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Extendibility and stable extendibility of the square of the normal bundle associated to an immersion of the real projective space

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ABSTRACT. The purpose of this paper is to study the extendibility and the stable extendibility of the square of the normal bundle associated to an immersion of the real projective space and the extendibility of its complexification.

1. Introduction

Let X be a space and A be its subspace. A t-dimensional F-vector bundle ζ over A is called extendible (respectively stably extendible) to X, if there is a t-dimensional F-vector bundle over X whose restriction to A is equivalent (respectively stably equivalent) to ζ as F-vector bundles, where F is the real number field R, the complex number field C or the quaternion number field H (cf. [10] and [3]). Let R^n be the n-dimensional Euclidean space and RP^n be the n-dimensional real projective space.

We study the question: Determine the dimension n for which a vector bundle over RP^n is extendible (or stably extendible) to RP^m for every $m > n$. The answers have been obtained for the tangent bundle of RP^n (cf. [5] and [7]), for the square of the tangent bundle of RP^n (cf. [4]) and for the normal bundle associated to an immersion of RP^n in R^{n+k} (cf. [8] and [9]).

Denote by $\phi(n)$ the number of integers s such that $0 < s \le n$ and $s \equiv 0, 1, 2$ or 4 mod 8. For the square of the normal bundle associated to an immersion of RP^n in R^{n+k}, we have

THEOREM 1. *Let ν^2 be the square of the normal bundle ν associated to an immersion of RP^n in R^{n+k}, where $k > 0$.*

(1) *Assume that there is an integer a such that*

$$2(n+1)^2 + 2k(n+1) \le a2^{\phi(n)} \le 2(n+1)^2 + 2k(n+1) + k^2.$$

Then ν^2 is stably extendible to RP^m for every $m > n$, and if $n < k^2$ in addition to the above condition, ν^2 is extendible to RP^m for every $m > n$.

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(2) *Assume that there is an integer a such that*

$$2(n+1)^2 + 2k(n+1) + k^2 < a2^{\phi(n)} < 2(n+1)^2 + 2k(n+1) + 2^{\phi(n)}.$$

Then v^2 is not stably extendible to RP^m for $m = a2^{\phi(n)} - 2(n+1)^2 - 2k(n+1)$.

THEOREM 2. *Let v^2 be the square of the normal bundle v associated to an immersion of RP^n in R^{2n+1}. Then v^2 is extendible to RP^m for every $m > n$ if and only if $1 \leq n \leq 17$ or $n = 20, 21$.*

Denote by $\lfloor x \rfloor$ the integral part of a positive real number x. For the complexification of the square of the normal bundle associated to an immersion of RP^n in R^{n+k}, we have

THEOREM 3. *Let cv^2 be the complexification of the square of the normal bundle v associated to an immersion of RP^n in R^{n+k}, where $k > 0$.*

(1) *Assume that there is an integer b such that*

$$2(n+1)^2 + 2k(n+1) \leq b2^{\lfloor n/2 \rfloor} \leq 2(n+1)^2 + 2k(n+1) + k^2.$$

Then cv^2 is stably extendible to RP^m for every $m > n$, and if $n \leq 2k^2$ in addition to the above condition, cv^2 is extendible to RP^m for every $m > n$.

(2) *Assume that there is an integer b such that*

$$2(n+1)^2 + 2k(n+1) + k^2 < b2^{\lfloor n/2 \rfloor} < 2(n+1)^2 + 2k(n+1) + 2^{\lfloor n/2 \rfloor}.$$

Then cv^2 is not stably extendible to RP^m for $m = b2^{\lfloor n/2 \rfloor + 1} - 4(n+1)^2 - 4k \cdot (n+1)$.

THEOREM 4. *Let cv^2 be the complexification of the square of the normal bundle v associated to an immersion of RP^n in R^{2n+1}. Then cv^2 is extendible to RP^m for every $m > n$ if and only if $1 \leq n \leq 18$ or $n = 20, 21$.*

This note is arranged as follows. Theorem 1 is proved in Section 2. In Section 3 we study the Whitney sum decomposition of the square of the normal bundle associated to an immersion of RP^n in R^{2n+1} and prove Theorem 2. Theorem 3 is proved in Section 4. In Section 5 we study the Whitney sum decomposition of the complexification of the square of the normal bundle associated to an immersion of RP^n in R^{2n+1} and prove Theorem 4.

2. Proof of Theorem 1

Let ξ_n be the canonical R-line bundle over RP^n. The ring structure of $KO(RP^n)$ is determined in [1]. We recall the results that are necessary for our proofs. We use the same letter for a vector bundle and its isomorphism class.

(2.1) [1, Theorem 7.4]. (1) *The reduced KO-group* $\widetilde{KO}(RP^n)$ *is isomorphic to the cyclic group* $Z/2^{\phi(n)}$, *generated by* $\xi_n - 1$.

(2) $(\xi_n)^2 = 1$.

Lemma 2.2. *Let* $v^2 = v(f) \otimes v(f)$ *be the square of the normal bundle* $v(f)$ *associated to an immersion* f *of* RP^n *in* R^{n+k}. *Then the equality*

$$v^2 = \{a2^{\phi(n)} - 2(n+1)^2 - 2k(n+1)\}\xi_n + 2(n+1)^2 + 2k(n+1) + k^2 - a2^{\phi(n)}$$

holds in $KO(RP^n)$, *where* a *is any integer*.

Proof. Let $\tau = \tau(RP^n)$ be the tangent bundle of RP^n and let $\oplus$ denote the Whitney sum. Since $\tau \oplus 1 = (n+1)\xi_n$ and $\tau \oplus v = n+k$, $v = -(n+1)\xi_n + n + k + 1$. Hence we have, by (2.1),

$$\begin{aligned} v^2 &= (n+1)^2(\xi_n)^2 - 2(n+1)(n+k+1)\xi_n + (n+k+1)^2 \\ &= \{a2^{\phi(n)} - 2(n+1)(n+k+1)\}\xi_n + (n+1)^2 + (n+k+1)^2 - a2^{\phi(n)} \\ &= \{a2^{\phi(n)} - 2(n+1)^2 - 2k(n+1)\}\xi_n + 2(n+1)^2 + 2k(n+1) + k^2 - a2^{\phi(n)} \end{aligned}$$

in $KO(RP^n)$ for any integer a. q.e.d.

Let d denote $\dim_R F$, where $F = R, C$ or H. For a real number x, let $\lceil x \rceil$ denote the smallest integer n with $x \leq n$. The following fact is well-known (cf. [2, Theorem 1.5, p. 100]).

(2.3). *If* α *and* β *are two t-dimensional F-vector bundles over an n-dimensional CW-complex X such that* $\lceil (n+2)/d - 1 \rceil \leq t$ *and* $\alpha \oplus k = \beta \oplus k$ *for some k-dimensional trivial F-bundle k over X, then* $\alpha = \beta$.

Proof of Theorem 1(1). By Lemma 2.2, we have $v^2 = A\xi_n + B$, where $A = a2^{\phi(n)} - 2(n+1)^2 - 2k(n+1)$ and $B = 2(n+1)^2 + 2k(n+1) + k^2 - a2^{\phi(n)}$. By the assumption $A \geq 0$ and $B \geq 0$. For every $m > n$, $i^*(A\xi_m \oplus B) = A\xi_n \oplus B$, where $i : RP^n \to RP^m$ is the standard inclusion. Hence v^2 is stably extendible to RP^m for every $m > n$, since v^2 is stably equivalent to $A\xi_n \oplus B$.

If $n < k^2$, in addition, $\dim RP^n = n < k^2 = \dim v^2 = A + B$, and so we obtain $v^2 = A\xi_n \oplus B$ by (2.3). Thus v^2 is extendible to RP^m for every $m > n$. q.e.d.

The following result is Theorem 4.1 in [7] which is the *stable extendible version* of Theorem 6.2 in [5].

(2.4). *Let ζ be a t-dimensional R-vector bundle over RP^n. Assume that there is a positive integer ℓ such that ζ is stably equivalent to $(t+\ell)\xi_n$ and $t+\ell < 2^{\phi(n)}$. Then $n < t+\ell$ and ζ is not stably extendible to RP^m for any m with $m \geq t+\ell$.*

Proof of Theorem 1(2). Put $\zeta = \nu^2$, $t = k^2$ and $\ell = a2^{\phi(n)} - 2(n+1)^2 - 2k(n+1) - k^2$. Then ζ is stably equivalent to $(t+\ell)\xi_n$ by Lemma 2.2, and $t+\ell < 2^{\phi(n)}$ and $\ell > 0$ by the assumption. Hence, by (2.4), ν^2 is not stably extendible to RP^m for $m = a2^{\phi(n)} - 2(n+1)^2 - 2k(n+1)$. q.e.d.

3. Proof of Theorem 2

In this section we discuss the square $\nu^2 = \nu(f) \otimes \nu(f)$ of the normal bundle $\nu(f)$ associated to an immersion f of RP^n in R^{2n+1} in detail.

Theorem 3.1. *Let $\nu(f_n)^2 = \nu(f_n) \otimes \nu(f_n)$ be the square of the normal bundle $\nu(f_n)$ associated to an immersion f_n of RP^n in R^{2n+1}. Then we have*

$\nu(f_1)^2 = 4,$ $\nu(f_2)^2 = 9,$ $\nu(f_3)^2 = 16,$

$\nu(f_4)^2 = 4\xi_4 \oplus 21,$ $\nu(f_5)^2 = 36,$ $\nu(f_6)^2 = 4\xi_6 \oplus 45,$

$\nu(f_7)^2 = 64,$ $\nu(f_8)^2 = 12\xi_8 \oplus 69,$ $\nu(f_9)^2 = 16\xi_9 \oplus 84,$

$\nu(f_{10})^2 = 28\xi_{10} \oplus 93,$ $\nu(f_{11})^2 = 128\xi_{11} \oplus 16,$ $\nu(f_{12})^2 = 92\xi_{12} \oplus 77,$

$\nu(f_{13})^2 = 112\xi_{13} \oplus 84,$ $\nu(f_{14})^2 = 124\xi_{14} \oplus 101,$ $\nu(f_{15})^2 = 256,$

$\nu(f_{16})^2 = 124\xi_{16} \oplus 165,$ $\nu(f_{17})^2 = 240\xi_{17} \oplus 84,$ $\nu(f_{18})^2 = 604\xi_{18} - 243,$

$\nu(f_{19})^2 = 448\xi_{19} - 48,$ $\nu(f_{20})^2 = 284\xi_{20} \oplus 157$ *and* $\nu(f_{21})^2 = 112\xi_{21} \oplus 372.$

Proof. In Lemma 2.2, let $k = n+1$ and put a as in the following table for $1 \leq n \leq 21$:

n	1	2	3	4	5	6	7	8	9	10	11
a	8	9	16	13	18	25	32	21	13	8	11

n	12	13	14	15	16	17	18	19	20	21
a	6	7	8	8	5	3	2	2	1	1

Then the equalities above follow from Lemma 2.2, (2.1) and (2.3). q.e.d.

Lemma 3.2. *Define*

$$\ell(n) = \begin{cases} 2^{\phi(n)+1} - 5(n+1)^2, & \text{if } n = 18, 19, 22 \text{ or } 23, \\ 2^{\phi(n)} - 5(n+1)^2, & \text{otherwise.} \end{cases}$$

Then $(n+1)^2 + \ell(n) < 2^{\phi(n)}$ *if* $n \geq 18$, *and* $\ell(n) > 0$ *if* $n = 18, 19$ *or* $n \geq 22$.

Proof. If $n = 18, 19, 22$ or 23, the inequality $(n+1)^2 + \ell(n) < 2^{\phi(n)}$ and $\ell(n) > 0$ are easily verified. Otherwise, $(n+1)^2 + \ell(n) = 2^{\phi(n)} - 4(n+1)^2 < 2^{\phi(n)}$.

As is easily seen, the inequality $\ell(n) > 0$ holds for $24 \leq n = 8s + i \leq 31$. For larger values of n, we have the inequality $\ell(n) > 0$ by induction on s. q.e.d.

Theorem 3.3. *Let* $\nu(f_n)^2 = \nu(f_n) \otimes \nu(f_n)$ *be the square of the normal bundle* $\nu(f_n)$ *associated to an immersion* f_n *of* RP^n *in* R^{2n+1}. *Then, if* $n = 18, 19$ *or* $n \geq 22$, ν^2 *is not stably extendible to* RP^m *for any* m *with* $m \geq a2^{\phi(n)} - 4(n+1)^2$, *where* $a = 2$ *if* $n = 18, 19, 22$ *or* 23, *or* $a = 1$ *otherwise*.

Proof. Put $\zeta = \nu(f_n)^2$, $t = (n+1)^2$ and $\ell = a2^{\phi(n)} - 5(n+1)^2$. Then ζ is stably equivalent to $(t+\ell)\xi_n$ by Lemma 2.2, and $t + \ell < 2^{\phi(n)}$ and $\ell > 0$ by Lemma 3.2. Hence the result follows from (2.4). q.e.d.

Proof of Theorem 2. ξ_n and the trivial bundles over RP^n are extendible to RP^m for every $m > n$. Hence the *if* part follows from Theorem 3.1.

The *only if* part is a consequence of Theorem 3.3. q.e.d.

4. Proof of Theorem 3

We recall the ring structure of $K(RP^n)$ determined in [1].

(4.1) [1, Theorem 7.3]. *The reduced K-group* $\tilde{K}(RP^n)$ *is isomorphic to the cyclic group* $Z/2^{\lfloor n/2 \rfloor}$, *generated by* $c\xi_n - 1$.

Lemma 4.2. *Let* $c\nu^2 = c(\nu(f) \otimes \nu(f))$ *be the complexification of the square of the normal bundle* $\nu(f)$ *associated to an immersion* f *of* RP^n *in* R^{n+k}. *Then the equality*

$$c\nu^2 = \{b2^{\lfloor n/2 \rfloor} - 2(n+1)^2 - 2k(n+1)\}c\xi_n + 2(n+1)^2 + 2k(n+1) + k^2 - b2^{\lfloor n/2 \rfloor}$$

holds in $K(RP^n)$, *where* b *is any integer*.

Proof. Complexifying the equality in Lemma 2.2 and using (4.1), we have the equality above, since $\lfloor n/2 \rfloor \leq \phi(n)$. q.e.d.

Proof of Theorem 3(1). By Lemma 4.2, we have $c\nu^2 = Ac\xi_n + B$, where $A = b2^{\lfloor n/2 \rfloor} - 2(n+1)^2 - 2k(n+1)$ and $B = 2(n+1)^2 + 2k(n+1) + k^2 - b2^{\lfloor n/2 \rfloor}$.

By the assumption $A \geq 0$ and $B \geq 0$. For every $m > n$, $i^*(Ac\xi_m \oplus B) = Ac\xi_n \oplus B$, where $i: RP^n \to RP^m$ is the standard inclusion. Hence cv^2 is stably extendible to RP^m for every $m > n$, since cv^2 is stably equivalent to $Ac\xi_n \oplus B$.

If $n \leq 2k^2$, in addition, $\lceil(\dim RP^n + 2)/2 - 1\rceil = \lceil n/2 \rceil \leq k^2 = \dim v^2 = A + B$, and so we obtain $cv^2 = Ac\xi_n \oplus B$ by (2.3). Thus cv^2 is extendible to RP^m for every $m > n$. q.e.d.

The following result is Theorem 2.1 in [7] which is the *stably extendible version* of Theorem 4.2 for $d = 1$ in [6].

(4.3). *Let ζ be a t-dimensional C-vector bundle over RP^n. Assume that there is a positive integer ℓ such that ζ is stably equivalent to $(t+\ell)c\xi_n$ and $t + \ell < 2^{\lfloor n/2 \rfloor}$. Then $\lfloor n/2 \rfloor < t + \ell$ and ζ is not stably extendible to RP^m for any m with $m \geq 2t + 2\ell$.*

PROOF OF THEOREM 3(2). Put $\zeta = cv^2$, $t = k^2$ and $\ell = b2^{\lfloor n/2 \rfloor} - 2(n+1)^2 - 2k(n+1) - k^2$. Then ζ is stably equivalent to $(t+\ell)c\xi_n$ by Lemma 4.2, and $t + \ell < 2^{\lfloor n/2 \rfloor}$ and $\ell > 0$ by the assumption. Hence, by (4.3), cv^2 is not stably extendible to RP^m for $m = b2^{\lfloor n/2 \rfloor + 1} - 4(n+1)^2 - 4k(n+1)$. q.e.d.

5. Proof of Theorem 4

In this section we discuss the complexification $cv^2 = c(v(f) \otimes v(f))$ of the square of the normal bundle $v(f)$ associated to an immersion f of RP^n in R^{2n+1} in detail.

THEOREM 5.1. *Let $cv(f_n)^2 = c(v(f_n) \otimes v(f_n))$ be the complexification of the square of the normal bundle $v(f_n)$ associated to an immersion f_n of RP^n in R^{2n+1}. Then we have*

$$cv(f_1)^2 = 4, \quad cv(f_2)^2 = 9, \quad cv(f_3)^2 = 16, \quad cv(f_4)^2 = 25,$$
$$cv(f_5)^2 = 36, \quad cv(f_6)^2 = 4c\xi_6 \oplus 45, \quad cv(f_7)^2 = 64,$$
$$cv(f_8)^2 = 12c\xi_8 \oplus 69, \quad cv(f_9)^2 = 100, \quad cv(f_{10})^2 = 28c\xi_{10} \oplus 93,$$
$$cv(f_{11})^2 = 128c\xi_{11} \oplus 16, \quad cv(f_{12})^2 = 92c\xi_{12} \oplus 77, \quad cv(f_{13})^2 = 112c\xi_{13} \oplus 84,$$
$$cv(f_{14})^2 = 124c\xi_{14} \oplus 101, \quad cv(f_{15})^2 = 256, \quad cv(f_{16})^2 = 124c\xi_{16} \oplus 165,$$
$$cv(f_{17})^2 = 240c\xi_{17} \oplus 84, \quad cv(f_{18})^2 = 92c\xi_{18} \oplus 269, \quad cv(f_{19})^2 = 448c\xi_{19} - 48,$$
$$cv(f_{20})^2 = 284c\xi_{20} \oplus 157 \quad \text{and} \quad cv(f_{21})^2 = 112c\xi_{21} \oplus 372.$$

PROOF. Complexifying the equalities in Theorem 3.1, we have the equalities above except for $n = 4, 9$ and 18. Using relations $4c\xi_4 = 4$ for $n = 4$, $16c\xi_9 = 16$ for $n = 9$ and $512c\xi_{18} = 512$ for $n = 18$, we have the equalities above from those in Theorem 3.1 by (2.3). q.e.d.

Lemma 5.2. *Define*

$$\ell(n) = \begin{cases} 2^{\lfloor n/2 \rfloor + 2} - 5(n+1)^2, & \text{if } n = 19, \\ 2^{\lfloor n/2 \rfloor + 1} - 5(n+1)^2, & \text{if } 20 \le n \le 23, \text{ and} \\ 2^{\lfloor n/2 \rfloor} - 5(n+1)^2, & \text{otherwise}. \end{cases}$$

Then $(n+1)^2 + \ell(n) < 2^{\lfloor n/2 \rfloor}$ *if* $n \ge 19$, *and* $\ell(n) > 0$ *if* $n = 19$ *or* $n \ge 22$.

Proof. If $n = 19, 22$ or 23, the inequality $(n+1)^2 + \ell(n) < 2^{\lfloor n/2 \rfloor}$ is easily verified. Otherwise, $(n+1)^2 + \ell(n) = 2^{\lfloor n/2 \rfloor} - 4(n+1)^2 < 2^{\lfloor n/2 \rfloor}$.

As is easily seen, the inequality $\ell(n) > 0$ holds for $n = 19, 22$ and 23. For larger values of n, we have the inequality $\ell(n) > 0$ by induction. q.e.d.

Theorem 5.3. *Let* $cv(f_n)^2 = c(v(f_n) \otimes v(f_n))$ *be the complexification of the square of the normal bundle associated to an immersion of* RP^n *in* R^{2n+1}. *Then, if* $n = 19$ *or* $n \ge 22$, $cv(f_n)^2$ *is not stably extendible to* RP^m *for any* m *with* $m \ge b2^{\lfloor n/2 \rfloor + 1} - 8(n+1)^2$, *where* $b = 4$ *if* $n = 19$, $b = 2$ *if* $n = 22$ *or* 23, *and* $b = 1$ *otherwise*.

Proof. Put $\zeta = cv(f_n)^2$, $t = (n+1)^2$ and $\ell = b2^{\lfloor n/2 \rfloor} - 5(n+1)^2$. Then ζ is stably equivalent to $(t+\ell)c\xi_n$ by Lemma 4.2, and $t + \ell < 2^{\lfloor n/2 \rfloor}$ and $\ell > 0$ by Lemma 5.2. Hence the result follows from (4.3). q.e.d.

Proof of Theorem 4. $c\xi_n$ and the trivial bundles over RP^n are extendible to RP^m for every $m > n$. Hence the *if* part follows from Theorem 5.1.

The *only if* part is a consequence of Theorem 5.3. q.e.d.

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