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A strong limit theorem expressed by inequalities for the sequences of absolutely continuous random variables

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Abstract. Let $\{X_n, n \geq 1\}$ be an arbitrary sequence of dependent absolutely continuous random viariables, $\{B_n, n \geq 1\}$ be Borel sets on the real line, and $I_{B_n}(x)$ be the indicator function of B_n. In this paper, the limit properties of $\{I_{B_n}(X_n), n \geq 1\}$ are studied, and a kind of strong limit theorem represented by inequalities with random bounds is obtained.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequences of absolutely continuous random variables on the probability space $(\Omega, \mathscr{F}, P)$ with the joint density function $g_n(x_1, \ldots, x_n)$, $n = 1, 2, \ldots$. Let $f_k(x_k)$, $k = 1, 2, \ldots$, be an arbitrary sequence of density functions, and call $\prod_{k=1}^{n} f_k(x_k)$ the reference product density. Let

$$r_n(\omega) = \begin{cases} \left[\prod_{k=1}^{n} f_k(X_k)\right] \Big/ g_n(X_1, \ldots, X_n) & \text{if the denominator} > 0; \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where ω is a sample point. In statistical terms, $r_n(\omega)$ is called the likelihood ratio, which is of fundamental importance in the theory of testing the statistical hypotheses (cf. [**1**, p. 483]; [**3**, p. 388]). Let

$$r(\omega) = -\liminf_{n} \frac{1}{n} \ln r_n(\omega) \tag{2}$$

with $\ln 0 = -\infty$. $r(\omega)$ is called asymptotic log-likelihood ratio. Obviously, $r_n(\omega) \equiv 0$ if

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$$g_n(x_1, \ldots, x_n) = \prod_{k=1}^{n} f_k(x_k), \qquad n \geq 1,$$

and it will be shown in (13) that $r(\omega) \geq 0$ a.e. in any case. Hence $r(\omega)$ can be used as a random measure of the deviation between the true joint density $g_n(x_1, \ldots, x_n)$ $(n = 1, 2, \ldots)$ and the reference product density $\prod_{k=1}^{n} f_k(x_k)$. Roughly speaking, this deviation may be regarded as the one between $\{X_n, n \geq 1\}$ and the independence case. The smaller $r(\omega)$ is, the smaller the deviation is. The purpose of this paper is to establish a kind of strong limit theorem represented by inequalities with random bounds for the dependent random variables, by using the notion of asymptotic log-likelihood and the martingale convergence theorem, and to extend the analytic technique proposed by Liu [**4**], [**5**], and Liu and Yang [**6**] to the case of absolutely continuous random variables.

2. Main result

THEOREM. *Let $\{X_n, n \geq 1\}$, $r_n(\omega)$, $r(\omega)$ be given as above, $\{B_n, n \geq 1\}$ be a sequence of Borel sets of the real line, and I_{B_n} be the indicator function of B_n. Let*

$$b = \limsup_{n} \frac{1}{n} \sum_{k=1}^{n} \int_{B_k} f_k(x_k) dx_k \tag{3}$$

and

$$D_1 = \{\omega : r(\omega) \leq b\}, \qquad D_2 = \{\omega : r(\omega) \geq b\}.$$

Then

$$\text{(a)} \quad \limsup_{n} \frac{1}{n} \sum_{k=1}^{n} \left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k) dx_k \right] \leq 2\sqrt{br(\omega)} + r(\omega) \ \textit{a.e.}; \tag{4}$$

$$\text{(b)} \quad \liminf_{n} \frac{1}{n} \sum_{k=1}^{n} \left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k) dx_k \right] \geq -2\sqrt{br(\omega)} \ \textit{a.e. on } D_1, \tag{5}$$

and

$$\liminf_{n} \frac{1}{n} \sum_{k=1}^{n} \left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k) dx_k \right] \geq -b - r(\omega) \ \textit{a.e. on } D_2. \tag{6}$$

PROOF. Let $\lambda > 0$ be a constant, and let

$$h_k(x_k) = \begin{cases} \dfrac{\lambda f_k(x_k)}{1 + (\lambda - 1)\int_{B_k} f_k(x_k)dx_k} & x_k \in B_k; \\ \dfrac{f_k(x_k)}{1 + (\lambda - 1)\int_{B_k} f_k(x_k)dx_k} & x_k \notin B_k. \end{cases} \tag{7}$$

It is easy to see that $\prod_{k=1}^{n} h_k(x_k)$ is a product density function of n variables. Let

$$t_n(\lambda, \omega) = \begin{cases} \left[\prod_{k=1}^{n} h_k(X_k)\right] \Big/ g_n(X_1, \dots, X_n) & \text{if the denominator} > 0; \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Then $t_n(\lambda, \omega)$ is a nonnegative supermartingale that converges a.e. Hence there exists $A(\lambda) \in \mathscr{F}$, $P(A(\lambda)) = 1$, such that

$$\limsup_n \frac{1}{n} \ln t_n(\lambda, \omega) \le 0, \qquad \omega \in A(\lambda). \tag{9}$$

Letting $\lambda = 1$ in (9), we obtain

$$\limsup_n \frac{1}{n} \ln r_n(\omega) \le 0, \qquad \omega \in A(1). \tag{10}$$

This implies that

$$r(\omega) \ge 0, \qquad \omega \in A(1). \tag{11}$$

We have by (7)

$$\begin{aligned} \prod_{k=1}^{n} h_k(X_k) &= \prod_{k=1}^{n} \frac{\lambda^{I_{B_k}(x_k)} f_k(x_k)}{1 + (\lambda - 1)\int_{B_k} f_k(x_k)dx_k} \\ &= \lambda^{\sum_{k=1}^{n} I_{B_k}(x_k)} \prod_{k=1}^{n} \frac{f_k(x_k)}{1 + (\lambda - 1)\int_{B_k} f_k(x_k)dx_k}. \end{aligned} \tag{12}$$

It follows from (1), (8), and (12) that

$$\ln t_n(\lambda, \omega) = \sum_{k=1}^{n} I_{B_k}(X_k) \ln \lambda - \sum_{k=1}^{n} \ln\left[1 + (\lambda - 1)\int_{B_k} f_k(x_k)dx_k\right] + \ln r_n(\omega). \tag{13}$$

We have by (9), and (13)

$$\limsup_n \frac{1}{n}\left(\sum_{k=1}^n I_{B_k}(X_k)\ln\lambda - \sum_{k=1}^n \ln\left[1+(\lambda-1)\int_{B_k} f_k(x_k)dx_k\right] + \ln r_n(\omega)\right) \le 0,$$

$$\omega \in A(\lambda). \qquad (14)$$

(a) Let $\lambda > 1$. Dividing the two sides of (16) by $\ln\lambda$, we obtain

$$\limsup_n \frac{1}{n}\left(\sum_{k=1}^n I_{B_k}(X_k) - \sum_{k=1}^n \frac{\ln[1+(\lambda-1)\int_{B_k} f_k(x_k)dx_k]}{\ln\lambda} + \frac{\ln r_n(\omega)}{\ln\lambda}\right) \le 0,$$

$$\omega \in A(\lambda). \qquad (15)$$

By (15) and (2), we have

$$\limsup_n \frac{1}{n}\left(\sum_{k=1}^n I_{B_k}(X_k) - \sum_{k=1}^n \frac{\ln[1+(\lambda-1)\int_{B_k} f_k(x_k)dx_k]}{\ln\lambda}\right) \le \frac{r(\omega)}{\ln\lambda}, \quad \omega \in A(\lambda) \qquad (16)$$

By (16), (3), the property of the superior limit

$$\limsup_n (a_n - b_n) \le d \Rightarrow \limsup_n (a_n - c_n) \le \limsup_n (b_n - c_n) + d,$$

and the inequality $0 \le \ln(1+x) \le x \ (x \ge 0)$, we have

$$\begin{aligned}
&\limsup_n \frac{1}{n}\sum_{k=1}^n \left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k)dx_k\right] \\
&\quad \le \limsup_n \frac{1}{n}\sum_{k=1}^n \left(\frac{\ln[1+(\lambda-1)\int_{B_k} f_k(x_k)dx_k]}{\ln\lambda} - \int_{B_k} f_k(x_k)dx_k\right) + \frac{r(\omega)}{\ln\lambda} \\
&\quad \le \limsup_n \frac{1}{n}\sum_{k=1}^n \left(\frac{(\lambda-1)\int_{B_k} f_k(x_k)dx_k}{\ln\lambda} - \int_{B_k} f_k(x_k)dx_k\right) + \frac{r(\omega)}{\ln\lambda} \\
&\quad \le b\left(\frac{\lambda-1}{\ln\lambda} - 1\right) + \frac{r(\omega)}{\ln\lambda}, \qquad \omega \in A(\lambda). \qquad (17)
\end{aligned}$$

By using the inequality $1 - \lambda^{-1} < \ln\lambda \ (\lambda > 1)$, we have by (17),

$$\limsup_n \frac{1}{n}\sum_{k=1}^n \left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k)dx_k\right] \le b(\lambda-1) + \frac{\lambda r(\omega)}{\lambda-1}, \qquad \omega \in A(\lambda). \qquad (18)$$

Let Q^* be the set of rational numbers in the interval $(1, +\infty)$, and let $A^* = \bigcap_{\lambda \in Q^*} A(\lambda)$, $g(\lambda, r) = b(\lambda - 1) + \lambda r/(\lambda - 1)$. Then we have by (20),

$$\limsup_n \frac{1}{n} \sum_{k=1}^{n} \left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k) dx_k \right] \le g(\lambda, r(\omega)), \qquad \omega \in A^*, \lambda \in Q^*. \tag{19}$$

Let $b > 0$. It is easy to see that if $r > 0$, then $g(\lambda, r)$ as a function of λ attains its smallest value $g(1 + \sqrt{r/b}, r) = 2\sqrt{br} + r$ on the interval $(1, +\infty)$, and $g(\lambda, 0)$ is increasing on the interval $(1, +\infty)$ and $\lim_{\lambda \to 1+0} g(\lambda, 0) = 0$. For each $\omega \in A^* \cap A(1)$, if $r(\omega) \ne \infty$, take $\lambda_n(\omega) \in Q^*$, $n = 1, 2, \ldots$, such that $\lambda_n(\omega) \to 1 + \sqrt{r(\omega)/b}$. We have by the continuity of g with respect to λ,

$$\lim_{n \to +\infty} g(\lambda_n(\omega), r(\omega)) = 2\sqrt{br(\omega)} + r(\omega). \tag{20}$$

By (19),

$$\limsup_n \frac{1}{n} \sum_{k=1}^{n} \left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k) dx_k \right] \le g(\lambda_n(\omega), r(\omega)), \qquad n = 1, 2, \ldots. \tag{21}$$

By (20) and (21),

$$\limsup_n \frac{1}{n} \sum_{k=1}^{n} \left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k) dx_k \right] \le 2\sqrt{br(\omega)} + r(\omega), \qquad \omega \in A^* \cap A(1). \tag{22}$$

If $r(\omega) = \infty$, (22) holds trivially. Since $P(A^* \cap A(1)) = 1$, (4) holds by (22) when $b > 0$.

When $b = 0$, we have by letting $\lambda = e$ in (19),

$$\limsup_n \frac{1}{n} \sum_{k=1}^{n} \left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k) dx_k \right] \le r(\omega), \qquad \omega \in A(e). \tag{23}$$

Since $P(A(e)) = 1$, (4) also holds by (23) when $b = 0$.

(b) Let $0 < \lambda < 1$. Dividing the two sides of (14) by $\ln \lambda$, we obtain

$$\liminf_n \frac{1}{n} \left(\sum_{k=1}^{n} I_{B_k}(X_k) - \sum_{k=1}^{n} \frac{\ln[1 + (\lambda - 1) \int_{B_k} f_k(x_k) dx_k]}{\ln \lambda} + \frac{\ln r_n(\omega)}{\ln \lambda} \right) \ge 0, \qquad \omega \in A(\lambda). \tag{24}$$

By (24) and (2), we have

$$\liminf_n \frac{1}{n}\left(\sum_{k=1}^n I_{B_k}(X_k) - \sum_{k=1}^n \frac{\ln[1+(\lambda-1)\int_{B_k} f_k(x_k)dx_k]}{\ln\lambda}\right) \geq \frac{r(\omega)}{\ln\lambda}, \quad \omega \in A(\lambda). \tag{25}$$

By (25), (3), the property of the inferior limit

$$\liminf_n (a_n - b_n) \geq d \Rightarrow \liminf_n (a_n - c_n) \geq \liminf_n (b_n - c_n) + d,$$

and the inequality $\ln(1+x) \leq x \ (-1 < x \leq 0)$, we have

$$\begin{aligned}
&\liminf_n \frac{1}{n}\sum_{k=1}^n \left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k)dx_k\right] \\
&\quad \geq \liminf_n \frac{1}{n}\sum_{k=1}^n \left(\frac{\ln[1+(\lambda-1)\int_{B_k} f_k(x_k)dx_k]}{\ln\lambda} - \int_{B_k} f_k(x_k)dx_k\right) + \frac{r(\omega)}{\ln\lambda} \\
&\quad \geq \liminf_n \frac{1}{n}\sum_{k=1}^n \left(\frac{(\lambda-1)\int_{B_k} f_k(x_k)dx_k}{\ln\lambda} - \int_{B_k} f_k(x_k)dx_k\right) + \frac{r(\omega)}{\ln\lambda} \\
&\quad \geq b\left(\frac{\lambda-1}{\ln\lambda} - 1\right) + \frac{r(\omega)}{\ln\lambda}, \qquad \omega \in A(\lambda).
\end{aligned} \tag{26}$$

By using the inequalities $1 - \lambda^{-1} < \ln\lambda < 0$ and $\ln\lambda < \lambda - 1 < 0 \ (0 < \lambda < 1)$, we have by (26),

$$\liminf_n \frac{1}{n}\sum_{k=1}^n \left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k)dx_k\right] \geq b(\lambda-1) + \frac{r(\omega)}{\lambda-1}, \qquad \omega \in A(\lambda) \cap A(1). \tag{27}$$

Let Q_* be the set of rational numbers in the interval $(0,1)$, and let $A_* = \bigcap_{\lambda \in Q_*} A(\lambda)$, $h(\lambda, r) = b(\lambda-1) + r/(\lambda-1)$. Then we have by (27),

$$\liminf_n \frac{1}{n}\sum_{k=1}^n \left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k)dx_k\right] \geq h(\lambda, r(\omega)), \qquad \omega \in A_* \cap A(1), \lambda \in Q_*. \tag{28}$$

Let $b > 0$. It is easy to see that if $0 < r < b$, then $h(\lambda, r)$ as a function of λ attains its largest value $h(1-\sqrt{r/b}, r) = -2\sqrt{br}$ on the interval $(0,1)$, and $h(\lambda, 0)$ is increasing on the interval $(0,1)$ and $\lim_{\lambda\to 1-0} h(\lambda, 0) = 0$, and $h(\lambda, b) = b[\lambda - 1 + 1/(\lambda-1)]$ is decreasing on the interval $(0,1)$ and $\lim_{\lambda\to 0^+} h(\lambda, b) = -2b$. For each $\omega \in A_* \cap A(1) \cap D_1$, take $\tau_n(\omega) \in Q_*$, $n = 1, 2, \ldots$, such that $\tau_n(\omega) \to 1 - \sqrt{r(\omega)/b}$. We have

$$\lim_{n\to+\infty} h(\tau_n(\omega), r(\omega)) = -2\sqrt{br(\omega)}. \tag{29}$$

By (28), we have

$$\liminf_n \frac{1}{n}\sum_{k=1}^{n}\left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k)dx_k\right] \geq h(\tau_n(\omega), r(\omega)), \qquad n = 1, 2, \ldots. \tag{30}$$

By (29) and (30),

$$\liminf_n \frac{1}{n}\sum_{k=1}^{n}\left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k)dx_k\right] \geq -2\sqrt{br(\omega)}, \qquad \omega \in A_* \cap A(1) \cap D_1. \tag{31}$$

Since $P(A_* \cap A(1)) = 1$, (5) holds by (31) when $b > 0$.

When $b = 0$, $r(\omega) = 0$ for $\omega \in D_1 \cap A(1)$. Hence we have by (28),

$$\liminf_n \frac{1}{n}\sum_{k=1}^{n}\left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k)dx_k\right] \geq 0, \qquad \omega \in A(\lambda) \cap A(1) \cap D_1, 0 < \lambda < 1. \tag{32}$$

Since $P(A(\lambda) \cap A(1)) = 1$, (5) also holds by (32) when $b = 0$.

It is easy to see that when $r > b \geq 0$, $h(\lambda, r)$ as a function of λ is decreasing on the interval $(0, 1)$ and $\lim_{\lambda\to 0^+} h(\lambda, r) = -(r + b)$. For each $\omega \in A_* \cap A(1) \cap D_2$, when $r(\omega) \neq \infty$, take $\lambda_n(\omega) \in Q_*$, $n = 1, 2, \ldots$, such that $\lambda_n(\omega) \to 0$. We have

$$\lim_{n\to\infty} h(\lambda_n(\omega), r(\omega)) = -r(\omega) - b. \tag{33}$$

By (28), we have

$$\liminf_n \frac{1}{n}\sum_{k=1}^{n}\left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k)dx_k\right] \geq h(\lambda_n(\omega), r(\omega)), \qquad n = 1, 2, \ldots. \tag{34}$$

It follows from (33) and (34) that,

$$\liminf_n \frac{1}{n}\sum_{k=1}^{n}\left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k)dx_k\right] \geq -r(\omega) - b, \qquad \omega \in A_* \cap A(1) \cap D_2. \tag{35}$$

Obviously, (35) also holds when $r(\omega) = \infty$. Since $P(A_* \cap A(1)) = 1$, (6) follows from (35) directly.

3. Some corollaries

COROLLARY 1. *Let B be a Borel set of the real line, $S_n(B,\omega)$ be the number of occurrence of $X_k (1 \le k \le n)$ in B, that is,*

$$S_n(B,\omega) = \sum_{k=1}^{n} I_B(X_k).$$

Then under the conditions of the theorem, we have

$$\limsup_n \frac{1}{n} \sum_{k=1}^{n} \left[S_n(B,\omega) - \int_B f_k(x_k)dx_k \right] \le 2\sqrt{br(\omega)} + r(\omega) \ \ a.e.,$$

$$\liminf_n \frac{1}{n} \sum_{k=1}^{n} \left[S_n(B,\omega) - \int_B f_k(x_k)dx_k \right] \ge -2\sqrt{br(\omega)} \ \ a.e. \ on \ D_1,$$

$$\liminf_n \frac{1}{n} \sum_{k=1}^{n} \left[S_n(B,\omega) - \int_B f_k(x_k)dx_k \right] \ge -b - r(\omega) \ \ a.e. \ on \ D_2.$$

PROOF. Letting $B_k = B$ $(k = 1, 2, \ldots)$, the corollary follows from the above theorem directly.

The strong law of large numbers for $I_{B_n}(X_n)$, $n \ge 1$, is a corollary of the above theorem.

COROLLARY 2. *If $\{X_k, k \ge 1\}$ is independent random variables with density function $f_k(x_k)$, then*

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \left[I_{B_k}(X_k) - \int_{B_k} f_k(x_k)dx_k \right] = 0 \ \ a.e. \tag{36}$$

PROOF. In this case, $g_n(x_1, \ldots, x_n) = \prod_{k=1}^{n} f_k(x_k)$, and $r(\omega) = 0$. Hence (38) follows from (4) and (5) directly.

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