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Intermediate dynamics of internal layers for a nonlocal reaction-diffusion equation

Koji Okada

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Abstract. A singular perturbation problem for a reaction-diffusion equation with a nonlocal term is treated. We derive an interface equation which describes the dynamics of internal layers in the intermediate time scale, i.e., in the time scale after the layers are generated and before the interfaces are governed by the volume-preserving mean curvature flow. The unique existence of solutions for the interface equation is demonstrated. A continuum of equilibria for the interface equation are identified and the stability of the equilibria is established. We rigorously prove that layer solutions of the nonlocal reaction-diffusion equation converge to solutions of the interface equation on a finite time interval as the singular perturbation parameter tends to zero.

1. Introduction

1.1. Nonlocal reaction-diffusion equation. As a model describing phase separation in binary mixtures, Novick-Cohen proposed the following equation, called the viscous Cahn-Hilliard equation (cf. [16, 17]):

$$\text{(VCH)} \qquad \begin{cases} \alpha u_t^\varepsilon = -\Delta(\varepsilon^2 \Delta u^\varepsilon + f(u^\varepsilon) - u_t^\varepsilon), & t > 0,\ x \in \Omega, \\ \partial u^\varepsilon / \partial \mathbf{n} = \partial \Delta u^\varepsilon / \partial \mathbf{n} = 0, & t > 0,\ x \in \partial\Omega. \end{cases}$$

In (VCH), Ω is a smooth bounded domain in $\mathbf{R}^N$ $(N \geq 2)$ and $\mathbf{n}$ stands for the outward unit normal vector on the boundary $\partial\Omega$. The function f is derived from a smooth double-well potential W; $f(u) = -W'(u)$, a typical example being $f(u) = u - u^3$. The constants α and ε are some small positive parameters. In particular, $u^\varepsilon = u^\varepsilon(t, x)$ represents, for instance, an order parameter (or the concentration of one of the components) in the mixture, and the term Δu_t^ε is regarded as a viscous effect. If the viscous effect is negligible, then (VCH) is reduced to the well-known Cahn-Hilliard equation.

For (VCH), Rubinstein and Sternberg treated in [18] the case where $\alpha \to 0$, and they derived, by formally setting $\alpha = 0$, the following nonlocal reaction-diffusion equation

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$$
(1.1) \qquad \begin{cases} u_t^\varepsilon = \varepsilon^2 \Delta u^\varepsilon + f(u^\varepsilon) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f(u^\varepsilon)dx, & t > 0,\ x \in \Omega, \\ \partial u^\varepsilon/\partial \mathbf{n} = 0, & t > 0,\ x \in \partial\Omega, \end{cases}
$$

where $|\Omega|$ stands for the volume of Ω. Because of the presence of the nonlocal term and no-flux boundary conditions, the spatial average of the solution u^ε is preserved:

$$
\frac{1}{|\Omega|}\int_\Omega u^\varepsilon(t,x)dx \equiv \frac{1}{|\Omega|}\int_\Omega u^\varepsilon(0,x)dx, \qquad t > 0.
$$

Rubinstein and Sternberg discussed in [18] the dynamics of the solution u^ε for (1.1) by employing the method of matched asymptotic expansions and the method of multiple time scales. According to their results, the dynamics of u^ε consists of three stages and is roughly summarized as follows.

1: The solution for an appropriate initial condition generates sharp internal transition layer in a narrow region of $O(\varepsilon)$ near an interface.

2: The interface begins to evolve according to a certain motion law, called an interface equation. The interface equation is given by (2.15) in [18].

3: Further evolution of the interface is governed by the so-called volume-preserving mean curvature flow (cf. (3.2) in [18]). The interface is driven in such a way that the volume enclosed by the interface is preserved and the area of the interface decreases. Eventually, the interface tends to a single sphere.

Let us refer to the dynamics in the stage 2 above as *intermediate* in the sense that it occurs *after* the formation of layers and *before* the volume-preserving mean curvature flow is effective. Our results in this paper are concerned with this intermediate dynamics.

Remark. As the interface in the stage 3 eventually approaches a sphere, it is known that the corresponding layer solution with spherical shape (called the bubble solution) drifts toward the boundary of domain $\partial\Omega$ with exponentially slow speed without changing its shape. Such a motion is called a bubble motion. For more detail of the motion, we refer to [23, 24] by Ward and the references therein.

1.2. Interface equation. In order to capture the intermediate dynamics for (1.1), it is adequate to rescale t in (1.1) by $t \to \varepsilon^{-1}t$ and consider the following problem:

$$
(\mathrm{RD}) \qquad \begin{cases} \varepsilon u_t^\varepsilon = \varepsilon^2 \Delta u^\varepsilon + f(u^\varepsilon) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f(u^\varepsilon)dx, & t > 0,\ x \in \Omega, \\ \partial u^\varepsilon/\partial \mathbf{n} = 0, & t > 0,\ x \in \partial\Omega. \end{cases}
$$

We present in this subsection a formulation of the interface equation which corresponds to (2.15) in [18]. Throughout the remaining part of this paper, an "interface" means a smooth, closed, $(N-1)$-dimensional hypersurface embedded in $\Omega \subset \mathbf{R}^N$, staying uniformly away from $\partial\Omega$.

To give a precise expression of the interface equation for (RD), we recast the equation in (RD), by introducing an auxiliary variable $v \in \mathbf{R}$, as

$$\varepsilon u_t^\varepsilon(t,x) = \varepsilon^2 \Delta u^\varepsilon(t,x) + f(u^\varepsilon(t,x)) - v^\varepsilon(t) \tag{1.2}$$

with

$$v^\varepsilon(t) := \frac{1}{|\Omega|}\int_\Omega f(u^\varepsilon(t,x))dx. \tag{1.3}$$

In this paper, we will work under the following conditions for the nonlinear term $f(u) - v$ as a function of $(u,v) \in \mathbf{R}^2$.

(A1) The function f is C^∞ on $\mathbf{R}$ and the nullcline $\{(u,v) \mid f(u) - v = 0\}$ has three branches of solutions

$$\mathscr{C}^- = \{(u,v) \mid u = h^-(v),\ v \in I^- := (\underline{v}, \infty)\},$$

$$\mathscr{C}^+ = \{(u,v) \mid u = h^+(v),\ v \in I^+ := (-\infty, \bar{v})\},$$

$$\mathscr{C}^0 = \{(u,v) \mid u = h^0(v),\ v \in I^v := I^- \cap I^+ = (\underline{v}, \bar{v})\},$$

with $h^-(v) < h^0(v) < h^+(v)$ for $v \in I^v$.

(A2) The following inequalities hold:

$$f'(h^\pm(v)) < 0 \text{ on } I^\pm, \text{ or equivalently } h_v^\pm(v) < 0 \text{ on } I^\pm.$$

(A3) Define $\mathscr{J}(v)$ by

$$\mathscr{J}(v) := \int_{h^-(v)}^{h^+(v)} f(u) - v\,du, \qquad v \in I^v.$$

Then there exists a unique point $v^* \in I^v$ such that $\mathscr{J}(v^*) = 0$ and $\mathscr{J}'(v^*) < 0$.

Under the assumptions **(A1)** and **(A2)**, it is known [9] that the following problem

$$\begin{cases} Q_{zz} + cQ_z + f(Q) - v = 0, & z \in (-\infty, \infty), \\ Q(\pm\infty) = h^\pm(v), & Q(0) = 0 \end{cases} \tag{1.4}$$

has a unique smooth solution $(Q(z;v), c(v))$, where $v \in I^v$ is regarded as a parameter. Along the line of arguments employed in Fife [8], the interface equation turns out to be

$$\mathbf{v}(x;\Gamma(t)) = c(v(t)), \qquad t > 0,\ x \in \Gamma(t). \tag{1.5}$$

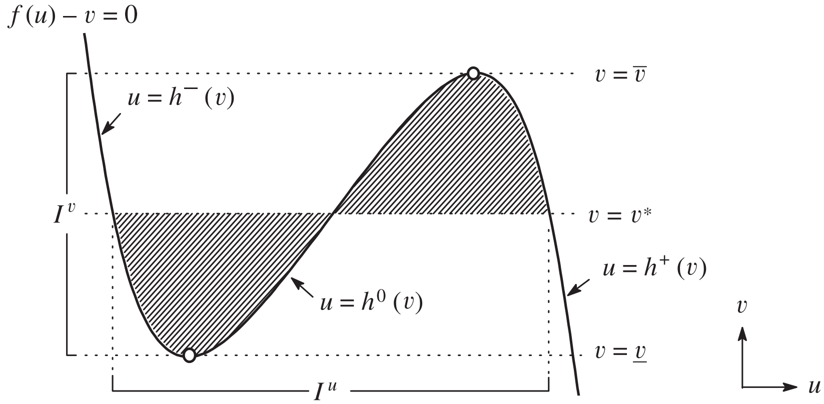

Fig. 1. Profiles of nullcline $\{(u, v) \mid f(u) - v = 0\}$.

In (1.5), Γ is the interface deviding Ω into two subregions $\Omega^{\pm}$ such as $\Omega = \Omega^- \cup \Gamma \cup \Omega^+$, and $\mathbf{v}(x; \Gamma)$ stands for the normal velocity of Γ at $x \in \Gamma$ in ν-direction with the unit normal vector $\nu(x; \Gamma)$ on Γ at $x \in \Gamma$ pointing into the interior of Ω^+. The function $v(t) \in I^v$ is interpreted as the limit of the nonlocal term $v^\varepsilon(t)$ as $\varepsilon \to 0$.

Since the interface $\Gamma(t)$ driven by (1.5) is regulated by the unknown function $v(t)$, we need to derive another equation for $\Gamma(t)$ and $v(t)$. To this end, we employ the following conservation property of the solution u^ε to (RD):

$$\frac{d}{dt}\left(\frac{1}{|\Omega|}\int_\Omega u^\varepsilon(t, x)dx\right) \equiv 0. \tag{1.6}$$

According to [8], the profile of the solution u^ε with $\varepsilon \ll 1$ is expected to be of the form

$$u^\varepsilon(t, x) \approx h^{\pm}(v(t)), \qquad t > 0,\ x \in \Omega^{\pm}(t).$$

Substituting this into (1.6) and using (1.5), we can calculate

$$\begin{aligned}
0 &= \frac{d}{dt}\int_\Omega u^\varepsilon(t, x)dx \\
&= \frac{d}{dt}\int_{\Omega^-(t)} h^-(v(t))dx + \frac{d}{dt}\int_{\Omega^+(t)} h^+(v(t))dx \\
&= \int_{\Omega^-(t)} h_v^-(v(t))\dot{v}(t)dx + \int_{\Gamma(t)} h^-(v(t))\mathbf{v}(x; \Gamma(t))dS_x^{\Gamma(t)} \\
&\quad + \int_{\Omega^+(t)} h_v^+(v(t))\dot{v}(t)dx - \int_{\Gamma(t)} h^+(v(t))\mathbf{v}(x; \Gamma(t))dS_x^{\Gamma(t)} \qquad \left(\dot{} := \frac{d}{dt}\right) \\
&= [h_v^-(v(t))|\Omega^-(t)| + h_v^+(v(t))|\Omega^+(t)|]\dot{v}(t) - [h^+(v(t)) - h^-(v(t))]c(v(t))|\Gamma(t)|,
\end{aligned}$$

where dS_x^{Γ}, $|\Gamma|$ and $|\Omega^{\pm}|$ are the volume element of Γ at $x \in \Gamma$, the surface area of Γ and the volume of $\Omega^{\pm}$, respectively. Since $h_v^{\pm}(v(t)) < 0$ for $v(t) \in I^v$ (cf. **(A2)**), we obtain

$$\dot{v}(t) = \frac{h^+(v(t)) - h^-(v(t))}{h_v^-(v(t))|\Omega^-(t)| + h_v^+(v(t))|\Omega^+(t)|} c(v(t))|\Gamma(t)|, \qquad t > 0. \tag{1.7}$$

The interface equation for (RD) is now explicitly represented as the following *system* of equations:

$$\text{(IE)} \qquad \begin{cases} \mathbf{v}(x; \Gamma(t)) = c(v(t)), & t > 0,\ x \in \Gamma(t), \\ \dot{v}(t) = h(v(t); \Gamma(t))c(v(t))|\Gamma(t)|, & t > 0, \\ \Gamma(0) = \Gamma_0, \quad v(0) = v_0, \end{cases}$$

where the function $h(v; \Gamma)$ is defined by

$$h(v; \Gamma) := \frac{h^+(v) - h^-(v)}{h_v^-(v)|\Omega^-| + h_v^+(v)|\Omega^+|}, \qquad v \in I^v. \tag{1.8}$$

The interface equation (IE) is essentially an initial value problem for a system of ordinary differential equations (cf. (2.7) below), and also arises as the lowest order compatibility condition in our construction of approximate solutions (cf. §3.4 below).

Let us heuristically describe the interface dynamics for (IE). By the property of $c(v)$

$$c(v) = -\left(\int_{-\infty}^{\infty} [Q_z(z; v)]^2 dz\right)^{-1} \mathscr{J}(v) \begin{cases} > 0 & \text{on } (v^*, \bar{v}), \\ = 0 & \text{at } v = v^*, \\ < 0 & \text{on } (\underline{v}, v^*), \end{cases} \tag{1.9}$$

and the fact that $h(\cdot\,; \Gamma) < 0$ on I^v (cf. **(A1)**–**(A2)**), we find the following:

(i) $v \in (v^*, \bar{v}) \Rightarrow \mathbf{v} > 0,\ \dot{v} < 0$;
$\Gamma(t)$ evolves in such a way that the bulk region $\Omega^-(t)$ grows uniformly, and $v(t)$ decreases monotonously toward v^*.

(ii) $v \in (\underline{v}, v^*) \Rightarrow \mathbf{v} < 0,\ \dot{v} > 0$;
$\Gamma(t)$ evolves in such a way that the bulk region $\Omega^-(t)$ shrinks uniformly, and $v(t)$ increases monotonously toward v^*.

(iii) $v = v^* \Rightarrow \mathbf{v} = 0,\ \dot{v} = 0$;
$\Gamma(t)$ and $v(t)$ do not evolve.

This description associated with the intermediate dynamics is the same as that in [18]. Then a natural question arises:

Does the interface equation (IE) have any solutions? Does a layer solution of reaction-diffusion equation (RD) converge to a solution of the interface equation (IE) as $\varepsilon \to 0$?

We will show that the answer to this question is affirmative.

1.3. Main results. We are now in a position to state our main results. The first result is concerned with the unique existence of solutions and the stability of equilibrium solutions to the interface equation (IE).

THEOREM 1.1. *Suppose that the initial pair* (Γ_0, v_0) *satisfies*

(S1) Γ_0 *is smooth and divides* Ω *into two subdomains* $\Omega_0^{\pm}$ *such as* $\Omega = \Omega_0^- \cup \Gamma_0 \cup \Omega_0^+$,

(S2) v_0 *lies in the open interval* $I^v = (\underline{v}, \bar{v})$,

(S3) m_0 *given by*

$$m_0 := h^-(v_0)\frac{|\Omega_0^-|}{|\Omega|} + h^+(v_0)\frac{|\Omega_0^+|}{|\Omega|}$$

lies in the open interval $I^u := (h^-(v^*), h^+(v^*))$.

Then the following statements hold:

(1) *There exists a constant* $T > 0$ *such that* (IE) *has a unique smooth solution* (Γ, v) *satisfying* $|\Omega^{\pm}| > 0$ *on the time interval* $[0, T]$.

(2) *There exists a neighborhood* $I^* \subset I^v$ *of* v^* *such that for* (Γ_0, v_0) *satisfying* **(S1)–(S3)** *with* $v_0 \in I^*$, *the unique solution* (Γ, v) *in* (1) *is defined on* $[0, \infty)$. *Furthermore, there exists a smooth interface* Γ^* *such that*

$$\lim_{t \to \infty} (\Gamma(t), v(t)) = (\Gamma^*, v^*).$$

(3) *A pair* (Γ_0, v_0) *is an equilibrium solution of* (IE) *if and only if* $v_0 = v^*$. *Moreover, the equilibrium solution* (Γ_0, v^*) *is asymptotically stable (relative to the system of ordinary differential equations* (2.7) *below).*

For small $\delta > 0$, let $\Gamma(t)^{\delta}$ denote the δ-neighborhood of the interface $\Gamma(t)$:

$$\Gamma(t)^{\delta} := \{x \in \Omega \mid \operatorname{dist}(x, \Gamma(t)) < \delta\}.$$

We also let

$$\Gamma_T^{\delta} := \bigcup_{t \in [0,T]} \{t\} \times \Gamma(t)^{\delta},$$

$$\Omega_T^{\pm} := \bigcup_{t \in [0,T]} \{t\} \times \Omega^{\pm}(t).$$

The second result is concerned with the existence of layer solutions of (RD) which converge to a solution of (IE) on a finite time interval.

Theorem 1.2. *Let* (Γ, v) *be the smooth solution of* (IE) *on a time interval* $[0, T]$. *Then there exists a family of smooth solutions* u^{ε} *to* (RD) *which satisfies the following property:*

$$\lim_{\varepsilon \to 0} u^{\varepsilon} = h^{\pm}(v) \qquad \text{uniformly on } \bar{\Omega}_T^{\pm} \backslash \Gamma_T^{\delta} \text{ for each } \delta > 0.$$

This paper is organized as follows. In §2, Theorem 1.1 is demonstrated. §3 and §4 are devoted to the proof of Theorem 1.2. In §3, approximate solutions to the problem (RD) with an arbitrarily high degree of accuracy are constructed by means of matched asymptotic expansions. More precisely, the following proposition is demonstrated:

Proposition 1.3. *Let* (Γ, v) *be the smooth solution of* (IE) *on a time interval* $[0, T]$. *Then for each integer* $k \geq 1$, *there exists a family of smooth approximate solutions* u_A^{ε} *of* (RD) *which enjoys the following properties:*

$$\max_{[0,T]} \left\| \varepsilon \frac{\partial u_A^{\varepsilon}}{\partial t} - \varepsilon^2 \Delta u_A^{\varepsilon} - f(u_A^{\varepsilon}) + \frac{1}{|\Omega|} \int_{\Omega} f(u_A^{\varepsilon}) dx \right\|_{L^{\infty}(\Omega)} = O(\varepsilon^{k+1}), \tag{1.10}$$

$$\frac{\partial u_A^{\varepsilon}}{\partial \mathbf{n}} = 0 \qquad \text{on } [0, T] \times \partial\Omega, \tag{1.11}$$

$$\lim_{\varepsilon \to 0} u_A^{\varepsilon} = h^{\pm}(v) \qquad \text{uniformly on } \bar{\Omega}_T^{\pm} \backslash \Gamma_T^{\delta} \text{ for each } \delta > 0. \tag{1.12}$$

The construction of approximate solutions consists of five parts;
(1) outer expansion (§3.1),
(2) inner expansion (§3.2),
(3) expansion of nonlocal relation (§3.3),
(4) C^1-matching (§3.4),
(5) uniform approximation (§3.5).

In §4, it is shown that there exist true solutions of (RD) near the approximate solutions constructed in §3. Namely, the following proposition is established:

Proposition 1.4. *Let* u_A^{ε} *be the family of approximate solutions in Proposition* 1.3. *Then there exists a family of smooth solutions* u^{ε} *of* (RD) *such that*

$$\max_{[0,T]} \|u^{\varepsilon} - u_A^{\varepsilon}\|_{L^{\infty}(\Omega)} \leq M \varepsilon^{k - 3N/2p}, \tag{1.13}$$

where $M > 0$ *is a constant independent of* $\varepsilon > 0$, *and* p *is a constant satisfying* $p \geq 3N$.

Theorem 1.2 follows from these two propositions, Proposition 1.3 and Proposition 1.4. Since the comparison principle is not applicable to the problem (RD) (cf. [3, 12]), we will establish Proposition 1.4 by employing a method based upon a spectral analysis as in §4.

Finally, we give in §5 an overview of application of our approximation method to the dynamics in the stage 3 (cf. §1.1). It is described by the following time-rescaled equation with *slower* time scale

$$\text{(RD-s)} \qquad \begin{cases} \varepsilon^2 u_t^\varepsilon = \varepsilon^2 \Delta u^\varepsilon + f(u^\varepsilon) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f(u^\varepsilon)dx, & t > 0,\ x \in \Omega, \\ \partial u^\varepsilon/\partial \mathbf{n} = 0, & t > 0,\ x \in \partial\Omega, \end{cases}$$

and the corresponding interface equation is the volume-preserving mean curvature flow

$$\text{(IE-s)} \qquad \mathbf{v}(x; \Gamma(t)) = -\kappa(x; \Gamma(t)) + \frac{1}{|\Gamma(t)|}\int_{\Gamma(t)} \kappa(x; \Gamma(t)) dS_x^{\Gamma(t)}.$$

Here, the symbol $\kappa(x; \Gamma)$ stands for the sum of the principal curvatures of Γ at $x \in \Gamma$ and its sign is chosen so that it is positive if the center of the curvature sphere lies in Ω^-.

The convergence of (RD-s) to (IE-s) as $\varepsilon \to 0$ in a radially symmetric setting was earlier established successfully by Bronsard and Stoth [3], in which a variational method was employed. Our approximation method developed in this paper gives another approach to some problems with nonlocal effects, as well as to higher order equations such as the viscous Cahn-Hilliard equation (VCH).

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2. Analysis of interface equation

In this section we prove Theorem 1.1. For $t \geq 0$, we assume that $\Gamma(t)$ is expressed as a smooth embedding from a fixed $(N-1)$-dimensional reference manifold $\mathscr{M}$ to $\mathbf{R}^N$:

$$\text{(2.1)} \qquad \gamma(t, \cdot) : \mathscr{M} \to \Gamma(t) \subset \Omega, \quad \mathscr{M} \ni y \mapsto x = \gamma(t, y) \in \Gamma(t).$$

Let $\nu(t,y) \in \mathbf{R}^N$ be the unit normal vector on $\Gamma(t)$ at $x = \gamma(t,y)$ pointing into the interior of $\Omega^+(t)$. We normalize the parametrization (2.1) in such a way that γ_t is always parallel to ν (cf. [6]). A point $x \in \Gamma(t)^\delta$ is uniquely represented as

$$x = \mathbf{F}(t,r,y) := \gamma(t,y) + r\nu(t,y) \tag{2.2}$$

by the diffeomorphism $\mathbf{F}(t,\cdot,\cdot) : (-\delta,\delta) \times \mathscr{M} \to \Gamma(t)^\delta$. In particular, (2.2) gives the transformation of coodinate systems $(t,x) \leftrightarrow (t,r,y)$.

Let $\mathbf{G}(t,r) = (\mathbf{G}_{ij}(t,r))$ $(i,j = 1,\dots,N-1)$ be the Riemannian metric tensor on $\mathscr{M}$ induced from the metric on $\Gamma(t)^\delta$ by $\mathbf{F}(t,r,\cdot)$, and the contravariant metric tensor is denoted by $\mathbf{G}(t,r)^{-1} = (\mathbf{G}^{ij}(t,r))$. We set

$$J(t,r,y) := \prod_{i=1}^{N-1}(1 + r\kappa_i(t,y)) = \sum_{i=0}^{N-1} H_i(t,y)r^i.$$

Here $\kappa_i(t,y)$ $(i = 1,\dots,N-1)$ stand for the principal cuvatures of $\Gamma(t)$ at $x = \gamma(t,y)$, and H_i $(i = 0,\dots,N-1)$ are the fundamental symmetric functions of $\kappa_1,\dots,\kappa_{N-1}$:

$$H_0 \equiv 1, \quad H_1 = \kappa := \kappa_1 + \cdots + \kappa_{N-1}, \dots, \quad H_{N-1} = \kappa_1 \cdots \kappa_{N-1}. \tag{2.3}$$

(1) Following the treatment of Sakamoto [19], we recast (IE) as an initial value problem for a system of ordinary differential equations.

For a given initial interface Γ_0, let us express the interface $\Gamma(t)$ as the graph of a function $r(t,y)$ over Γ_0:

$$\Gamma(t) = \{x \in \Omega \mid x = \gamma(t,y) = \gamma(0,y) + r(t,y)\nu(0,y),\, y \in \mathscr{M}\}. \tag{2.4}$$

Then an elementary calculation yields that $\nu(t,y) \equiv \nu(0,y)$ and $r(t,y) \equiv r(t)$. Since $\mathbf{v}(x;\Gamma(t)) = \gamma_t(t,y) \cdot \nu(t,y)$, the first equation in (IE) is recast as $\dot r(t) = c(v(t))$.

By (2.4) and $r(t,y) \equiv r(t)$, the interface $\Gamma(t)$ is expressed as

$$\Gamma(t) = \{x \in \Omega \mid x = \gamma(0,y) + r(t)\nu(0,y),\ y \in \mathscr{M}\} = \mathbf{F}(0,r(t),\mathscr{M}). \tag{2.5}$$

On the other hand, the surface area of the interface $\mathbf{F}(0,r,\mathscr{M})$ is given by

$$|\mathbf{F}(0,r,\mathscr{M})| = g(r) := \int_{\mathscr{M}} J(0,r,y)\,dS_y^{\gamma(0,\cdot)} = \sum_{i=0}^{N-1}\left(\int_{\mathscr{M}} H_i(0,y)\,dS_y^{\gamma(0,\cdot)}\right)r^i, \tag{2.6}$$

where $dS_y^{\gamma(t,\cdot)}$ stands for the volume element on $\mathscr{M}$ induced from $dS_x^{\Gamma(t)}$ on the interface $\Gamma(t)$ at $x = \gamma(t,y)$ by the embedding $\gamma(t,\cdot)$. Thus we have $|\Gamma(t)| = g(r(t))$ from (2.5) and (2.6).

Furthermore, thanks to the relation

$$\frac{d}{dt}|\Omega^-(t)| = -\frac{d}{dt}|\Omega^+(t)| = \int_{\Gamma(t)} \mathbf{v}(x;\Gamma(t))dS_x^{\Gamma(t)},$$

it is easy to verify that the interface equation (IE) gives rise to

$$\frac{d}{dt}\left[h^-(v(t))\frac{|\Omega^-(t)|}{|\Omega|} + h^+(v(t))\frac{|\Omega^+(t)|}{|\Omega|}\right] \equiv 0.$$

Therefore, we obtain the conservation property

$$h^-(v(t))\frac{|\Omega^-(t)|}{|\Omega|} + h^+(v(t))\frac{|\Omega^+(t)|}{|\Omega|} \equiv m_0, \qquad t > 0.$$

This, together with $|\Omega^-(t)| + |\Omega^+(t)| \equiv |\Omega|$, implies that $|\Omega^\pm(t)|$ are represented in term of $v(t)$ alone:

$$|\Omega^-(t)| = \frac{h^+(v(t)) - m_0}{h^+(v(t)) - h^-(v(t))}|\Omega| > 0, \qquad |\Omega^+(t)| = \frac{m_0 - h^-(v(t))}{h^+(v(t)) - h^-(v(t))}|\Omega| > 0,$$

from which $h(v(t);\Gamma(t))$ is rewritten as $h(v(t))$, where the function $h(v)$ is defined by

$$h(v) := \frac{1}{|\Omega|}\frac{[h^+(v) - h^-(v)]^2}{h_v^-(v)[h^+(v) - m_0] + h_v^+(v)[m_0 - h^-(v)]}.$$

Thus the interface equation (IE) is recast as the following initial value problem for (r, v):

$$\text{(2.7)} \qquad \begin{cases} \dot{r} = c(v), & t > 0 \\ \dot{v} = h(v)c(v)g(r), & t > 0, \\ r(0) = 0, \quad v(0) = v_0. \end{cases}$$

The statement (1) immediately follows from (2.7).

(2) Let $R^* > 0$ be a constant such that the interface $\mathbf{F}(0, r, \mathscr{M})$ is smooth for all $r \in [-R^*, R^*]$. Then there exists a constant $g^* > 0$ such that

$$\text{(2.8)} \qquad g(r) \geq g^*, \quad r \in [-R^*, R^*].$$

We now introduce the interval $[v^* - \eta, v^* + \eta] \subset I^v$ for some $\eta > 0$. Since $h(v) < 0$ for all $v \in I^v$, there exists a constant $h^* > 0$ such that

$$\text{(2.9)} \qquad h(v) \leq -h^*, \quad v \in [v^* - \eta, v^* + \eta].$$

Furthermore, the relation in (1.9) and **(A3)** yield that $c(v^*) = 0$ and

$$c'(v^*) = -\left(\int_{-\infty}^{\infty} [Q_z(z;v^*)]^2 dz\right)^{-1} \mathscr{I}'(v^*) > 0.$$

Therefore, there exist constants k^*, $K^* > 0$ such that

$$k^*(v - v^*) \le c(v) \le K^*(v - v^*), \qquad v \in [v^*, v^* + \eta], \tag{2.10}$$

$$K^*(v - v^*) \le c(v) \le k^*(v - v^*), \qquad v \in [v^* - \eta, v^*]. \tag{2.11}$$

The estimates (2.8), (2.9) and (2.10) yield that

$$h(v)c(v)g(r) \le -\omega^*(v - v^*), \qquad v \in [v^*, v^* + \eta],$$

where $\omega^* := h^* k^* g^* > 0$. This estimate and the equation for v in (2.7) imply that if $v_0 \in [v^*, v^* + \eta]$, then $v(t)$ satisfies $v(t) - v^* \le (v_0 - v^*)e^{-\omega^* t}$. By (2.11) and the same argument as above, we find that $v(t)$ for $v_0 \in [v^* - \eta, v^*]$ satisfies $v(t) - v^* \ge (v_0 - v^*)e^{-\omega^* t}$. Therefore, the solution $v(t)$ starting from $v_0 \in [v^* - \eta, v^* + \eta]$ satisfies

$$|v(t) - v^*| \le |v_0 - v^*| e^{-\omega^* t}. \tag{2.12}$$

Using (2.10), (2.11) and (2.12) in the equation for r, we obtain

$$|r(t)| \le \frac{K^*}{\omega^*} |v_0 - v^*| (1 - e^{-\omega^* t}). \tag{2.13}$$

By (2.12) and (2.13), we find that the solution (r, v) of (2.7) for $v_0 \in [v^* - \eta, v^* + \eta]$ enjoys

$$|v(t) - v^*| \le |v_0 - v^*| e^{-\omega^* t}, \qquad |r(t)| \le L^* |v_0 - v^*|,$$

where $L^* := K^*/\omega^* > 0$.

Set $\eta := R^*/L^* > 0$ and $I^* := (v^* - \eta, v^* + \eta)$. We immediately find that if $v_0 \in I^*$, then the solution $(r(t), v(t))$ satisfies $|r(t)| \le R^*$ for $t \in [0, \infty)$. Hence the corresponding interface $\Gamma(t) = \mathbf{F}(0, r(t), \mathscr{M})$ is smooth for all $t > 0$. Moreover, it is also easy to show that there exists $r^* \in [-R^*, R^*]$ such that $(r(t), v(t)) \to (r^*, v^*)$ as $t \to \infty$. Hence the smooth interface defined by $\Gamma^* := \mathbf{F}(0, r^*, \mathscr{M})$ is the limit interface as $t \to \infty$.

(3) The relation in (1.9) together with **(A3)** proves the first statement. To prove the second statement, we linearize (2.7) arround the corresponding equilibrium solution $(0, v^*)$. Then we obtain the eigenvalues 0 and $h(v^*)c'(v^*)g(0) < 0$ because of the fact that $h(v^*) < 0$, $c'(v^*) > 0$ and $g(0) = |\Gamma_0| > 0$. This completes the proof of Theorem 1.1. □

Remark. The equation (2.7) implies $dv/dr = h(v)g(r)$, from which we have

$$G(r^*) = \int_{v_0}^{v^*} \frac{dv}{h(v)}, \tag{2.14}$$

where

$$G(r) := \int_0^r g(\rho)d\rho = \sum_{i=0}^{N-1}\left(\int_{\mathscr{M}} H_i(0, y)dS_y^{\gamma(0,\cdot)}\right)\frac{r^{i+1}}{i+1}.$$

The relation (2.14) is uniquely solvable with respect to $r^* \in [-R^*, R^*]$, and we obtain

$$r^* = G^{-1}\left(\int_{v_0}^{v^*}\frac{dv}{h(v)}\right).$$

3. Construction of approximate solutions

In this section, we prove Proposition 1.3. Let u^ε be a solution of (RD) with an appropriate initial condition $u^\varepsilon(0,\cdot) = u_0^\varepsilon$ (appropriate initial functions u_0^ε will be chosen later, cf. §4.2):

$$\varepsilon u_t^\varepsilon(t,x) = \varepsilon^2 \Delta u^\varepsilon(t,x) + f(u^\varepsilon(t,x)) - v^\varepsilon(t), \qquad t > 0,\ x \in \Omega. \tag{3.1}$$

Note that v^ε is related to u^ε as

$$v^\varepsilon(t) = \frac{1}{|\Omega|}\int_\Omega f(u^\varepsilon(t,x))dx, \qquad t \geq 0. \tag{3.2}$$

For $t \geq 0$, we define the interface $\Gamma^\varepsilon(t)$ as a level set of the solution u^ε. Transition layers are expected to develop near $\{x \in \Omega \mid u^\varepsilon(t,x) \approx h^0(v^*)\}$ and we may identify the point $(h^0(v^*), v^*)$ with $(0, v^*)$ in $\mathbf{R}^2$ by an appropriate translation. Therefore, we define the family of ε-dependent interfaces by

$$\Gamma^\varepsilon(t) := \{x \in \Omega \mid u^\varepsilon(t,x) = 0\}. \tag{3.3}$$

We also expect that $\Gamma^\varepsilon(t)$ is expressed as the graph of a function over $\Gamma(t)$:

$$\Gamma^\varepsilon(t) = \{x \in \Omega \mid x = \gamma(t,y) + \varepsilon R^\varepsilon(t,y)\nu(t,y),\ y \in \mathscr{M}\}. \tag{3.4}$$

In terms of (t, r, y), the differential operators $\partial/\partial t$, Δ and the volume element dx transform as follows:

$$\begin{aligned}
&\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \gamma_t \cdot \nu\frac{\partial}{\partial r} - r\nu_t \cdot (D_y\mathbf{F})\mathbf{G}^{-1}\nabla_y,\\
&\Delta = \frac{\partial^2}{\partial r^2} + K\frac{\partial}{\partial r} + \frac{1}{\sqrt{\det \mathbf{G}}}\sum_{i,j=1}^{N-1}\frac{\partial}{\partial y^i}\left(\sqrt{\det \mathbf{G}}\mathbf{G}^{ij}\frac{\partial}{\partial y^j}\right),\\
&dx = J\, drdS_y \qquad (dS_y = dS_y^{\gamma(t,\cdot)}).
\end{aligned} \tag{3.5}$$

Here $K(t, r, y)$ is the sum of the principal curvatures of the interface $\mathbf{F}(t, r, \mathcal{M})$ at $x = \mathbf{F}(t, r, y)$, and is given by

$$K(t, r, y) = \sum_{i=1}^{N-1} \frac{\kappa_i(t, y)}{1 + r\kappa_i(t, y)}.$$

We note that $K(t, 0, y) = \kappa(t, y)$.

3.1. Outer expansion. Let $\Omega^{\varepsilon,\pm}(t)$ be the components of Ω separated by the interface $\Gamma^{\varepsilon}(t)$ such as $\Omega = \Omega^{\varepsilon,-}(t) \cup \Gamma^{\varepsilon}(t) \cup \Omega^{\varepsilon,+}(t)$. We substitute the formal expansions

$$U^{\varepsilon}(t, x) = U^{\varepsilon,\pm}(t, x) = \sum_{j \ge 0} \varepsilon^j U^{j,\pm}(t, x), \qquad v^{\varepsilon}(t) = \sum_{j \ge 0} \varepsilon^j v^j(t) \tag{3.6}$$

into (3.1) in order to approximate the solution away from the layer region. Equating the coefficient of each power of ε in the resulting equation, we obtain the following equations:

$$f(U^{0,\pm}) - v^0 = 0, \tag{3.7}$$

$$f'(U^{0,\pm})U^{1,\pm} = v^1 + U_t^{0,\pm}, \tag{3.8}$$

$$f'(U^{0,\pm})U^{j,\pm} = v^j + F_j^{\pm}, \qquad j \ge 2. \tag{3.9}$$

Here $F_j^{\pm}$ stand for terms depending only on $U^{m,\pm}$ $(0 \le m < j)$, and are explicitly given by

$$F_j^{\pm} := U_t^{j-1,\pm} - \Delta U^{j-2,\pm} - \frac{1}{j!}\frac{d^j}{d\varepsilon^j} f\left(\sum_{m \ge 0} \varepsilon^m U^{m,\pm}\right)\Bigg|_{\varepsilon=0} + f'(U^{0,\pm})U^{j,\pm}.$$

As a solution of (3.7), we choose

$$U^{0,\pm}(t, x) = U^{0,\pm}(t) = h^{\pm}(v^0(t)), \qquad x \in \Omega^{\pm}(t). \tag{3.10}$$

Once we make this choice, $U^{j,\pm}$ $(j \ge 1)$ can be successively expressed, by (3.8) and (3.9), as

$$U^{j,\pm}(t, x) = U^{j,\pm}(t) = h_v^{\pm}(v^0(t))v^j(t) + V_j^{\pm}(t), \qquad x \in \Omega^{\pm}(t), \tag{3.11}$$

with $V_j^{\pm}$ being some functions depending only on v^m $(0 \le m < j)$. Note that v^j $(j \ge 0)$ are unknown at this stage and will be determined in § 3.4.

Setting

$$U^{\varepsilon,\pm}(t) := \sum_{j \ge 0} \varepsilon^j U^{j,\pm}(t),$$

and extending $U^{\varepsilon,\pm}$ up to the interface $\Gamma^{\varepsilon}(t)$, we define

$$U^{\varepsilon}(t,x) := U^{\varepsilon,\pm}(t), \quad x \in \Omega^{\varepsilon,\pm}(t). \tag{3.12}$$

3.2. Inner expansion. To describe layer phenomena near $r = \varepsilon R^{\varepsilon}(t,y)$ (cf. (3.4)), we introduce a stretched variable $z := \varepsilon^{-1}[r - \varepsilon R^{\varepsilon}(t,y)]$. Under this change of variables, the differential operators and the volume element in the right hand side of (3.5) are replaced as

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \frac{\partial R^{\varepsilon}}{\partial t}\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial r} = \frac{1}{\varepsilon}\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial y^i} \to \frac{\partial}{\partial y^i} - \frac{\partial R^{\varepsilon}}{\partial y^i}\frac{\partial}{\partial z},$$

$$J\, drdS_y = \varepsilon J^{\varepsilon}\, dzdS_y,$$

where $J^{\varepsilon}(t,z,y)$ is defined by

$$J^{\varepsilon}(t,z,y) := J(t, \varepsilon z + \varepsilon R^{\varepsilon}(t,y), y). \tag{3.13}$$

Our problem (3.1) is then recast in terms of (t,z,y) as follows:

$$\begin{gathered}\tilde{u}^{\varepsilon}_{zz} + (\gamma_t \cdot \nu)\tilde{u}^{\varepsilon}_z + f(\tilde{u}^{\varepsilon}) + \varepsilon R^{\varepsilon}_t \tilde{u}^{\varepsilon}_z - v^{\varepsilon} + \mathscr{D}^{\varepsilon}\tilde{u}^{\varepsilon} = 0,\\ z \in (-\delta/\varepsilon - R^{\varepsilon}, \delta/\varepsilon - R^{\varepsilon}).\end{gathered} \tag{3.14}$$

In the equation above, $\mathscr{D}^{\varepsilon}$ stands for the differential operator defined by

$$\begin{aligned}\mathscr{D}^{\varepsilon} :=\ & \varepsilon K^{\varepsilon}\frac{\partial}{\partial z} + \varepsilon^2(z + R^{\varepsilon})\nu_t \cdot (D\mathbf{F})^{\varepsilon}(\mathbf{G}^{\varepsilon})^{-1}\left(\nabla - \nabla R^{\varepsilon}\frac{\partial}{\partial z}\right)\\ & + \frac{\varepsilon^2}{\sqrt{\det \mathbf{G}^{\varepsilon}}}\sum_{i,j=1}^{N-1}\left(\frac{\partial}{\partial y^i} - R^{\varepsilon}_{y^i}\frac{\partial}{\partial z}\right)\left[\sqrt{\det \mathbf{G}^{\varepsilon}}(\mathbf{G}^{\varepsilon})^{ij}\left(\frac{\partial}{\partial y^j} - R^{\varepsilon}_{y^j}\frac{\partial}{\partial z}\right)\right] - \varepsilon\frac{\partial}{\partial t},\end{aligned}$$

where K^{ε}, $(D\mathbf{F})^{\varepsilon}$, and $\mathbf{G}^{\varepsilon}$ are functions of (t,z,y) defined by K, $D\mathbf{F}$ and $\mathbf{G}$ with r being replaced by $r = \varepsilon z + \varepsilon R^{\varepsilon}(t,y)$. In terms of (t,z,y), we write $U^{\varepsilon}(t,x)$ as

$$\tilde{U}^{\varepsilon}(t,z,y) := U^{\varepsilon}(t, \gamma(t,y) + (\varepsilon z + \varepsilon R^{\varepsilon}(t,y))\nu(t,y)).$$

By (3.12), it turns out that $\tilde{U}^{\varepsilon}$ is nothing but the function given by

$$\tilde{U}^{\varepsilon}(t,z,y) = \begin{cases} U^{\varepsilon,-}(t), & z \in (-\delta/\varepsilon - R^{\varepsilon}(t,y), 0),\\ U^{\varepsilon,+}(t), & z \in (0, \delta/\varepsilon - R^{\varepsilon}(t,y)).\end{cases}$$

Let us now seek an asymptotic solution to (3.14) of the form

$$\tilde{u}^{\varepsilon}(t,z,y) = \tilde{U}^{\varepsilon}(t,z,y) + \tilde{\phi}^{\varepsilon}(t,z,y), \tag{3.15}$$

where $\tilde{\phi}^{\varepsilon}$ is a layer correction. For this, we expand R^{ε} and $\tilde{\phi}^{\varepsilon}$ as

$$R^{\varepsilon}(t,y) = R^1(t,y) + \varepsilon R^2(t,y) + \varepsilon^2 R^3(t,y) + \cdots, \tag{3.16}$$

$$\begin{aligned} \tilde{u}^{\varepsilon}(t,z,y) = \tilde{u}^{\varepsilon,\pm}(t,z,y) &= \tilde{U}^{\varepsilon,\pm}(t,z,y) + \tilde{\phi}^{\varepsilon,\pm}(t,z,y) \\ &= \sum_{j\ge 0} \varepsilon^j U^{j,\pm}(t) + \sum_{j\ge 0} \varepsilon^j \tilde{\phi}^{j,\pm}(t,z,y) \\ &=: \sum_{j\ge 0} \varepsilon^j \tilde{u}^{j,\pm}(t,z,y), \end{aligned} \tag{3.17}$$

and determine $\tilde{\phi}^{j,\pm}$ in such a way that $\tilde{u}^{\varepsilon,\pm}$ in (3.17) asymptotically satisfy (3.14) for $\pm z \in (0,\infty)$. We also expand $\mathscr{D}^{\varepsilon}$ as $\mathscr{D}^{\varepsilon} = \sum_{j\ge 0} \varepsilon^j \mathscr{D}_j$. It is easy to find that $\mathscr{D}_0 \equiv 0$, $\mathscr{D}_1 = \kappa\partial/\partial z - \partial/\partial t$, and $\mathscr{D}_j$ $(j \ge 2)$ are differential operators with respect to t, z, y depending only on Γ, R^m $(1 \le m < j)$. Substituting the formal expansion (3.17) and the expansion for v^{ε} in (3.6) into (3.14), and equating the coefficient of each power of ε in the resulting equation, we obtain the following equations for $\tilde{u}^{j,\pm}$ in $\pm z \in (0,\infty)$:

$$\tilde{u}^{0,\pm}_{zz} + (\gamma_t \cdot v)\tilde{u}^{0,\pm}_z + f(\tilde{u}^{0,\pm}) - v^0 = 0, \tag{3.18}$$

$$\tilde{u}^{1,\pm}_{zz} + (\gamma_t \cdot v)\tilde{u}^{1,\pm}_z + f'(\tilde{u}^{0,\pm})\tilde{u}^{1,\pm} + (R^1_t \tilde{u}^{0,\pm}_z - v^1 + \tilde{\mathscr{F}}^{\pm}_1) = 0, \tag{3.19}$$

$$\tilde{u}^{j,\pm}_{zz} + (\gamma_t \cdot v)\tilde{u}^{j,\pm}_z + f'(\tilde{u}^{0,\pm})\tilde{u}^{j,\pm} + (R^j_t \tilde{u}^{0,\pm}_z - v^j + \tilde{\mathscr{F}}^{\pm}_j) = 0, \qquad j \ge 2. \tag{3.20}$$

Here $\tilde{\mathscr{F}}^{\pm}_j$ $(j \ge 1)$ are functions defined by

$$\tilde{\mathscr{F}}^{\pm}_1 := \kappa \tilde{u}^{0,\pm}_z - \tilde{u}^{0,\pm}_t, \tag{3.21}$$

$$\begin{aligned} \tilde{\mathscr{F}}^{\pm}_j := &\sum_{m=1}^{j-1} R^m_t \tilde{u}^{j-m,\pm}_z + \sum_{m=1}^{j} \mathscr{D}_m \tilde{u}^{j-m,\pm} \\ &+ \frac{1}{j!}\frac{d^j}{d\varepsilon^j} f\left(\sum_{m\ge 0} \varepsilon^m \tilde{u}^{m,\pm}\right)\Bigg|_{\varepsilon=0} - f'(\tilde{u}^{0,\pm})\tilde{u}^{j,\pm}, \qquad j \ge 2. \end{aligned} \tag{3.22}$$

Note that $\tilde{\mathscr{F}}^{\pm}_j$ depend only on Γ, R^m $(1 \le m < j)$ and $\tilde{u}^{m,\pm}$ $(0 \le m < j)$.

Using (3.15) and the fact that $\tilde{U}^{\varepsilon}$ satisfies (3.14), we also obtain the equations for $\tilde{\phi}^{j,\pm}$ in $\pm z \in (0,\infty)$:

$$\tilde{\phi}^{0,\pm}_{zz} + (\gamma_t \cdot v)\tilde{\phi}^{0,\pm}_z + f(\tilde{\phi}^{0,\pm} + h^{\pm}(v^0)) - v^0 = 0, \tag{3.23}$$

$$\tilde{\phi}^{j,\pm}_{zz} + (\gamma_t \cdot v)\tilde{\phi}^{j,\pm}_z + f'(\tilde{\phi}^{0,\pm} + h^{\pm}(v^0))\tilde{\phi}^{j,\pm} + \tilde{\mathscr{G}}^{\pm}_j = 0, \qquad j \ge 1, \tag{3.24}$$

where $\tilde{\mathscr{G}}^{\pm}_j$ are functions given by

$$\tilde{\mathscr{G}}_j^{\pm} := \sum_{m=1}^{j} R_t^m \tilde{\phi}_z^{j-m,\pm} + \sum_{m=1}^{j} \mathscr{D}_m \tilde{\phi}^{j-m,\pm}$$
$$+ \frac{1}{j!}\frac{d^j}{d\varepsilon^j}\left[f\left(\sum_{m\geq 0} \varepsilon^m (\tilde{\phi}^{m,\pm} + U^{m,\pm})\right) - f\left(\sum_{m\geq 0} \varepsilon^m U^{m,\pm}\right)\right]\Bigg|_{\varepsilon=0}$$
$$- f'(\tilde{\phi}^{0,\pm} + h^{\pm}(v^0))\tilde{\phi}^{j,\pm}.$$

We impose the following boundary conditions for $j \geq 0$.

(i) Boundary conditions at $z = 0$ (interface conditions, cf. (3.3)):

$$\tilde{u}^{j,\pm}(t,0,y) = U^{j,\pm}(t) + \tilde{\phi}^{j,\pm}(t,0,y) = 0. \tag{3.25}$$

(ii) Boundary conditions at $z = \pm\infty$ (outer-inner matching conditions):

$$\tilde{\phi}^{j,\pm}(t,z,y) \to 0 \qquad \text{exponentially as } z \to \pm\infty. \tag{3.26}$$

Lemma 3.1. *Let $(Q(z;v), c(v))$ be the unique solution pair of* (1.4). *If the condition $\gamma_t \cdot v = c(v^0)$ is satisfied and the solutions $\tilde{\phi}^{0,\pm}$ of* (3.23) *are chosen so that*

$$\tilde{\phi}^{0,\pm}(t,z,y) = Q(z;v^0(t)) - h^{\pm}(v^0(t)), \qquad \pm z \in (0,\infty), \tag{3.27}$$

then the boundary conditions (3.25) *and* (3.26) *are valid for $j = 0$. Moreover, there exist unique solutions $\tilde{\phi}^{j,\pm}$ of* (3.24) *satisfying* (3.25) *and* (3.26) *for each $j \geq 1$.*

Proof. When $\gamma_t \cdot v = c(v^0)$ and $\tilde{\phi}^{0,\pm}$ are given by (3.27), they satisfy the boundary conditions (3.25) and (3.26) ($j = 0$) thanks to (1.4).

We next consider the equations (3.24) ($j = 1$). The equations are recast as

$$\tilde{\phi}_{zz}^{1,\pm} + c(v^0)\tilde{\phi}_z^{1,\pm} + f'(Q(\cdot\,;v^0))\tilde{\phi}^{1,\pm} + \tilde{\mathscr{G}}_1^{\pm} = 0, \tag{3.28}$$

where

$$\tilde{\mathscr{G}}_1^{\pm} = (R_t^1 + \kappa)\tilde{\phi}_z^{0,\pm} - \tilde{\phi}_t^{0,\pm} + [f'(Q(\cdot\,;v^0)) - f'(h^{\pm}(v^0))]U^{1,\pm}. \tag{3.29}$$

To treat this, we use the following result without proof:

Lemma 3.2. *Let $\mathscr{H}^{\pm}(z)$ be continuous functions and consider the following initial value problems:*

$$\begin{cases} u_{zz}^{\pm} + cu_z^{\pm} + f'(Q)u^{\pm} + \mathscr{H}^{\pm} = 0, & \pm z \in (0,\infty), \\ u^{\pm}(\pm\infty) = 0, \quad u^{\pm}(0) = u_0^{\pm}, \end{cases} \tag{3.30}$$

where (Q,c) satisfies (1.4). *Then the solutions $u^{\pm}(z)$ of* (3.30) *uniquely exist, and are given by*

$$u^{\pm}(z) = Q_z(z)\left[\frac{u_0^{\pm}}{Q_z(0)} - \int_0^z \frac{e^{-c\zeta}}{[Q_z(\zeta)]^2}\int_{\pm\infty}^{\zeta} e^{c\eta} Q_z(\eta)\mathscr{H}^{\pm}(\eta)\,d\eta d\zeta\right].$$

If $\mathscr{H}^{\pm}(z)$ decay exponentially as $z \to \pm\infty$, then $u^{\pm}(z)$ decay exponentially as $z \to \pm\infty$.

Since $\tilde{\mathscr{G}}_1^{\pm}$ in (3.29) decay exponentially as $z \to \pm\infty$, we can apply Lemma 3.2 by setting as $\mathscr{H}^{\pm} := \tilde{\mathscr{G}}_1^{\pm}$ and $u_0^{\pm} := -U^{1,\pm}$ so that we obtain unique solutions $\tilde{\phi}^{1,\pm}$ of (3.28) which decay exponentially as $z \to \pm\infty$.

For $j \geq 2$, we proceed by induction on j. Consider the j-th equations in (3.24). The equations are recast as

$$\tilde{\phi}_{zz}^{j,\pm} + c(v^0)\tilde{\phi}_z^{j,\pm} + f'(Q(\cdot\,;v^0))\tilde{\phi}^{j,\pm} + \tilde{\mathscr{G}}_j^{\pm} = 0 \tag{3.31}$$

with

$$\begin{aligned}\tilde{\mathscr{G}}_j^{\pm} = {} & \sum_{m=1}^{j} R_t^m \tilde{\phi}_z^{j-m,\pm} + \sum_{m=1}^{j} \mathscr{D}_m \tilde{\phi}^{j-m,\pm} \\ & + \frac{1}{j!}\frac{d^j}{d\varepsilon^j}\left[f\left(\sum_{m\geq 0}\varepsilon^m(\tilde{\phi}^{m,\pm} + U^{m,\pm})\right) - f\left(\sum_{m\geq 0}\varepsilon^m U^{m,\pm}\right)\right]\Bigg|_{\varepsilon=0} \\ & - f'(Q(\cdot\,;v^0))\tilde{\phi}^{j,\pm}.\end{aligned} \tag{3.32}$$

Let us suppose that $\tilde{\phi}^{m,\pm}$ $(0 \leq m < j)$ decay exponentially as $z \to \pm\infty$. We note that the first line of (3.32) depends polynomially on $\tilde{\phi}^{m,\pm}$ $(0 \leq m < j)$, while the second and third lines consist of the term $[f'(Q(\cdot\,;v^0)) - f'(h^{\pm}(v^0))]U^{j,\pm}$ and a polynomial in $\tilde{\phi}^{m,\pm}$ $(0 \leq m < j)$. Therefore, $\tilde{\mathscr{G}}_j^{\pm}$ decay exponentially as $z \to \pm\infty$. Lemma 3.2 with $\mathscr{H}^{\pm} := \tilde{\mathscr{G}}_j^{\pm}$ and $u_0^{\pm} := -U^{j,\pm}$ again implies the unique existence of solutions $\tilde{\phi}^{j,\pm}$ to (3.31) with exponential decay as $z \to \pm\infty$. $\square$

3.3. Expansion of nonlocal relation. In this subsection, we deal with the nonlocal relation (3.2).

Lemma 3.3. *The relation* (3.2) *is recast as*

$$\dot{U}^{\varepsilon,-}|\Omega^-| + \dot{U}^{\varepsilon,+}|\Omega^+| = (\dot{U}^{\varepsilon,+} - \dot{U}^{\varepsilon,-})\mathscr{P}^{\varepsilon} + \mathscr{I}^{\varepsilon} + O(\varepsilon^{-1}e^{-\eta/\varepsilon}) \tag{3.33}$$

for some $\eta > 0$. The functions $\mathscr{P}^{\varepsilon}(t)$ and $\mathscr{I}^{\varepsilon}(t)$ are given by

$$\mathscr{P}^{\varepsilon} := \sum_{i\geq 0}\int_{\mathscr{M}} \frac{H_i}{i+1}(\varepsilon R^{\varepsilon})^{i+1} dS_y, \tag{3.34}$$

$$\mathscr{I}^{\varepsilon} := \int_{\mathscr{M}}\int_{-\infty}^{\infty} [\tilde{\phi}_{zz}^{\varepsilon} + (\gamma_t \cdot \nu)\tilde{\phi}_z^{\varepsilon} + \varepsilon R_t^{\varepsilon}\tilde{\phi}_z^{\varepsilon} + \mathscr{D}^{\varepsilon}\tilde{\phi}^{\varepsilon}] J^{\varepsilon}\, dz dS_y. \tag{3.35}$$

Proof. We first rewrite the right hand side of (3.2) as

$$\frac{1}{|\Omega|}\int_{\Omega} f(u^{\varepsilon}(t,x))dx = \frac{1}{|\Omega|}\int_{\Omega} f(U^{\varepsilon}(t,x))dx + \frac{1}{|\Omega|}\int_{\Gamma(t)^{\delta}}[f(u^{\varepsilon}(t,x)) - f(U^{\varepsilon}(t,x))]dx$$
$$+\frac{1}{|\Omega|}\int_{\Omega\setminus\Gamma(t)^{\delta}}[f(u^{\varepsilon}(t,x)) - f(U^{\varepsilon}(t,x))]dx$$
$$=: N_1^{\varepsilon} + N_2^{\varepsilon} + N_3^{\varepsilon},$$

and explicitly compute N_j^{ε} as follows.

By (3.12) and the fact that $U^{\varepsilon,\pm}$ are spatially homogeneous and satisfy (3.1) in $\Omega^{\varepsilon,\pm}$, N_1^{ε} becomes

$$\begin{aligned}
N_1^{\varepsilon} &= \frac{1}{|\Omega|}\int_{\Omega^{\varepsilon,-}(t)} f(U^{\varepsilon,-}(t))dx + \frac{1}{|\Omega|}\int_{\Omega^{\varepsilon,+}(t)} f(U^{\varepsilon,+}(t))dx \\
&= \frac{1}{|\Omega|}f(U^{\varepsilon,-}(t))|\Omega^{\varepsilon,-}(t)| + \frac{1}{|\Omega|}f(U^{\varepsilon,+}(t))|\Omega^{\varepsilon,+}(t)| \\
&= \frac{1}{|\Omega|}(v^{\varepsilon}(t) + \varepsilon\dot{U}^{\varepsilon,-}(t))\left(|\Omega^{-}(t)| + \int_{\mathcal{M}}\int_0^{\varepsilon R^{\varepsilon}(t,y)} J(t,r,y)drdS_y\right) \\
&\quad + \frac{1}{|\Omega|}(v^{\varepsilon}(t) + \varepsilon\dot{U}^{\varepsilon,+}(t))\left(|\Omega^{+}(t)| - \int_{\mathcal{M}}\int_0^{\varepsilon R^{\varepsilon}(t,y)} J(t,r,y)drdS_y\right) \\
&= v^{\varepsilon}(t) + \frac{\varepsilon}{|\Omega|}(\dot{U}^{\varepsilon,-}(t)|\Omega^{-}(t)| + \dot{U}^{\varepsilon,+}(t)|\Omega^{+}(t)|) \\
&\quad - \frac{\varepsilon}{|\Omega|}(\dot{U}^{\varepsilon,+}(t) - \dot{U}^{\varepsilon,-}(t))\mathscr{P}^{\varepsilon}(t).
\end{aligned}$$

As for N_2^{ε}, we can compute it by rewriting in terms of (t,z,y) so that

$$\begin{aligned}
N_2^{\varepsilon} &= \frac{\varepsilon}{|\Omega|}\int_{\mathcal{M}}\int_{-\delta/\varepsilon - R^{\varepsilon}(t,y)}^{\delta/\varepsilon - R^{\varepsilon}(t,y)} [f(\tilde{u}^{\varepsilon}(t,z,y)) - f(\tilde{U}^{\varepsilon}(t,z,y))]J^{\varepsilon}(t,z,y)dzdS_y \\
&= -\frac{\varepsilon}{|\Omega|}\int_{\mathcal{M}}\int_{-\delta/\varepsilon - R^{\varepsilon}}^{\delta/\varepsilon - R^{\varepsilon}} [\tilde{\phi}_{zz}^{\varepsilon} + (\gamma_t\cdot v)\tilde{\phi}_z^{\varepsilon} + \varepsilon R_t^{\varepsilon}\tilde{\phi}_z^{\varepsilon} + \mathscr{D}^{\varepsilon}\tilde{\phi}^{\varepsilon}]J^{\varepsilon}\, dzdS_y \\
&= -\frac{\varepsilon}{|\Omega|}\mathscr{I}^{\varepsilon}(t) + O(e^{-\eta/\varepsilon}),
\end{aligned}$$

since $\tilde{u}^{\varepsilon}$ and $\tilde{U}^{\varepsilon}$ satisfy (3.14) and $\tilde{\phi}^{\varepsilon} = O(e^{-\eta|z|})$ for some $\eta > 0$. We immediately have $N_3^{\varepsilon} = O(e^{-\eta/\varepsilon})$ since the solution u^{ε} is $O(e^{-\eta/\varepsilon})$-near of outer solutions $U^{\varepsilon,\pm}$ on $\Omega\setminus\Gamma(t)^{\delta}$. The relation $v^{\varepsilon}(t) = N_1^{\varepsilon} + N_2^{\varepsilon} + N_3^{\varepsilon}$ yields (3.33). $\square$

Using the expansion (3.16) and the definitions (3.13) and (3.34), we first expand J^ε and $\mathscr{P}^\varepsilon$ as

$$J^\varepsilon(t,z,y) = \sum_{i\ge 0} H_i(t,y)\left(\varepsilon z + \sum_{j\ge 1}\varepsilon^j R^j(t,y)\right)^i =: \sum_{j\ge 0}\varepsilon^j J^j(t,z,y), \tag{3.36}$$

$$\mathscr{P}^\varepsilon(t) = \sum_{i\ge 0}\int_{\mathscr{M}} \frac{H_i(t,y)}{i+1}\left(\sum_{j\ge 1}\varepsilon^j R^j(t,y)\right)^{i+1} dS_y =: \sum_{j\ge 0}\varepsilon^j \mathscr{P}^j(t), \tag{3.37}$$

where H_i $(0\le i\le N-1)$ are the same as in (2.3) and we define as $H_i\equiv 0$ for $i\ge N$. We immediately find in (3.36) and (3.37) that

$$J^0\equiv 1,\quad J^1 = \kappa R^1 + \kappa z,$$

$$\mathscr{P}^0\equiv 0,\quad \mathscr{P}^1 = \int_{\mathscr{M}} R^1\, dS_y.$$

For each $j\ge 2$, one can find that J^j and $\mathscr{P}^j$ depend on Γ, R^m $(1\le m\le j)$, including R^j as κR^j and $\int_{\mathscr{M}} R^j\, dS_y$, respectively. We expand $\mathscr{I}^\varepsilon$ in (3.35) as $\mathscr{I}^\varepsilon(t) = \sum_{j\ge 0}\varepsilon^j\mathscr{I}^j(t)$, where $\mathscr{I}^j$ $(j\ge 0)$ are given by

$$\begin{aligned}\mathscr{I}^0 :&= \int_{\mathscr{M}}\int_{-\infty}^{0}[\tilde\phi^{0,-}_{zz} + (\gamma_t\cdot\nu)\tilde\phi^{0,-}_z]J^0\, dzdS_y\\ &\quad+\int_{\mathscr{M}}\int_{0}^{\infty}[\tilde\phi^{0,+}_{zz} + (\gamma_t\cdot\nu)\tilde\phi^{0,+}_z]J^0\, dzdS_y,\end{aligned}\tag{3.38}$$

and for $j\ge 1$,

$$\begin{aligned}\mathscr{I}^j &= \int_{\mathscr{M}}\int_{-\infty}^{0}[\tilde\phi^{0,-}_{zz} + (\gamma_t\cdot\nu)\tilde\phi^{0,-}_z]J^j\, dzdS_y\\ &\quad+\int_{\mathscr{M}}\int_{0}^{\infty}[\tilde\phi^{0,+}_{zz} + (\gamma_t\cdot\nu)\tilde\phi^{0,+}_z]J^j\, dzdS_y\\ &\quad+\sum_{m=1}^{j}\int_{\mathscr{M}}\int_{-\infty}^{0}\left[\tilde\phi^{m,-}_{zz} + (\gamma_t\cdot\nu)\tilde\phi^{m,-}_z + \sum_{l=1}^{m} R^l_t\tilde\phi^{m-l,-}_z\right.\\ &\qquad\left.+\sum_{l=1}^{m}\mathscr{D}_l\tilde\phi^{m-l,-}\right]J^{j-m}\, dzdS_y\\ &\quad+\sum_{m=1}^{j}\int_{\mathscr{M}}\int_{0}^{\infty}\left[\tilde\phi^{m,+}_{zz} + (\gamma_t\cdot\nu)\tilde\phi^{m,+}_z + \sum_{l=1}^{m} R^l_t\tilde\phi^{m-l,+}_z\right.\\ &\qquad\left.+\sum_{l=1}^{m}\mathscr{D}_l\tilde\phi^{m-l,+}\right]J^{j-m}\, dzdS_y.\end{aligned}\tag{3.39}$$

Substituting the expansions for U^ε, $\mathscr{P}^\varepsilon$ and $\mathscr{I}^\varepsilon$ into (3.33), and equating the coefficient of each power of ε in the resulting equation, we obtain the following equations for $j \geq 0$:

$$\dot{U}^{j,-}|\Omega^-| + \dot{U}^{j,+}|\Omega^+| = \sum_{m=0}^{j} (\dot{U}^{m,+} - \dot{U}^{m,-})\mathscr{P}^{j-m} + \mathscr{I}^j. \tag{3.40}$$

3.4. C^1-matching. We show in this subsection that (Γ, v^0) and (R^j, v^j) can be chosen in such a way that the following C^1-matching conditions

$$\tilde{u}_z^{j,-}(t,0,y) = \tilde{u}_z^{j,+}(t,0,y) \tag{3.41}$$

and the equations (3.40) are both satisfied for all $j \geq 0$.

Let us first begin with $j = 0$ to determine (Γ, v^0). Since the problem (1.4) has the unique smooth solution pair $(Q(z;v), c(v))$ for $v \in I^v$, the equations (3.18) together with the boundary conditions (3.25) and (3.26) $(j = 0)$ have unique solutions which satisfy $\tilde{u}_z^{0,-}(t,0,y) = \tilde{u}_z^{0,+}(t,0,y)$ if and only if

$$\gamma_t \cdot v = c(v^0), \qquad v^0 \in I^v. \tag{3.42}$$

When (3.42) is satisfied, the unique solutions are given by

$$\tilde{u}^{0,\pm}(t,z,y) = Q(z; v^0(t)), \qquad \pm z \in [0,\infty). \tag{3.43}$$

Therefore, $\phi^{0,\pm}$ are given by (3.27). By these facts and (3.10), (3.38), $J^0 \equiv 1$ and $\mathscr{P}^0 \equiv 0$, the equation (3.40) $(j = 0)$ becomes

$$\begin{aligned}[h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|]\dot{v}^0 &= \int_{\mathscr{M}} \int_{-\infty}^{\infty} (Q_{zz} + cQ_z)\,dz\,dS_y \\ &= [h^+(v^0) - h^-(v^0)]c(v^0)|\Gamma|.\end{aligned}$$

Thanks to the fact that $h_v^\pm(v) < 0$ for $v \in I^v$ (cf. **(A2)**), this yields

$$\dot{v}^0 = h(v^0;\Gamma)c(v^0)|\Gamma|, \tag{3.44}$$

where $h(v;\Gamma)$ is the function in (1.8). Thus (3.40) and (3.41) $(j = 0)$ are recast as (3.42) and (3.44), which are nothing but the interface equation (IE). Once the initial condition $(\Gamma(0), v^0(0)) = (\Gamma_0, v_0^0)$ is given so that the assumptions in Theorem 1.1 are satisfied, (Γ, v^0) is uniquely determined on a finite time interval $[0,T]$.

We move on to determine (R^1, v^1). The equations (3.19) are recast as

$$\tilde{u}_{zz}^{1,\pm} + c\tilde{u}_z^{1,\pm} + f'(Q)\tilde{u}^{1,\pm} + (R_t^1 Q_z - v^1 + \tilde{\mathscr{F}}_1) = 0, \tag{3.45}$$

where $\tilde{\mathscr{F}}_1 := \kappa Q_z - Q_v \dot{v}^0$ by (3.21) and (3.43). Applying Lemma 3.2 to (3.45) and (3.25) with $j = 1$, we find that the equations (3.45), together with the boundary conditions (3.25), have unique solutions $\tilde{u}^{1,\pm}$ satisfying $\tilde{u}_z^{1,-}(t,0,y) = \tilde{u}_z^{1,+}(t,0,y)$ if and only if

$$\int_{-\infty}^{\infty} e^{cz} Q_z (R_t^1 Q_z - v^1 + \tilde{\mathscr{F}}_1) dz = 0.$$

This condition turns out to be equivalent to

$$R_t^1(t,y) = c'(v^0(t))v^1(t) + \rho_1(t,y), \tag{3.46}$$

where ρ_1 is a function depending only on (Γ, v^0) given explicitly by

$$\rho_1 := -\kappa + \frac{\int_{-\infty}^{\infty} e^{cz} Q_z Q_v \, dz}{\int_{-\infty}^{\infty} e^{cz} (Q_z)^2 dz} \dot{v}^0.$$

On the other hand, the equation (3.40) $(j = 1)$ is

$$\dot{U}^{1,-}|\Omega^-| + \dot{U}^{1,+}|\Omega^+| = (\dot{U}^{0,+} - \dot{U}^{0,-})\mathscr{P}^1 + \mathscr{I}^1 \tag{3.47}$$

with

$$\begin{aligned} \mathscr{I}^1 = & \int_{\mathscr{M}} \int_{-\infty}^{0} (\tilde{\phi}_{zz}^{0,-} + c\tilde{\phi}_z^{0,-})\kappa(R^1 + z) dz dS_y \\ & + \int_{\mathscr{M}} \int_{0}^{\infty} (\tilde{\phi}_{zz}^{0,+} + c\tilde{\phi}_z^{0,+})\kappa(R^1 + z) dz dS_y \\ & + \int_{\mathscr{M}} \int_{-\infty}^{0} [\tilde{\phi}_{zz}^{1,-} + c\tilde{\phi}_z^{1,-} + R_t^1 \tilde{\phi}_z^{0,-} + (\kappa\tilde{\phi}_z^{0,-} - \tilde{\phi}_t^{0,-})] dz dS_y \\ & + \int_{\mathscr{M}} \int_{0}^{\infty} [\tilde{\phi}_{zz}^{1,+} + c\tilde{\phi}_z^{1,+} + R_t^1 \tilde{\phi}_z^{0,+} + (\kappa\tilde{\phi}_z^{0,+} - \tilde{\phi}_t^{0,+})] dz dS_y \\ =: & \mathscr{I}_1^1 + \mathscr{I}_2^1 + \mathscr{I}_3^1 + \mathscr{I}_4^1. \end{aligned} \tag{3.48}$$

Employing (3.46) and $\tilde{\phi}_z^{1,-}(t,0,y) = \tilde{\phi}_z^{1,+}(t,0,y)$ (which are equivalent to the C^1-matching conditions $\tilde{u}_z^{1,-}(t,0,y) = \tilde{u}_z^{1,+}(t,0,y)$), we closely examine (3.47).

By the expression of outer solutions (3.11) $(j = 1)$, we find that the left hand side of (3.47) is expressed as follows:

$$\begin{aligned} \dot{U}^{1,-}|\Omega^-| + \dot{U}^{1,+}|\Omega^+| = & [h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|]\dot{v}^1 \\ & + [(h_{vv}^-(v^0)|\Omega^-| + h_{vv}^+(v^0)|\Omega^+|)\dot{v}^0]v^1 \\ & + [\dot{V}_1^-|\Omega^-| + \dot{V}_1^+|\Omega^+|]. \end{aligned}$$

In the right hand side of (3.47), the first term becomes

$$(\dot{U}^{0,+} - \dot{U}^{0,-})\mathscr{P}^1 = [h_v^+(v^0) - h_v^-(v^0)]\dot{v}^0 \int_{\mathscr{M}} R^1 \, dS_y.$$

For the second term $\mathscr{I}^1$, the term $\mathscr{I}_1^1 + \mathscr{I}_2^1$ in (3.48) is expressed as

$$\begin{aligned}
\mathscr{I}_1^1 + \mathscr{I}_2^1 &= \int_{\mathscr{M}} \int_{-\infty}^{\infty} (Q_{zz} + cQ_z)\kappa R^1 \, dzdS_y + \int_{\mathscr{M}} \int_{-\infty}^{\infty} (Q_{zz} + cQ_z)\kappa z \, dzdS_y \\
&= [h^+(v^0) - h^-(v^0)]c(v^0) \int_{\mathscr{M}} \kappa R^1 \, dS_y - [h^+(v^0) - h^-(v^0)] \int_{\mathscr{M}} \kappa \, dS_y \\
&\quad + c(v^0)\left(\int_{-\infty}^{\infty} zQ_z \, dz\right) \int_{\mathscr{M}} \kappa \, dS_y,
\end{aligned}$$

while $\mathscr{I}_3^1 + \mathscr{I}_4^1$ becomes

$$\begin{aligned}
\mathscr{I}_3^1 + \mathscr{I}_4^1 &= (U^{1,+} - U^{1,-})c(v^0)|\Gamma| + \int_{\mathscr{M}} \int_{-\infty}^{\infty} [(c'(v^0)v^1 + \rho_1) + \kappa]Q_z \, dzdS_y \\
&\quad - \int_{\mathscr{M}} \int_{-\infty}^{0} (Q - h^-(v^0))_t dzdS_y - \int_{\mathscr{M}} \int_{0}^{\infty} (Q - h^+(v^0))_t dzdS_y \\
&= [h^+(v^0) - h^-(v^0)]c'(v^0)|\Gamma|v^1 + [h_v^+(v^0) - h_v^-(v^0)]c(v^0)|\Gamma|v^1 \\
&\quad + (V_1^+ - V_1^-)c(v^0)|\Gamma| + [h^+(v^0) - h^-(v^0)] \frac{\int_{-\infty}^{\infty} e^{cz} Q_z Q_v \, dz}{\int_{-\infty}^{\infty} e^{cz} (Q_z)^2 dz} \dot{v}^0 |\Gamma| \\
&\quad - \left(\int_{-\infty}^{0} (Q_v - h_v^-(v^0))dz + \int_{0}^{\infty} (Q_v - h_v^+(v^0))dz\right) \dot{v}^0 |\Gamma|.
\end{aligned}$$

Then we arrive at the following equation:

$$\dot{v}^1(t) = \int_{\mathscr{M}} a(t,y)R^1(t,y)dS_y + b(t)v^1(t) + \sigma_1(t). \tag{3.49}$$

Here a and b are some functions depending only on (Γ, v^0), defined by

$$a := h(v^0;\Gamma)c(v^0)\kappa + \frac{h_v^+(v^0) - h_v^-(v^0)}{h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|} \dot{v}^0, \tag{3.50}$$

$$\begin{aligned}
b &:= h(v^0;\Gamma)c'(v^0)|\Gamma| \\
&\quad + \frac{(h_v^+(v^0) - h_v^-(v^0))c(v^0)|\Gamma| - (h_{vv}^-(v^0)|\Omega^-| + h_{vv}^+(v^0)|\Omega^+|)\dot{v}^0}{h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|}.
\end{aligned} \tag{3.51}$$

The term σ_1, also depending only on (Γ, v^0), is given by

$$
\begin{aligned}
\sigma_1 := & -h(v^0;\Gamma)\int_{\mathscr{M}} \kappa\, dS_y \\
& + [h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|]^{-1} \times \Bigg[c(v^0)\left(\int_{-\infty}^{\infty} zQ_z\, dz\right)\int_{\mathscr{M}} \kappa\, dS_y \\
& - \left(\int_{-\infty}^{0} (Q_v - h_v^-(v^0))dz + \int_0^{\infty} (Q_v - h_v^+(v^0))dz\right)\dot{v}^0|\Gamma| \\
& + (h^+(v^0) - h^-(v^0))\frac{\int_{-\infty}^{\infty} e^{cz} Q_z Q_v\, dz}{\int_{-\infty}^{\infty} e^{cz}(Q_z)^2 dz}\dot{v}^0|\Gamma| \\
& + (V_1^+ - V_1^-)c(v^0)|\Gamma| - (\dot{V}_1^-|\Omega^-| + \dot{V}_1^+|\Omega^+|)\Bigg].
\end{aligned}
$$

Hence the equations (3.40) and (3.41) ($j = 1$) are recast as (3.46) and (3.49).

Once the initial condition $(R^1(0, y), v^1(0)) = (R_0^1(y), v_0^1)$ is given, the equations (3.46) and (3.49) determines (R^1, v^1) uniquely. Indeed, this problem is reduced to an initial value problem for a system of linear non-homogeneous ordinary differential equations as follows. From (3.46), it turns out that $R_t^1(t, y) - \rho_1(t, y)$ is independent of $y \in \mathscr{M}$. Defining $\bar{R}^1(t)$ by

$$
\bar{R}^1(t) := R^1(t, y) - R_0^1(y) - \int_0^t \rho_1(s, y)ds \tag{3.52}
$$

and substituting this into (3.46) and (3.49), we obtain

$$
\begin{aligned}
\frac{d}{dt}\bar{R}^1(t) &= c'(v^0(t))v^1(t) =: B(t)v^1(t), \\
\frac{d}{dt}v^1(t) &= \left(\int_{\mathscr{M}} a(t, y)dS_y\right)\bar{R}^1(t) + b(t)v^1(t) \\
&\quad + \int_{\mathscr{M}} a(t, y)\left(R_0^1(y) + \int_0^t \rho_1(s, y)ds\right)dS_y + \sigma_1(t) \\
&=: C(t)\bar{R}^1(t) + D(t)v^1(t) + E_1(t).
\end{aligned}
$$

Therefore our problem is now rewritten as

$$
\begin{cases}
\dfrac{d\bar{R}^1(t)}{dt} = B(t)v^1(t), \\
\dfrac{dv^1(t)}{dt} = C(t)\bar{R}^1(t) + D(t)v^1(t) + E_1(t), \\
\bar{R}^1(0) = 0, \quad v^1(0) = v_0^1.
\end{cases} \tag{3.53}
$$

This problem has a unique solution pair $(\bar{R}^1, v^1)$ on the time interval $[0, T]$, from which and (3.52) the pair (R^1, v^1) is uniquely determined. Consequently, $\tilde{u}^{1,\pm}$ are smoothly joined at $z = 0$ and give rise to the function $\tilde{u}^1$ defined by $\tilde{u}^1(z) = \tilde{u}^{1,\pm}(z)$ $(\pm z \in [0, \infty))$.

We will show that the same procedure as above works for each $j \geq 2$, namely, the pair (R^j, v^j) is chosen so that the condition $\tilde{u}_z^{j,-}(t, 0, y) = \tilde{u}_z^{j,+}(t, 0, y)$ and the j-th equation in (3.40) are satisfied. Let us assume that (R^m, v^m) $(1 \leq m < j)$ have been already determined in order to join $\tilde{u}^{m,\pm}$ smoothly at $z = 0$. We now deal with the j-th equation in (3.20). Since the functions $\tilde{\mathscr{F}}_j^{\pm}$ in (3.22) depend only on Γ, R^m $(1 \leq m < j)$ and $\tilde{u}^{m,\pm}$ $(0 \leq m < j)$, they give rise to the smooth known function $\tilde{\mathscr{F}}_j$ defined on $(-\infty, \infty)$. Thus the equations (3.20) are recast as

$$\tilde{u}_{zz}^{j,\pm} + c\tilde{u}_z^{j,\pm} + f'(Q)\tilde{u}^{j,\pm} + (R_t^j Q_z - v^j + \tilde{\mathscr{F}}_j) = 0. \tag{3.54}$$

Applying Lemma 3.2 to (3.54) and (3.25), we find that the equations (3.54) with (3.25) have unique solutions $\tilde{u}^{j,\pm}$ satisfying $\tilde{u}_z^{j,-}(t, 0, y) = \tilde{u}_z^{j,+}(t, 0, y)$ if and only if

$$\int_{-\infty}^{\infty} e^{cz} Q_z (R_t^j Q_z - v^j + \tilde{\mathscr{F}}_j) dz = 0.$$

This condition is equivalent to

$$R_t^j(t, y) = c'(v^0(t))v^j(t) + \rho_j(t, y), \tag{3.55}$$

where ρ_j is computed in terms of functions which have been already known.

Let us next rewrite the j-th equation in (3.40) by employing (3.55) and the condition $\tilde{\phi}_z^{j,-}(t, 0, y) = \tilde{\phi}_z^{j,+}(t, 0, y)$. For this purpose, only the terms including the pair (R^j, v^j) are expressed explicitly, and we simply denote by "$\cdots$" the other terms with indices less than j. Then the left hand side of (3.40) is represented, by (3.11), as

$$\begin{aligned}\dot{U}^{j,-}|\Omega^-| + \dot{U}^{j,+}|\Omega^+| = {} & [h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|]\dot{v}^j \\ & + [(h_{vv}^-(v^0)|\Omega^-| + h_{vv}^+(v^0)|\Omega^+|)\dot{v}^0]v^j + \cdots.\end{aligned}$$

In the right hand side of (3.40), the first term is expressed, by $\mathscr{P}^j = \int_{\mathscr{M}} R^j \, dS_y + \cdots$, as

$$\sum_{m=0}^{j} (\dot{U}^{m,+} - \dot{U}^{m,-})\mathscr{P}^{j-m} = [h_v^+(v^0) - h_v^-(v^0)]\dot{v}^0 \int_{\mathscr{M}} R^j \, dS_y + \cdots,$$

while $\mathscr{I}^j$ has the following expression thanks to (3.39) and $J^j = \kappa R^j + \cdots$:

$$\begin{aligned}
\mathscr{I}^j &= \int_{\mathscr{M}}\int_{-\infty}^{0}(\tilde{\phi}_{zz}^{0,-}+c\tilde{\phi}_{z}^{0,-})J^j\,dzdS_y+\int_{\mathscr{M}}\int_{0}^{\infty}(\tilde{\phi}_{zz}^{0,+}+c\tilde{\phi}_{z}^{0,+})J^j\,dzdS_y\\
&\quad+\int_{\mathscr{M}}\int_{-\infty}^{0}(\tilde{\phi}_{zz}^{j,-}+c\tilde{\phi}_{z}^{j,-}+R_t^j\tilde{\phi}_{z}^{0,-}+\cdots)J^0\,dzdS_y\\
&\quad+\int_{\mathscr{M}}\int_{0}^{\infty}(\tilde{\phi}_{zz}^{j,+}+c\tilde{\phi}_{z}^{j,+}+R_t^j\tilde{\phi}_{z}^{0,+}+\cdots)J^0\,dzdS_y+\cdots\\
&=\int_{\mathscr{M}}\int_{-\infty}^{0}(\tilde{\phi}_{zz}^{0,-}+c(v^0)\tilde{\phi}_{z}^{0,-})(\kappa R^j+\cdots)dzdS_y\\
&\quad+\int_{\mathscr{M}}\int_{0}^{\infty}(\tilde{\phi}_{zz}^{0,+}+c(v^0)\tilde{\phi}_{z}^{0,+})(\kappa R^j+\cdots)dzdS_y\\
&\quad+\int_{\mathscr{M}}\int_{-\infty}^{0}(\tilde{\phi}_{zz}^{j,-}+c(v^0)\tilde{\phi}_{z}^{j,-}+c'(v^0)v^j\tilde{\phi}_{z}^{0,-}+\cdots)dzdS_y\\
&\quad+\int_{\mathscr{M}}\int_{0}^{\infty}(\tilde{\phi}_{zz}^{j,+}+c(v^0)\tilde{\phi}_{z}^{j,+}+c'(v^0)v^j\tilde{\phi}_{z}^{0,+}+\cdots)dzdS_y+\cdots\\
&=[h^+(v^0)-h^-(v^0)]c(v^0)\int_{\mathscr{M}}\kappa R^j\,dS_y\\
&\quad+[h^+(v^0)-h^-(v^0)]c'(v^0)|\Gamma|v^j+[h_v^+(v^0)-h_v^-(v^0)]c(v^0)|\Gamma|v^j+\cdots.
\end{aligned}$$

Hence we have the following equation from (3.40):

$$\dot{v}^j(t)=\int_{\mathscr{M}}a(t,y)R^j(t,y)dS_y+b(t)v^j(t)+\sigma_j(t). \tag{3.56}$$

Here a, b are the functions as in (3.50) and (3.51), while σ_j is a function calculated by using functions which have been already determined.

By the same argument for $j=1$ as above, the equations (3.55) and (3.56) with an initial condition $(R^j(0,y),v^j(0))=(R_0^j(y),v_0^j)$ are recast as the following linear non-homogeneous equations for $v^j(t)$ and

$$\bar{R}^j(t):=R^j(t,y)-R_0^j(y)-\int_0^t\rho_j(s,y)ds:$$

$$\begin{cases}\dfrac{d\bar{R}^j(t)}{dt}=B(t)v^j(t),\\[2mm] \dfrac{dv^j(t)}{dt}=C(t)\bar{R}^j(t)+D(t)v^j(t)+E_j(t),\\[2mm] \bar{R}^j(0)=0,\quad v^j(0)=v_0^j.\end{cases} \tag{3.57}$$

Here the coefficient functions B, C and D are all the same as in (3.53) and the non-homogeneous term E_j can be treated as a known function. Therefore, once the initial condition $(R^j(0,y), v^j(0)) = (R_0^j(y), v_0^j)$ is given, the problem (3.57) determines $(\bar{R}^j, v^j)$ and (R^j, v^j) uniquely on the time interval $[0,T]$.

3.5. Uniform approximation. We are now ready to construct an approximate solution u_A^ε. For each $k \geq 1$, we determine (Γ, v^0), $(R^1, v^1), \dots, (R^k, v^k)$ by the procedure described in the previous subsection, and define

$$R_A^\varepsilon(t,y) := R^1(t,y) + \varepsilon R^2(t,y) + \cdots + \varepsilon^{k-1} R^k(t,y),$$

$$v_A^\varepsilon(t) := \sum_{j=0}^{k} \varepsilon^j v^j(t). \tag{3.58}$$

We also define the approximate interface $\Gamma_A^\varepsilon(t)$ by

$$\Gamma_A^\varepsilon(t) := \{x \in \Omega \mid x = \gamma(t,y) + \varepsilon R_A^\varepsilon(t,y)\nu(t,y),\ y \in \mathscr{M}\},$$

and denote by $\Omega_A^{\varepsilon,\pm}(t)$ the subregions divided by $\Gamma_A^\varepsilon(t)$ as $\Omega = \Omega_A^{\varepsilon,-}(t) \cup \Gamma_A^\varepsilon(t) \cup \Omega_A^{\varepsilon,+}(t)$. Using the functions $U^{j,\pm}$ and $\tilde{\phi}^{j,\pm}$ $(j = 0,\dots,k)$ determined in the outer and inner expansions, we set $U_A^{\varepsilon,\pm}$ and $\tilde{\phi}_A^{\varepsilon,\pm}$ as

$$U_A^{\varepsilon,\pm}(t) := \sum_{j=0}^{k} \varepsilon^j U^{j,\pm}(t), \qquad \tilde{\phi}_A^{\varepsilon,\pm}(t,z,y) := \sum_{j=0}^{k} \varepsilon^j \tilde{\phi}^{j,\pm}(t,z,y).$$

Let $\Theta(r)$ be a smooth cut-off function defined by

$$\Theta(r) := \begin{cases} 1, & |r| \leq \delta/2, \\ 0, & |r| \geq \delta, \end{cases} \qquad 0 \leq \Theta \leq 1,$$

and $(r(t,x), y(t,x)) \in (-\delta,\delta) \times \mathscr{M}$ the inverse map of $\mathbf{F}(t,\cdot,\cdot)$ in (2.2). We now define our approximate solution u_A^ε on $[0,T] \times \bar{\Omega}$ as follows:

$$u_A^\varepsilon(t,x) := U_A^\varepsilon(t,x) + \phi_A^\varepsilon(t,x)\Theta(r(t,x)), \qquad (t,x) \in [0,T] \times \bar{\Omega}. \tag{3.59}$$

Here U_A^ε and ϕ_A^ε are given by

$$U_A^\varepsilon(t,x) := U_A^{\varepsilon,\pm}(t), \qquad x \in \Omega_A^{\varepsilon,\pm}(t),$$

$$\phi_A^\varepsilon(t,x) := \tilde{\phi}_A^{\varepsilon,\pm}\left(t, \frac{r(t,x)}{\varepsilon} - R_A^\varepsilon(t, y(t,x)), y(t,x)\right).$$

$\phi_A^\varepsilon(t,x)$ is also denoted by $\tilde{\phi}_A^\varepsilon(t,z,y)$ in terms of (t,z,y), where $z = \varepsilon^{-1}[r - \varepsilon R_A^\varepsilon(t,y)]$:

$$\phi_A^\varepsilon(t,x) = \tilde{\phi}_A^{\varepsilon,\pm}\left(t, \frac{r}{\varepsilon} - R_A^\varepsilon(t,y), y\right) = \tilde{\phi}_A^\varepsilon(t,z,y).$$

We define J_A^ε, $\mathscr{D}_A^\varepsilon, \dots$ by the same formula as J^ε, $\mathscr{D}^\varepsilon, \dots$ with R^ε being replaced by R_A^ε. In particular, $\mathscr{P}_A^\varepsilon$ and $\mathscr{I}_A^\varepsilon$ are defined as

$$\mathscr{P}_A^\varepsilon := \sum_{i \geq 0} \int_{\mathscr{M}} \frac{H_i}{i+1} (\varepsilon R_A^\varepsilon)^{i+1} dS_y,$$

$$\mathscr{I}_A^\varepsilon := \int_{\mathscr{M}} \int_{-\infty}^{\infty} [(\tilde{\phi}_A^\varepsilon)_{zz} + (\gamma_t \cdot v)(\tilde{\phi}_A^\varepsilon)_z + \varepsilon (R_A^\varepsilon)_t (\tilde{\phi}_A^\varepsilon)_z + \mathscr{D}_A^\varepsilon \tilde{\phi}_A^\varepsilon] J_A^\varepsilon \, dz dS_y.$$

By our way of construction and the Taylor-Cauchy formula with integral remainder

$$m(\varepsilon) = \sum_{j=0}^{k} \varepsilon^j \frac{1}{j!} m^{(j)}(0) + \varepsilon^{k+1} \int_0^1 \frac{1}{k!} (1-s)^k m^{(k+1)}(s\varepsilon) ds,$$

we can easily find that the following approximations are valid:

(i) Outer approximation (cf. §3.1):

$$\max_{[0,T]} \left\| \varepsilon \frac{\partial U_A^\varepsilon}{\partial t} - \varepsilon^2 \Delta U_A^\varepsilon - f(U_A^\varepsilon) + v_A^\varepsilon \right\|_{L^\infty(\Omega)} = O(\varepsilon^{k+1}), \tag{3.60}$$

(ii) Inner approximation (cf. §3.2):

$$\max_{[0,T]} \left\| \varepsilon \frac{\partial u_A^\varepsilon}{\partial t} - \varepsilon^2 \Delta u_A^\varepsilon - f(u_A^\varepsilon) + v_A^\varepsilon \right\|_{L^\infty(\Gamma^{\delta/2})} = O(\varepsilon^{k+1}), \tag{3.61}$$

(iii) Approximation of nonlocal relation (cf. §3.3):

$$\max_{[0,T]} |\dot{U}_A^{\varepsilon,-}|\Omega^-| + \dot{U}_A^{\varepsilon,+}|\Omega^+| - (\dot{U}_A^{\varepsilon,+} - \dot{U}_A^{\varepsilon,-})\mathscr{P}_A^\varepsilon - \mathscr{I}_A^\varepsilon| = O(\varepsilon^{k+1}). \tag{3.62}$$

Using these results, we have the following

Lemma 3.4. *Let u_A^ε, v_A^ε be the functions defined as in* (3.59), (3.58), *respectively. Then the following estimates are valid:*

$$\max_{[0,T]} \left\| \varepsilon \frac{\partial u_A^\varepsilon}{\partial t} - \varepsilon^2 \Delta u_A^\varepsilon - f(u_A^\varepsilon) + v_A^\varepsilon \right\|_{L^\infty(\Omega)} = O(\varepsilon^{k+1}), \tag{3.63}$$

$$\max_{[0,T]} \left| v_A^\varepsilon - \frac{1}{|\Omega|} \int_\Omega f(u_A^\varepsilon) dx \right| = O(\varepsilon^{k+1}). \tag{3.64}$$

Proof. Let us first prove (3.63). Since $u_A^\varepsilon(t,x) = U_A^\varepsilon(t,x)$ on $\Omega \backslash \Gamma(t)^\delta$, the estimate (3.60) immediately yields that

$$\max_{[0,T]}\left\|\varepsilon\frac{\partial u_A^\varepsilon}{\partial t}-\varepsilon^2\Delta u_A^\varepsilon-f(u_A^\varepsilon)+v_A^\varepsilon\right\|_{L^\infty(\Omega\setminus\Gamma^\delta)}=O(\varepsilon^{k+1}). \tag{3.65}$$

In $\Gamma(t)^\delta\setminus\Gamma(t)^{\delta/2}$, we can compute as

$$\begin{aligned}
&\varepsilon^2\Delta u_A^\varepsilon+f(u_A^\varepsilon)-v_A^\varepsilon-\varepsilon\frac{\partial u_A^\varepsilon}{\partial t}\\
&\quad=\varepsilon^2\Delta U_A^\varepsilon+f(U_A^\varepsilon)-v_A^\varepsilon-\varepsilon\frac{\partial U_A^\varepsilon}{\partial t}\\
&\qquad+\varepsilon^2\Delta(\phi_A^\varepsilon\Theta(r))+f(U_A^\varepsilon+\phi_A^\varepsilon\Theta(r))-f(U_A^\varepsilon)-\varepsilon\frac{\partial(\phi_A^\varepsilon\Theta(r))}{\partial t}\\
&\quad=\varepsilon^2\Delta U_A^\varepsilon+f(U_A^\varepsilon)-v_A^\varepsilon-\varepsilon\frac{\partial U_A^\varepsilon}{\partial t}\\
&\qquad+\varepsilon^2\Theta(r)\Delta\phi_A^\varepsilon+2\varepsilon^2\Theta'(r)\nabla_\nu\phi_A^\varepsilon+\varepsilon^2\phi_A^\varepsilon(\Theta''(r)+\Theta'(r)\Delta r)\\
&\qquad+\phi_A^\varepsilon\Theta(r)\int_0^1 f'(U_A^\varepsilon+s\phi_A^\varepsilon\Theta(r))ds+\varepsilon\Theta'(r)\phi_A^\varepsilon\mathbf{v}-\varepsilon\Theta(r)\frac{\partial\phi_A^\varepsilon}{\partial t},
\end{aligned}$$

where the following identities are employed:

$$r_t(t,x)=-\mathbf{v}(x;\Gamma(t)),\qquad \nabla r(t,x)=\nu(x;\Gamma(t)).$$

By the estimate (3.60) and the fact that ϕ_A^ε and its derivatives with respect to t, x of any order are $O(e^{-\eta/\varepsilon})$ for some $\eta>0$, we obtain

$$\max_{[0,T]}\left\|\varepsilon\frac{\partial u_A^\varepsilon}{\partial t}-\varepsilon^2\Delta u_A^\varepsilon-f(u_A^\varepsilon)+v_A^\varepsilon\right\|_{L^\infty(\Gamma^\delta\setminus\Gamma^{\delta/2})}=O(\varepsilon^{k+1}). \tag{3.66}$$

Combining (3.61), (3.65) and (3.66) together, we obtain (3.63).

Let us next prove (3.64). The terms in the left hand side of (3.64) are recast as

$$\begin{aligned}
\frac{1}{|\Omega|}\int_\Omega f(u_A^\varepsilon)dx-v_A^\varepsilon&=\frac{1}{|\Omega|}\int_\Omega[f(U_A^\varepsilon)-v_A^\varepsilon]dx+\frac{1}{|\Omega|}\int_\Omega[f(u_A^\varepsilon)-f(U_A^\varepsilon)]dx\\
&=\frac{1}{|\Omega|}\int_\Omega[f(U_A^\varepsilon)-v_A^\varepsilon]dx+\frac{1}{|\Omega|}\int_{\Gamma^{\delta/2}}[f(u_A^\varepsilon)-f(U_A^\varepsilon)]dx\\
&\quad+\frac{1}{|\Omega|}\int_{\Gamma^\delta\setminus\Gamma^{\delta/2}}[f(u_A^\varepsilon)-f(U_A^\varepsilon)]dx\\
&\quad+\frac{1}{|\Omega|}\int_{\Omega\setminus\Gamma^\delta}[f(u_A^\varepsilon)-f(U_A^\varepsilon)]dx\\
&=:N_A^1+N_A^2+N_A^3+N_A^4.
\end{aligned}$$

Using (3.60), (3.62) and (3.63), we treat $N_A^1 + N_A^2$ as follows.

$$
\begin{aligned}
&N_A^1 + N_A^2 \\
&\quad = \frac{1}{|\Omega|}\int_{\Omega}[\varepsilon(U_A^\varepsilon)_t - \varepsilon^2 \Delta U_A^\varepsilon]dx + \frac{1}{|\Omega|}\int_{\Gamma^{\delta/2}}[\varepsilon(\phi_A^\varepsilon)_t - \varepsilon^2 \Delta \phi_A^\varepsilon]dx + O(\varepsilon^{k+1}) \\
&\quad = \frac{\varepsilon}{|\Omega|}(\dot{U}_A^{\varepsilon,-}|\Omega_A^{\varepsilon,-}| + \dot{U}_A^{\varepsilon,+}|\Omega_A^{\varepsilon,+}|) \\
&\qquad - \frac{\varepsilon}{|\Omega|}\int_{\mathscr{M}}\int_{-\delta/2\varepsilon - R_A^\varepsilon}^{\delta/2\varepsilon - R_A^\varepsilon}[(\tilde{\phi}_A^\varepsilon)_{zz} + (\gamma_t \cdot v)(\tilde{\phi}_A^\varepsilon)_z + \varepsilon(R_A^\varepsilon)_t(\tilde{\phi}_A^\varepsilon)_z + \mathscr{D}_A^\varepsilon \tilde{\phi}_A^\varepsilon]J_A^\varepsilon \, dz dS_y \\
&\qquad + O(\varepsilon^{k+1}) \\
&\quad = \frac{\varepsilon}{|\Omega|}[\dot{U}_A^{\varepsilon,-}|\Omega^-| + \dot{U}_A^{\varepsilon,+}|\Omega^+| - (\dot{U}_A^{\varepsilon,+} - \dot{U}_A^{\varepsilon,-})\mathscr{P}_A^\varepsilon - \mathscr{I}_A^\varepsilon] + O(e^{-\eta/\varepsilon}) \\
&\qquad + O(\varepsilon^{k+1}) \\
&\quad = O(\varepsilon^{k+2}) + O(e^{-\eta/\varepsilon}) + O(\varepsilon^{k+1}) \\
&\quad = O(\varepsilon^{k+1}).
\end{aligned}
$$

On the other hand, N_A^3 is computed as

$$
\begin{aligned}
N_A^3 &= \frac{1}{|\Omega|}\int_{\Gamma^\delta \setminus \Gamma^{\delta/2}}[f(U_A^\varepsilon + \phi_A^\varepsilon \Theta(r)) - f(U_A^\varepsilon)]dx \\
&= \frac{1}{|\Omega|}\int_{\Gamma^\delta \setminus \Gamma^{\delta/2}} \phi_A^\varepsilon \Theta(r)\left(\int_0^1 f'(U_A^\varepsilon + s\phi_A^\varepsilon \Theta(r))ds\right)dx \\
&= O(e^{-\eta/\varepsilon}),
\end{aligned}
$$

and $N_A^4 \equiv 0$ since $u_A^\varepsilon(t,x) = U_A^\varepsilon(t,x)$ on $\Omega \setminus \Gamma(t)^\delta$. Therefore, we obtain (3.64). □

It is easily verified that our approximate solution u_A^ε is smooth on $\bar{\Omega}$. From Lemma 3.4, we immediately obtain (1.10). It also turns out that u_A^ε satisfies the boundary conditions (1.11) since $u_A^\varepsilon(t,x) = U_A^{\varepsilon,\pm}(t)$ on $\bar{\Omega}\setminus\Gamma(t)^\delta$. Furthermore, we can verify that

$$
\lim_{\varepsilon \to 0} u_A^\varepsilon = h^\pm(v^0) \qquad \text{uniformly on } \bar{\Omega}_T^\pm \setminus \Gamma_T^\delta,
$$

in which (Γ, v^0) is the solution of (IE) on $[0,T]$. This means that (1.12) is satisfied, and therefore our u_A^ε defined as in (3.59) is the desired approximate solution. This completes the proof of Proposition 1.3. □

Remark. The spatial average of the approximate solution u_A^ε above is *not* preserved. However, it is *approximately* preserved in the following sense:

$$\frac{1}{|\Omega|}\int_\Omega u_A^\varepsilon(t,x)dx = \frac{1}{|\Omega|}\int_\Omega u_A^\varepsilon(0,x)dx + O(\varepsilon^k), \qquad t \in [0,T].$$

4. Estimates for perturbation

In this section, we prove Proposition 1.4 based on the idea presented in [11, 20]. For each $t \in [0,T]$ fixed, let $\mathscr{L}^\varepsilon(t)$ be the linearized operator of (RD) arround the approximate solution u_A^ε obtained in § 3:

$$\mathscr{L}^\varepsilon(t)\varphi := \varepsilon\Delta\varphi + \varepsilon^{-1}[f'(u_A^\varepsilon(t,\cdot))\varphi - \langle f'(u_A^\varepsilon(t,\cdot))\varphi\rangle].$$

Here the symbol $\langle\cdot\rangle$ stands for the statial average over Ω. We rescale the time t in $\mathscr{L}^\varepsilon(t)$ by $t = \varepsilon^2\tau$ and seek a true solution u^ε of (RD) as follows:

$$u^\varepsilon(\varepsilon^2\tau,\cdot) = u_A^\varepsilon(\varepsilon^2\tau,\cdot) + \varphi^\varepsilon(\tau)(\cdot), \qquad \tau \in [0, T/\varepsilon^2].$$

Our equation (RD) is recast as an evolution equation for $\varphi^\varepsilon(\tau)$

$$\dot\varphi^\varepsilon(\tau) = \mathscr{A}^\varepsilon(\tau)\varphi^\varepsilon(\tau) + \mathscr{N}^\varepsilon(\tau,\varphi^\varepsilon(\tau)) + \mathscr{R}^\varepsilon(\tau), \tag{4.1}$$

where dot "·" stands for the differentiation with respect to τ and

$$\begin{aligned}
\mathscr{A}^\varepsilon(\tau)\varphi &:= \varepsilon^2\mathscr{L}^\varepsilon(\varepsilon^2\tau)\varphi = \varepsilon^3\Delta\varphi + \varepsilon[f'(u_A^\varepsilon(\varepsilon^2\tau,\cdot))\varphi - \langle f'(u_A^\varepsilon(\varepsilon^2\tau,\cdot))\varphi\rangle],\\
\mathscr{N}^\varepsilon(\tau,\varphi) &:= \varepsilon[f(u_A^\varepsilon(\varepsilon^2\tau,\cdot)+\varphi) - f(u_A^\varepsilon(\varepsilon^2\tau,\cdot)) - f'(u_A^\varepsilon(\varepsilon^2\tau,\cdot))\varphi\\
&\qquad - \langle f(u_A^\varepsilon(\varepsilon^2\tau,\cdot)+\varphi) - f(u_A^\varepsilon(\varepsilon^2\tau,\cdot)) - f'(u_A^\varepsilon(\varepsilon^2\tau,\cdot))\varphi\rangle],\\
\mathscr{R}^\varepsilon(\tau) &:= \varepsilon\left[\varepsilon^2\Delta u_A^\varepsilon(\varepsilon^2\tau,\cdot) + f(u_A^\varepsilon(\varepsilon^2\tau,\cdot)) - \langle f(u_A^\varepsilon(\varepsilon^2\tau,\cdot))\rangle - \varepsilon\frac{\partial u_A^\varepsilon}{\partial t}(\varepsilon^2\tau,\cdot)\right].
\end{aligned}$$

Notice that the following estimates are valid for $\tau \in [0, T/\varepsilon^2]$:

$$\mathscr{N}^\varepsilon(\tau,\varphi)/\varepsilon = O(|\varphi|^2), \tag{4.2}$$

$$\|\mathscr{R}^\varepsilon(\tau)\|_{L^\infty(\Omega)} = O(\varepsilon^{k+2}). \tag{4.3}$$

The unique existence of smooth solutions to (4.1) is known, and therefore our task is only to have an estimate on $\|\varphi^\varepsilon(\tau)\|_{L^\infty(\Omega)}$.

We now decompose a solution $\varphi^\varepsilon(\tau)$ to (4.1) as

$$\varphi^\varepsilon(\tau)(\cdot) = \varphi_1^\varepsilon(\tau)(\cdot) + \varphi_2^\varepsilon(\tau), \tag{4.4}$$

in which $\varphi_1^\varepsilon(\tau)$ satisfies $\langle \varphi_1^\varepsilon(\tau)\rangle \equiv 0$ while $\varphi_2^\varepsilon(\tau) \in \mathbf{R}$ is spatially homogeneous, and also decompose the equation (4.1) for $\varphi^\varepsilon(\tau)$ as an *evolution equation* for $\varphi_1^\varepsilon(\tau)$ and an *ordinary differential equation* for $\varphi_2^\varepsilon(\tau)$:

$$\dot{\varphi}_1^\varepsilon(\tau) = \mathscr{A}^\varepsilon(\tau)\varphi_1^\varepsilon(\tau) + \mathscr{N}^\varepsilon(\tau, \varphi_1^\varepsilon(\tau) + \varphi_2^\varepsilon(\tau)) + \bar{\mathscr{R}}^\varepsilon(\tau, \varphi_2^\varepsilon(\tau)), \tag{4.5}$$

$$\dot{\varphi}_2^\varepsilon(\tau) = \langle \mathscr{R}^\varepsilon(\tau)\rangle. \tag{4.6}$$

Here $\bar{\mathscr{R}}^\varepsilon(\tau, \varphi_2)$ is given by

$$\begin{aligned}\bar{\mathscr{R}}^\varepsilon(\tau, \varphi_2) &:= \mathscr{R}^\varepsilon(\tau) - \langle \mathscr{R}^\varepsilon(\tau)\rangle + \mathscr{A}^\varepsilon(\tau)\varphi_2 \\ &= \mathscr{R}^\varepsilon(\tau) - \langle \mathscr{R}^\varepsilon(\tau)\rangle + \varepsilon[f'(u_A^\varepsilon(\varepsilon^2\tau, \cdot)) - \langle f'(u_A^\varepsilon(\varepsilon^2\tau, \cdot))\rangle]\varphi_2\end{aligned} \tag{4.7}$$

with φ_2 being spatially homogeneous.

4.1. Preliminaries. In order to deal with the evolution equation (4.5), let us now set up appropriate function spaces.

Let $p \geq 2$ and we define the basic space X_0^ε and the domain X_1^ε of $\mathscr{A}^\varepsilon(\tau)$ by

$$X_0^\varepsilon := L^p(\Omega) \cap \mathbf{M}, \qquad X_1^\varepsilon := W^{2,p}_{\varepsilon,\mathscr{B}}(\Omega) \cap \mathbf{M},$$

where $\mathbf{M}$ stands for the space consisting of average-zero functions, and $W^{2,p}_{\varepsilon,\mathscr{B}}(\Omega)$ is the same as the usual Sobolev space

$$W^{2,p}_{\mathscr{B}}(\Omega) := \{u \in W^{2,p}(\Omega) \mid \partial u/\partial \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

as a set, with the weighted norm

$$\|u\|_{W^{2,p}_{\varepsilon,\mathscr{B}}(\Omega)} := (\|u\|^p_{L^p(\Omega)} + (\varepsilon^{3/2}\|Du\|_{L^p(\Omega)})^p + (\varepsilon^3\|D^2u\|_{L^p(\Omega)})^p)^{1/p}. \tag{4.8}$$

For $\alpha \in (0,1)$, let X_α^ε be the real interpolation spaces between X_0^ε and X_1^ε (cf. [5, 22]):

$$X_\alpha^\varepsilon := (X_0^\varepsilon, X_1^\varepsilon)_{\alpha,p} = (L^p(\Omega), W^{2,p}_{\varepsilon,\mathscr{B}}(\Omega))_{\alpha,p} \cap \mathbf{M},$$

where $(\cdot,\cdot)_{\alpha,p}$ stands for the standard real interpolation method (functor). Thanks to the weighted norm (4.8), X_α^ε enjoy some continuous embedding properties with embedding constants being *independent* of $\varepsilon > 0$:

$$0 \leq \alpha < \beta \leq 1 \Rightarrow u \in X_\beta^\varepsilon \hookrightarrow X_\alpha^\varepsilon, \qquad \|u\|_\alpha \leq M\|u\|_\beta.$$

We note that the spaces $(L^p(\Omega), W^{2,p}_{\varepsilon,\mathscr{B}}(\Omega))_{\alpha,p}$ have the following characterization *as sets* (cf. [22], Theorem 4.3.3):

$$(L^p(\Omega), W^{2,p}_{\varepsilon,\mathscr{B}}(\Omega))_{\alpha,p} = B^{2\alpha}_{p,p,\mathscr{B}}(\Omega), \qquad 0 \leq \alpha \leq 1 \ \left(\alpha \neq \frac{1}{2} + \frac{1}{2p}\right). \tag{4.9}$$

Here $B^s_{p,p,\mathscr{B}}(\Omega)$ $(0 \le s \le 2)$ stand for the spaces defined by

$$B^s_{p,p,\mathscr{B}}(\Omega) := \begin{cases} \{u \in B^s_{p,p}(\Omega) \mid \partial u/\partial \mathbf{n} = 0 \text{ on } \partial\Omega\}, & 1 + 1/p < s \le 2, \\ B^s_{p,p}(\Omega), & 0 \le s < 1 + 1/p, \end{cases}$$

with $B^s_{p,p}(\Omega)$ being the usual Besov spaces. As for the characterization in the case where $\alpha = 1/2 + 1/2p$, we refer to [22].

We also set up weighted Hölder spaces $C^\alpha_{\varepsilon,p}(\bar{\Omega})$ for $\alpha \in (0,1)$, which plays an important role in the treatment of the nonlinear term $\mathcal{N}^\varepsilon(\tau,\varphi)$ in (4.5). These spaces are the same as the usual Hölder spaces $C^\alpha(\bar{\Omega})$ as sets, with the weighted norm

$$\|u\|_{C^\alpha_{\varepsilon,p}(\bar{\Omega})} := (\varepsilon^{3/2})^{N/p}\|u\|_{L^\infty(\Omega)} + (\varepsilon^{3/2})^{\alpha+N/p}[u]_{C^\alpha(\bar{\Omega})}, \tag{4.10}$$

where

$$[u]_{C^\alpha(\bar{\Omega})} := \sup_{x,x' \in \bar{\Omega}, x \neq x'} \frac{|u(x) - u(x')|}{|x - x'|^\alpha}.$$

In particular, if the relation $2\alpha - N/p > \beta$ is valid for some $\alpha, \beta \in (0,1)$, then X^ε_α is continuously embedded in $C^\beta_{\varepsilon,p}(\bar{\Omega})$ with embedding constants being *independent* of $\varepsilon > 0$ (thanks to the weighted norms):

$$2\alpha - \frac{N}{p} > \beta \Rightarrow u \in X^\varepsilon_\alpha \hookrightarrow C^\beta_{\varepsilon,p}(\bar{\Omega}), \qquad \|u\|_{C^\beta_{\varepsilon,p}(\bar{\Omega})} \le M\|u\|_\alpha. \tag{4.11}$$

4.2. Proof of Proposition 1.4. Let us first treat the ordinary differential equation (4.6) together with appropriate initial data $\varphi^\varepsilon_2(0)$. We immediately find that $\varphi^\varepsilon_2(\tau)$ is uniquely determined as

$$\varphi^\varepsilon_2(\tau) = \varphi^\varepsilon_2(0) + \int_0^\tau \langle \mathscr{R}^\varepsilon(\sigma) \rangle d\sigma. \tag{4.12}$$

We now choose $\varphi^\varepsilon_2(0)$ so small that

$$|\varphi^\varepsilon_2(0)| = O(\varepsilon^{k+1}). \tag{4.13}$$

Then (4.12) and (4.3) yield the estimates

$$\begin{aligned} |\varphi^\varepsilon_2(\tau)| &\le |\varphi^\varepsilon_2(0)| + \int_0^\tau \|\mathscr{R}^\varepsilon(\sigma)\|_{L^\infty(\Omega)} d\sigma \\ &\le M\varepsilon^{k+1} + M\varepsilon^{k+2} \cdot \frac{T}{\varepsilon^2}. \end{aligned}$$

Therefore, the solution $\varphi^\varepsilon_2(\tau)$ of (4.6) with (4.13) satisfies

$$|\varphi^\varepsilon_2(\tau)| = O(\varepsilon^k), \qquad \tau \in [0, T/\varepsilon^2]. \tag{4.14}$$

Substituting the solution $\varphi_2^\varepsilon(\tau)$ satisfying (4.14) into (4.5), we move on to deal with (4.5). We simply denote by $\|\cdot\|_\alpha$ and $\|\cdot\|_{\alpha,\beta}$ the norm of X_α^ε and the operator norm of a bounded linear operator $X_\alpha^\varepsilon \to X_\beta^\varepsilon$, respectively. One can find that the operator $\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma)$ $(0 \le \tau, \sigma \le T/\varepsilon^2)$ consists of a multiplication operator and an integral operator. In particular, it does not involve any differential operator. Thanks to this fact, the operator norm of $\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma)$ has the following characterization.

Lemma 4.1. *Let $\alpha \in [0, 1/2)$. Then there exists a constant $M > 0$ such that the following estimate holds for $0 \le \sigma \le \tau \le T/\varepsilon^2$:*

$$\|\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma)\|_{1,\alpha} \le M\varepsilon^2(\tau - \sigma). \tag{4.15}$$

On the other hand, by examining the principal eigenvalue of $\mathscr{L}^\varepsilon(t)$, we obtain the following

Lemma 4.2. *The operator $\mathscr{A}^\varepsilon(\tau)$ is sectorial for all $\tau \in [0, T/\varepsilon^2]$. More precisely, there exist some constants $\lambda_* > 0$, $\theta_* \in (0, \pi/2)$ and $M_* > 0$ such that*

$$\rho(\mathscr{A}^\varepsilon(\tau)) \supset S_* := \{\lambda \in \mathbf{C} \mid \lambda \ne \varepsilon^2\lambda_*, |\arg(\lambda - \varepsilon^2\lambda_*)| < \pi/2 + \theta_*\}$$

and the following resolvent estimate is valid for all $\tau \in [0, T/\varepsilon^2]$:

$$\|(\lambda - \mathscr{A}^\varepsilon(\tau))^{-1}\|_{0,0} \le \frac{M_*}{|\lambda - \varepsilon^2\lambda_*|}, \qquad \lambda \in S_*. \tag{4.16}$$

Lemma 4.1 and Lemma 4.2 allow us to obtain some estimates on $\Phi^\varepsilon(\tau, \sigma) : X_\alpha^\varepsilon \to X_\beta^\varepsilon$ with $\Phi^\varepsilon(\cdot\,,\cdot)$ being the evolution operator associated with the family $\{\mathscr{A}^\varepsilon(\tau)\}_{\tau \in [0, T/\varepsilon^2]}$.

Lemma 4.3. *For $0 \le \alpha \le \beta \le 1$, there exists a constant $M > 0$ such that the following estimate holds for $0 \le \sigma \le \tau \le T/\varepsilon^2$:*

$$\|\Phi^\varepsilon(\tau, \sigma)\|_{\alpha,\beta} \le M(\tau - \sigma)^{\alpha-\beta} e^{\varepsilon^2(\lambda_* + K)(\tau - \sigma)}, \qquad (\alpha, \beta) \ne (0, 1), \tag{4.17}$$

where $K > 0$ is a certain constant.

We postpone the proof of Lemma 4.1 through Lemma 4.3 to §4.3, and treat the equation (4.5) with appropriate initial data $\varphi_1^\varepsilon(0)$. Applying the variation of constants formula to (4.5), we obtain

$$\begin{aligned} \varphi_1^\varepsilon(\tau) = \Phi^\varepsilon(\tau, 0)\varphi_1^\varepsilon(0) &+ \int_0^\tau \Phi^\varepsilon(\tau, \sigma)\mathscr{N}^\varepsilon(\sigma, \varphi_1^\varepsilon(\sigma) + \varphi_2^\varepsilon(\sigma))\,d\sigma \\ &+ \int_0^\tau \Phi^\varepsilon(\tau, \sigma)\bar{\mathscr{R}}^\varepsilon(\sigma, \varphi_2^\varepsilon(\sigma))\,d\sigma. \end{aligned} \tag{4.18}$$

Let $p \geq 3N$ and choose $\alpha \in (1/2, 1)$. Then there exists $\beta \in (0,1)$ such that $2\alpha - N/p > \beta$. Therefore, by (4.2), (4.10), (4.11), (4.14) and $X_\alpha^\varepsilon \hookrightarrow X_0^\varepsilon$, we have the following estimates for $\sigma \in [0, T/\varepsilon^2]$:

$$\begin{aligned}
&\|\mathcal{N}^\varepsilon(\sigma, \varphi_1^\varepsilon(\sigma) + \varphi_2^\varepsilon(\sigma))\|_0 \\
&\quad \leq M\varepsilon\|\varphi_1^\varepsilon(\sigma) + \varphi_2^\varepsilon(\sigma)\|_{L^\infty(\Omega)}\|\varphi_1^\varepsilon(\sigma) + \varphi_2^\varepsilon(\sigma)\|_0 \\
&\quad \leq M\varepsilon(\|\varphi_1^\varepsilon(\sigma)\|_{L^\infty(\Omega)} + |\varphi_2^\varepsilon(\sigma)|)(\|\varphi_1^\varepsilon(\sigma)\|_0 + |\varphi_2^\varepsilon(\sigma)|) \\
&\quad \leq M\varepsilon(\varepsilon^{-3N/2p}\|\varphi_1^\varepsilon(\sigma)\|_\alpha + O(\varepsilon^k))(\|\varphi_1^\varepsilon(\sigma)\|_\alpha + O(\varepsilon^k)) \\
&\quad \leq M(\varepsilon^{1-3N/2p}\|\varphi_1^\varepsilon(\sigma)\|_\alpha^2 + M\varepsilon^{k+1-3N/2p}(1 + \varepsilon^{3N/2p})\|\varphi_1^\varepsilon(\sigma)\|_\alpha + M\varepsilon^{2k+1}) \\
&\quad \leq M(\varepsilon^{1-3N/2p}\|\varphi_1^\varepsilon(\sigma)\|_\alpha^2 + \varepsilon^{k+1-3N/2p}\|\varphi_1^\varepsilon(\sigma)\|_\alpha + \varepsilon^{2k+1}).
\end{aligned}$$

Moreover, by using (4.3) and (4.14) in (4.7), we have for $\sigma \in [0, T/\varepsilon^2]$

$$\begin{aligned}
\|\bar{\mathscr{R}}^\varepsilon(\sigma, \varphi_2^\varepsilon(\sigma))\|_0 &\leq \|\mathscr{R}^\varepsilon(\sigma) - \langle \mathscr{R}^\varepsilon(\sigma)\rangle\|_0 \\
&\qquad + \varepsilon\|f'(u_A^\varepsilon(\varepsilon^2\sigma, \cdot)) - \langle f'(u_A^\varepsilon(\varepsilon^2\sigma, \cdot))\rangle\|_0|\varphi_2^\varepsilon(\sigma)| \\
&\leq 2(\|\mathscr{R}^\varepsilon(\sigma)\|_{L^\infty(\Omega)} + \varepsilon\|f'(u_A^\varepsilon(\varepsilon^2\sigma, \cdot))\|_{L^\infty(\Omega)}|\varphi_2^\varepsilon(\sigma)|) \\
&= O(\varepsilon^{k+2}) + \varepsilon O(1)O(\varepsilon^k) \\
&\leq M\varepsilon^{k+1}.
\end{aligned}$$

Using these estimates in (4.18), we find that

$$\begin{aligned}
\|\varphi_1^\varepsilon(\tau)\|_\alpha &\leq \|\Phi^\varepsilon(\tau, 0)\|_{\alpha,\alpha}\|\varphi_1^\varepsilon(0)\|_\alpha \\
&\quad + M\varepsilon^{1-3N/2p}\int_0^\tau \|\Phi^\varepsilon(\tau,\sigma)\|_{0,\alpha}\|\varphi_1^\varepsilon(\sigma)\|_\alpha^2 d\sigma \\
&\quad + M\varepsilon^{k+1-3N/2p}\int_0^\tau \|\Phi^\varepsilon(\tau,\sigma)\|_{0,\alpha}\|\varphi_1^\varepsilon(\sigma)\|_\alpha d\sigma \\
&\quad + M\varepsilon^{k+1}(\varepsilon^k + 1)\int_0^\tau \|\Phi^\varepsilon(\tau,\sigma)\|_{0,\alpha} d\sigma.
\end{aligned} \tag{4.19}$$

Let $r^\varepsilon(\tau)$ be the function defined by

$$r^\varepsilon(\tau) := \|\varphi_1^\varepsilon(\tau)\|_\alpha e^{-\varepsilon^2(\lambda_* + K)\tau}, \qquad \tau \in [0, T/\varepsilon^2]. \tag{4.20}$$

Then, by the estimates (4.17), we can compute (4.19) in terms of $r^\varepsilon(\tau)$ so that

$$
\begin{aligned}
(4.21)\qquad r^{\varepsilon}(\tau) &\le M\biggl(r^{\varepsilon}(0)+e^{(\lambda*+K)T}\varepsilon^{1-3N/2p}\int_0^{\tau}(\tau-\sigma)^{-\alpha}r^{\varepsilon}(\sigma)^2d\sigma \\
&\qquad +\varepsilon^{k+1-3N/2p}\int_0^{\tau}(\tau-\sigma)^{-\alpha}r^{\varepsilon}(\sigma)d\sigma+M\varepsilon^{k+1}\frac{T^{1-\alpha}}{1-\alpha}\varepsilon^{-2(1-\alpha)}\biggr) \\
&\le M\biggl(r^{\varepsilon}(0)+\varepsilon^{1-3N/2p}\int_0^{\tau}(\tau-\sigma)^{-\alpha}r^{\varepsilon}(\sigma)^2d\sigma \\
&\qquad +\varepsilon^{k+1-3N/2p}\int_0^{\tau}(\tau-\sigma)^{-\alpha}r^{\varepsilon}(\sigma)d\sigma+\varepsilon^{k+2\alpha-1}\biggr).
\end{aligned}
$$

We now choose $\varphi_1^{\varepsilon}(0)$ so small that

$$
(4.22)\qquad r^{\varepsilon}(0)=\|\varphi_1^{\varepsilon}(0)\|_{\alpha}=O(\varepsilon^{k+1}).
$$

Then from the continuity of $r^{\varepsilon}(\tau)$ it follows that

$$
(4.23)\qquad r^{\varepsilon}(\tau)\le\varepsilon^{k}
$$

for small $\tau>0$. Setting $T_0:=\sup\{\tau\in[0,T/\varepsilon^2]\,|\,r^{\varepsilon}(\sigma)\le\varepsilon^k \text{ for all } \sigma\in[0,\tau]\}$, we have one of the alternatives $r^{\varepsilon}(T_0)=\varepsilon^k$ or $T_0=T/\varepsilon^2$. Assuming the former situation is realized, and noting that $k\ge 1$, $\alpha\in(1/2,1)$ and $p\ge 3N$, we can compute by employing (4.22) in (4.21) so that

$$
\begin{aligned}
\varepsilon^k=r^{\varepsilon}(T_0) &\le M\biggl(\varepsilon^{k+1}+\varepsilon^{1-3N/2p}\varepsilon^{2k}\frac{T^{1-\alpha}}{1-\alpha}\varepsilon^{-2(1-\alpha)} \\
&\qquad +\varepsilon^{k+1-3N/2p}\varepsilon^{k}\frac{T^{1-\alpha}}{1-\alpha}\varepsilon^{-2(1-\alpha)}+\varepsilon^{k+2\alpha-1}\biggr) \\
&\le \varepsilon^k\biggl(M\varepsilon+\frac{2MT^{1-\alpha}}{1-\alpha}\varepsilon^{k-3N/2p+2\alpha-1}+M\varepsilon^{2\alpha-1}\biggr) \\
&\le \frac{\varepsilon^k}{2},
\end{aligned}
$$

which is a contradiction. Thus (4.23) is valid for $\tau\in[0,T/\varepsilon^2]$, and by (4.20) we have

$$
\|\varphi_1^{\varepsilon}(\tau)\|_{\alpha}\le Me^{(\lambda_*+K)T}\varepsilon^k=O(\varepsilon^k),\qquad \tau\in[0,T/\varepsilon^2].
$$

By employing (4.10) and (4.11), it follows that

$$
(4.24)\qquad \|\varphi_1^{\varepsilon}(\tau)\|_{L^{\infty}(\Omega)}=O(\varepsilon^{k-3N/2p}),\qquad \tau\in[0,T/\varepsilon^2].
$$

Combining (4.14) and (4.24) in (4.4), we have

$$\begin{aligned}\|\varphi^{\varepsilon}(\tau)\|_{L^{\infty}(\Omega)} &\le \|\varphi_1^{\varepsilon}(\tau)\|_{L^{\infty}(\Omega)} + |\varphi_2^{\varepsilon}(\tau)| \\ &= O(\varepsilon^{k-3N/2p}) + O(\varepsilon^{k}) \\ &= O(\varepsilon^{k-3N/2p}), \qquad \tau \in [0, T/\varepsilon^2].\end{aligned}$$

Thus (1.13) is obtained, which completes the proof of Proposition 1.4. □

4.3. Proof of key lemmas. In this subsection, we prove Lemma 4.1, Lemma 4.2 and Lemma 4.3.

Proof of Lemma 4.1. For $\varphi \in X_1^{\varepsilon}$ and $0 \le \sigma \le \tau \le T/\varepsilon^2$, we set

$$u_{\tau,\sigma}^{\varepsilon} := (\mathscr{A}^{\varepsilon}(\tau) - \mathscr{A}^{\varepsilon}(\sigma))\varphi.$$

Then an elementary calculation gives

$$u_{\tau,\sigma}^{\varepsilon}(x) = [F_{\tau,\sigma}^{\varepsilon}(x) G_{\tau,\sigma}^{\varepsilon}(x)\varphi(x) - \langle F_{\tau,\sigma}^{\varepsilon} G_{\tau,\sigma}^{\varepsilon}\varphi\rangle]\varepsilon^2(\tau - \sigma), \tag{4.25}$$

where

$$F_{\tau,\sigma}^{\varepsilon}(x) := \int_0^1 f''(u_A^{\varepsilon}(\varepsilon^2\sigma, x) + s(u_A^{\varepsilon}(\varepsilon^2\tau, x) - u_A^{\varepsilon}(\varepsilon^2\sigma, x)))ds,$$

$$G_{\tau,\sigma}^{\varepsilon}(x) := \int_0^1 \varepsilon \frac{\partial u_A^{\varepsilon}}{\partial t}(\varepsilon^2(\sigma + s(\tau - \sigma)), x)ds.$$

Since it is easily verified that $F_{\tau,\sigma}^{\varepsilon}$ and $G_{\tau,\sigma}^{\varepsilon}$ for $0 \le \sigma \le \tau \le T/\varepsilon^2$ satisfy

$$\begin{aligned}\|F_{\tau,\sigma}^{\varepsilon}\|_{L^{\infty}(\Omega)} &= O(1), \\ \|G_{\tau,\sigma}^{\varepsilon}\|_{L^{\infty}(\Omega)} &= O(1),\end{aligned} \tag{4.26}$$

we obtain, by (4.25), (4.26) and the embedding $X_1^{\varepsilon} \hookrightarrow X_0^{\varepsilon}$,

$$\|u_{\tau,\sigma}^{\varepsilon}\|_0 \le M\varepsilon^2(\tau - \sigma)\|\varphi\|_1, \tag{4.27}$$

which establishes (4.15) with $\alpha = 0$.

We move on to prove (4.15) for $\alpha \in (0, 1/2)$. By virtue of (4.9) and the relation between Besov and Sobolev-Slobodeckii spaces [1, 22], it turns out that the space $(L^p(\Omega), W_{\varepsilon,\mathscr{B}}^{2,p}(\Omega))_{\alpha,p}$ coincides with $W_{\varepsilon}^{2\alpha,p}(\Omega)$, where $W_{\varepsilon}^{2\alpha,p}(\Omega) = W^{2\alpha,p}(\Omega)$ as a set, equipped with the following weighted norm equivalent to the norm of $(L^p(\Omega), W_{\varepsilon,\mathscr{B}}^{2,p}(\Omega))_{\alpha,p}$:

$$\|u\|_{W_{\varepsilon}^{2\alpha,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)}^p + (\varepsilon^{3/2})^{2\alpha p} \int\int_{\Omega\times\Omega} \frac{|u(x) - u(x')|^p}{|x - x'|^{N+2\alpha p}} dx dx' \right)^{1/p}.$$

Let us now compute $\|u^{\varepsilon}_{\tau,\sigma}\|_{W^{2\alpha,p}_{\varepsilon}(\Omega)}$. From (4.25), we can calculate $u^{\varepsilon}_{\tau,\sigma}(x) - u^{\varepsilon}_{\tau,\sigma}(x')$ so that

$$\begin{aligned} u^{\varepsilon}_{\tau,\sigma}(x) - u^{\varepsilon}_{\tau,\sigma}(x') = [(& F^{\varepsilon}_{\tau,\sigma}(x) - F^{\varepsilon}_{\tau,\sigma}(x'))G^{\varepsilon}_{\tau,\sigma}(x)\varphi(x) \\ &+ F^{\varepsilon}_{\tau,\sigma}(x')(G^{\varepsilon}_{\tau,\sigma}(x) - G^{\varepsilon}_{\tau,\sigma}(x'))\varphi(x) \\ &+ F^{\varepsilon}_{\tau,\sigma}(x')G^{\varepsilon}_{\tau,\sigma}(x')(\varphi(x) - \varphi(x'))]\varepsilon^2(\tau - \sigma). \end{aligned}$$

By (4.26) we thus easily find that

$$\begin{aligned} (\varepsilon^{3/2})^{2\alpha p} \iint_{\Omega\times\Omega} & \frac{|u^{\varepsilon}_{\tau,\sigma}(x) - u^{\varepsilon}_{\tau,\sigma}(x')|^p}{|x - x'|^{N+2\alpha p}}\, dxdx' \\ &\le M[\varepsilon^2(\tau - \sigma)]^p (I^{\varepsilon}_1 + I^{\varepsilon}_2 + I^{\varepsilon}_3) \end{aligned} \tag{4.28}$$

with

$$I^{\varepsilon}_1 := \varepsilon^{3\alpha p} \iint_{\Omega\times\Omega} \frac{|\varphi(x)|^p |F^{\varepsilon}_{\tau,\sigma}(x) - F^{\varepsilon}_{\tau,\sigma}(x')|^p}{|x - x'|^{N+2\alpha p}}\, dxdx',$$

$$I^{\varepsilon}_2 := \varepsilon^{3\alpha p} \iint_{\Omega\times\Omega} \frac{|\varphi(x)|^p |G^{\varepsilon}_{\tau,\sigma}(x) - G^{\varepsilon}_{\tau,\sigma}(x')|^p}{|x - x'|^{N+2\alpha p}}\, dxdx',$$

$$I^{\varepsilon}_3 := \varepsilon^{3\alpha p} \iint_{\Omega\times\Omega} \frac{|\varphi(x) - \varphi(x')|^p}{|x - x'|^{N+2\alpha p}}\, dxdx'.$$

We first examine I^{ε}_1. Let $D^{\delta} := (\Omega\times\Omega)\backslash(\Gamma^{\delta/2}\times\Gamma^{\delta/2})$, namely,

$$D^{\delta} = [(\Omega\backslash\Gamma^{\delta/2})\times(\Omega\backslash\Gamma^{\delta/2})] \cup [(\Omega\backslash\Gamma^{\delta/2})\times\Gamma^{\delta/2}] \cup [\Gamma^{\delta/2}\times(\Omega\backslash\Gamma^{\delta/2})].$$

We also define $S^{\delta} \subset D^{\delta}$ by $S^{\delta} := \{(x, x') \in D^{\delta};\, |x - x'| < \delta/4\}$ and introduce

$$[u]_{\mathrm{Lip}(S^{\delta})} := \sup_{x, x' \in S^{\delta},\, x \ne x'} \frac{|u(x) - u(x')|}{|x - x'|}.$$

It is easily verified that $F^{\varepsilon}_{\tau,\sigma}$ for $0 \le \sigma \le \tau \le T/\varepsilon^2$ enjoys the following properties:

$$[F^{\varepsilon}_{\tau,\sigma}]_{\mathrm{Lip}(S^{\delta})} = O(1), \tag{4.29}$$

$$[F^{\varepsilon}_{\tau,\sigma}]_{C^{\beta}(\bar{\Gamma}^{\delta/2})} = O(\varepsilon^{-\beta}) \qquad \text{for all } \beta \in (0, 1). \tag{4.30}$$

Since $\alpha \in (0, 1/2)$, we can choose $\theta > 0$ so small that $0 < 2\alpha + \theta < 1$. Using

(4.29), (4.30) with $\beta := 2\alpha + \theta$ and the embedding $X_1^\varepsilon \hookrightarrow X_0^\varepsilon$ together with spherical coodinates, we can compute I_1^ε as

$$
\begin{aligned}
I_1^\varepsilon &= \varepsilon^{3\alpha p} \iint_{S^\delta} \frac{|\varphi(x)|^p |F_{\tau,\sigma}^\varepsilon(x) - F_{\tau,\sigma}^\varepsilon(x')|^p}{|x - x'|^{N+2\alpha p}}\, dx dx' \\
&\quad + \varepsilon^{3\alpha p} \iint_{D^\delta \setminus S^\delta} \frac{|\varphi(x)|^p |F_{\tau,\sigma}^\varepsilon(x) - F_{\tau,\sigma}^\varepsilon(x')|^p}{|x - x'|^{N+2\alpha p}}\, dx dx' \\
&\quad + \varepsilon^{3\alpha p} \iint_{\Gamma^{\delta/2} \times \Gamma^{\delta/2}} \frac{|\varphi(x)|^p |F_{\tau,\sigma}^\varepsilon(x) - F_{\tau,\sigma}^\varepsilon(x')|^p}{|x - x'|^{N+2\alpha p}}\, dx dx' \\
&\le \varepsilon^{3\alpha p} [F_{\tau,\sigma}^\varepsilon]_{\mathrm{Lip}(S^\delta)}^p \iint_{S^\delta} \frac{|\varphi(x)|^p}{|x - x'|^{N+(2\alpha-1)p}}\, dx dx' \\
&\quad + M\varepsilon^{3\alpha p} \|F_{\tau,\sigma}^\varepsilon\|_{L^\infty(\Omega)}^p \iint_{\{(x,x') \in D^\delta;\, |x-x'| \ge \delta/4\}} \frac{|\varphi(x)|^p}{|x - x'|^{N+2\alpha p}}\, dx dx' \\
&\quad + \varepsilon^{3\alpha p} [F_{\tau,\sigma}^\varepsilon]_{C^\beta(\bar{\Gamma}^{\delta/2})}^p \iint_{\Gamma^{\delta/2} \times \Gamma^{\delta/2}} \frac{|\varphi(x)|^p}{|x - x'|^{N+(2\alpha-\beta)p}}\, dx dx' \\
&\le M\varepsilon^{3\alpha p} \int_\Omega |\varphi(x)|^p \left(\int_\Omega \frac{dx'}{|x - x'|^{N+(2\alpha-1)p}} \right) dx \\
&\quad + M\varepsilon^{3\alpha p} \int_\Omega |\varphi(x)|^p \left(\int_{\Omega \cap \{|x-x'| \ge \delta/4\}} \frac{dx'}{|x - x'|^{N+2\alpha p}} \right) dx \\
&\quad + M\varepsilon^{(3\alpha-\beta)p} \int_\Omega |\varphi(x)|^p \left(\int_\Omega \frac{dx'}{|x - x'|^{N+(2\alpha-\beta)p}} \right) dx \\
&\le M\varepsilon^{3\alpha p} \|\varphi\|_1^p + M\varepsilon^{(\alpha-\theta)p} \|\varphi\|_1^p.
\end{aligned}
$$

Choosing $\theta > 0$ so small that $\alpha - \theta > 0$, we obtain

$$I_1^\varepsilon \le M\|\varphi\|_1^p.$$

For I_2^ε, we find that $G_{\tau,\sigma}^\varepsilon$ enjoys the same estimates as in (4.29) and (4.30) for $F_{\tau,\sigma}^\varepsilon$, and therefore I_2^ε can be estimated, by the same computations as above, so that $I_2^\varepsilon \le M\|\varphi\|_1^p$. The estimate $I_3^\varepsilon \le M\|\varphi\|_1^p$ follows from the embedding $X_1^\varepsilon \hookrightarrow X_\alpha^\varepsilon \subset W_\varepsilon^{2\alpha,p}(\Omega)$.

By substituting these three estimates for I_j^ε into (4.28), the resultant estimate together with (4.27) implies

$$\|(\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma))\varphi\|_\alpha \le M\|u^\varepsilon_{\tau,\sigma}\|_{W^{2\alpha,p}_\varepsilon(\Omega)} \le M\varepsilon^2(\tau-\sigma)\|\varphi\|_1,$$

which completes the proof of Lemma 4.1. □

Proof of Lemma 4.2. We first treat the case where $p = 2$. It is easy to verify that $\mathscr{L}^\varepsilon(t)$ under the Neumann boundary condition is formally self-adjoint in $L^2(\Omega) \cap \mathbf{M}$, and therefore eigenvalues are real. We also obtain by the variational characterization for the principal eigenvalue λ^ε of $\mathscr{L}^\varepsilon(t)$ that

$$\begin{aligned}
\lambda^\varepsilon &= \sup_{\varphi\in W^{1,2}(\Omega),\,\varphi\neq 0,\,\langle\varphi\rangle=0} \frac{\int_\Omega -\varepsilon|\nabla\varphi|^2 + \varepsilon^{-1}f'(u^\varepsilon_A)|\varphi|^2dx}{\|\varphi\|^2_{L^2(\Omega)}} \\
&\le \sup_{\varphi\in W^{1,2}(\Omega),\,\varphi\neq 0} \frac{\int_\Omega -\varepsilon|\nabla\varphi|^2 + \varepsilon^{-1}f'(u^\varepsilon_A)|\varphi|^2dx}{\|\varphi\|^2_{L^2(\Omega)}} \\
&\le \sup_{\varphi\in W^{1,2}(\Gamma^\delta),\,\varphi\neq 0} \frac{\int_{\Gamma^\delta} -\varepsilon|\nabla\varphi|^2 + \varepsilon^{-1}f'(u^\varepsilon_A)|\varphi|^2dx}{\|\varphi\|^2_{L^2(\Gamma^\delta)}} =: \mu^\varepsilon,
\end{aligned}$$

due to $f'(u^\varepsilon_A) < 0$ in $\Omega\backslash\Gamma^\delta$ and $\|\varphi\|_{L^2(\Omega)} \ge \|\varphi\|_{L^2(\Gamma^\delta)}$. This says that λ^ε is estimated from above by the principal eigenvalue μ^ε of the Allen-Cahn operator $\varepsilon\Delta + \varepsilon^{-1}f'(u^\varepsilon_A)$ in a neighborhood Γ^δ. According to the results established by Alikakos et al. [2] and Chen [4], it is known that μ^ε is bounded above for $\varepsilon > 0$ and $t \in [0, T]$. Thus we have $\lambda^\varepsilon \le \mu^\varepsilon \le \mu_*$ for some $\mu_* > 0$.

For $\lambda \in \mathbf{C}$ and a complex-valued function v with zero average, let us now consider the resolvent equation

$$\lambda u - \mathscr{L}^\varepsilon(t)u = v, \qquad \frac{\partial u}{\partial \mathbf{n}} = 0. \tag{4.31}$$

Multiplying the equation in (4.31) by the complex conjugate $\bar{u}$ of u and integrating over Ω, we have

$$\lambda\|u\|^2_{L^2(\Omega)} = (\mathscr{L}^\varepsilon(t)u, \bar{u})_{L^2(\Omega)} + (\bar{u}, v)_{L^2(\Omega)}, \tag{4.32}$$

where the symbol $(\cdot,\cdot)_{L^2(\Omega)}$ stands for the usual L^2-inner product. We decompose $\lambda \in \mathbf{C}$, $u : \Omega \to \mathbf{C}$ and $v : \Omega \to \mathbf{C}$ so that

$$\lambda = \lambda^R + i\lambda^I, \qquad u = u^R + iu^I, \qquad v = v^R + iv^I. \tag{4.33}$$

We note that the real-valued functions $u^R : \Omega \to \mathbf{R}$ and $u^I : \Omega \to \mathbf{R}$ also have zero average and satisfy the Neumann boundary conditions. Associated with the decomposition in (4.33), the real part of (4.32) is computed as

$$\begin{aligned}
\lambda^R \|u\|^2_{L^2(\Omega)} &= (\mathscr{L}^\varepsilon(t)u^R, u^R)_{L^2(\Omega)} + (\mathscr{L}^\varepsilon(t)u^I, u^I)_{L^2(\Omega)} \\
&\quad + (u^R, v^R)_{L^2(\Omega)} + (u^I, v^I)_{L^2(\Omega)} \\
&\le \mu_*(\|u^R\|^2_{L^2(\Omega)} + \|u^I\|^2_{L^2(\Omega)}) \\
&\quad + \|u^R\|_{L^2(\Omega)}\|v^R\|_{L^2(\Omega)} + \|u^I\|_{L^2(\Omega)}\|v^I\|_{L^2(\Omega)} \\
&\le \mu_*(\|u^R\|^2_{L^2(\Omega)} + \|u^I\|^2_{L^2(\Omega)}) \\
&\quad + (\|u^R\|^2_{L^2(\Omega)} + \|u^I\|^2_{L^2(\Omega)})^{1/2}(\|v^R\|^2_{L^2(\Omega)} + \|v^I\|^2_{L^2(\Omega)})^{1/2} \\
&\le \mu_*\|u\|^2_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}
\end{aligned}$$

to obtain

$$(\lambda^R - \mu_*)\|u\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)}. \tag{4.34}$$

On the other hand, the imaginary part of (4.32) becomes

$$\begin{aligned}
\lambda^I \|u\|^2_{L^2(\Omega)} &= -(\mathscr{L}^\varepsilon(t)u^R, u^I)_{L^2(\Omega)} + (\mathscr{L}^\varepsilon(t)u^I, u^R)_{L^2(\Omega)} \\
&\quad + (u^R, v^I)_{L^2(\Omega)} - (u^I, v^R)_{L^2(\Omega)} \\
&= (u^R, v^I)_{L^2(\Omega)} - (u^I, v^R)_{L^2(\Omega)},
\end{aligned}$$

where integration by parts and the Neumann boundary conditions are used. Thus we have

$$|\lambda^I| \, \|u\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)}. \tag{4.35}$$

Setting $\lambda_* := \mu_* > 0$, we obtain, from (4.34) and (4.35), the estimate

$$[(\lambda^R - \lambda_*) + |\lambda^I|]\|u\|_{L^2(\Omega)} \le 2\|v\|_{L^2(\Omega)},$$

which implies that

$$\|u\|_{L^2(\Omega)} \le \frac{M_*}{|\lambda - \lambda_*|}\|v\|_{L^2(\Omega)} \tag{4.36}$$

is valid for $\lambda \in \{\lambda \in \mathbf{C} \mid \lambda \ne \lambda_*, |\arg(\lambda - \lambda_*)| < \pi/2 + \theta_*\} \subset \rho(\mathscr{L}^\varepsilon(t))$ with $\theta_* \in (0, \pi/4)$ and $M_* := \sqrt{2}/\cos(\theta_* + \pi/4)$.

Once this is established, then we find, along the line of arguments in Tanabe [21], that the following L^p-version $(p > 2)$ of (4.36)

$$\|u\|_{L^p(\Omega)} \le \frac{M_*}{|\lambda - \lambda_*|}\|v\|_{L^p(\Omega)} \tag{4.37}$$

holds for all $\lambda \in \{\lambda \in \mathbf{C} \mid \lambda \neq \lambda_*, |\arg(\lambda - \lambda_*)| < \pi/2 + \theta_*\} \subset \rho(\mathscr{L}^\varepsilon(t))$ with the same $\lambda_* > 0$ in (4.36), replacing θ_* and M_* by other constants. The estimate (4.16) then follows from (4.37) and the rescale $t = \varepsilon^2\tau$, which completes the proof of Lemma 4.2. $\square$

Proof of Lemma 4.3. Let $\alpha_0 \in [0, 1/2)$. By Lemma 4.1, it follows that

$$\|\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma)\|_{1,\alpha_0} \le M_0\varepsilon^2(\tau - \sigma). \tag{4.38}$$

Moreover, from Lemma 4.2 above and Proposition 2.3.1 in Lunardi [14], we find that for $0 \le \alpha \le \beta \le 1$, there exists a constant $M = M_{\alpha,\beta} > 0$ such that the estimate

$$\|e^{(\tau-\sigma)\mathscr{A}^\varepsilon(\sigma)}\|_{\alpha,\beta} \le M(\tau - \sigma)^{\alpha-\beta}e^{\varepsilon^2\lambda_*(\tau-\sigma)} \tag{4.39}$$

is valid. We emphasize that M can be chosen independent of $\varepsilon > 0$ thanks to the weighted norm (4.8). We now define the operator $k_1^\varepsilon(\cdot,\cdot)$ by

$$k_1^\varepsilon(\tau,\sigma) := (\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma))e^{(\tau-\sigma)\mathscr{A}^\varepsilon(\sigma)}.$$

Then we can estimate k_1^ε by employing (4.38) and (4.39) as

$$\|k_1^\varepsilon(\tau,\sigma)\|_{0,\alpha_0} \le \|\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma)\|_{1,\alpha_0}\|e^{(\tau-\sigma)\mathscr{A}^\varepsilon(\sigma)}\|_{0,1} \le M_0M\varepsilon^2e^{\varepsilon^2\lambda_*(\tau-\sigma)}.$$

For this k_1^ε, it is known [5] that the evolution operator Φ^ε is the unique solution of the integral equation

$$\Phi^\varepsilon(\tau,\sigma) = e^{(\tau-\sigma)\mathscr{A}^\varepsilon(\sigma)} + \int_\sigma^\tau \Phi^\varepsilon(\tau,s)k_1^\varepsilon(s,\sigma)ds,$$

and that the solution Φ^ε has the unique representation

$$\Phi^\varepsilon(\tau,\sigma) = e^{(\tau-\sigma)\mathscr{A}^\varepsilon(\sigma)} + \int_\sigma^\tau e^{(\tau-s)\mathscr{A}^\varepsilon(s)}k^\varepsilon(s,\sigma)ds \tag{4.40}$$

with resolvent kernel $k^\varepsilon(\cdot,\cdot)$. This kernel can be successively constructed starting from k_1^ε. We inductively define $k_m^\varepsilon(\cdot,\cdot)$ $(m \ge 2)$ by

$$k_m^\varepsilon(\tau,\sigma) := \int_\sigma^\tau k_{m-1}^\varepsilon(\tau,s)k_1^\varepsilon(s,\sigma)ds.$$

By the repeated application of the following estimates

$$\|k_m^\varepsilon(\tau,\sigma)\|_{0,\alpha_0} \le \int_\sigma^\tau \|k_{m-1}^\varepsilon(\tau,s)\|_{0,\alpha_0}\|k_1^\varepsilon(s,\sigma)\|_{0,0}ds,$$

we find, by induction, that

$$\|k_m^\varepsilon(\tau,\sigma)\|_{0,\alpha_0} \le \frac{(M_0 M\varepsilon^2)^m}{(m-1)!}(\tau-\sigma)^{m-1}e^{\varepsilon^2\lambda_*(\tau-\sigma)}, \qquad m \ge 1.$$

This immediately implies that the series

$$k^\varepsilon(\tau,\sigma) := \sum_{m=1}^{\infty} k_m^\varepsilon(\tau,\sigma) \tag{4.41}$$

converges and that it can be estimated as

$$\begin{aligned}\|k^\varepsilon(\tau,\sigma)\|_{0,\alpha_0} &\le M_0 M\varepsilon^2 e^{\varepsilon^2\lambda_*(\tau-\sigma)}\left[\sum_{m=1}^{\infty}\frac{(M_0 M\varepsilon^2(\tau-\sigma))^{m-1}}{(m-1)!}\right] \\ &= M_0 M\varepsilon^2 e^{\varepsilon^2(\lambda_*+M_0 M)(\tau-\sigma)}.\end{aligned}$$

Therefore, there exist some constants M, $K > 0$ such that the resolvent kernel k^ε defined in (4.41) satisfies the estimate

$$\|k^\varepsilon(\tau,\sigma)\|_{0,\alpha_0} \le M\varepsilon^2 e^{\varepsilon^2(\lambda_*+K)(\tau-\sigma)}. \tag{4.42}$$

Let us now examine the norm $\|\Phi^\varepsilon(\tau,\sigma)\|_{\alpha,\beta}$ by using the estimates (4.39) and (4.42) in (4.40). Suppose that $0 \le \alpha \le \beta < 1$. Then, by using (4.42) with $\alpha_0 = 0$, we have

$$\begin{aligned}\|\Phi^\varepsilon(\tau,\sigma)\|_{\alpha,\beta} &\le \|e^{(\tau-\sigma)\mathscr{A}^\varepsilon(\sigma)}\|_{\alpha,\beta} + \int_\sigma^\tau \|e^{(\tau-s)\mathscr{A}^\varepsilon(s)}\|_{0,\beta}\|k^\varepsilon(s,\sigma)\|_{\alpha,0}ds \\ &\le M(\tau-\sigma)^{\alpha-\beta}e^{\varepsilon^2(\lambda_*+K)(\tau-\sigma)} \\ &\quad + M\int_\sigma^\tau (\tau-s)^{-\beta}e^{\varepsilon^2(\lambda_*+K)(\tau-s)}\|k^\varepsilon(s,\sigma)\|_{0,0}ds \\ &\le M(\tau-\sigma)^{\alpha-\beta}e^{\varepsilon^2(\lambda_*+K)(\tau-\sigma)} + M\varepsilon^2 e^{\varepsilon^2(\lambda_*+K)(\tau-\sigma)}\frac{(\tau-\sigma)^{1-\beta}}{1-\beta} \\ &\le M(\tau-\sigma)^{\alpha-\beta}e^{\varepsilon^2(\lambda_*+K)(\tau-\sigma)} + \frac{MT^{1-\alpha}}{1-\beta}\varepsilon^{2\alpha}(\tau-\sigma)^{\alpha-\beta}e^{\varepsilon^2(\lambda_*+K)(\tau-\sigma)} \\ &\le M(\tau-\sigma)^{\alpha-\beta}e^{\varepsilon^2(\lambda_*+K)(\tau-\sigma)}.\end{aligned}$$

In the case where $0 < \alpha \le \beta = 1$, we choose $\alpha_0 > 0$ so small that $\alpha > \alpha_0$. Then we have

$$
\begin{aligned}
\|\Phi^\varepsilon(\tau,\sigma)\|_{\alpha,1} &\le \|e^{(\tau-\sigma)\mathscr{A}^\varepsilon(\sigma)}\|_{\alpha,1} + \int_\sigma^\tau \|e^{(\tau-s)\mathscr{A}^\varepsilon(s)}\|_{\alpha_0,1}\|k^\varepsilon(s,\sigma)\|_{\alpha,\alpha_0}ds \\
&\le M(\tau-\sigma)^{\alpha-1}e^{\varepsilon^2(\lambda_*+K)(\tau-\sigma)} \\
&\quad + M\int_\sigma^\tau (\tau-s)^{\alpha_0-1}e^{\varepsilon^2(\lambda_*+K)(\tau-s)}\|k^\varepsilon(s,\sigma)\|_{0,\alpha_0}ds \\
&\le M(\tau-\sigma)^{\alpha-1}e^{\varepsilon^2(\lambda_*+K)(\tau-\sigma)} + M\varepsilon^2 e^{\varepsilon^2(\lambda_*+K)(\tau-\sigma)}\frac{(\tau-\sigma)^{\alpha_0}}{\alpha_0} \\
&\le M(\tau-\sigma)^{\alpha-1}e^{\varepsilon^2(\lambda_*+K)(\tau-\sigma)} \\
&\quad + \frac{MT^{(1-\alpha)+\alpha_0}}{\alpha_0}\varepsilon^{2(\alpha-\alpha_0)}(\tau-\sigma)^{\alpha-1}e^{\varepsilon^2(\lambda_*+K)(\tau-\sigma)} \\
&\le M(\tau-\sigma)^{\alpha-1}e^{\varepsilon^2(\lambda_*+K)(\tau-\sigma)}.
\end{aligned}
$$

Thus (4.17) is obtained, which completes the proof of Lemma 4.3. □

REMARK. *Only* the estimates $\|\Phi^\varepsilon(\tau,\sigma)\|_{0,\alpha}$ and $\|\Phi^\varepsilon(\tau,\sigma)\|_{\alpha,\alpha}$ are used in the proof of Proposition 1.4. Even if $(\alpha,\beta)=(0,1)$, the norm $\|\Phi^\varepsilon(\tau,\sigma)\|_{0,1}$ can be also estimated, by employing (4.39) and (4.42) with $\alpha_0>0$, so that

$$\|\Phi^\varepsilon(\tau,\sigma)\|_{0,1} \le Me^{\varepsilon^2(\lambda_*+K)(\tau-\sigma)}[1+(\tau-\sigma)^{-1}].$$

5. Discussion

In order to capture the dynamics occurs in the stage 3 mentioned in §1.1, it is adequate to employ the following equation with the slower time scale:

$$\text{(RD-s)} \qquad \varepsilon^2 u_t^\varepsilon = \varepsilon^2\Delta u^\varepsilon + f(u^\varepsilon) - \frac{1}{|\Omega|}\int_\Omega f(u^\varepsilon)dx.$$

Let us recast (RD-s) as

$$\text{(5.1)} \qquad \varepsilon^2 u_t(t,x) = \varepsilon^2\Delta u^\varepsilon + f(u^\varepsilon(t,x)) - v^\varepsilon(t),$$

where $v^\varepsilon(t)$ is given by

$$\text{(5.2)} \qquad v^\varepsilon(t) = \frac{1}{|\Omega|}\int_\Omega f(u^\varepsilon(t,x))dx.$$

Through the representation $\Gamma^\varepsilon(t)=\{x\in\Omega \mid x=\gamma(t,y)+\varepsilon R^\varepsilon(t,y)\nu(t,y),\ y\in\mathscr{M}\}$, we substitute the expansions

$$R^\varepsilon = R^1 + \varepsilon R^2 + \varepsilon^2 R^3 + \cdots, \qquad v^\varepsilon = \sum_{j \geq 0} \varepsilon^j v^j$$

into (5.1) and (5.2). Then C^1-matching conditions and the nonlocal relation (5.2) also give rise to a series of equations. The lowest order equation is $v^0 \equiv v^*$, which means that the interface dynamics of (IE) is in equilibrium in this time scale.

The first order equation is

$$\tag{5.3} \begin{cases} \gamma_t \cdot \nu = -\kappa + c'(v^*)v^1, \\ c'(v^*)v^1 = \dfrac{1}{|\Gamma|}\displaystyle\int_{\mathscr{M}} \kappa \, dS_y, \end{cases}$$

which is nothing but the volume-preserving mean curvature flow:

$$\tag{IE-s} \mathbf{v}(x; \Gamma(t)) = -\kappa(x; \Gamma(t)) + \frac{1}{|\Gamma(t)|}\int_{\Gamma(t)} \kappa(x; \Gamma(t)) dS_x^{\Gamma(t)}.$$

In contract to the *system* of equations (IE), we can see in (5.3) that one of the equations involves the other and consequently the *scalar* equation (IE-s) is obtained. It is known [7, 10, 13] that if the initial interface is uniformly convex, then the solution $\Gamma(t)$ of (IE-s) exists globally in time and it converges to a sphere as $t \to \infty$. We also mention that the flow (IE-s) can derive interfaces to self-intersections in finite time (cf. [15]) because of the failure of comparison principles for (IE-s).

The j-th $(j \geq 2)$ order equation is

$$\begin{cases} R_t^{j-1} = \left(\Delta^{\mathscr{M}} + \displaystyle\sum_{i=0}^{N-1} \kappa_i^2\right) R^{j-1} + c'(v^*)v^j + \cdots, \\ c'(v^*)v^j = \dfrac{1}{|\Gamma|}\displaystyle\int_{\mathscr{M}} \alpha R^{j-1} \, dS_y + \cdots, \end{cases}$$

which is equivalent to the *scalar* equation

$$\tag{5.4} R_t^j = \left(\Delta^{\mathscr{M}} + \sum_{i=0}^{N-1} \kappa_i^2\right) R^j + \frac{1}{|\Gamma|}\int_{\mathscr{M}} \alpha R^j \, dS_y + \beta_j, \qquad j \geq 1.$$

Here the symbol $\Delta^{\mathscr{M}}$ stands for the Laplace-Beltrami operator on $\mathscr{M}$ induced from that on Γ, α is a function depending only on Γ, and the non-homogeneous term β_j depends only on Γ and R^m $(1 \leq m < j)$. Since the equation (5.4) for R^j is non-homogeneous linear parabolic equation with nonlocal term, the unique existence of solutions are known. Our method of approximations developed for the equation (RD) also works for the rescaled version (RD-s).

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Koji Okada
Department of Mathematical and Life Sciences
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739-8526, *Japan*

Current address: Institut für Mathematik
Universität Zürich
Winterthurerstarasse 190
CH-8057 *Zürich, Switzerland*
e-mail: koji.okada@math.unizh.ch