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Extendibility and stable extendibility of vector bundles over lens spaces mod 3

Dedicated to the Memory of Professor Masahiro Sugawara

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Abstract. In this paper, we prove that the tangent bundle $\tau(L^n(3))$ of the $(2n+1)$-dimensional mod 3 standard lens space $L^n(3)$ is stably extendible to $L^m(3)$ for every $m \geq n$ if and only if $0 \leq n \leq 3$. Combining this fact with the results obtained in [6], we see that $\tau(L^2(3))$ is stably extendible to $L^3(3)$, but is not extendible to $L^3(3)$. Furthermore, we prove that the t-fold power of $\tau(L^n(3))$ and its complexification are extendible to $L^m(3)$ for every $m \geq n$ if $t \geq 2$, and have a necessary and sufficient condition that the square ν^2 of the normal bundle ν associated to an immersion of $L^n(3)$ in the Euclidean $(4n+3)$-space is extendible to $L^m(3)$ for every $m \geq n$.

1. Definitions and results

The extension problem is one of the fundamental problems in topology. We study the problem for F-vector bundles over standard lens spaces mod 3, where F is either the real number field R or the complex number field C.

First, we recall the definitions of extendibility and stable extendibility according to [12] and [2]. Let X be a space and A be its subspace. A k-dimensional F-vector bundle ζ over A is said to be extendible (respectively stably extendible) to X, if there is a k-dimensional F-vector bundle over X whose restriction to A is equivalent (respectively stably equivalent) to ζ as F-vector bundles, that is, if ζ is equivalent (respectively stably equivalent) to the induced bundle $i^*\alpha$ of a k-dimensional F-vector bundle α over X under the inclusion map $i : A \to X$. For simplicity, we use the same letter for an F-vector bundle and its equivalence class, and use a non-negative integer k for the k-dimensional trivial F-vector bundle.

For a non-negative integer n and an integer $q > 1$, let $L^n(q)$ denote the $(2n+1)$-dimensional standard lens space mod q and $L_0^n(q)$ its $2n$-skeleton (cf. [3], [4] and [11]). For a positive integer n, let η_n stand for the canonical C-line

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bundle over $L^n(q)$. For simplicity, we use the same symbol η_n for the restriction of η_n to $L_0^n(q)$. For a differentiable manifold M, let $\tau(M)$ denote the tangent bundle of M.

On extendibility and stable extendibility of tangent bundles over standard lens spaces mod 3, we have

Theorem 1. *As for the tangent bundle $\tau(L^n(3))$ of $L^n(3)$. The following three conditions are equivalent:*

(1) *$\tau(L^n(3))$ is stably extendible to $L^m(3)$ for every $m \geq n$.*
(2) *$\tau(L^n(3))$ is stably extendible to $L^{2n+2}(3)$.*
(3) $0 \leq n \leq 3$.

Combining Theorem 1 with Theorem 5.1 of [6], we obtain

Corollary 2. *$\tau(L^2(3))$ is stably extendible to $L^3(3)$, but is not extendible to $L^3(3)$.*

Another example of an R-vector bundle that is stably extendible but is not extendible is given by the tangent bundle $\tau(S^n)$ of the n-sphere S^n in the $(n+1)$-sphere S^{n+1} for $n \neq 1, 3, 7$ (cf. [10, Proof of Theorem 2.2]).

Let $c : K_R(X) \to K_C(X)$ be the complexification. Then we have

Theorem 3. *The complexification $c\tau(L^n(3))$ of $\tau(L^n(3))$ is stably extendible to $L^m(3)$ for every m with $n \leq m \leq 2n+1$, but $c\tau(L^n(3))$ is not stably extendible to $L^{2n+2}(3)$ if $n \geq 6$.*

For a positive integer t, let $\tau(L^n(q))^t = \tau(L^n(q)) \otimes \cdots \otimes \tau(L^n(q))$ (t-fold) be the t-fold power of $\tau(L^n(q))$, where $\otimes$ denotes the tensor product. Then we prove

Theorem 4. *$\tau(L^n(3))^t$ is extendible to $L^m(3)$ for every $m \geq n$ if $t \geq 2$.*

Theorem 5. *The complexification $c\tau(L^n(3))^t$ of $\tau(L^n(3))^t$ is extendible to $L^m(3)$ for every $m \geq n$ if $t \geq 2$.*

As for the normal bundle ν associated to an immersion of $L^n(3)$ in the Euclidean $(4n+3)$-space R^{4n+3}, it was proved in [9, Theorem B] that the following three conditions are equivalent:

(1) ν is extendible to $L^m(3)$ for every $m \geq n$.
(2) ν is stably extendible to $L^m(3)$ for every $m \geq n$.
(3) $0 \leq n \leq 5$.

For the square $\nu^2 = \nu \otimes \nu$ of ν, we have

Theorem 6. *Let ν be the normal bundle associated to an immersion of $L^n(3)$ in R^{4n+3} and ν^2 its square. Then the following three conditions are equivalent:*

(1) ν^2 *is extendible to* $L^m(3)$ *for every* $m \geq n$.
(2) ν^2 *is stably extendible to* $L^m(3)$ *for every* $m \geq n$.
(3) $0 \leq n \leq 13$ *or* $n = 15$.

Theorem 2 of [5] is the result which corresponds to Theorem 6 for extendibility of the square of the normal bundle associated to an immersion of the real projective n-space RP^n in R^{2n+1}.

This paper is arranged as follows. In Section 2 we recall results that are necessary for our proofs. In Section 3 we prove Theorem 1, Corollary 2 and Theorem 3. In Section 4 we prepare lemmas and prove Theorems 4 and 5. In Section 5 we give Whitney sum decompositions of the squares ν^2 of the normal bundles ν associated to immersions of $L^n(3)$ in R^{4n+3} for $0 \leq n \leq 13$ and $n = 15$ and prove Theorem 6.

2. Preliminaries

For a positive integer n, let η_n denote the canonical C-line bundle over $L^n(3)$ and $\sigma_n = \eta_n - 1$ its stable class. Let $r : K_C(X) \to K_R(X)$ be the forgetful map and Z/q denote the cyclic group of order q, where q is an integer > 1. For a real number x, let $\lfloor x \rfloor$ denote the largest integer s with $s \leq x$.

The ring structure of the reduced Grothendieck ring $\tilde{K}_R(L^n(3))$ is determined in [3] as follows (cf. [4] and [11]).

Theorem 2.1 (cf. [3, Theorem 2] and [4, Proposition 2.11]).

$$\tilde{K}_R(L^n(3)) \cong \begin{cases} \tilde{K}_R(L_0^n(3)) + Z/2 & \text{for } n \equiv 0 \bmod 4, \\ \tilde{K}_R(L_0^n(3)) & \text{otherwise,} \end{cases}$$

where $+$ *denotes the direct sum. The group* $\tilde{K}_R(L_0^n(3))$ *is isomorphic to the cyclic group* $Z/3^{\lfloor n/2 \rfloor}$ *of order* $3^{\lfloor n/2 \rfloor}$ *and is generated by* $r\sigma_n$. *Moreover, the ring structure is given by*

$$(r\sigma_n)^2 = -3r\sigma_n, \qquad \textit{namely} \quad (r\eta_n)^2 = r\eta_n + 2, \quad \textit{and} \quad (r\sigma_n)^{\lfloor n/2 \rfloor + 1} = 0.$$

The ring structure of the reduced Grothendieck ring $\tilde{K}_C(L^n(3))$ is determined in [3] as follows (cf. [4] and [11]).

Theorem 2.2 (cf. [3, Theorem 1] and [4, Lemma 2.4]).

$$\tilde{K}_C(L^n(3)) \cong \tilde{K}_C(L_0^n(3)) \cong \begin{cases} Z/3^{\lfloor n/2 \rfloor} + Z/3^{\lfloor n/2 \rfloor} & \text{for even } n, \\ Z/3^{\lfloor n/2 \rfloor + 1} + Z/3^{\lfloor n/2 \rfloor} & \text{for odd } n. \end{cases}$$

The first summand is generated by σ_n *and the second summand is generated by* σ_n^2. *Moreover, the ring structure is given by*

$$\sigma_n^3 = -3\sigma_n^2 - 3\sigma_n, \qquad \textit{namely} \quad \eta_n^3 = 1, \quad \textit{and} \quad \sigma_n^{n+1} = 0.$$

We recall two theorems on F-vector bundles over $L^n(p)$ which are useful for our proofs.

Theorem 2.3 (cf. [6, Theorem 1.1] and [8, Theorem 3.1]). *Let p be an odd prime and ζ be a k-dimensional R-vector bundle over $L^n(p)$. Assume that there is a positive integer ℓ such that ζ is stably equivalent to a sum of $\lfloor k/2 \rfloor + \ell$ non-trivial 2-dimensional R-vector bundles and $\lfloor k/2 \rfloor + \ell < p^{\lfloor n/(p-1) \rfloor}$. Then $n < 2\lfloor k/2 \rfloor + 2\ell$ and ζ is not stably extendible to $L^m(p)$ for every m with $m \geq 2\lfloor k/2 \rfloor + 2\ell$.*

Theorem 2.4 (cf. [7, Theorem 1.1] and [9, Theorem 4.5]). *Let p be a prime and ζ be a k-dimensional C-vector bundle over $L^n(p)$. Assume that there is a positive integer ℓ such that ζ is stably equivalent to a sum of $k + \ell$ non-trivial C-line bundles and $k + \ell < p^{\lfloor n/(p-1) \rfloor}$. Then $n < k + \ell$ and ζ is not stably extendible to $L^m(p)$ for every m with $m \geq k + \ell$.*

Let $d = 1$ or 2 according as $F = R$ or C. For a real number x, let $\lceil x \rceil$ denote the smallest integer s with $x \leq s$. The following results are known.

Theorem 2.5 (cf. [1, Theorem 1.2, p. 99]). *Let $m = \lceil (n+1)/d - 1 \rceil$. Then each k-dimensional F-vector bundle over an n-dimensional CW-complex X is equivalent to $\alpha \oplus (k - m)$ for some m dimensional F-vector bundle α over X if $m \leq k$.*

Theorem 2.6 (cf. [1, Theorem 1.5, p. 100]). *Let $m = \lceil (n+2)/d - 1 \rceil$. Then two k-dimensional F-vector bundles over an n-dimensional CW-complex which are stably equivalent are equivalent if $m \leq k$.*

3. Proofs of Theorem 1, Corollary 2 and Theorem 3

Let q be any integer > 1. As for extendibility of $\tau(L^n(q))$, the following result is obtained.

Theorem 3.1 ([6, Theorems 5.1 and 5.3]). *For any integer $q > 1$, the following three conditions are equivalent:*

(1) *$\tau(L^n(q))$ is extendible to $L^m(q)$ for every $m \geq n$.*
(2) *$\tau(L^n(q))$ is extendible to $L_0^{n+1}(q)$.*
(3) *$n = 0, 1$ or 3.*

As for stable extendibility of $\tau(L^n(q))$, the following result is obtained.

Theorem 3.2 ([8, Theorem 4.3]). *Let p be an odd prime. Then $\tau(L^n(p))$ is not stably extendible to $L^{2n+2}(p)$, if $n \geq 2p - 2$.*

Proof of Theorem 1. Obviously, (1) implies (2). It follows from Theorem 3.2 that (2) implies (3), since $n < 2p - 2$ for $p = 3$ if and only if

$0 \le n \le 3$. Hence it remains to prove that (3) implies (1). If $n = 0, 1$ or 3, (1) holds by Theorem 3.1. Let $n = 2$. Then $r\eta_2 - 2$ is of order 3 by Theorem 2.1, and so $3r\eta_2 = 6$ in $K_R(L^2(3))$. As is well-known,

$$\tau(L^2(3)) \oplus 1 = 3r\eta_2$$

So we have $\tau(L^2(3)) = 3r\eta_2 - 1 = 5$ in $K_R(L^2(3))$. Hence $\tau(L^2(3))$ is stably trivial, and so $\tau(L^2(3))$ is stably extendible to $L^m(3)$ for every $m \ge 2$, as desired. □

Proof of Corollary 2. The former part follows from Theorem 1. By Theorem 3.1, $\tau(L^2(3))$ is not extendible to $L_0^3(3)$, and hence is not extendible to $L^3(3)$. So we obtain the latter part. □

Proof of Theorem 3. As is well-known,

$$\tau(L^n(3)) \oplus 1 = (n+1)r\eta_n.$$

Applying the complexification $c : K_R(L^n(3)) \to K_C(L^n(3))$ to the both sides of the equality, we have

$$c\tau(L^n(3)) \oplus 1 = (n+1)cr\eta_n = (n+1)(\eta_n + \eta_n^2),$$

since $cr\eta_n = \eta_n + \eta_n^{-1}$ (cf. [1, Proposition 11.3, p. 191]) and $\eta_n^3 = 1$ (cf. Theorem 2.2).

Suppose $m \le 2n+1$. Then $\dim\{(n+1)(\eta_m + \eta_m^2)\} - \lceil (2m+1+1)/2 - 1 \rceil = 2n+2-m \ge 1$. Hence, by Theorem 2.5, there is a $(2n+1)$-dimensional C-vector bundle α over $L^m(3)$ such that

$$(n+1)(\eta_m + \eta_m^2) = \alpha \oplus 1.$$

Let $n \le m$ and $i : L^n(3) \to L^m(3)$ be the standard inclusion. Then, applying i^* to the both sides of the equality above, we have

$$(n+1)(\eta_n + \eta_n^2) = i^*\alpha \oplus 1,$$

since $i^*\eta_m = \eta_n$. Hence $c\tau(L^n(3))$ is stably equivalent to $i^*\alpha$. Now, both $c\tau(L^n(3))$ and $i^*\alpha$ are $(2n+1)$-dimensional. So $c\tau(L^n(3))$ is stably extendible to $L^m(3)$. Thus the former part of the theorem is proved.

Put $p = 3$, $\zeta = c\tau(L^n(3))$, $k = 2n+1$ and $\ell = 1$ in Theorem 2.4. Then the latter part of the theorem follows from Theorem 2.4, since $2n+2 < 3^{\lfloor n/2 \rfloor}$ if and only if $n \ge 6$. □

4. Proofs of Theorems 4 and 5

In Sections 4 and 5, η denotes the canonical C-line bundle η_n over $L^n(3)$ and N the set of all positive integers. We prepare some lemmas for our proofs.

LEMMA 4.1. *Let t be any positive integer. Then there is a function $g : N \to N$ such that*

$$\tau(L^n(3))^t = g(t)r\eta + (2n+1)^t - 2g(t) \qquad \text{in } K_R(L^n(3)),$$

namely $\tau(L^n(3))^t - (2n+1)^t = g(t)(r\eta - 2)$ *in* $\tilde{K}_R(L^n(3))$. *Furthermore, the function $g(t)$ is uniquely determined modulo* $3^{\lfloor n/2 \rfloor}$.

PROOF. We prove the first part of the lemma by induction on t. Since $\tau(L^n(3)) = (n+1)r\eta - 1$ in $K_R(L^n(3))$, we may define $g(1) = n+1$. Assume that there exists $g(t)$ for every $t \geq 1$ such that $\tau(L^n(3))^t = g(t)r\eta + (2n+1)^t - 2g(t)$ and $g(1) = n+1$. Then, by Theorem 2.1,

$$\begin{aligned}
\tau(L^n(3))^{t+1} &= \{g(t)r\eta + (2n+1)^t - 2g(t)\}\{(n+1)r\eta - 1\} \\
&= g(t)(n+1)(r\eta)^2 + \{(2n+1)^t(n+1) - 2g(t)(n+1) - g(t)\}r\eta \\
&\quad - (2n+1)^t + 2g(t) \\
&= \{(2n+1)^t(n+1) - g(t)(n+2)\}r\eta - (2n+1)^t + 2g(t)(n+2).
\end{aligned}$$

Now set

$$g(t+1) = (2n+1)^t(n+1) - g(t)(n+2).$$

Then we have $-(2n+1)^t + 2g(t)(n+2) = (2n+1)^{t+1} - 2g(t+1)$, as desired.

Suppose there are two functions $f, g : N \to N$ such that

$$f(t)r\eta + (2n+1)^t - 2f(t) = g(t)r\eta + (2n+1)^t - 2g(t).$$

Then $(f(t) - g(t))(r\eta - 2) = 0$, and so $f(t) - g(t) \equiv 0 \mod 3^{\lfloor n/2 \rfloor}$ by Theorem 2.1. So we have the latter part. □

LEMMA 4.2. *There is a function $g : N \to N$ defined in Lemma* 4.1 *which satisfies the inequalities:*

$$(2n+1)^{t-1} < g(t) < 2^{-1}(2n+1)^t \qquad \text{for } n \geq 3 \text{ and } t \geq 2.$$

PROOF. We prove the lemma by induction on t. Define $g(1) = n+1$. Next, by Theorem 2.1,

$$\tau(L^n(3))^2 = \{(n+1)r\eta - 1\}^2 = (n^2-1)r\eta + 2n^2 + 4n + 3.$$

Define $g(2) = n^2 - 1$. Then clearly $2n+1 < g(2) < 2^{-1}(2n+1)^2$ for $n \geq 3$. Assume that there exists $g(t)$ for $t \geq 2$ which satisfies the inequalities: $(2n+1)^{t-1} < g(t) < 2^{-1}(2n+1)^t$ for $n \geq 3$. As in the proof of Lemma 4.1, set $g(t+1) = (2n+1)^t(n+1) - g(t)(n+2)$. Then, by the inductive assumption,

$$\begin{aligned}
g(t+1) &> (2n+1)^t(n+1) - 2^{-1}(2n+1)^t(n+2) \\
&= 2^{-1}(2n+1)^t n > (2n+1)^t
\end{aligned}$$

and

$$\begin{aligned} g(t+1) &< (2n+1)^t(n+1) - (2n+1)^{t-1}(n+2) \\ &= (2n+1)^{t-1}(2n^2+2n-1) \\ &< (2n+1)^{t-1}(2n^2+2n+1/2) = 2^{-1}(2n+1)^{t+1}. \end{aligned}$$

Thus the inequalities: $(2n+1)^t < g(t+1) < 2^{-1}(2n+1)^{t+1}$ hold. □

Proof of Theorem 4. If $n = 0, 1$ or 3, $\tau(L^n(3))$ is extendible to $L^m(3)$ for every $m \geq n$ by Theorem 3.1. Hence $\tau(L^n(3))^t$ is extendible to $L^m(3)$ for every $m \geq n$, where $t \geq 1$. If $n = 2$, we see in the proof of Theorem 1 that $\tau(L^2(3))$ is stably trivial. So $\tau(L^2(3))^t$ is stably trivial, where $t \geq 1$. If, in addition, $t \geq 2$, $\tau(L^2(3))^t$ is trivial by Theorem 2.6, since $\lceil(\dim L^2(3)+2) - 1\rceil = 6 \leq 5^t = \dim \tau(L^2(3))^t$. Hence $\tau(L^2(3))^t$ is extendible to $L^m(3)$ for every $m \geq n$, if $t \geq 2$. We may therefore devote our attention to the case where $n \geq 4$.

According to Lemmas 4.1 and 4.2, there is a positive integer $g(t)$ such that $\tau(L^n(3))^t = g(t)r\eta + (2n+1)^t - 2g(t)$ in $K_R(L^n(3))$ and $(2n+1)^t - 2g(t) > 0$ for $n \geq 3$ and $t \geq 2$. Since $\lceil(\dim L^n(3)+2) - 1\rceil = 2n+2 \leq (2n+1)^t = \dim \tau(L^n(3))^t$ for $n \geq 1$ and $t \geq 2$, we have the equality

$$\tau(L^n(3))^t = g(t)r\eta \oplus \{(2n+1)^t - 2g(t)\}$$

of R-vector bundles by Theorem 2.6. Since $r\eta$ and the trivial R-vector bundle over $L^n(3)$ are extendible to $L^m(3)$ for every $m \geq n$, $\tau(L^n(3))^t$ is extendible to $L^m(3)$ for every $m \geq n$, as desired. □

Complexifying the equality in Lemma 4.1, we have

Lemma 4.3. *For the function $g: N \to N$ in Lemmas 4.1 and 4.2,*

$$c\tau(L^n(3))^t = g(t)(\eta+\eta^2) + (2n+1)^t - 2g(t) \qquad \text{in } K_C(L^n(3)).$$

Proof. Since $cr\eta = \eta + \eta^2$, the result follows from the equality in Lemma 4.1. □

Proof of Theorem 5. As is well-known, $\tau(L^1(3))$ is trivial. In the proof of Theorem 1, we see that $\tau(L^2(3))$ is stably trivial. So $c\tau(L^1(3))^t$ and $c\tau(L^2(3))^t$ are stably trivial for any $t \geq 1$. Furthermore, $c\tau(L^1(3))^t$ and $c\tau(L^2(3))^t$ are trivial for any $t \geq 1$ by Theorem 2.6, since $\lceil(\dim L^1(3)+2)/2 - 1\rceil = 2 \leq 3^t = \dim c\tau(L^1(3))^t$ and $\lceil(\dim L^2(3)+2)/2-1\rceil = 3 \leq 5^t = \dim c\tau(L^2(3))^t$ hold for any $t \geq 1$. Hence we have the results for $n = 1$ and $n = 2$, since the trivial C-bundle over $L^n(3)$ is extendible to $L^m(3)$ for every $m \geq n$.

Suppose $n \geq 3$. Then, by Lemma 4.2, $(2n+1)^t - 2g(t) > 0$ for $t \geq 2$. Since $\lceil(\dim L^n(3) + 2)/2 - 1\rceil = n + 1 \leq (2n+1)^t = \dim c\tau(L^n(3))^t$ holds for any $t \geq 1$, it follows from Lemma 4.3 that the equality

$$c\tau(L^n(3))^t = g(t)(\eta \oplus \eta^2) \oplus \{(2n+1)^t - 2g(t)\}$$

of C-vector bundles holds by Theorem 2.6. Since η, η^2 and the trivial C-vector bundle over $L^n(3)$ are extendible to $L^m(3)$ for every $m \geq n$, $c\tau(L^n(3))^t$ is extendible to $L^m(3)$ for every $m \geq n$, as desired. □

5. Proof of Theorem 6

First, we study the square of the normal bundle associated to an immersion of $L^n(3)$ in R^{4n+3}.

THEOREM 5.1. *Let $\nu = \nu(f_n)$ be the normal bundle associated to an immersion $f_n : L^n(3) \to R^{4n+3}$ and $\nu^2 = \nu(f_n)^2$ its square. Then we have the Whitney sum decompositions:*

$$\nu(f_0)^2 = 4, \qquad \nu(f_1)^2 = 16, \qquad \nu(f_2)^2 = 36,$$

$$\nu(f_3)^2 = 2r\eta_3 \oplus 60, \qquad \nu(f_4)^2 = 5r\eta_4 \oplus 90, \qquad \nu(f_5)^2 = 144,$$

$$\nu(f_6)^2 = 8r\eta_6 \oplus 180, \qquad \nu(f_7)^2 = 11r\eta_7 \oplus 234, \qquad \nu(f_8)^2 = 324,$$

$$\nu(f_9)^2 = 29r\eta_9 \oplus 342, \qquad \nu(f_{10})^2 = 125r\eta_{10} \oplus 234,$$

$$\nu(f_{11})^2 = 207r\eta_{11} \oplus 162, \qquad \nu(f_{12})^2 = 275r\eta_{12} \oplus 126,$$

$$\nu(f_{13})^2 = 86r\eta_{13} \oplus 612, \qquad \nu(f_{15})^2 = 395r\eta_{15} \oplus 234.$$

PROOF. If $n = 0$, the result is clear. Hence we assume that $n > 0$. Let $\tau = \tau(L^n(3))$ denote the tangent bundle of $L^n(3)$. Then $\tau \oplus 1 = (n+1)r\eta$ and $\tau \oplus \nu = 4n + 3$. Hence $\nu = -(n+1)r\eta + 4n + 4$. By Theorem 2.1, we have

$$\begin{aligned} \nu^2 &= (n+1)^2(r\eta)^2 - 2(n+1)(4n+4)r\eta + (4n+4)^2 \\ &= (n+1)^2(r\eta + 2) - 8(n+1)^2 r\eta + 16(n+1)^2 \\ &= \{a3^{\lfloor n/2 \rfloor} - 7(n+1)^2\}r\eta + 18(n+1)^2 - 2a3^{\lfloor n/2 \rfloor} \end{aligned}$$

in $K_R(L^n(3))$, where a is any integer. If $a3^{\lfloor n/2 \rfloor} - 7(n+1)^2 \geq 0$ and $18(n+1)^2 - 2a3^{\lfloor n/2 \rfloor} \geq 0$, then we have the equality

$$\nu^2 = \{a3^{\lfloor n/2 \rfloor} - 7(n+1)^2\}r\eta \oplus \{18(n+1)^2 - 2a3^{\lfloor n/2 \rfloor}\}$$

of R-vector bundles, since $\lceil(\dim L^n(3) + 2) - 1\rceil = 2n + 2 \leq (2n+2)^2 = \dim \nu^2$ by Theorem 2.6.

Put $a=28$ for $n=1$, $a=21$ for $n=2$, $a=38$ for $n=3$, $a=20$ for $n=4$, $a=28$ for $n=5$, $a=13$ for $n=6$, $a=17$ for $n=7$, $a=7$ for $n=8$, $a=9$ for $n=9$, $a=4$ for $n=10$, $a=5$ for $n=11$, $a=2$ for $n=12$, $a=2$ for $n=13$ and $a=1$ for $n=15$. Then we can check easily that the two inequalities $a3^{\lfloor n/2\rfloor}-7(n+1)^2\geq 0$ and $18(n+1)^2-2a3^{\lfloor n/2\rfloor}\geq 0$ hold, if $1\leq n\leq 13$ or $n=15$. □

Theorem 3.1 of [5] is the result corresponding to Theorem 5.1 for the square of the normal bundle associated to an immersion of RP^n in R^{2n+1}.

Theorem 5.2. *Under the assumption of Theorem* 5.1, *the following two equalities hold in* $K_R(L^{14}(3))$ *and* $K_R(L^{16}(3))$, *respectively.*

$$\nu(f_{14})^2=612r\eta_{14}-324, \qquad \nu(f_{16})^2=4538r\eta_{16}-7920.$$

Proof. Putting $a=1$ for $n=14$ and 16 in the proof of Theorem 5.1, we have the desired equalities. □

Using Theorem 2.3, we prove

Theorem 5.3. *Let* ν *be the normal bundle associated to an immersion of* $L^n(3)$ *in* R^{4n+3}. *Then the square* ν^2 *of* ν *is not stably extendible to* $L^m(3)$ *for* $m=2\{3^{\lfloor n/2\rfloor}-7(n+1)^2\}$, *if* $n=14$ *or* $n\geq 16$.

Proof. We see in the proof of Theorem 5.1 that ν^2 is stably equivalent to $\{3^{\lfloor n/2\rfloor}-7(n+1)^2\}r\eta$. Note that $3^{\lfloor n/2\rfloor}-9(n+1)^2>0$ if $n=14$ or $n\geq 16$. Then, putting $p=3$, $\zeta=\nu^2$, $k=4(n+1)^2$ and $\ell=3^{\lfloor n/2\rfloor}-9(n+1)^2$ in Theorem 2.3, we see that ν^2 is not stably extendible to $L^m(3)$ for $m=2\{3^{\lfloor n/2\rfloor}-7(n+1)^2\}$, if $n=14$ or $n\geq 16$, by Theorem 2.3. □

Corollary 5.4. *Under the assumption of Theorem* 5.1, $\nu(f_{14})^2$ *and* $\nu(f_{16})^2$ *are not stably extendible to* $L^{1224}(3)$ *and* $L^{9076}(3)$, *respectively.*

Proof. The results follow from Theorems 5.2 and 5.3. □

Proof of Theorem 6. Clearly (1) implies (2). It follows from Theorem 5.3 that (2) implies (3). It follows from Theorem 5.1 that (3) implies (1), since $r\eta$ and trivial R-vector bundles over $L^n(3)$ are extendible to $L^m(3)$ for every $m\geq n$. □

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