma &= 2, \\ 2\alpha + 2\beta &= 2, \\ 2\alpha + 2s\gamma &= 2, \\ 2t\beta + 2u\gamma &= 2. \end{aligned}$$

In particular, from the above figure, we know that a vertex of type $2\alpha + 2s\gamma = 2$ actually exists.

Now, we first consider the case where a vertex of type $2\alpha + 2\beta = 2$ exists. In this case, from the condition $\beta > \gamma$, we have $s \geq 2$. If $q \geq 2$, then the following two figures must contact at some place:

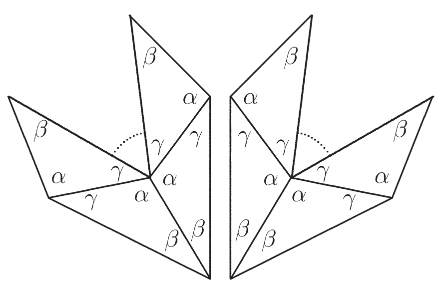


Fig. 81

Then, a vertex containing $2\alpha + 2\gamma$ appears, and this vertex must be of type $2\alpha + 2s\gamma = 2$ $(s \geq 2)$. But $2s$ γ's must appear adjacently around this vertex, which is a contradiction. Therefore, we have $q = 1$. Then, from the equalities $\alpha + 3\beta + \gamma = 2$, $2\alpha + 2\beta = 2$, $2\alpha + 2s\gamma = 2$ and $\alpha + \beta + \gamma = 1 + 4/F$, we have

$$\alpha = \frac{s+1}{2s+1}, \qquad \beta = \frac{s}{2s+1}, \qquad \gamma = \frac{1}{2s+1}, \qquad F = 8s+4.$$

In this case, starting from the vertex $\alpha + 3\beta + \gamma = 2$, we can uniquely draw the development map as follows:

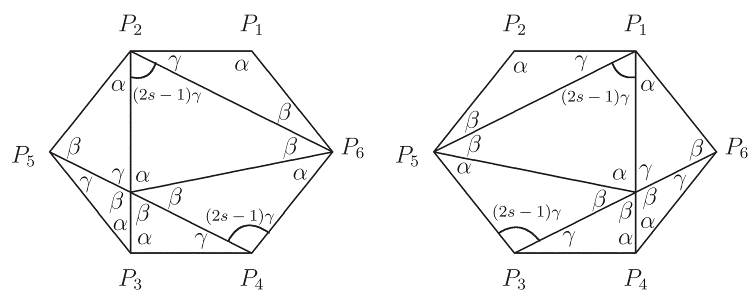


Fig. 82

This tiling is MTG^{II}_{8s+4} $(s \geq 2)$.

Next, we consider the case where a vertex of type $2\alpha + 2\beta = 2$ does not appear in the tiling. In this case, possible types of vertices are

$$\alpha + (2q+1)\beta + \gamma = 2,$$

$$2\alpha + 2s\gamma = 2,$$

$$2t\beta + 2u\gamma = 2.$$

As we indicated above, there exists a vertex of type $2\alpha + 2s\gamma = 2$. If $s \geq 2$, then starting from the vertex $\alpha + (2q+1)\beta + \gamma = 2$, we can draw the development map as follows:

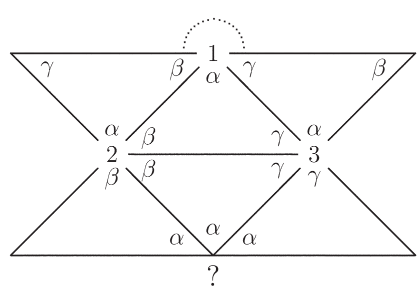


Fig. 83

Then, a vertex containing the angles 3α appears, and this is a contradiction. Hence, we have $s = 1$. Combined with the equalities $\alpha + (2q+1)\beta + \gamma = 2$

and $\alpha + \beta + \gamma = 1 + 4/F$, we have $\beta = 1/(2q+1)$, $\gamma = 1 - \alpha$ and $F = 8q + 4$. Then, from the condition $\beta > \gamma > 2/F$, we have

$$\frac{2q}{2q+1} < \alpha < \frac{4q+1}{4q+2}.$$

In this situation, starting from the vertex $\alpha + (2q+1)\beta + \gamma = 2$, we can uniquely draw the development map as follows:

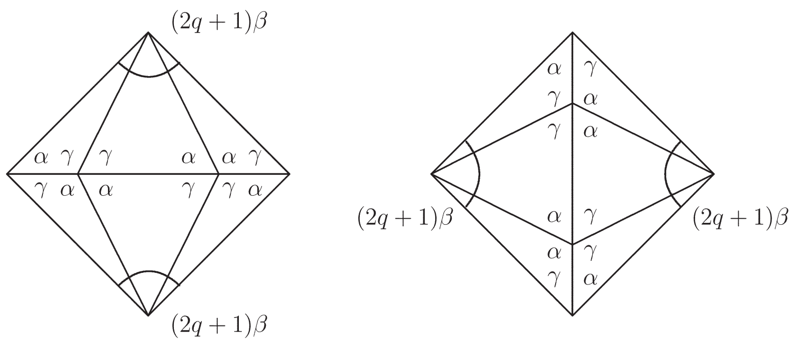


Fig. 84

By exchanging the angles β and γ, this tiling corresponds to TG_{8q+4} with

$$\alpha + \beta = 1, \qquad \gamma = \frac{1}{2q+1}, \qquad \frac{2q}{2q+1} < \alpha < \frac{4q+1}{4q+2} \qquad \text{and} \qquad q \geq 1.$$

Combining these results, we complete the proof of Proposition 12. q.e.d.

8. Case of scalene triangles IV: Even tilings

In this last section, we classify monohedral tilings by scalene triangles with $F > 4$, consisting of only even-valent vertices. The results are summarized in the following proposition.

PROPOSITION 17. *Monohedral even tilings of the* 2*-dimensional sphere by scalene triangles with* $F > 4$ *are given by the following*:

(i) F_{48}, TF_{48}, F_{120},

(ii) G_{4n} $\left(\alpha + \beta = 1, \gamma = \dfrac{1}{n}, \dfrac{1}{2} < \alpha < \dfrac{n-1}{n}, n \geq 3\right)$,

(iii) G_{4n} $\left(\alpha+\beta=1, \gamma=\dfrac{1}{n}, \dfrac{n-1}{n}<\alpha<\dfrac{2n-1}{2n}, n \geq 2\right)$,

(iv) I_{8n} $(n \geq 4)$,

(v) TI_{16n+8} $(n \geq 2)$.

Note that in the case (iii), the inequality $\beta<\gamma<\alpha$ holds instead of $\gamma<\beta<\alpha$. But in the following proof, we always assume $\gamma<\beta<\alpha$ as before, unless otherwise stated.

Proof. We divide the proof into the following four cases:

(a) The case $\alpha=1/2$ and the angle α appears only in vertices of type $4\alpha=2$.

(b) The case $\alpha=1/2$ and there is a vertex containing α which is not of type $4\alpha=2$.

(c) The case $\alpha \neq 1/2$ and there is a vertex of type $2\alpha+2\beta=2$.

(d) The case $\alpha \neq 1/2$ and a vertex of type $2\alpha+2\beta=2$ does not exist. (Note that the cases (a) and (b) are already treated in our previous paper [10]. But we give here a new proof for the sake of completeness.)

(a) In this case, we can use the results in Propositions 5 and 6. In fact, since the angle $\alpha=1/2$ appears only in the vertex of type $4\alpha=2$, the sphere is tiled by the following rhombus:

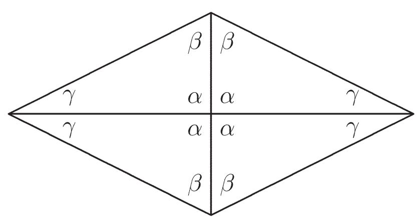


Fig. 85

By deleting a horizontal line in this figure, we obtain a tiling on the sphere consisting of isosceles or equilateral triangles with angles β, β, 2γ. These angles satisfy the condition $\frac{1}{2}\cdot 2\gamma<\beta<\frac{1}{2}$. In case $\beta \neq 2\gamma$, this triangle is isosceles and by using Proposition 6, we can easily list up the isosceles triangles satisfying the above condition. They are exhausted by I_{24}, F_{60}^{II}, H_{4n} $(n=4$, $n \geq 6)$ and TH_{8n+4} $(n \geq 3)$. And by dividing these tilings, we obtain the tilings F_{48}, F_{120}, I_{8n} $(n=4, n \geq 6)$ and TI_{16n+8} $(n \geq 3)$, respectively. These tilings satisfy the assumption in Proposition 17.

In the special case $\beta=2\gamma$, this triangle is equilateral whose angles are smaller than $1/2$. Then, by Proposition 5, this triangle is restricted to the one in H_{20}. By dividing equilateral triangles in H_{20} into two isosceles triangles, we obtain combinatorially two types of tilings: I_{40} and TI_{40}. We can easily verify this fact by considering the existence or non-existence of a vertex of type $10\gamma = 2$. Therefore, in conclusion, we obtain the tilings F_{48}, F_{120}, I_{8n} $(n \geq 4)$ and TI_{16n+8} $(n \geq 2)$ in the case (a).

(b) In this case, since $\alpha = 1/2$ and $\alpha > \beta$, a vertex of type $2\alpha + 2\beta = 2$ cannot exist. Hence, from Proposition 11, possible types of vertices are exhausted by

$$\begin{aligned} &4\alpha = 2, \\ &2\alpha + 2s\gamma = 2 \qquad s \geq 1, \\ &2t\beta + 2u\gamma = 2 \qquad t, u \geq 0. \end{aligned}$$

From the assumption, a vertex of type $2\alpha + 2s\gamma = 2$ actually exists, and hence, we have $\gamma = 1/2s$ $(s \geq 2)$.

Now, we show that there exists a vertex of type $2t\beta + 2u\gamma = 2$ with $t \geq 3$. Assume that a vertex of this type does not exist. Then, since there is a vertex containing the angle β, a vertex of type $4\beta + 2u\gamma = 2$ or $2\beta + 2u\gamma = 2$ must exist in the tiling. We assume that a vertex of type $4\beta + 2u\gamma = 2$ exists. Then, combined with the equality $\alpha + \beta + \gamma = 1 + 4/F$, we have $\beta = (2s - u)/4s$ and $(u-2)F + 16s = 0$. And from the second equality we have $u = 0$ or 1. But, if $u = 0$, we have $\beta = 1/2$, which is a contradiction. Hence, $u = 1$ and we have

$$\alpha = \frac{1}{2}, \qquad \beta = \frac{2s-1}{4s}, \qquad \gamma = \frac{1}{2s}, \qquad F = 16s.$$

In this case, it is easy to check that possible types of vertices are exhausted by

$$\begin{aligned} &4\alpha = 2, \\ &2\alpha + 2s\gamma = 2 \qquad s \geq 2, \\ &4\beta + 2\gamma = 2, \\ &4s\gamma = 2. \end{aligned}$$

Starting from the vertex $2\alpha + 2s\gamma = 2$, we draw the development map:

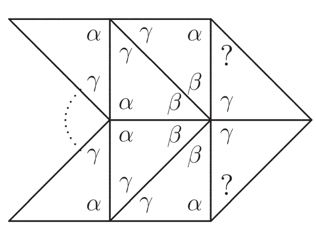


Fig. 86

(Note that two α's in $2\alpha + 2s\gamma = 2$ must appear adjacently.) But, as this figure shows, a contradiction occurs, and therefore a vertex of type $4\beta + 2u\gamma = 2$ cannot exist. Hence, a vertex of type $2\beta + 2u\gamma = 2$ necessarily exists, and possible types of vertices are exhausted by

$$\begin{aligned} &4\alpha = 2, \\ &2\alpha + 2s\gamma = 2 \qquad s \geq 2, \\ &2\beta + 2u\gamma = 2, \\ &4s\gamma = 2. \end{aligned}$$

Then, starting from the vertex $2\alpha + 2s\gamma = 2$, we draw the development map:

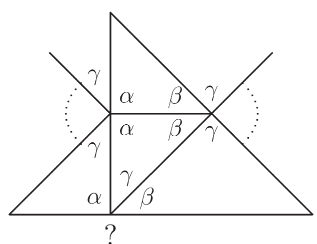


Fig. 87

But a vertex containing the angles $\alpha + \beta + \gamma$ cannot exist, and this is a contradiction. Therefore, as a conclusion, there exists a vertex of type $2t\beta + 2u\gamma = 2$ satisfying $t \geq 3$.

Then, from the equalities $\alpha = 1/2$, $\gamma = 1/2s$ $(s \geq 2)$, $2t\beta + 2u\gamma = 2$ $(t \geq 3)$ and $\alpha + \beta + \gamma = 1 + 4/F$, we have $\beta = (2s - u)/2st$, $(st - 2s - t + u)F + 8st = 0$. In particular, we have $st - 2s - t + u = (s-1)(t-2) + u - 2 < 0$. Since $s \geq 2$ and $t \geq 3$, we have $u = 0$, $s = 2$, $t = 3$. This implies $\alpha = 1/2$, $\beta = 1/3$, $\gamma = 1/4$ and $F = 48$. Then, possible types of vertices are $4\alpha = 2$, $2\alpha + 4\gamma = 2$, $6\beta = 2$, $8\gamma = 2$. Starting from the vertex $2\alpha + 4\gamma = 2$, we can uniquely draw the development map as follows:

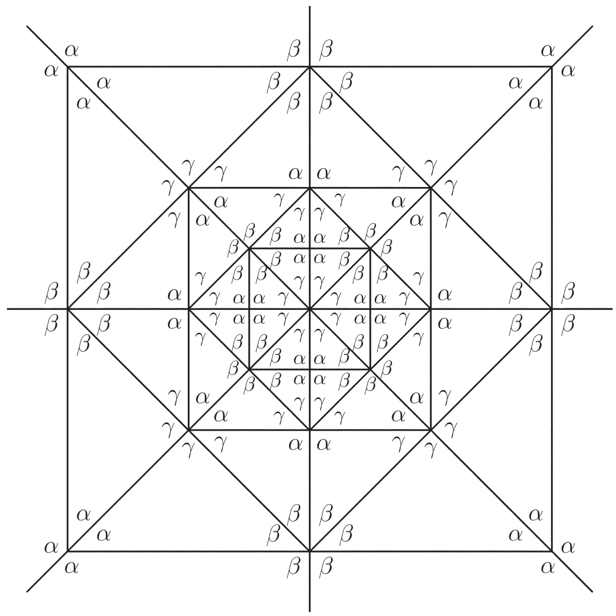


Fig. 88

This tiling is TF_{48}.

(c) Next we consider the case $\alpha \neq 1/2$ and there exists a vertex of type $2\alpha + 2\beta = 2$. In this case, from Proposition 11, possible types of vertices are

$$2\alpha + 2\beta = 2,$$

$$2\alpha + 2s\gamma = 2,$$

$$2t\beta + 2u\gamma = 2.$$

It is easy to see that the vertex of type $2\alpha + 2\beta = 2$ must be expressed in the following form

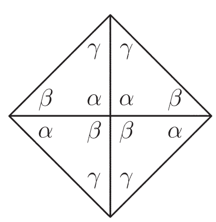


Fig. 89

Since a vertex containing the angles $\alpha + \beta$ is necessarily of type $2\alpha + 2\beta = 2$, this tiling must be obtained by joining the following figures:

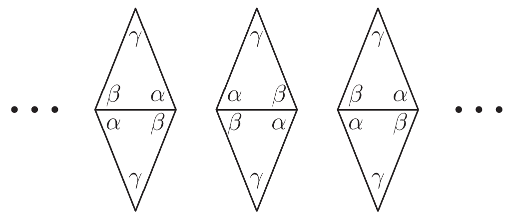


Fig. 90

Hence, this tiling is G_{4n} for some n. We have clearly $\alpha+\beta=1$, $\gamma=1/n$ and from the condition $\gamma<\beta<\alpha$, the inequality $1/2<\alpha<(n-1)/n$ $(n\geq 3)$ follows.

(d) Assume $\alpha\neq 1/2$ and a vertex of type $2\alpha+2\beta=2$ does not exist. In this case, possible types of vertices are

$$2\alpha+2s\gamma=2,$$

$$2t\beta+2u\gamma=2.$$

Clearly, both types must exist in the tiling. Now assume $s\geq 2$. Then starting from the vertex $2\alpha+2s\gamma=2$, we draw the following development map:

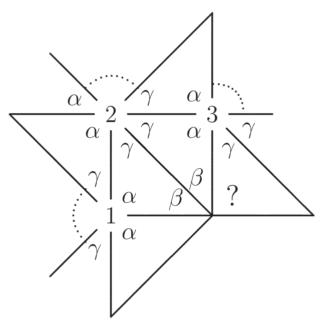


Fig. 91

Then, a contradiction occurs, and hence we have $s=1$. Then, as in the above case (c), we can easily see that this tiling is G_{4n} $(n\geq 2)$ with $\alpha+\gamma=1$, $\beta=1/n$. From the conditions $\gamma<\beta$ and $\alpha+\beta<1+\gamma$, we have $(n-1)/n<\alpha<(2n-1)/2n$. By exchanging the angles β and γ, we obtain the tiling (iii).

Combining these results, we complete the proof of Proposition 17. q.e.d.

By Propositions 5, 6, 12 and 17, we have completed the proof of Theorem 1.

Appendix

In this appendix, we give examples of tilings on non-compact spaces of constant positive curvature with boundary possessing a five valent vertex $3\alpha + \beta + \gamma = 2$ in its interior. And as its special case, we construct dihedral tilings on the usual 2-dimensional sphere, consisting of $10n$ congruent triangles and two regular n-gons ($3 \le n \le 7$).

First, we put $\beta = 3\alpha - 1$, $\gamma = 3 - 6\alpha$, where α satisfies the conditions $3/7 < \alpha < 1/2$. Then, we can easily show that these angles satisfy the conditions

$$1 < \alpha + \beta + \gamma, \qquad \alpha + \beta < 1 + \gamma, \qquad \beta + \gamma < 1 + \alpha, \qquad \gamma + \alpha < 1 + \beta.$$

Hence by Proposition 4, there exists a triangle on the sphere with angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$. Note that this triangle is scalene unless $\alpha = 4/9$, and the largest angle is $\alpha\pi$.

In this situation, we consider the following figure on the sphere consisting of ten congruent triangles.

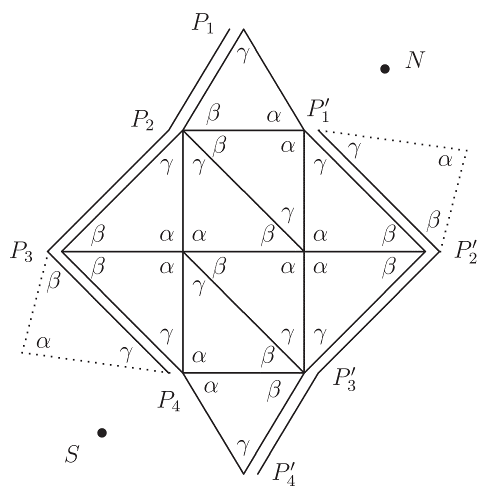


Fig. 92

Clearly α, β, γ satisfies the conditions

$$3\alpha + \beta + \gamma = 2, \qquad 4\beta + 2\gamma = 2, \qquad 2\alpha + 2\gamma < 2,$$

and hence we can connect these figures repeatedly along the double lines indicated in the above figure. Then, we obtain a tiling on the infinitely spreaded strip, which is a non-compact (simply connected) space of constant positive curvature with boundary. (The symbols N and S indicate the north and the south poles, respectively.) As the above figure shows, this tiling contains a vertex of type $3\alpha + \beta + \gamma = 2$ in its interior.

This tiling is deformable because α can move in the range $3/7 < \alpha < 1/2$ ($\alpha \neq 4/9$). In general, this tiling does not close after connecting these figures on the sphere. But in special cases, these figures constitute a closed dihedral tiling on the sphere.

PROPOSITION A. *The tiling closes after connecting the above figure n times if and only if the angle α satisfies the following equality.*

$$\frac{\cos\frac{2\pi}{n} + \cos^2(5\alpha - 2)\pi}{\sin^2(5\alpha - 2)\pi} = \frac{\cos(3\alpha - 1)\pi + \cos\alpha\pi\cos(3 - 6\alpha)\pi}{\sin\alpha\pi\sin(3 - 6\alpha)\pi}.$$

This equation on α possesses a solution in the interval $3/7 < \alpha < 1/2$ ($\alpha \neq 4/9$) only in the cases $n = 3, 4, 5, 6, 7$. In these closed cases, the complementary set in the sphere consists of two regular n-gons. Hence the sphere is tiled by $10n$ congruent triangles and two regular n-gons, giving a dihedral tiling with $10n + 2$ faces ($3 \leq n \leq 7$).

PROOF. Clearly, in case the tiling closes, one of the complementary set is expressed in the following form:

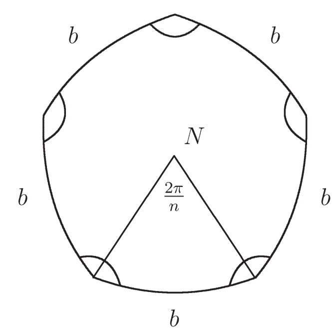


Fig. 93

The interior angle of this figure is equal to $2\pi - (2\alpha + 2\gamma)\pi = (10\alpha - 4)\pi$. Therefore, by using the cosine rule, we have

$$\cos b = \frac{\cos\frac{2\pi}{n} + \cos^2(5\alpha - 2)\pi}{\sin^2(5\alpha - 2)\pi}.$$

On the other hand, from the cosine rule for the initial triangle, we have

$$\cos b = \frac{\cos \beta\pi + \cos \alpha\pi \cos \gamma\pi}{\sin \alpha\pi \sin \gamma\pi}$$
$$= \frac{\cos(3\alpha - 1)\pi + \cos \alpha\pi \cos(3 - 6\alpha)\pi}{\sin \alpha\pi \sin(3 - 6\alpha)\pi}.$$

Therefore, we obtain the desired equality in Proposition A. We can show that two functions

$$f(x) = \frac{\cos\frac{2\pi}{n} + \cos^2(5x - 2)\pi}{\sin^2(5x - 2)\pi},$$
$$g(x) = \frac{\cos(3x - 1)\pi + \cos x\pi \cos(3 - 6x)\pi}{\sin x\pi \sin(3 - 6x)\pi}$$

are both decreasing in the interval $3/7 < x < 1/2$, and the equation $f(x) = g(x)$ admits a solution if and only if $f\left(\frac{1}{2}\right) < \lim_{x\to 1/2} g(x)$, i.e., $\cos\frac{2\pi}{n} < \frac{2}{3}$. Therefore, we have $n = 3, 4, 5, 6, 7$. In addition, by substituting the value $\alpha = 4/9$, we know that $\alpha = 4/9$ is not a solution of this equation. (The length b increases as α goes to $1/2$, and its limit is $\lim_{\alpha\to 1/2} b = \cos^{-1}\left(\frac{2}{3}\right) > 0$. Therefore, if n is sufficiently large, the value of b exceeds (length of the equator)$/n = 2\pi/n$, which implies that n must have an upper bound. By the above argument, we showed that this upper bound is 7.) q.e.d.

As an example, we give here a figure of this closed tiling in the case $n = 5$, where α takes value $\alpha = 0.4691\ldots$. The 8-gons surrounded by thick lines are the figure indicated in Figure 92.

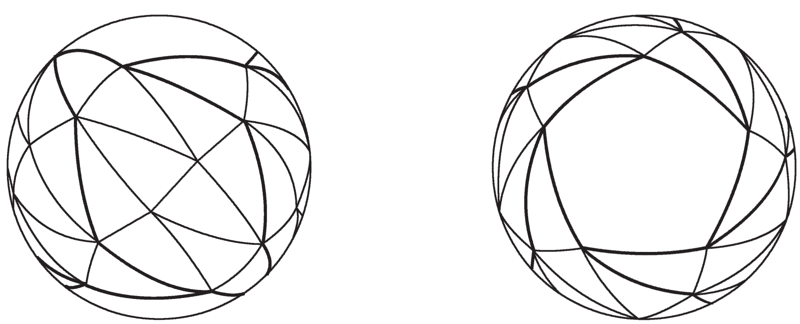


Fig. 94

We remark that in the case $2/5 < \alpha < 3/7$, we can also construct a tiling completely in the same way as above. But in this case, the largest angle is $\gamma\pi$. In the boundary case $\alpha = 2/5$, we have the equality $2\alpha + 2\gamma = 2$, and we obtaine a closed tiling on the sphere consisting of 20 congruent triangles. We can easily check that this tiling just coincides with MTG_{20}^{II}, corresponding to the case $n = 2$ in Proposition A.

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