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Error bounds for asymptotic expansions of the distribution of multivariate scale mixture

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Abstract. This paper is concerned with error bounds for asymptotic expansions of the distribution of a multivariate scale mixture variate defined by $\boldsymbol{X} = \boldsymbol{S}\boldsymbol{Z}$, where $\boldsymbol{Z} = (Z_1, \dots, Z_p)'$, $Z_1, \dots, Z_p$ are *i.i.d.* random variables, and $\boldsymbol{S}$ is a symmetric positive definite random matrix independent of $\boldsymbol{Z}$. Recently Fujikoshi, Ulyanov and Shimizu (2005) obtained L_1-norm error bounds for asymptotic expansions of the density function of $\boldsymbol{X}$ when $\boldsymbol{S} = \mathrm{diag}(S_1, \dots, S_p)$. In this paper, first we obtain uniform error bounds for asymptotic expansions of the distribution function of $\boldsymbol{X}$ under the same diagonal structure of $\boldsymbol{S}$. Next we extend the L_1-norm error bounds to tha case when $\boldsymbol{S}$ is a symmetric positive definite random matrix provided Z_1 is distributed as the standard normal distribution $N(0,1)$.

1. Introduction

Let $\boldsymbol{Z} = (Z_1, \dots, Z_p)'$ be a random vector, where $Z_1, \dots, Z_p$ are *i.i.d.* random variables, and G and g be the distribution function and the density function of Z_1, respectively. Further, let $\boldsymbol{S}$ be a symmetric positive definite random matrix independent of $\boldsymbol{Z}$. Our interest is to obtain error bounds for asymptotic expansions of the distribution of

$$\boldsymbol{X} = \boldsymbol{S}\boldsymbol{Z} \tag{1.1}$$

which is called a multivariate scale mixture of $\boldsymbol{Z}$. Here it is tacitly assumed that the scale factor $\boldsymbol{S}$ is close to $\boldsymbol{I}_p$ in some sense. Some important applications appear in two cases when Z_1 is distributed as the standard normal distribution $N(0,1)$ or a gamma distribution. Having in mind statistical applications and a unified treatment of our results we consider a transformation given by

$$\boldsymbol{S} = \boldsymbol{Y}^{\delta\rho} \qquad \text{or} \qquad \boldsymbol{Y} = \boldsymbol{S}^{\delta/\rho}, \tag{1.2}$$

where $\delta = 1$ or -1 and ρ is a positive constant. The notation δ is used for two types of asymptotic expansions. In practical applications the positive constant

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ρ is chosen as $\rho = 1/2$ or 1 according to that Z_1 is distributed as the standard normal distribution or a gamma distribution.

It may be noted that a relatively wide class of statistics can be expressed as a mixture of the standard normal or a chi-square distribution and its multivariate versions as in (1.1). On the other hand, a scale mixture appears as a basic statistical distribution. Then one of the important problems in the former case is to study asymptotic expansions of the distributions of such mixture variates and theier error bounds. In the latter case we are interesting in the distance of a mixture from its parent, see, example, Keilson and Steutel (1974).

In this paper we are interesting in asymptotic expansions of the distribution of $\boldsymbol{X}$ in (1.1) and their error bounds. Asymptotic expansions have been studied for a function of the sum of *i.i.d.* random vectors, see, for example, Bhattacharya and Ghosh (1978). Our class of statistics may be not large in the the class of statistics in Bhattacharya and Ghosh (1978). However, our class is not a subset of the latter class. Furthermore, it may be noted that our error estimate has been done by deriving error bounds in explicit and computable forms.

Asymptotic expansions and their error bounds in the univariate case of (1.1) have been extensively studied. For the results, see, e.g., Hall (1979), Fujikoshi and Shimizu (1990), Fujikoshi (1993), Shimizu and Fujikoshi (1997), Ulyanov, Fujikoshi and Shimizu (1999), etc. However, for multivariate scale mixtures, some special cases have been studied. As for results on the distribution function, Fujikoshi and Shimizu (1989a) treated the case $\boldsymbol{S} = s\boldsymbol{I}_p$. Fujikoshi and Shimizu (1989b) treated the case $\boldsymbol{S} - \boldsymbol{I}_p \geq O$, $G = \Phi$, $\delta = 1$ and $\rho = 1/2$, where Φ is the distribution function of $N(0,1)$. As for results on the density function, Shimizu (1995) obtained L_1-error bound when $G = \Phi$, $\delta = 1$ and $\rho = 1/2$. Recently Fujikoshi, Ulyanov and Shimizu (2005) obtained L_1-norm error bounds for asymptotic expansions of the density function of $\boldsymbol{X}$ when $\boldsymbol{S} = \mathrm{diag}(S_1, \ldots, S_p)$.

In this paper, first we obtain uniform error bounds for asymptotic expansions of the distribution function of $\boldsymbol{X}$ when $\boldsymbol{S} = \mathrm{diag}(S_1, \ldots, S_p)$. We note that the results have improved error bounds in the comparison with the ones derived from the L_1-norm error bounds due to Fujikoshi, Ulyanov and Shimizu (2005). Next we extend the L_1-norm error bounds due to Fujikoshi, Ulyanov and Shimizu (2005) to tha case when $\boldsymbol{S}$ is a symmetric positive definite random matrix provided Z_1 is distributed as the standard normal distribution $N(0,1)$.

2. Uniform error bounds

The multivariate scale mixture variate $\boldsymbol{X}$ in (1.1) is written for $p = 1$ as

$$X = SZ \tag{2.1}$$

where S is a positive random variable, and Z and S are independent. Let F and G be the distribution fuctions of X and Z, respectively. We assume that for a given positive integer k,

A1. G is k times continuously differentiable on D,

where $D = \{x \in \mathbf{R} : g(x) > 0\}$, and g is the density function of Z. Consider the transformation $Y = S^{\delta/\rho}$ as in (1.2). The distribution function of $X = SZ$ given $Y = y$ is expressed as $G(xy^{-\delta\rho})$. For $j = 1, \dots, k$, let $c_{\delta,j}(x)$ be defined by

$$\frac{\partial^j}{\partial y^j} G(xy^{-\delta\rho}) = y^{-j} c_{\delta,j}(xy^{-\delta\rho}) g(xy^{-\delta\rho}), \tag{2.2}$$

for $x \in D$, and $c_{\delta,j}(x) = 0$ for $x \notin D$, and write

$$\alpha_{\delta,j} \equiv \begin{cases} 1, & \text{if } j = 0, \\ (1/j!) \sup_x |c_{\delta,j}(x)| g(x), & \text{if } j \geq 1. \end{cases}$$

Note that if $p = 1$ we can take

$$\alpha_{\delta,0} = \min\{G(0), 1 - G(0)\}.$$

However, $\alpha_{\delta,0} = 1$ for all $p \geq 2$. The functions $c_{\delta,j}(x)$ may be defined also by

$$\left. \frac{\partial^j}{\partial y^j} G(xy^{-\delta\rho}) \right|_{y=1} = c_{\delta,j}(x) g(x).$$

For explicit expressions of $c_{\delta,j}(x)$ in normal or Gamma distribution, see, e.g., Fujikoshi and Shimizu (1990), Fujikoshi (1993), etc.

In this section we consider the distribution function of $\boldsymbol{X} = (X_1, \dots, X_p)$ in (1.1) with $\boldsymbol{S} = \mathrm{diag}(S_1, \dots, S_p)$. Then $X_i = S_i Z_i = Y_i^{\delta\rho} Z_i$, $i = 1, \dots, p$, and the distribution function of $\boldsymbol{X}$ can be written as

$$\begin{aligned} F_p(\boldsymbol{x}) &= \mathrm{P}(X_1 \leq x_1, \dots, X_p \leq x_p) \\ &= \mathrm{E}[G(x_1 Y_1^{-\delta\rho}) \dots G(x_p Y_p^{-\delta\rho})], \end{aligned}$$

where $\boldsymbol{x} = (x_1, \dots, x_p)'$. Let $G_p(\boldsymbol{x}) = G(x_1) \dots G(x_p)$ and $g_p(\boldsymbol{x}) = g(x_1) \dots g(x_p)$. We consider an approximation for $F_p(\boldsymbol{x})$,

$$\begin{aligned} G_{\delta,k,p}(\boldsymbol{x}) = \mathrm{E}\Bigg[G_p(\boldsymbol{x}) + \sum_{j=1}^{k-1} \frac{1}{j!} \left\{ (Y_1 - 1) \frac{\partial}{\partial y_1} + \cdots + (Y_p - 1) \frac{\partial}{\partial y_p} \right\}^j \\ \times G(x_1 y_1^{-\delta\rho}) \dots G(x_p y_p^{-\delta\rho}) \Bigg|_{y_1 = \cdots = y_p = 1} \Bigg] \end{aligned}$$

$$= G_p(\boldsymbol{x}) + \sum_{j=1}^{k-1} \sum_{(j)} \frac{1}{j_1! \dots j_p!} c_{\delta, j_1}(x_1) \dots c_{\delta, j_p}(x_p) g_p(\boldsymbol{x}) \times \mathrm{E}[(Y_1 - 1)^{j_1} \dots (Y_p - 1)^{j_p}], \tag{2.3}$$

where the sum $\sum_{(j)}$ is taken over all p-tupbs of non-negative integers such that $j_1 + \cdots + j_p = j$.

Now we give two types of error bounds for an asymptotic expansion (2.3) of $F_p(\boldsymbol{x})$, which are given in Theorems 2.1 and 2.2. The results can be proved by arguments similar to ones as in Fujikoshi, Ulyanov and Shimizu (2005) and Shimizu (1995), respectively. In Section 4 we give an outline of the proofs. Our error bounds are expressed as explicit functions of $\alpha_{\delta, j}$, $j = 1, \dots, k$. More presicely, one of the error bounds depend on

$$w_{\delta, j, p} = \sum_{[j]} \frac{(p-1)!}{i_1! \dots i_m!} \alpha_{\delta, j_1} \dots \alpha_{\delta, j_p}, \tag{2.4}$$

where the summation $\sum_{[j]}$ is taken over all p-tupbs of non-negative integers $0 \le j_1 \le \cdots \le j_p$ such that $j_1 + \cdots + j_p = j$, and the constants m, $i_1, \dots, i_m$ are positive integers such that

$$0 \le j_1 = \cdots = j_{i_1} < j_{i_1+1} = \cdots = j_{i_1+i_2} < \cdots < j_{i_1+\cdots+i_{m-1}+1} = \cdots = j_{i_1+\cdots+i_m} (= j_p) \le j.$$

In particular, we have

$$\begin{aligned} w_{\delta,1,p} &= \alpha_{\delta,1}, \\ w_{\delta,2,p} &= \alpha_{\delta,2} + \frac{1}{2}(p-1)\alpha_{\delta,1}^2, \\ w_{\delta,3,p} &= \alpha_{\delta,3} + (p-1)\alpha_{\delta,1}\alpha_{\delta,2} + \frac{1}{6}(p-1)(p-2)\alpha_{\delta,1}^3, \\ w_{\delta,4,p} &= \alpha_{\delta,4} + \frac{1}{2}(p-1)\alpha_{\delta,2}^2 + (p-1)\alpha_{\delta,1}\alpha_{\delta,3} \\ &\quad + \frac{1}{2}(p-1)(p-2)\alpha_{\delta,1}^2\alpha_{\delta,2} + \frac{1}{24}(p-1)(p-2)(p-3)\alpha_{\delta,1}^4. \end{aligned} \tag{2.5}$$

THEOREM 2.1. *Let $\boldsymbol{X} = \boldsymbol{SZ}$ be a multivariate scale mixture in* (1.1) *with $\boldsymbol{S} = \mathrm{diag}(S_1, \dots, S_p)$, and $Y_i = S_i^{\delta/\rho}$, $i = 1, \dots, p$, where $\delta = 1$ or -1 and $\rho > 0$. Suppose that the distribution function G of Z_1 satisfies* A1 *and* $\mathrm{E}(Y_i^k) < \infty$, *$i = 1, \dots, p$ for a given integer k. Then we have*

$$|F_p(\boldsymbol{x}) - G_{\delta,k,p}(\boldsymbol{x})| \le \beta_{\delta,k,p} \sum_{i=1}^{p} \mathrm{E}[|Y_i - 1|^k], \tag{2.6}$$

where $\beta_{\delta,1,p} = 1 + w_{\delta,1,p}$ *and for* $k \ge 2$

$$\beta_{\delta,k,p} = \left\{ w_{\delta,k,p}^{1/k} + \left(1 + p \sum_{j=1}^{k-1} w_{\delta,j,p}\right)^{1/k} \right\}^k. \tag{2.7}$$

Theorem 2.2. *Suppose that the conditions of Theorem* 2.1 *are satisfied. Then we have*

$$|F_p(\boldsymbol{x}) - G_{\delta,k,p}(\boldsymbol{x})| \le \gamma_{\delta,k,p} \sum_{i=1}^{p} \mathrm{E}[|Y_i - 1|^k], \tag{2.8}$$

where $\gamma_{\delta,k,p}$ *are defined recursively by the relation*

$$\gamma_{\delta,k,p} = p^{-1} \left\{ \beta_{\delta,k} + (p-1) \sum_{q=0}^{k-1} \gamma_{\delta,k-q,p-1} \alpha_{\delta,q} \right\}, \qquad \text{for } k \ge 2, \tag{2.9}$$

with $\gamma_{\delta,1,p} = \beta_{\delta,1}$, $\gamma_{\delta,k,0} = 0$ *and* $\gamma_{\delta,k,1} = \beta_{\delta,k}$ *for all* $k \ge 1$; *here*

$$\beta_{\delta,k} = \beta_{\delta,k,1} = \{\alpha_{\delta,k}^{1/k} + (\alpha_{\delta,0} + \cdots + \alpha_{\delta,k-1})^{1/k}\}^k.$$

From the relation (2.9) the constants $\gamma_{\delta,k,p}$ for $k = 1, \ldots, 4$ are determined recursively as follows.

$$\begin{aligned}
\gamma_{\delta,1,p} &= \beta_{\delta,1}, \\
\gamma_{\delta,2,p} &= \beta_{\delta,2} + \frac{1}{2}(p-1)\alpha_{\delta,1}\beta_{\delta,1}, \\
\gamma_{\delta,3,p} &= \beta_{\delta,3} + \frac{1}{2}(p-1)\{\alpha_{\delta,1}\beta_{\delta,2} + \beta_{\delta,2}\beta_{\delta,1}\} \\
&\quad + \frac{1}{6}(p-1)(p-2)\alpha_{\delta,1}\beta_{\delta,1}, \\
\gamma_{\delta,4,p} &= \beta_{\delta,4} + \frac{1}{2}(p-1)\{\alpha_{\delta,1}\beta_{\delta,3} + \alpha_{\delta,2}\beta_{\delta,2} + \alpha_{\delta,3}\beta_{\delta,1}\} \\
&\quad + \frac{1}{6}(p-1)(p-2)\{\alpha_{\delta,1}^2\beta_{\delta,2} + 2\alpha_{\delta,1}\alpha_{\delta,2}\beta_{\delta,1}\} \\
&\quad + \frac{1}{24}(p-1)(p-2)(p-3)\alpha_{\delta,1}^3\beta_{\delta,1}.
\end{aligned} \tag{2.10}$$

Combining Theorems 2.1 and 2.2 we have

$$|F_p(\boldsymbol{x}) - G_{\delta,k,p}(\boldsymbol{x})| \le \min(\beta_{\delta,k,p}, \gamma_{\delta,k,p}) \sum_{i=1}^{p} \mathrm{E}[|Y_i - 1|^k]. \tag{2.11}$$

Note that

$$\beta_{\delta,1,p} = \gamma_{\delta,1,p}, \qquad \beta_{\delta,2,p} \ge \gamma_{\delta,2,p}. \tag{2.12}$$

In a special case $\delta = 1$, $\rho = 1/2$ and $Z_1 \sim N(0,1)$ we have the following property.

$$\frac{\partial^j}{\partial y^j} \Phi(xy^{-1/2})\bigg|_{y=1} = -2^{-j} H_{2j-1}(x)\phi(x), \tag{2.13}$$

where $H_n(x)$ is Hermite polynomial of degree n defined by the equality

$$H_n(x) = (-1)^n \{\phi(x)\}^{-1} \frac{d^n}{dx^n} \phi(x).$$

It follows from (2.13) that

$$\frac{\partial^j}{\partial y^j} \Phi(xy^{-1/2})\bigg|_{y=1} = 2^{-j} \frac{d^{2j}}{dx^{2j}} \Phi(x). \tag{2.14}$$

Therefore from (2.3) we can write $G_{1,k,p}$ in the form

$$G_{1,k,p}(\boldsymbol{x}) = \mathrm{E}\left[\Phi_p(\boldsymbol{x}) + \sum_{j=1}^{k-1} \frac{1}{2^j j!} \{\partial_{\boldsymbol{x}}'(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)\partial_{\boldsymbol{x}}\}^j \Phi_p(\boldsymbol{x})\right], \tag{2.15}$$

where $\Phi_p(\boldsymbol{x}) = \Phi(x_1)\dots\Phi(x_p)$ and $\partial_{\boldsymbol{x}} = (\partial/\partial x_1, \dots, \partial/\partial x_p)'$. The approximation expressed by the right-hande side of (2.5) was considered by Fujikoshi and Shimizu (1989b) in a special case when

$$\boldsymbol{S} - \boldsymbol{I}_p \ge O, \qquad \delta = 1, \qquad \rho = \frac{1}{2}, \qquad G = \Phi.$$

3. L_1-norm error bounds

Fujikoshi, Ulyanov and Shimizu (2005) obtained L_1-norm error bounds for asymptotic expansions of the density function of $\boldsymbol{X}$ in (1.1) with $\boldsymbol{S} = \mathrm{diag}(S_1, \dots, S_p)$. In this section we extend their results to the case when $\boldsymbol{S}$ is a general symmetric positive definite random matrix provided Z_1 is distributed as $N(0,1)$. First we review their results with the help of a fundamental property that the density function $f_p(\boldsymbol{x})$ of $\boldsymbol{X}$ can be expressed in term of the distribution function as

$$f_p(\boldsymbol{x}) = \frac{\partial^p}{\partial x_1 \dots \partial x_p} G_p(\boldsymbol{x}).$$

Put $g_p(\boldsymbol{x}) = g(x_1)\dots g(x_p)$, where g is the density function of Z_1. Assume that

A2. g is k times continuously differentiable on D,

where $D = \{x \in \mathbf{R} : g(x) > 0\}$. For $j = 1, \dots, k$, let $b_{\delta,j}(x)$ be defined by

$$\frac{d}{dx}(c_{\delta,j}(x)g(x)) = b_{\delta,j}(x)g(x). \tag{3.1}$$

The function may alternatively be also defined for $j \geq 1$ and for $x \in D$, by formula

$$\frac{\partial^j}{\partial y^j}\{y^{-\delta\rho}g(xy^{-\delta\rho})\}\bigg|_{y=1} = b_{\delta,j}(x)g(x),$$

and $b_{\delta,j}(x) = 0$ for $x \notin D$. We define also for $j \geq 0$

$$\xi_{\delta,j} = \frac{1}{j!}\left\|b_{\delta,j}(x)g(x)\right\|_1, \tag{3.2}$$

where for any integrable function $h(x)$,

$$\|h(x)\|_1 = \int_{-\infty}^{\infty} |h(x)|dx.$$

It is natural ro approximate $f_p(\boldsymbol{x})$ by

$$\begin{aligned} g_{\delta,k,p}(\boldsymbol{x}) &= \frac{\partial^p}{\partial x_1 \dots \partial x_p} G_{\delta,k,p}(\boldsymbol{x}) \\ &= g_p(\boldsymbol{x}) + \sum_{j=1}^{k-1} \sum_{(j)} \frac{1}{j_1! \dots j_p!} b_{\delta,j_1}(x_1)\dots b_{\delta,j_p}(x_p) g_p(\boldsymbol{x}) \\ &\quad \times \mathrm{E}[(Y_1 - 1)^{j_1} \dots (Y_p - 1)^{j_p}]. \end{aligned} \tag{3.3}$$

One of our error bounds depends on the quantity $\eta_{\delta,k,p}$ defined as follows. Put $\eta_{\delta,1,p} = 2 + v_{\delta,1,p}$ and for $k \geq 2$

$$\eta_{\delta,k,p} = \left\{ v_{\delta,k,p}^{1/k} + \left(2 + p\sum_{j=1}^{k-1} v_{\delta,j,p}\right)^{1/k} \right\}^k, \tag{3.4}$$

where

$$v_{\delta,j,p} = \sum_{[j]} \frac{(p-1)!}{i_1! \dots i_m!} \xi_{\delta,j_1} \dots \xi_{\delta,j_p}. \tag{3.5}$$

Here the summation $\sum_{[j]}$ is taken in the sense of (2.4). Note that $v_{\delta,j,p}$ is expressed in the same form as the expression (2.5) for $w_{\delta,j,p}$, i.e., the one replaced $\alpha_{\delta,j}$ by $\xi_{\delta,j}$ in (2.5). Then we have the results corresponding to Theorems 2.1 and 2.2, which were proved by Fujikoshi, Ulyanov and Shimizu (2005).

THEOREM 3.1. *Let* $\boldsymbol{X} = \boldsymbol{SZ}$ *be a multivariate scale mixture in* (1.1) *with* $\boldsymbol{S} = \mathrm{diag}(S_1, \dots, S_p)$, *and* $Y_i = S_i^{\delta/\rho}$, $i = 1, \dots, p$, *where* $\delta = 1$ *or* -1 *and* $\rho > 0$. *Suppose that the density function* g *of* Z_1 *satisfies* A2 *and* $\mathrm{E}(Y_i^k) < \infty$, $i = 1, \dots, p$ *for a given integer* k. *Then we have for any Borel set* $A \subset \mathbf{R}^p$

$$\left|\mathrm{P}(\boldsymbol{X} \in A) - \int_A g_{\delta,k,p}(\boldsymbol{x})d\boldsymbol{x}\right| \leq \frac{1}{2}\eta_{\delta,k,p}\sum_{i=1}^{p}\mathrm{E}[|Y_i - 1|^k]. \tag{3.6}$$

THEOREM 3.2. *Under the same condition as in Theorem* 3.1 *we have for any Borel set* $A \subset \mathbf{R}^p$

$$\left|\mathrm{P}(\boldsymbol{X} \in A) - \int_A g_{\delta,k,p}(\boldsymbol{x})d\boldsymbol{x}\right| \leq \frac{1}{2}\nu_{\delta,k,p}\sum_{i=1}^{p}\mathrm{E}[|Y_i - 1|^k], \tag{3.7}$$

where $\nu_{\delta,k,p}$ *are determined recursively by the relation*

$$\nu_{\delta,k,p} = p^{-1}\left\{\eta_{\delta,k} + (p-1)\sum_{q=0}^{k-1}\nu_{\delta,k-q,p-1}\xi_{\delta,q}\right\}, \qquad \text{for } k \geq 2, \tag{3.8}$$

with $\nu_{\delta,1,p} = \eta_{\delta,1}$, $\nu_{\delta,k,0} = 0$ *and* $\nu_{\delta,k,1} = \eta_{\delta,k}$ *for all* $k \geq 1$.

Note that $\nu_{\delta,j,p}$ is expressed in the same form as the expression (2.10) for $\gamma_{\delta,j,p}$, i.e., the one replaced $\alpha_{\delta,j}$ and $\beta_{\delta,j}$ by $\xi_{\delta,j}$ and $\eta_{\delta,j}$, respectively, in (2.10). Combining Theorems 3.1 and 3.2 we have

$$\left|\mathrm{P}(\boldsymbol{X} \in A) - \int_A g_{\delta,k,p}(\boldsymbol{x})d\boldsymbol{x}\right| \leq \frac{1}{2}\min(\eta_{\delta,k,p}, \nu_{\delta,k,p})\sum_{i=1}^{p}\mathrm{E}[|Y_i - 1|^k]. \tag{3.9}$$

Further, it is known (Fujikoshi, Ulyanov and Shimizu (2005)) that

$$\eta_{\delta,1,p} = \nu_{\delta,1,p}, \qquad \eta_{\delta,2,p} \geq \nu_{\delta,2,p}. \tag{3.10}$$

In the following we extend the results (3.9) for the case when $\boldsymbol{S}$ is a symmetric positive definite matrix, assuming that Z_1 is distributed as $N(0,1)$. For $\delta = 1$ and $\rho = 1/2$, we have an identity (2.15). Differentiating both sides of (2.15) with respect to $x_1, \dots, x_p$ we have

$$g_{1,k,p}(\boldsymbol{x}) = \phi_{1,k,p}(\boldsymbol{x})$$
$$= \mathrm{E}\left[\phi_p(\boldsymbol{x}) + \sum_{j=1}^{k-1} \frac{1}{2^j j!} \{\partial'_{\boldsymbol{x}}(\boldsymbol{Y} - \boldsymbol{I}_p)\partial_{\boldsymbol{x}}\}^j \phi_p(\boldsymbol{x})\right], \tag{3.11}$$

where $\phi_p(\boldsymbol{x}) = \phi(x_1)\dots\phi(x_p)$ and $\boldsymbol{Y} = \mathrm{diag}(Y_1, \dots, Y_p)$. As it was shown in Shimizu (1995) (see the proof of Theorem 2, p. 135) the alternative expression of $\phi_{1,k,p}$ in the form (3.11) enables us to extend Theorem 3.2 to the general case when the scale matrix $\boldsymbol{S}$ may not necessarily be diagonal. In the following we give a more general extension as well as the result.

Fix any Borel set $A \subset \mathbf{R}^p$. We have

$$\mathrm{P}(\boldsymbol{X} \in A) = \mathrm{E}_{\boldsymbol{S}}[\mathrm{P}(\boldsymbol{X} \in A \mid \boldsymbol{S})],$$

where $\mathrm{E}_{\boldsymbol{S}}$ denotes expectation with respect to $\boldsymbol{S}$. It means we can construct at first approximation for $\mathrm{P}(\boldsymbol{X} \in A)$ for any given value of $\boldsymbol{S}$ and then taking expectation with respect to $\boldsymbol{S}$ we get result for $\mathrm{P}(\boldsymbol{X} \in A)$. Under the assumption on $\boldsymbol{S}$ there exists an orthogonal matrix $\boldsymbol{T}$ such that $\boldsymbol{S} = \boldsymbol{T}\boldsymbol{L}\boldsymbol{T}'$, where $\boldsymbol{L} = \mathrm{diag}(L_1, \dots, L_p)$. Then we have

$$\mathrm{P}(\boldsymbol{S}\boldsymbol{Z} \in A) = \mathrm{P}(\boldsymbol{L}\boldsymbol{T}'\boldsymbol{Z} \in T'A) = \mathrm{P}(\boldsymbol{L}\boldsymbol{Z} \in \boldsymbol{T}'A), \tag{3.12}$$

since $\boldsymbol{T}'\boldsymbol{Z}$ has also the standard multivariate normal distribution in $\mathbf{R}^p$. Consider the transformation

$$Y_i = L_i^{\delta/\rho}, \qquad i = 1, \dots, p \tag{3.13}$$

as in (1.2) or Theorems 3.1 and 3.2. Applying the result (3.9) to the right-hand side of (3.12), we have

$$\left|\mathrm{P}(\boldsymbol{X} \in A) - \mathrm{E}\left[\int_{T'A} \phi_{\delta,k,p}(\boldsymbol{x}, \boldsymbol{Y}) d\boldsymbol{x}\right]\right| \le \frac{1}{2}\min(\eta_{\delta,k,p}, \nu_{\delta,k,p}) \sum_{i=1}^{p} \mathrm{E}[|Y_i - 1|^k], \tag{3.14}$$

where

$$\phi_{\delta,k,p}(\boldsymbol{x}, \boldsymbol{Y}) = \phi_p(\boldsymbol{x}) + \sum_{j=1}^{k-1} \frac{1}{j!}\left\{(Y_1 - 1)\frac{\partial}{\partial y_1} + \dots + (Y_p - 1)\frac{\partial}{\partial y_p}\right\}^j$$
$$\times\, y_1^{-\delta\rho}\phi(x_1 y_1^{-\delta\rho}) \dots y_p^{-\delta\rho}\phi(x_p y_p^{-\delta\rho})\Big|_{y_1=\dots=y_p=1} \tag{3.15}$$

For the case $\delta = 1$, $\rho = 1/2$, we can simplify (3.14) as follows.

Theorem 3.3. *Let* $\boldsymbol{X} = \boldsymbol{SZ}$ *be a multivariate scale mixture in* (1.1) *with* $Z_1 \sim N(0,1)$. *Suppose that for a given integer* $k > 0$, $(\boldsymbol{S}^2 - \boldsymbol{I}_p)^k$ *is positive semi-definite and* $\mathrm{E}[\mathrm{tr}(\boldsymbol{S}^2 - \boldsymbol{I}_p)^k] < \infty$. *Then we have for any Borel set* $A \subset \mathbf{R}^p$

$$\left| \mathrm{P}(\boldsymbol{X} \in A) - \int_A \mathrm{E}\left[\phi_p(\boldsymbol{x}) + \sum_{j=1}^{k-1} \frac{1}{2^j j!} \{\partial_{\boldsymbol{x}}'(\boldsymbol{S}^2 - \boldsymbol{I}_p)\partial_{\boldsymbol{x}}\}^j \phi_p(\boldsymbol{x}) \right] d\boldsymbol{x} \right|$$
$$\leq \frac{1}{2} \min(\eta_{1,k,p}, \nu_{1,k,p}) \mathrm{E}[\mathrm{tr}(\boldsymbol{S}^2 - \boldsymbol{I}_p)^k]. \tag{3.16}$$

Proof. Differentiating both sides of (2.14) with respect to x, we have

$$\left. \frac{\partial^j}{\partial y^j} y^{-1/2} \phi(xy^{-1/2}) \right|_{y=1} = 2^{-j} \frac{d^{2j}}{dx^{2j}} \phi(x).$$

This implies that

$$\left\{ (Y_1 - 1)\frac{\partial}{\partial y_1} + \cdots + (Y_p - 1)\frac{\partial}{\partial y_p} \right\}^j h(\boldsymbol{x}, \boldsymbol{y}) \Big|_{y_1 = \cdots = y_p = 1}$$
$$= \sum_{(j)} \frac{j!}{j_1! \dots j_p!} \left(\frac{\partial}{\partial x_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial x_p}\right)^{j_p} h(\boldsymbol{x}, \boldsymbol{y}) \Big|_{y_1 = \cdots = y_p = 1}$$
$$\times (Y_1 - 1)^{j_1} \dots (Y_p - 1)^{j_p}$$
$$= \frac{1}{2^j} \{\partial_{\boldsymbol{x}}'(\boldsymbol{Y} - \boldsymbol{I}_p)\partial_{\boldsymbol{x}}\}^j \phi_p(\boldsymbol{x}), \tag{3.17}$$

where $h(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^p y_i^{-\delta/p} \phi(x_i y_i^{-\delta/p})$. Note that

$$\int_{T'A} \phi_p(\boldsymbol{x}) d\boldsymbol{x} = \int_A \phi_p(\boldsymbol{x}) d\boldsymbol{x}, \tag{3.18}$$

as the standard multivariate normal distribution is invariant with respect to orthogonal transformations. Moreover, if we put $\boldsymbol{v} = \boldsymbol{Tx}$, then $\boldsymbol{T}\partial_{\boldsymbol{x}} = \partial_{\boldsymbol{v}}$ and therefore we have

$$\partial_{\boldsymbol{x}}'(\boldsymbol{Y} - \boldsymbol{I}_p)\partial_{\boldsymbol{x}} = \partial_{\boldsymbol{v}}'\boldsymbol{T}(\boldsymbol{Y} - \boldsymbol{I}_p)\boldsymbol{T}'\partial_{\boldsymbol{v}} = \partial_{\boldsymbol{v}}'(\boldsymbol{S}^2 - \boldsymbol{I}_p)\partial_{\boldsymbol{v}}.$$

Thus, we get for any $j = 1, 2, \dots, k-1$

$$\int_{T'A} \{\partial_{\boldsymbol{x}}'(\boldsymbol{Y} - \boldsymbol{I}_p)\partial_{\boldsymbol{x}}\}^j \phi_p(\boldsymbol{x}) d\boldsymbol{x} = \int_A \{\partial_{\boldsymbol{v}}'(\boldsymbol{S}^2 - \boldsymbol{I}_p)\partial_{\boldsymbol{v}}\}^j \phi_p(\boldsymbol{v}) d\boldsymbol{v}. \tag{3.19}$$

Note that

$$\mathrm{tr}(\boldsymbol{S}^2 - \boldsymbol{I}_p)^k = \sum_{k=1}^{p} (Y_i - 1)^k$$

since $(\boldsymbol{S}^2 - \boldsymbol{I}_p)^k$ is positive semi-definite.

Combining (3.14) and (3.18)–(3.19) we get (3.16).

Lemma 3.1. *Let*

$$h(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{p} y_i^{-\delta/\rho} \phi(x_i y_i^{-\delta/\rho}).$$

Then it holds that

$$(1) \quad \sum_{i=1}^{p} (Y_i - 1) \frac{\partial}{\partial y_i} h(\boldsymbol{x}, \boldsymbol{y}) \bigg|_{y_1 = \cdots = y_p} = \delta\rho \partial_{\boldsymbol{x}}' (\boldsymbol{Y} - \boldsymbol{I}_p) \partial_{\boldsymbol{x}} \phi_p(\boldsymbol{x}),$$

$$(2) \quad \left\{ \sum_{i=1}^{p} (Y_i - 1) \frac{\partial}{\partial y_i} \right\}^2 h(\boldsymbol{x}, \boldsymbol{y}) \bigg|_{y_1 = \cdots = y_p}$$

$$= [(\delta\rho)^2 \{\partial_{\boldsymbol{x}}' (\boldsymbol{Y} - \boldsymbol{I}_p) \partial_{\boldsymbol{x}}\}^2 + \delta\rho(2\delta\rho - 1) \partial_{\boldsymbol{x}}' (\boldsymbol{Y} - \boldsymbol{I}_p)^2 \partial_{\boldsymbol{x}}] \phi_p(\boldsymbol{x}),$$

$$(3) \quad \left\{ \sum_{i=1}^{p} (Y_i - 1) \frac{\partial}{\partial y_i} \right\}^3 h(\boldsymbol{x}, \boldsymbol{y}) \bigg|_{y_1 = \cdots = y_p}$$

$$= [(\delta\rho)^3 \{\partial_{\boldsymbol{x}}' (\boldsymbol{Y} - \boldsymbol{I}_p) \partial_{\boldsymbol{x}}\}^3$$

$$+ 3(\delta\rho)^2 (2\delta\rho - 1) \{\partial_{\boldsymbol{x}}' (\boldsymbol{Y} - \boldsymbol{I}_p)^2 \partial_{\boldsymbol{x}}\} \{\partial_{\boldsymbol{x}}' (\boldsymbol{Y} - \boldsymbol{I}_p) \partial_{\boldsymbol{x}}\}$$

$$+ 2\delta\rho(2\delta\rho - 1)(\delta\rho - 1) \{\partial_{\boldsymbol{x}}' (\boldsymbol{Y} - \boldsymbol{I}_p)^3 \partial_{\boldsymbol{x}}\}] \phi_p(\boldsymbol{x}).$$

Proof. The results follow by using that for $\delta = -1$ or $+1$ and for any positive ρ

$$(1) \quad \frac{\partial}{\partial y} \{ y^{-\delta\rho} \phi(x y^{-\delta\rho}) \} \bigg|_{y=1} = \delta\rho H_2(x) \phi(x),$$

$$(2) \quad \frac{\partial^2}{\partial y^2} \{ y^{-\delta\rho} \phi(x y^{-\delta\rho}) \} \bigg|_{y=1} = \{ (\delta\rho)^2 H_4(x) \phi(x) + \delta\rho(2\delta\rho - 1) H_2(x) \},$$

$$(3) \quad \frac{\partial^3}{\partial y^3} \{ y^{-\delta\rho} \phi(x y^{-\delta\rho}) \} \bigg|_{y=1} = \{ (\delta\rho)^3 H_6(x) \phi(x) + 3(\delta\rho)^2 (2\delta\rho - 1) H_4(x)$$

$$+ 2\delta\rho(2\delta\rho - 1)(\delta\rho - 1) H_2(x) \}.$$

Theorem 3.4. *Let* $k = 1, 2, 3$ *or* 4 *and* $\boldsymbol{X} = \boldsymbol{SZ}$ *be a multivariate scale mixture* (1.1) *with* $Z_1 \sim N(0,1)$ *and* $\boldsymbol{S}$ *be a symmetric positive definite matrix such that* $(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)^k$ *is symmetric positive semi-definite and* $\mathrm{E}[\mathrm{tr}(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)^k] < \infty$, *where* $\delta = 1$ *or* -1 *and* $\rho > 0$. *Then for any Borel set* $A \subset \mathbf{R}^p$ *we have*

(i) $k = 1$:

$$\left| \mathrm{P}(\boldsymbol{X} \in A) - \int_A \phi_p(\boldsymbol{x}) d\boldsymbol{x} \right| \le \frac{1}{2} \nu_{\delta,1,p} \mathrm{E}[\mathrm{tr}(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)],$$

(ii) $k = 2$:

$$\left| \mathrm{P}(\boldsymbol{X} \in A) - \int_A \mathrm{E}[\phi_p(\boldsymbol{x}) + (\delta\rho)\{\partial_{\boldsymbol{x}}'(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)\partial_{\boldsymbol{x}}\}\phi_p(\boldsymbol{x})] d\boldsymbol{x} \right|$$
$$\le \frac{1}{2} \nu_{\delta,2,p} \mathrm{E}[\mathrm{tr}(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)^2],$$

(iii) $k = 3$:

$$\left| \mathrm{P}(\boldsymbol{X} \in A) - \int_A \mathrm{E}\left[\phi_p(\boldsymbol{x}) + \sum_{j=1}^{2} \frac{(\delta\rho)^j}{j!} \{\partial_{\boldsymbol{x}}'(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)\partial_{\boldsymbol{x}}\}^j \phi_p(\boldsymbol{x}) \right.\right.$$
$$\left.\left. + \frac{1}{2}\delta\rho(2\delta\rho - 1)\{\partial_{\boldsymbol{x}}'(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)^2\partial_{\boldsymbol{x}}\}\phi_p(\boldsymbol{x}) \right] d\boldsymbol{x} \right|$$
$$\le \frac{1}{2} \min\{\eta_{\delta,3,p}, \nu_{\delta,3,p}\} \mathrm{E}[\mathrm{tr}(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)^3],$$

(iv) $k = 4$:

$$\left| \mathrm{P}(\boldsymbol{X} \in A) - \int_A \mathrm{E}\left[\phi_p(\boldsymbol{x}) + \sum_{j=1}^{3} \frac{(\delta\rho)^j}{j!} \{\partial_{\boldsymbol{x}}'(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)\partial_{\boldsymbol{x}}\}^j \phi_p(\boldsymbol{x}) \right.\right.$$
$$+ \frac{1}{2}\delta\rho(2\delta\rho - 1)\{\partial_{\boldsymbol{x}}'(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)^2\partial_{\boldsymbol{x}}\}\phi_p(\boldsymbol{x})$$
$$+ \frac{1}{2}(\delta\rho)^2(2\delta\rho - 1)\{\partial_{\boldsymbol{x}}'(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)^2\partial_{\boldsymbol{x}}\}\{\partial_{\boldsymbol{x}}'(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)\partial_{\boldsymbol{x}}\}\phi_p(\boldsymbol{x})$$
$$\left.\left. + \frac{1}{3}\delta\rho(2\delta\rho - 1)(\delta\rho - 1)\{\partial_{\boldsymbol{x}}'(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)^3\partial_{\boldsymbol{x}}\}\phi_p(\boldsymbol{x}) \right] d\boldsymbol{x} \right|$$
$$\le \frac{1}{2} \min\{\eta_{\delta,4,p}, \nu_{\delta,4,p}\} \mathrm{E}[\mathrm{tr}(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)^4].$$

Proof. Note that we show Lemma 2.1 (1)$\sim$(3) for $\delta = -1$ or $+1$ and for any positive ρ. Therefore the arguments simillar to ones in the proof of Theorem 3.3 imply the parts (i)$\sim$(iv).

The parts (ii) and (iv) in Theorem 3.4 hold without the assumption that $(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)$ is positive definite matrix. Moreover if $(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)$ is not positive definite, then in Theorem 3.4 (i) and (iii) we can replace $\mathrm{E}[\mathrm{tr}(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)]$ and $\mathrm{E}[\mathrm{tr}(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)^3]$ by $\mathrm{E}[\sum_{i=1}^p |Y_i - 1|]$ and $\mathrm{E}[\sum_{i=1}^p |Y_i - 1|^3]$. Further, we can use inequalities

$$\mathrm{E}\left[\sum_{i=1}^p |Y_i - 1|\right] \le p^{1/2}\left(\mathrm{E}\left[\sum_{i=1}^p |Y_i - 1|^2\right]\right)^{1/2} = p^{1/2}(\mathrm{E}[\mathrm{tr}(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)^2])^{1/2},$$

and

$$\mathrm{E}\left[\sum_{i=1}^p |Y_i - 1|^3\right] \le p^{1/4}\left(\mathrm{E}\left[\sum_{i=1}^p |Y_i - 1|^4\right]\right)^{3/4} = p^{1/4}(\mathrm{E}[\mathrm{tr}(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)^4])^{3/4},$$

provided that $\mathrm{E}[\mathrm{tr}(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)^2] < \infty$ and $\mathrm{E}[\mathrm{tr}(\boldsymbol{S}^{\delta/\rho} - \boldsymbol{I}_p)^4] < \infty$, respectively. The inequlities follow from Hölder's inequality.

4. Proofs of Theorems 2.1 and 2.2

Proof of Theorems 2.1. We see that the result can be proved in the same line as in Fujikoshi, Ulyanov and Shimizu (2005). Note that

$$F_p(\boldsymbol{x}) = \mathrm{E}[Q(\boldsymbol{x}, \boldsymbol{Y})],$$

where $Q(\boldsymbol{x}, \boldsymbol{Y}) = G(x_1 Y_1^{-\delta\rho}) \dots G(x_p Y_p^{-\delta\rho})$. Here $\boldsymbol{Y}$ is used for a vector notation such that $\boldsymbol{Y} = (Y_1, \dots, Y_p)'$. We use a Taylor formula for a function $Q(y) = G(xy^{-\delta\rho})$ with $k \ge 1$ continuous derivatives

$$Q(y) = Q(1) + \sum_{j=1}^{k-1} \frac{1}{j!} Q^{(j)}(1)(y-1)^j + \frac{1}{k!} Q^{(k)}(1 + \tau(y-1))(y-1)^k, \quad (4.1)$$

where τ is a number on $(0, 1)$. We construct an expansion for $Q(\boldsymbol{x}, \boldsymbol{Y})$ using (4.1) sequentially. Namely, at first we apply (4.1) to $G(x_1 y_1^{-\delta\rho})$. We get

$$Q(\boldsymbol{x}, \boldsymbol{y}) = \left[G(x_1) + \sum_{j=1}^{k-1} \frac{1}{j!} c_{\delta,j}(x_1) g(x_1)(y_1 - 1)^j + R_1(y_1 - 1)^k\right] Q_2(\boldsymbol{x}, \boldsymbol{y}), \quad (4.2)$$

where

$$R_1 = \frac{1}{k!} \frac{\partial}{\partial y^k}(G(x_1 y^{-\delta\rho}))\bigg|_{y=1+\tau(y_1-1)}, \qquad \text{and} \qquad Q_2(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=2}^p G(x_i y_i^{-\delta\rho}).$$

Now we apply (4.1) for a function $G(x_2 y_2^{-\delta\rho})$ so that for a summand

$$\frac{1}{j!} c_{\delta,j}(x_1) g(x_1)(y_1 - 1)^j Q_2(\boldsymbol{x}, \boldsymbol{y})$$

we apply (4.1) with k replaced by $k - j$. At last we obtain the following expansion

$$Q(\boldsymbol{x}, \boldsymbol{y}) = G(x_1) \dots G(x_p) + \sum_{j=1}^{k-1} \sum_{(j)} \prod_{i=1}^{p} \frac{1}{j_i!} c_{\delta,j_i}(x_i) g(x_i)(y_i - 1)^{j_i} + R_{\delta,k,p}, \quad (4.3)$$

where $R_{\delta,k,p}$ is a sum of terms each of which can be written in the form

$$(y_1 - 1)^{k_1} \dots (y_p - 1)^{k_p} M_{k_1}(y_1) \dots M_{k_p}(y_p) \quad (4.4)$$

with $k_i \geq 0$ for $i = 1, 2, \dots, p$ and $k_1 + \dots + k_p = k$. Each factor M_j in (4.4) has one of the following form:

$$M_k(y) = \frac{1}{k!} \frac{\partial^k}{\partial y_1^k} (G(x y_1^{-\delta\rho}))\bigg|_{y_1 = 1 + \tau(y-1)}, \quad (4.5)$$

$M_0(y) = G(x)$ or $M_0(y) = G(xy^{-\delta\rho})$ and when $1 \leq j \leq k - 1$, we have for $M_j(y)$ one of the two representations:

$$\frac{1}{j!} c_{\delta,j}(x) g(x) \qquad \text{or} \qquad \frac{1}{j!} \frac{\partial^j}{\partial y_1^j} (G(x y_1^{-\delta\rho}))\bigg|_{y_1 = 1 + \tau(y-1)}. \quad (4.6)$$

Put

$$\varphi_1 = (w_{\delta,k,p} / \eta_{\delta,k,p})^{1/k}. \quad (4.7)$$

At first we consider the case when $0 < \min(y_1, \dots, y_p) \leq \varphi_1$. Assume that y_1 is such that $0 < y_1 \leq \varphi_1$. We have for any j $(1 \leq j \leq k)$,

$$\begin{aligned} &|1 - y_1|^j + \dots + |1 - y_p|^j \\ &\quad \leq \frac{1}{(1 - \varphi_1)^{k-j}} (|1 - y_1|^k + |1 - y_1|^{k-j} |1 - y_2|^j + \dots + |1 - y_1|^{k-j} |1 - y_p|^j) \\ &\quad \leq \frac{p}{(1 - \varphi_1)^{k-j}} (|1 - y_1|^k + \dots + |1 - y_p|^k). \quad (4.8) \end{aligned}$$

Therefore, using Lemma 5.2 in Fujikoshi, Ulyanov and Shimizu (2005) and (4.3) we get

$$\begin{aligned}|R_{\delta,k,p}| &\le 2 + \sum_{j=1}^{k-1}(|1-y_1|^j + \cdots + |1-y_p|^j)w_{\delta,j,p} \\ &\le \frac{1}{(1-\varphi_1)^k}(|1-y_1|^k + \cdots + |1-y_p|^k)\left(2 + \sum_{j=1}^{k-1} w_{\delta,j,p}\right) \\ &= \eta_{\delta,k,p}[|1-y_1|^k + \cdots + |1-y_p|^k]. \end{aligned} \tag{4.9}$$

If $\min(y_1, \ldots, y_p) > \varphi_1$ then using Lemma 5.2 in Fujikoshi, Ulyanov and Shimizu (2005) and representations for summands contained in $R_{\delta,k,p}$ we get

$$\begin{aligned}|R_{\delta,k,p}| &\le \frac{w_{\delta,k,p}}{\varphi_1^k}[|1-y_1|^k + \cdots + |1-y_p|^k] \\ &= \eta_{\delta,k,p}[|1-y_1|^k + \cdots + |1-y_p|^k]. \end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10) we finish the proof of Theorem 2.1.

Proof of Theorems 2.2. The result can be proved by using arguments similar to the proof of Lemma 2 in Shimizu (1995). In order to prove (2.8) it is enough as usual to show that

$$\left|\prod_{i=1}^{p} G(x_i y_i^{-\delta\rho}) - G_{\delta,k,p}(x)\right| \le \gamma_{\delta,k,p} \sum_{i=1}^{p} |y_i - 1|^k, \tag{4.11}$$

where $G_{\delta,k,p}$ is defined by (2.3) but Y_i, $i = 1, \ldots, p$, are considered as positive real numbers y_i.

We prove (4.11) by mathematical induction with respect to p. In the case $p = 1$ the inequality (4.11) was proved in Theorem 2.1 of Shimizu and Fujikoshi (1997). Therefore, we can write for $p \ge 2$

$$\prod_{i=1}^{p} G(x_i y_i^{-\delta\rho}) = \left[G(x_p) + \sum_{j=1}^{k-1} \frac{1}{j!}(y_j - 1)^j c_{\delta,j}(x) g(x) + R_{\delta,p}\right] \prod_{i=1}^{p-1} G(x_i y_i^{-\delta\rho}), \tag{4.12}$$

where $|R_{\delta,p}| \le \beta_{\delta,k}|y_p - 1|^k$. Assume that (4.11) holds for $p - 1$. Then we apply (4.11) to $\prod_{i=1}^{p-1} G(x_i y_i^{-\delta\rho})$ with p replaced by $p - 1$ and k replaced by $k - j$ when $\prod_{i=1}^{p-1} G(x_i y_i^{-\delta\rho})$ is a factor by $(y_p - 1)^j$ in (4.12). Thus, we get

$$\left|\prod_{i=1}^{p} G(x_i y_i^{-\delta\rho}) - G_{\delta,k,p}(x)\right| \le \beta_{\delta,k}|y_p - 1|^k$$

$$+ \sum_{q=0}^{k-1} \alpha_{\delta,q}|y_p - 1|^q \gamma_{\delta,k-q,p-1} \sum_{i=1}^{p} |y_i - 1|^{k-q}. \tag{4.13}$$

We got (4.13) from (4.12) applying induction hypothesis to $\prod_{i=1}^{p-1} G(x_i y_i^{-\delta\rho})$. It is clear we could use the same arguments to the function $\prod_{i=1, i\neq j}^{p} G(x_i y_i^{-\delta\rho})$ with any $j = 1, \ldots, p$. Then we could get (4.13) with $|y_p - 1|$ replaced by $|y_j - 1|$. Since in all these inequalities the left-hand sides will coincide, summing up the inequalities for $j = 1, \ldots, p$ and using

$$\sum_{i\neq j}^{p} |y_i - 1|^{k-q}|y_j - 1|^q \le (p-1)\sum_{i=1}^{p} |y_i - 1|^k,$$

(cf. the proof of Lemma 2 in Shimizu (1995)) we come to (4.11) and recurrence formula for $\gamma_{\delta,k,p}$ stated in Theorem 2.2.

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