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On a nonlinear diffusion system with resource-consumer interaction

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ABSTRACT. This article is devoted to the study of a resource-consumer type reaction-diffusion system arising in chemistry, biology and in other applied sciences. We prove that the problem is well-posed and describe the large time behavior of the solutions. A key ingredient is to obtain a uniform in time L^∞-bound for the solutions. We also present numerical simulations describing the transient behavior of the solution which show very unstable interfaces.

1. Introduction

Among a lot of reaction-diffusion (RD) equations, a class of RD equations with consumer and resource interaction have been thoroughly investigated by many authors. A typical but suggestive example is the following two component system where u and v act as a consumer and its resource, respectively:

$$\begin{cases} u_t = d_u \Delta u + u^m v, \\ v_t = d_v \Delta v - u^m v, \end{cases} \tag{1.1}$$

where d_u and d_v are the diffusion coefficients of u and v and m is a positive integer. The main results concern the well-posedness of the parabolic problems, L^∞-bounds on solutions which do not depend on time and their asymptotic behavior ([Ali, Mas, HaYo, Hos, Kan] for instance). A characteristics of this system under the zero-flux boundary condition is that the spatial average of $u + v$ is conserved in time. It is shown that

$$\lim_{t\to\infty}(u(t), v(t)) = (\langle u(0) + v(0)\rangle, 0),$$

where $\langle f \rangle$ is the spatial average of f.

In view of the result on the asymptotic behavior of the solutions above, one used to believe that resource-consumer systems without feeding process such as (1.1) are not interesting from the pattern formation viewpoint. However, recent numerical simulations have revealed that it is not necessarily

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true, that is, solutions of (1.1) may exhibit spatio-temporal patterns in transient time [HiMiWe]. This suggests that resource-consumer systems should be revisited from the pattern formation viewpoint.

From the mathematical modelling viewpoint, one encounters several resource-consumer systems in the fields of biology and chemistry. An example is bacterial growth in biology. One easily notices that bacteria and nutrients obviously correspond to a consumer and its (finite) resource. The resulting model [Kit] is

$$\begin{cases} u_t = d_u \Delta(u^k) + uv - au, \\ v_t = d_v \Delta v - uv, \\ w_t = au, \end{cases} \tag{1.2}$$

where u and w are the densities of active and inactive bacteria respectively, and v is the concentration of nutrients, k (>1) is a positive integer, a is the conversion rate of u into w. The first two equations of (1.2) with $a = 0$ coincide with (1.1) with $m = 1$ except for the diffusion term for u. From experimental requirements, we may take $k = 1$ (linear diffusion) in (1.2) when the medium (agar) is soft, while when it is hard, we may say that $k = 2$ (nonlinear degenerate diffusion) is plausible. It is obvious that the first two equations are closed for u and v and that w can be obtained by solving them. Biologically the total bacterial density, which is given by

$$u(x,t) + w(x,t) = u(x,t) + a\int_0^t u(x,s)ds,$$

is an important parameter.

Another model involves the following autocatalytic reactions for the intermediate component U and the reactant V

$$\begin{cases} mU + V \to (m+1)U \\ nU \to P. \end{cases}$$

If these processes happen in a porous medium, a suitable model can be given by

$$\begin{cases} u_t = d_u \Delta(u^k) + u^m v - au^n \\ v_t = d_v \Delta(v^l) - u^m v. \end{cases} \tag{1.3}$$

When $m = n = 1$ and $k = l = 1$, (1.3) is a familiar system in epidemics where u and v are the densities of susceptible and infective species, respectively. When $m = 2$, $n = 1$ and $k = l = 1$, (1.3) is called the Gray-Scott model in chemistry. A special case for (1.2), (1.3) is the scalar equation

$$u_t = \Delta(u^k), \tag{1.4}$$

which is called the porous medium equation for $k > 1$ (see for instance [Pel]). This equation has been fully investigated.

Let us present some numerical simulations of (1.2) under the following initial and boundary conditions in a bounded domain Ω in $\mathbf{R}^2$

$$(u(x,0), v(x,0)) = (u_0(x), v_0) \qquad \text{for } x \in \Omega, \tag{1.5}$$

where $u_0(x)$ is an approximation of the Dirac measure at one point, v_0 is a positive constant and

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0, T), \tag{1.6}$$

where ν is the outer normal vector to the boundary $\partial\Omega$.

The first case deals with $k = 1$ (linear diffusion). The resulting patterns of u and $u + w$ are shown in Fig. 1.1; u generates an expanding ring and the corresponding w is an expanding disc. After a large time, the ring pattern of u disappears, while w occupies the whole domain so that it becomes spatially constant. This indicates that there does not develop any pattern asymptotically. The second case is that where $k = 2$. As in Fig. 1.2, for suitable values of d_u, d_v and v_0, the ring pattern of u breaks into several spots, each of which splits into smaller spots repeatedly and eventually all the spots fade away. However, the sum $u + w$ surprisingly forms very complex patterns asymptotically. These numerical results clearly indicate that the effect of nonlinear diffusion creates spatio-temporal patterns in consumer-resource systems.

Motivated by the above results, we consider the following nonlinear diffusion system with resource-consumer interaction:

$$\begin{cases} u_t = \Delta(u^k) + u^m v - a(u,v)u^n & \text{in } Q_T := \Omega \times (0, T) \\ v_t = d\Delta(v^l) - u^m v & \text{in } Q_T \end{cases} \tag{1.7}$$

where k, l, m and n are positive integers, d is a positive constant and $a(u, v)$ is a strictly positive function of u and v. Ω is a smooth domain of $\mathbf{R}^N$ and $T > 0$. We associate to the system (1.7) the boundary and initial conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0, T), \tag{1.8}$$

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \qquad \text{for all } x \in \Omega. \tag{1.9}$$

The organization of this paper is as follows: In Section 2, we state the main results. In Section 3, we define the notion of weak solutions of the initial-boundary problem (1.7)–(1.9) (we refer to it as Problem (P) hereafter) and present a sequence of related uniformly parabolic problem (P^ε) and denote

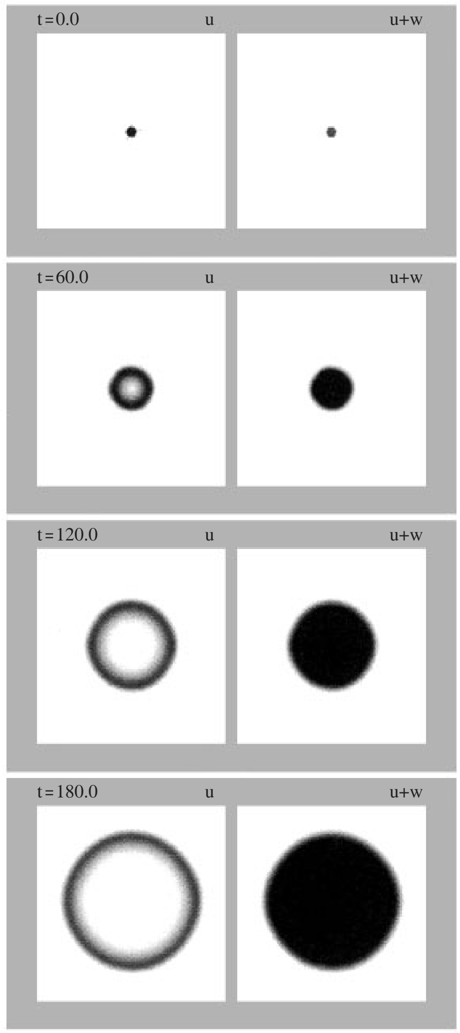

Fig. 1.1. Time evolution of solutions to (1.3), (1.5) and (1.6) where $d_u = 0.01$, $d_v = 1.0$, $k = l = 1$, $m = n = 1$, $a = 0.15$ and $v_0 = 1.0$.

their solution by $(u^\varepsilon, v^\varepsilon)$. In Section 4, we state some auxiliary results which are useful in the sequel. We present in Section 5 L^p-bounds (with p arbitrary) for the function u^ε which depend neither on ε nor on time. In Section 6, we prove an L^∞-bound for u^ε uniformly in time. Then the existence, uniqueness and continuity of the weak solution of Problem (P) follow in Section 7. Finally we describe in Section 8 the large time behavior of the solution of Problem (P): there exists a pair of nonnegative constants (u^∞, v^∞) such that

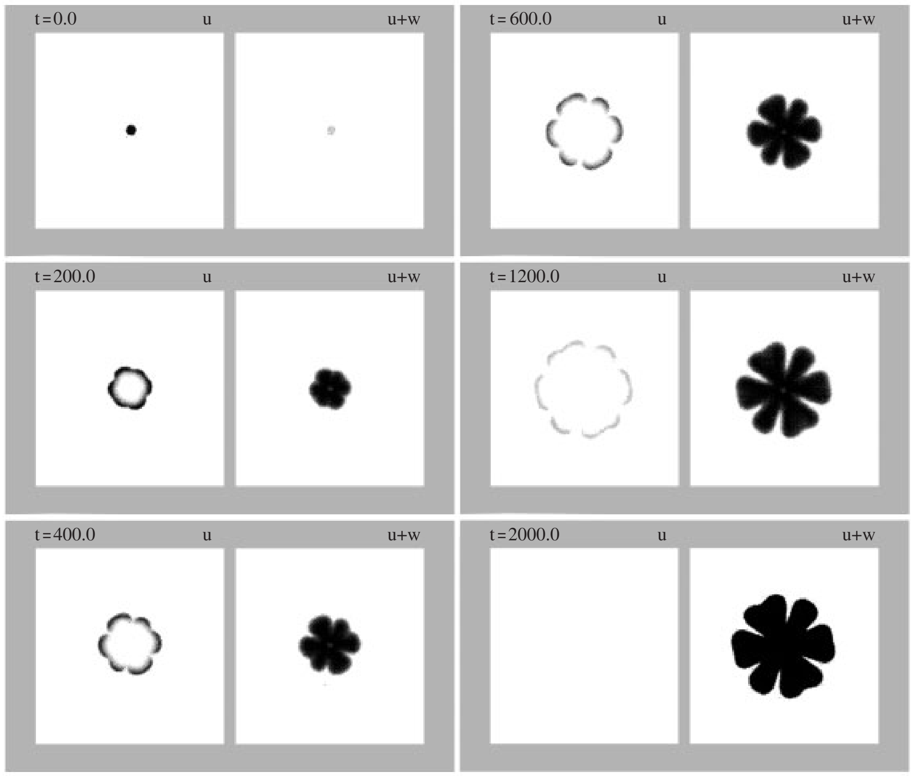

Fig. 1.2. Time evolution of solutions to (1.3), (1.5) and (1.6) where the parameters are the same ones as in Fig. 1.1 except for $k = 2$.

$(u, v)(t)$ tends to (u^∞, v^∞) as $t \to \infty$. The case that $a = 0$ is also discussed in a similar way.

2. Main results

We suppose that the following hypothesis holds:

$$H_0 : u_0, v_0 \in C(\overline{\Omega}), \qquad 0 \le u_0, v_0 \le M,$$

for some constant $M > 0$,

H_a: a is a stricly positive locally Lipschitz continuous function on $\mathbf{R}^+ \times \mathbf{R}^+$ or $a = 0$, and

$$1 \le m < \begin{cases} k + 2/N & \text{if } N \ge 3, \\ k + 1 & \text{if } N = 1, 2. \end{cases}$$

The definition of weak solutions of Problem (P) is stated in Definition 3.1 in Section 3. The results are, in the case that $a \ne 0$,

(i) Problem (P) admits a unique weak solution (u, v) satisfying

$$0 \le u(x,t) \le C_0, \quad \text{and} \quad 0 \le v(x,t) \le M \quad \text{for all } (x,t) \in \Omega \times (0,T),$$

for some constant $C_0 > 0$.

(ii) There exists a constant v^∞ such that

$$\lim_{t\to\infty} (u(t), v(t)) = (0, v^\infty) \qquad \text{uniformly in } \bar{\Omega}.$$

Furthermore, if $1 \le m < n$, $v^\infty = 0$, while, if $1 \le n \le m$ then $v^\infty > 0$. Especially if $m = n$, then $v^\infty \le a(0, v^\infty)$.

If we consider the third equation for the unknown w in addition to (1.7)

$$w_t = a(u,v)u^n \qquad \text{in } \Omega \times (0,T), \tag{2.1}$$

with

$$w(x,0) = 0 \qquad \text{for all } x \in \Omega,$$

then we find that there exists $w^\infty(x)$ such that

$$\lim_{t\to\infty} w(x,t) = w^\infty(x) \qquad \text{uniformly in } \bar{\Omega}.$$

For the special case that $a = 0$, we can also obtain a similar result as (i) and (ii)$'$ $(u(t), v(t))$ tends to $(\langle u_0 + v_0\rangle, 0)$ uniformly in $\bar{\Omega}$ as $t \to \infty$.

3. A sequence of approximate problems

Since in general a solution of Problem (P) is not smooth, we define a weak solution as follows

Definition 3.1. *We say that (u, v) is a weak solution of Problem (P) on $[0, T]$, if it satisfies:*

(i) *$u, v \in C(\bar{Q}_T)$ and $u, v \ge 0$,*

(ii) *For all $\varphi \in C^{2,1}(\bar{Q}_T)$ such that $\dfrac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times [0,T]$, we have for all $t \in [0,T]$:*

$$\int_\Omega u(t)\varphi(t) = \int_\Omega u_0\varphi(0) + \int_0^t \int_\Omega (u^k \Delta\varphi + u\varphi_t + (u^m v - a(u,v)u^n)\varphi), \tag{3.1}$$

$$\int_\Omega v(t)\varphi(t) = \int_\Omega v_0\varphi(0) + \int_0^t \int_\Omega (dv^l \Delta\varphi + v\varphi_t - u^m v\varphi). \tag{3.2}$$

In order to prove the existence of a solution of Problem (P), we introduce the sequence of approximate problems (P^ε)

$$(P^{\varepsilon})\quad \begin{cases} u_t = \Delta\phi_{\varepsilon}(u) + u^m v - a(u,v)u^n - \varepsilon u^{m+1} & \text{in } Q_T, \\ v_t = d\Delta\psi_{\varepsilon}(v) - u^m v & \text{in } Q_T, \\ \dfrac{\partial}{\partial v}\phi_{\varepsilon}(u) = \dfrac{\partial}{\partial v}\psi_{\varepsilon}(v) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0^{\varepsilon}(x) \;\; v(x,0) = v_0^{\varepsilon}(x) & \text{for all } x \in \Omega, \end{cases}$$

where $\phi_{\varepsilon}(s) := (s+\varepsilon)^k - \varepsilon^k$, $\psi_{\varepsilon}(s) := (s+\varepsilon)^l - \varepsilon^l$ and u_0^{ε} (resp. v_0^{ε}) is a smooth approximation of u_0 (resp. v_0) such that $\|u_0^{\varepsilon}\|_{L^{\infty}(\bar{\Omega})}, \|v_0^{\varepsilon}\|_{L^{\infty}(\bar{\Omega})} \le M$,

$$\frac{\partial}{\partial v}\phi_{\varepsilon}(u_0^{\varepsilon}) = \frac{\partial}{\partial v}\psi_{\varepsilon}(v_0^{\varepsilon}) = 0 \qquad \text{on } \partial\Omega,$$

and

$$\lim_{\varepsilon\to 0}(\|u_0^{\varepsilon} - u_0\|_{L^{\infty}(\bar{\Omega})} + \|v_0^{\varepsilon} - v_0\|_{L^{\infty}(\bar{\Omega})}) = 0.$$

We prove below the following result.

Theorem 3.2. *There exists a unique classical solution pair* $(u^{\varepsilon}, v^{\varepsilon})$ *of Problem* (P^{ε}).

Proof. We define $K := \{w \in C(\bar{Q}_T), 0 \le w \le M\}$ and suppose that $v^{\varepsilon} \in K$. Since Problem (P_u^{ε}) defined by

$$(P_u^{\varepsilon})\quad \begin{cases} u_t = \Delta\phi_{\varepsilon}(u) + u^m v^{\varepsilon} - a(u,v^{\varepsilon})u^n - \varepsilon u^{m+1} & \text{in } \Omega \times (0,T), \\ \dfrac{\partial}{\partial v}\phi_{\varepsilon}(u) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0^{\varepsilon}(x) & \text{for } x \in \Omega, \end{cases}$$

is uniformly parabolic, it possesses a unique classical solution u^{ε} (see [LSU, Chapter 5, Theorem 7.4]), and it follows from the standard comparison principle that $0 \le u^{\varepsilon} \le M/\varepsilon$. Furthermore we have the following stability property: if u_1^{ε} and u_2^{ε} are two solutions of Problem (P_u^{ε}) corresponding to v_1^{ε} and v_2^{ε} and the initial functions $u_{1,0}^{\varepsilon}, u_{2,0}^{\varepsilon}$, it follows as in [ACP, Corollary 11] that

$$\begin{aligned} \|u_1^{\varepsilon}(t) - u_2^{\varepsilon}(t)\|_{L^1(\Omega)} \le \|u_{1,0}^{\varepsilon} - u_{2,0}^{\varepsilon}\|_{L^1(\Omega)} + \int_0^t \|(u_1^{\varepsilon})^m v_1^{\varepsilon} - a(u_1^{\varepsilon}, v_1^{\varepsilon})(u_1^{\varepsilon})^n \\ - \varepsilon(u_1^{\varepsilon})^{m+1} - (u_2^{\varepsilon})^m v_2^{\varepsilon} + a(u_2^{\varepsilon}, v_2^{\varepsilon})(u_2^{\varepsilon})^n \\ + \varepsilon(u_2^{\varepsilon})^{m+1}\|_{L^1(\Omega)}(s)ds, \end{aligned}$$

so that there exists a positive constant C which depends on $\|u_1^{\varepsilon}\|_{L^{\infty}(\bar{Q}_T)}$, $\|u_2^{\varepsilon}\|_{L^{\infty}(\bar{Q}_T)}$ and on the data M, m, n and a such that

$$\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\|_{L^1(\Omega)} \le C\left(\|u_{1,0}^\varepsilon - u_{2,0}^\varepsilon\|_{L^1(\Omega)} + \int_0^t \|u_1^\varepsilon - u_2^\varepsilon\|_{L^1(\Omega)}(s)ds + \int_0^t \|v_1^\varepsilon - v_2^\varepsilon\|_{L^1(\Omega)}(s)ds\right) \tag{3.3}$$

$$\le C\left(\|u_{1,0}^\varepsilon - u_{2,0}^\varepsilon\|_{L^1(\Omega)} + \int_0^t \|u_1^\varepsilon - u_2^\varepsilon\|_{L^1(\Omega)}(s)ds + \int_0^T \|v_1^\varepsilon - v_2^\varepsilon\|_{L^1(\Omega)}(s)ds\right). \tag{3.4}$$

Then Gronwall's Lemma implies that

$$\int_0^T \|u_1^\varepsilon - u_2^\varepsilon\|_{L^1(\Omega)}(s)ds \le e^{CT}\left(\|u_{1,0}^\varepsilon - u_{2,0}^\varepsilon\|_{L^1(\Omega)} + \int_0^T \|v_1^\varepsilon - v_2^\varepsilon\|_{L^1(\Omega)}(s)ds\right). \tag{3.5}$$

Moreover, if u^ε is a solution of Problem (P_u^ε), there exists a unique classical solution $\hat{v}^\varepsilon$ of the problem

$$(P_v^\varepsilon) \quad \begin{cases} v_t = d\Delta\psi_\varepsilon(v) - (u^\varepsilon)^m v & \text{in } \Omega \times (0,T), \\ \dfrac{\partial}{\partial \nu}\psi_\varepsilon(v) = 0 & \text{on } \partial\Omega \times (0,T), \\ v(x,0) = v_0^\varepsilon(x) & \text{for } x \in \Omega. \end{cases}$$

By the standard comparison principle, we have that

$$0 \le \hat{v}^\varepsilon \le M.$$

Moreover if u_1^ε and u_2^ε are two given functions and $\hat{v}_1^\varepsilon, \hat{v}_2^\varepsilon$ the corresponding solutions with initial functions $\hat{v}_{1,0}^\varepsilon, \hat{v}_{2,0}^\varepsilon$, we have that

$$\|\hat{v}_1^\varepsilon(t) - \hat{v}_2^\varepsilon(t)\|_{L^1(\Omega)} \le C\left(\|\hat{v}_{1,0}^\varepsilon - \hat{v}_{2,0}^\varepsilon\|_{L^1(\Omega)} + \int_0^t \|\hat{v}_1^\varepsilon - \hat{v}_2^\varepsilon\|_{L^1(\Omega)}(s)ds + \int_0^t \|u_1^\varepsilon - u_2^\varepsilon\|_{L^1(\Omega)}(s)ds\right), \tag{3.6}$$

where $C = C(\|u_1^\varepsilon\|_{L^\infty(\bar{Q}_T)}, \|u_2^\varepsilon\|_{L^\infty(\bar{Q}_T)})$ is a positive constant, which by Gronwall's Lemma implies that

$$\int_0^T \|\hat{v}_1^\varepsilon - \hat{v}_2^\varepsilon\|_{L^1(\Omega)}(s)ds \le e^{CT}\left(\|\hat{v}_{1,0}^\varepsilon - \hat{v}_{2,0}^\varepsilon\|_{L^1(\Omega)} + \int_0^T \|u_1^\varepsilon - u_2^\varepsilon\|_{L^1(\Omega)}(s)ds\right). \tag{3.7}$$

Therefore, we have defined a map $\mathscr{F} : v^\varepsilon \mapsto \hat{v}^\varepsilon$ from K into itself. Let $v_1^\varepsilon, v_2^\varepsilon \in K$. We remark that we take $u_1^\varepsilon(0) = u_2^\varepsilon(0) = u_0^\varepsilon$ and $\hat{v}_1^\varepsilon(0) = \hat{v}_2^\varepsilon(0) = v_0^\varepsilon$ in the problems (P_u^ε) and (P_v^ε) when we define $\mathscr{F}$. We deduce from the inequalities (3.5) and (3.7) that there exists a constant $C = C(\|u_1^\varepsilon\|_{L^\infty(\bar{Q}_T)}, \|u_2^\varepsilon\|_{L^\infty(\bar{Q}_T)})$ such that for all $v_1^\varepsilon, v_2^\varepsilon \in K$ we have

$$\int_0^T \|\mathscr{F}(v_1^\varepsilon) - \mathscr{F}(v_2^\varepsilon)\|_{L^1(\Omega)}(s)ds = \int_0^T \|\hat{v}_1^\varepsilon - \hat{v}_2^\varepsilon\|_{L^1(\Omega)}(s)ds \tag{3.8}$$

$$\leq e^{2CT}\int_0^T \|v_1^\varepsilon - v_2^\varepsilon\|_{L^1(\Omega)}(s)ds. \tag{3.9}$$

Thus $\mathscr{F}$ is continuous in the $L^1(Q_T)$ norm. Furthermore, if $\{v_j^\varepsilon\}$ is a sequence of functions in K, a result of DiBenedetto [DiB] insures that the sequence $\{\mathscr{F}(v_j^\varepsilon)\}$ is precompact in $C(\bar{Q}_T)$. Suppose that $v_j^\varepsilon \to v^\varepsilon$ as $j \to \infty$ in $C(\bar{Q}_T)$, we deduce from the inequality (3.9) that $\mathscr{F}(v_j^\varepsilon) \to \mathscr{F}(v^\varepsilon)$ in $L^1(Q_T)$ as $j \to \infty$ and thus in $C(\bar{Q}_T)$. Thus the mapping $\mathscr{F} : v^\varepsilon \mapsto \hat{v}^\varepsilon$ is continuous and compact for the $C(\bar{Q}_T)$ topology from the closed convex set K into itself. We deduce from Schauder fixed point theorem (see [Sma, Theorem 4.1.1]) that there exists a function $v^\varepsilon \in K$ such that $v^\varepsilon = \hat{v}^\varepsilon$. This proves the existence of a solution of Problem (P^ε). Also, if $(u_1^\varepsilon, v_1^\varepsilon)$ and $(u_2^\varepsilon, v_2^\varepsilon)$ are two solution pairs, adding up (3.3) and (3.6) gives

$$\begin{aligned}
&\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\|_{L^1(\Omega)} + \|v_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{L^1(\Omega)} \\
&\quad \leq C\Bigg(\|u_1^\varepsilon(0) - u_2^\varepsilon(0)\|_{L^1(\Omega)} + \|v_1^\varepsilon(0) - v_2^\varepsilon(0)\|_{L^1(\Omega)} \\
&\qquad + \int_0^t (\|u_1^\varepsilon(s) - u_2^\varepsilon(s)\|_{L^1(\Omega)} + \|v_1^\varepsilon(s) - v_2^\varepsilon(s)\|_{L^1(\Omega)})ds\Bigg)
\end{aligned}$$

for all $t \in [0, T]$, where $C = C(\|u_1^\varepsilon\|_{L^\infty(\bar{Q}_T)}, \|u_2^\varepsilon\|_{L^\infty(\bar{Q}_T)})$. Then Gronwall's Lemma implies that

$$\begin{aligned}
&\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\|_{L^1(\Omega)} + \|v_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{L^1(\Omega)} \\
&\quad \leq Ce^{Ct}[\|u_1^\varepsilon(0) - u_2^\varepsilon(0)\|_{L^1(\Omega)} + \|v_1^\varepsilon(0) - v_2^\varepsilon(0)\|_{L^1(\Omega)}],
\end{aligned} \tag{3.10}$$

for all $t \in [0, T]$. The uniqueness of the solution of Problem (P^ε) follows from (3.10). □

Next we present some bounds for solution pairs $(u^\varepsilon, v^\varepsilon)$ of Problem (P^ε).

Lemma 3.3. *Let $(u^\varepsilon, v^\varepsilon)$ be the solution of Problem (P^ε), there exists a positive constant C_1 which does not depend on ε nor on T such that*
(i) $0 \leq u^\varepsilon \leq M/\varepsilon$, $0 \leq v^\varepsilon \leq M$;
(ii) $\|(u^\varepsilon)^m v^\varepsilon\|_{L^1(0,T;L^1(\Omega))} \leq C_1$;
(iii) $\|u^\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C_1$;
(iv) $\|a(u^\varepsilon, v^\varepsilon)(u^\varepsilon)^n\|_{L^1(0,T;L^1(\Omega))} \leq C_1$.

PROOF. (i) follows from the standard comparison principle. (ii) follows from

$$\int_0^t \int_\Omega (u^\varepsilon)^m v^\varepsilon = \int_\Omega v_0^\varepsilon - \int_\Omega v^\varepsilon(t) \le M|\Omega|, \tag{3.11}$$

for all $t \in [0, T]$. Integrating the first equation we deduce that

$$\int_\Omega u^\varepsilon(t) + \int_0^t \int_\Omega a(u^\varepsilon, v^\varepsilon)(u^\varepsilon)^n \le \int_\Omega u_0^\varepsilon + \int_0^t \int_\Omega (u^\varepsilon)^m v^\varepsilon, \tag{3.12}$$

for all $t \in [0, T]$, so that (iii) and (iv) follow. □

In the following sections, we will prove a bound for u^ε uniform in ε and in time.

4. Auxiliary results

In this section we present some preliminary results which will be useful in the following.

LEMMA 4.1. (i) *If* $N \ge 3$ *there exists a constant* $c_0 = c_0(\Omega)$ *such that for all* $\alpha \ge 1$ *and* $z \in H^1(\Omega)$

$$\left(\int_\Omega |z|^{2^\star}\right)^{2/2^\star} \le c_0\left[\int_\Omega |\nabla z|^2 dx + \left(\frac{2}{|\Omega|}\int_\Omega |z|^{2/\alpha} dx\right)^\alpha\right], \tag{4.1}$$

with $2^\star := \dfrac{2N}{N-2}$*;*

(ii) *If* $N = 1, 2$ *and* $q \ge 1$ *is arbitrary there exists a constant* $c_0 = c_0(q, \Omega)$ *such that for all* $\alpha \ge 1$ *and* $z \in H^1(\Omega)$

$$\left(\int_\Omega |z|^q\right)^{2/q} \le c_0\left[\int_\Omega |\nabla z|^2 dx + \left(\frac{2}{|\Omega|}\int_\Omega |z|^{2/\alpha} dx\right)^\alpha\right]; \tag{4.2}$$

(iii) *In addition, if* $N = 1$ *there exists a constant* $c_0 = c_0(\Omega)$ *such that for all* $\alpha \ge 1$ *and* $z \in H^1(\Omega)$

$$(\sup|z|)^2 \le c_0\left[\int_\Omega |\nabla z|^2 dx + \left(\frac{2}{|\Omega|}\int_\Omega |z|^{2/\alpha} dx\right)^\alpha\right]. \tag{4.3}$$

We refer to the Appendix for the proof of Lemma 4.1.

LEMMA 4.2 (Young's inequalities). *Let* $r \in (0, 1)$ *and* $\eta > 0$ *be arbitrary. Then*

(i) $a^r b^{1-r} \le \eta a + \eta^{-r/(1-r)} b$ *for all* $a, b > 0$*;*

(ii) $a^r B \le \eta a + \eta^{-r/(1-r)} B^{1/(1-r)}$ *for all* $a, B > 0$.

PROOF. (i) We first recall the proof of the classical Young inequality

$$a^r b^{1-r} \le a + b. \tag{4.4}$$

Set $a = e^A$, $b = e^B$. Then by the convexity of exponential function

$$\begin{aligned} a^r b^{1-r} &= e^{rA} e^{(1-r)B} = e^{rA+(1-r)B} \\ &\le re^A + (1-r)e^B \le e^A + e^B = a + b. \end{aligned}$$

In turn (4.4) implies that

$$a^r b^{1-r} = (\eta a)^r \left(\frac{b}{\eta^{r/(1-r)}} \right)^{1-r} \le \eta a + \eta^{-r/(1-r)} b.$$

(ii) We set $B = b^{1-r}$. □

We recall Hölder's inequality in a form which is used in the sequel.

LEMMA 4.3 (Hölder's inequalities). *Let f be a nonnegative measurable function on Ω and $a > 0$.*

(i) *if $0 < s < 1$ and $b, c > 0$ are such that $sb + (1-s)c = a$ then*

$$\int_\Omega f^a \le \left(\int_\Omega f^b \right)^s \left(\int_\Omega f^c \right)^{1-s}.$$

(ii) *if $r, s, t, b, c, d > 0$ are such that $r + s + t = 1$ and $rb + sc + td = a$ then*

$$\int_\Omega f^a \le \left(\int_\Omega f^b \right)^r \left(\int_\Omega f^c \right)^s \left(\int_\Omega f^d \right)^t.$$

PROOF. Let g, h be two nonnegative measurable functions and $0 < s < 1$. The Hölder inequality gives

$$\int_\Omega gh \le \left(\int_\Omega g^{1/s} \right)^s \left(\int_\Omega h^{1/(1-s)} \right)^{1-s}.$$

Now let b, c be as in (i); we have

$$\int_\Omega f^a = \int_\Omega f^{sb} f^{(1-s)c} \le \left(\int_\Omega f^{sb/s} \right)^s \left(\int_\Omega f^{(1-s)c/(1-s)} \right)^{1-s},$$

and thus

$$\int_\Omega f^a \le \left(\int_\Omega f^b \right)^s \left(\int_\Omega f^c \right)^{1-s}.$$

The proof of (ii) is similar and omitted. □

5. L^p bounds for the sequence $\{u^\varepsilon\}$

We present below uniform in time L^p-bounds for the solution u^ε of Problem (P_u^ε). We first prove the following property

LEMMA 5.1. *We have that for all $t > 0$, $\varepsilon > 0$ and for all $p \geq 1$*

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega (u^\varepsilon)^{p+1}(t) + \frac{4kp}{(k+p)^2}\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2 \leq M\int_\Omega (u^\varepsilon)^{m+p}(t). \quad (5.1)$$

PROOF. Multiplying by $(u^\varepsilon)^p(t)$ the partial differential equation in (P_u^ε) and integrating over Ω gives

$$\begin{aligned}\int_\Omega [u_t^\varepsilon (u^\varepsilon)^p](t) + \int_\Omega [-\{\Delta\phi_\varepsilon(u^\varepsilon)\}(u^\varepsilon)^p](t) &\leq \int_\Omega [v^\varepsilon (u^\varepsilon)^m (u^\varepsilon)^p](t)\\ &\leq \|v^\varepsilon(t)\|_{L^\infty(\Omega)}\int_\Omega (u^\varepsilon)^{m+p}(t)\\ &\leq M\int_\Omega (u^\varepsilon)^{m+p}(t).\end{aligned}$$

Since

$$\begin{aligned}\int_\Omega \{-\Delta\phi_\varepsilon(u^\varepsilon)(u^\varepsilon)^p\}(t) &= \int_\Omega \{-\Delta((\varepsilon+u^\varepsilon)^k - \varepsilon^k)(u^\varepsilon)^p\}(t)\\ &= \int_\Omega \{kp(\varepsilon+u^\varepsilon)^{k-1}(u^\varepsilon)^{p-1}|\nabla u^\varepsilon|^2\}(t)\\ &\geq \int_\Omega \{kp(u^\varepsilon)^{k+p-2}|\nabla u^\varepsilon|^2\}(t)\\ &\geq \frac{4kp}{(k+p)^2}\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2,\end{aligned}$$

we deduce that

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega (u^\varepsilon)^{p+1}(t) + \frac{4kp}{(k+p)^2}\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2 \leq M\int_\Omega (u^\varepsilon)^{m+p}(t),$$

which completes the proof. □

Next we prove the following result

LEMMA 5.2. *For all $p \geq 1$, there exists a constant C_p independent of ε and T such that*

$$\|u^\varepsilon(t)\|_{L^\infty(0,T;L^p(\Omega))} \leq C_p. \quad (5.2)$$

Proof. The existence of C_1 follows from Lemma 3.3 (iii). We first consider the case that $N \geq 3$ and then the cases that $N = 1$ and $N = 2$.

(i) **The case that** $N \geq 3$. Let $p \geq 1$ be fixed and $t \in (0, T)$. By Lemma 5.1 we have that

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega (u^\varepsilon)^{p+1}(t) + \frac{4kp}{(k+p)^2}\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2 \leq M\int_\Omega (u^\varepsilon)^{m+p}(t). \quad (5.3)$$

Furthermore Lemma 4.3 (i) yields

$$\int_\Omega (u^\varepsilon)^{m+p}(t) \leq \left(\int_\Omega (u^\varepsilon)^{2^\star(k+p)/2}(t)\right)^s \left(\int_\Omega u^\varepsilon(t)\right)^{1-s} \quad (5.4)$$

with $s = \dfrac{2(m+p-1)}{2^\star(k+p)-2}$. We remark that since $m < k + 2/N$, $r := 2^\star s/2 < 1$ which in turn implies that $0 < s < 1$. In what follows we will use that

$$\left(\int_\Omega (u^\varepsilon)^{2^\star(k+p)/2}(t)\right)^s = \left(\left(\int_\Omega (u^\varepsilon)^{2^\star(k+p)/2}(t)\right)^{2/2^\star}\right)^r. \quad (5.5)$$

Let η be a positive constant which will be chosen later. We substitute (5.5) into (5.4) and apply Young's inequality (Lemma 4.2 (ii)) to obtain

$$\int_\Omega (u^\varepsilon)^{m+p}(t) \leq \left(\left(\int_\Omega (u^\varepsilon)^{2^\star(k+p)/2}(t)\right)^{2/2^\star}\right)^r \left(\int_\Omega u^\varepsilon(t)\right)^{1-s} \quad (5.6)$$

$$\leq \eta\left(\int_\Omega (u^\varepsilon)^{2^\star(k+p)/2}(t)\right)^{2/2^\star} + \eta^{-r/(1-r)}\left(\int_\Omega u^\varepsilon(t)\right)^{(1-s)/(1-r)}. \quad (5.7)$$

Moreover, applying Lemma 4.1 (i) to $z = (u^\varepsilon)^{(k+p)/2}(t)$ with $\alpha = k + p$ gives

$$\left(\int_\Omega (u^\varepsilon)^{2^\star(k+p)/2}(t)\right)^{2/2^\star} \leq c_0\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2 + c_0\left(\frac{2}{|\Omega|}\int_\Omega u^\varepsilon(t)\right)^{k+p}. \quad (5.8)$$

Substituting (5.8) into (5.7) gives

$$\int_\Omega (u^\varepsilon)^{m+p}(t) \leq \eta c_0\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2 + \eta c_0\left(\frac{2}{|\Omega|}\int_\Omega u^\varepsilon(t)\right)^{k+p}$$
$$+ \eta^{-r/(1-r)}\left(\int_\Omega u^\varepsilon(t)\right)^{(1-s)/(1-r)} \quad (5.9)$$

where η will be chosen below. Substituting inequality (5.9) into (5.3) gives

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega (u^\varepsilon)^{p+1}(t) + \frac{4kp}{(k+p)^2}\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2$$
$$\leq M\eta c_0\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2 + M\eta c_0\left(\frac{2}{|\Omega|}\int_\Omega u^\varepsilon(t)\right)^{k+p}$$
$$+ M\eta^{-r/(1-r)}\left(\int_\Omega u^\varepsilon(t)\right)^{(1-s)/(1-r)}.$$

Choosing $\eta = \dfrac{2kp}{M(k+p)^2 c_0}$ yields

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega (u^\varepsilon)^{p+1}(t) + \frac{2kp}{(k+p)^2}\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2$$
$$\leq G_1(k,p)\left(\frac{2}{|\Omega|}\int_\Omega u^\varepsilon(t)\right)^{k+p} + G_2(\Omega, M, k, m, p)\left(\int_\Omega u^\varepsilon(t)\right)^{(1-s)/(1-r)}. \quad (5.10)$$

Applying again Lemma 4.3 (i), we have that

$$\int_\Omega (u^\varepsilon)^{p+1}(t) \leq \left(\int_\Omega (u^\varepsilon)^{2^\star(k+p)/2}(t)\right)^{s'}\left(\int_\Omega u^\varepsilon(t)\right)^{1-s'} \quad (5.11)$$

with $s' = \dfrac{2p}{2^\star(k+p)-2} \in (0,1)$. Next we substitute $r' := 2^\star s'/2 < 1$ into (5.11) and apply Young's inequality (cf. Lemma 4.2 (ii)) to deduce

$$\int_\Omega (u^\varepsilon)^{p+1}(t) \leq \left(\int_\Omega (u^\varepsilon)^{2^\star(k+p)/2}(t)\right)^{2r'/2^\star}\left(\int_\Omega u^\varepsilon(t)\right)^{1-s'} \quad (5.12)$$
$$\leq \eta\left(\int_\Omega (u^\varepsilon)^{2^\star(k+p)/2}(t)\right)^{2/2^\star}$$
$$+ \eta^{-r'/(1-r')}\left(\int_\Omega u^\varepsilon(t)\right)^{(1-s')/(1-r')}, \quad (5.13)$$

where $\eta > 0$ will be chosen below. Substituting inequality (5.8) into inequality (5.13) gives

$$\int_\Omega (u^\varepsilon)^{p+1}(t) \leq \eta c_0\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2 + \eta c_0\left(\frac{2}{|\Omega|}\int_\Omega u^\varepsilon(t)\right)^{k+p}$$
$$+ \eta^{-r'/(1-r')}\left(\int_\Omega u^\varepsilon(t)\right)^{(1-s')/(1-r')}.$$

Choosing $\eta = \dfrac{2kp(p+1)}{(k+p)^2 c_0}$ gives

$$\int_\Omega (u^\varepsilon)^{p+1}(t) \le \frac{2kp(p+1)}{(k+p)^2}\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2 + G_3(k,p)\left(\frac{2}{|\Omega|}\int_\Omega u^\varepsilon(t)\right)^{k+p}$$
$$+ G_4(\Omega,k,p)\left(\int_\Omega u^\varepsilon(t)\right)^{(1-s')/(1-r')}. \tag{5.14}$$

In view of (5.14) and then (5.10) we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u^\varepsilon)^{p+1}(t) + \int_\Omega (u^\varepsilon)^{p+1}(t)\\
&\quad\le \frac{d}{dt}\int_\Omega (u^\varepsilon)^{p+1}(t) + \frac{2kp(p+1)}{(k+p)^2}\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2\\
&\qquad + G_3\left(\frac{2}{|\Omega|}\int_\Omega u^\varepsilon(t)\right)^{k+p} + G_4\left(\int_\Omega u^\varepsilon(t)\right)^{(1-s')/(1-r')}\\
&\quad\le (p+1)G_1\left(\frac{2}{|\Omega|}\int_\Omega u^\varepsilon(t)\right)^{k+p} + (p+1)G_2\left(\int_\Omega u^\varepsilon(t)\right)^{(1-s)/(1-r)}\\
&\qquad + G_3\left(\frac{2}{|\Omega|}\int_\Omega u^\varepsilon(t)\right)^{k+p} + G_4\left(\int_\Omega u^\varepsilon(t)\right)^{(1-s')/(1-r')}.
\end{aligned}$$

By Lemma 3.3 (iii), $\int_\Omega u^\varepsilon(t)$ is bounded independently of t and ε so that

$$\frac{d}{dt}\int_\Omega (u^\varepsilon)^{p+1}(t) + \int_\Omega (u^\varepsilon)^{p+1}(t) \le G_5(\Omega, M, k, m, p). \tag{5.15}$$

We deduce from Gronwall's Lemma that

$$\begin{aligned}
\int_\Omega (u^\varepsilon)^{p+1}(t) &\le \int_\Omega (u_0^\varepsilon)^{p+1} + G_5\\
&\le |\Omega| M^{p+1} + G_5 := (C_{p+1})^{p+1}
\end{aligned}$$

which completes the proof.

(ii) **The case that** $N = 1$ **or** 2. By our assumption on k and m, we can choose $q \ge 2$ such that $m < k + 1 - 2/q$. The proof then goes as that of the case where $N \ge 3$ with $2^\star$ replaced by q and with remplacing inequality (4.1) by inequality (4.2). □

6. L^∞ bound for the sequence $\{u^\varepsilon\}$

The purpose of this section is to prove the following result:

THEOREM 6.1. *There exists a positive constant C_0 which does not depend either on ε nor on T such that*

$$\sup_{t\in[0,T]} \|u^{\varepsilon}(t)\|_{L^{\infty}(\Omega)} \leq C_0.$$

We first present a technical lemma. The proof consists in similar computations as those presented in the proof of Lemma 5.2. The estimates are now sharper, since we already know by Lemma 5.2 that the sequence $\{u^{\varepsilon}\}$ is bounded in the spaces $L^{\infty}(0,T;L^p(\Omega))$ for all $p \geq 1$ and not only in $L^{\infty}(0,T;L^1(\Omega))$ as in the beginning of the previous section.

LEMMA 6.2. *There exist some positive constants c_1, c_2 and ω which do not depend either on ε nor on T such that for all $p \geq 1$ we have for all $t \in (0,T)$*

$$\frac{d}{dt}\int_{\Omega}(u^{\varepsilon})^{p+1}(t) + c_1\int_{\Omega}|\nabla\{(u^{\varepsilon})^{(k+p)/2}\}(t)|^2 \leq c_2(1+p)^{\omega}\int_{\Omega}(u^{\varepsilon})^{p+1}(t). \quad (6.1)$$

PROOF. Let $p \geq 1$. We recall that by Lemma 5.1

$$\frac{1}{p+1}\frac{d}{dt}\int_{\Omega}(u^{\varepsilon})^{p+1}(t) + \frac{4kp}{(k+p)^2}\int_{\Omega}|\nabla\{(u^{\varepsilon})^{(k+p)/2}\}(t)|^2 \leq M\int_{\Omega}(u^{\varepsilon})^{p+m}(t). \quad (6.2)$$

The purpose is to decrease the power $p+m$ of u^{ε} in the right-hand-side of (6.2) to the power $p+1$ on the right-hand-side of (6.1). Therefore we may suppose that $m > 1$.

Let $q > 2(N+2)/N$ be a real number to be chosen later. We define

$$\theta_1 = (p+1)\left(1 - \frac{(m-1)}{2k+m-\frac{2}{q}(k+m+1)}\right), \quad (6.3)$$

$$\theta_2 = (m-1)\left(1 - \frac{(k-1)}{2k+m-\frac{2}{q}(k+m+1)}\right), \quad (6.4)$$

and

$$\theta_3 = \frac{(m-1)(k+p)}{2k+m-\frac{2}{q}(k+m+1)}. \quad (6.5)$$

Then we show that

$$\theta_1, \theta_2, \theta_3 > 0. \quad (6.6)$$

Since $q > 2$, we have that

$$\frac{2}{q}(k+m+1) < k+m+1 \leq 2k+m,$$

and then we deduce that

$$2k+m-\frac{2}{q}(k+m+1) > k-1 \geq 0,$$

which implies that $\theta_2 > 0$ and that $\theta_3 > 0$. Furthermore, since $m < k + 2/N$ for all $N \geq 1$, we have that

$$k+m+1 \leq 2k + \frac{N+2}{N},$$

and thus

$$\begin{aligned} 2k+1-\frac{2}{q}(k+m+1) &> 2k+1-\frac{2}{q}\left(2k+\frac{N+2}{N}\right) \\ &> 2k+1-\frac{N}{N+2}\left(2k+\frac{N+2}{N}\right) \\ &\geq 2k+1-2k\frac{N}{N+2}-1 \\ &\geq 2k\frac{2}{N+2} > 0. \end{aligned}$$

Then we have that $2k+m-\frac{2}{q}(k+m+1) > m-1$ and $\theta_1 > 0$. Next we remark that

$$\theta_1+\theta_2+\theta_3 = p+m$$

and that

$$\frac{\theta_1}{p+1}+\frac{\theta_2}{k+m+1}+\frac{2\theta_3}{q(k+p)} = 1. \tag{6.7}$$

Then by Lemma 4.3 (ii)

$$\int_\Omega (u^\varepsilon)^{m+p} = \int_\Omega (u^\varepsilon)^{\theta_1}(u^\varepsilon)^{\theta_2}(u^\varepsilon)^{\theta_3} \tag{6.8}$$

$$\leq \left(\int_\Omega (u^\varepsilon)^{p+1}\right)^{\theta_1/(p+1)} \left(\int_\Omega (u^\varepsilon)^{k+m+1}\right)^{\theta_2/(k+m+1)} \times \left(\int_\Omega (u^\varepsilon)^{q(k+p)/2}\right)^{2\theta_3/q(k+p)}, \tag{6.9}$$

Since by Lemma 5.2, u^ε is bounded in each space of the form $L^\infty(0,T;L^r(\Omega))$ uniformly on ε and T, we deduce that

$$\sup_{t\in[0,T]}\left(\int_\Omega (u^\varepsilon(t))^{k+m+1}\right)^{\theta_2/(k+m+1)} \le g_1(\Omega, M, k, m, q),$$

where the constant g_1 does not depend on ε. Therefore, we deduce from (6.9) that

$$\int_\Omega (u^\varepsilon(t))^{m+p} \le g_1\left(\int_\Omega (u^\varepsilon)^{p+1}(t)\right)^{\theta_1/(p+1)}\left(\int_\Omega (u^\varepsilon)^{q((k+p)/2)}(t)\right)^{2\theta_3/q(k+p)}. \quad (6.10)$$

Choosing $q = 2^\star$ if $N \ge 3$ and $q > 6$ arbitrary in the case that $N = 1$ or 2; so that $q > 2(N+2)/N$, we apply Lemma 4.1 to $z = (u^\varepsilon)^{(k+p)/2}$ to obtain

$$\left(\int_\Omega (u^\varepsilon)^{q((k+p)/2)}(t)\right)^{2/q} \le c_0\left(\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2 + \left(\frac{2}{|\Omega|}\int_\Omega (u^\varepsilon)^{(k+p)/\alpha}(t)\right)^\alpha\right) \quad (6.11)$$

where the positive constant $\alpha = \alpha(p) \ge 1$ will be chosen below. Next we substitute (6.11) into (6.10). We get

$$\int_\Omega (u^\varepsilon)^{m+p}(t) \le g_1(c_0)^{\theta_3/(k+p)}\left(\int_\Omega (u^\varepsilon)^{p+1}(t)\right)^{\theta_1/(p+1)}\left(\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2 + \left(\frac{2}{|\Omega|}\int_\Omega (u^\varepsilon)^{(k+p)/\alpha}(t)\right)^\alpha\right)^{\theta_3/(k+p)}.$$

Writing that $(a+b)^s \le a^s + b^s$ for all $a, b > 0$ and $s \in (0,1)$ and noticing that $\theta_3/(k+p) = 1 - \theta_1/(p+1)$, we deduce that

$$\begin{aligned}\int_\Omega (u^\varepsilon)^{m+p}(t) &\le g_2(\Omega, M, k, m, q)\Bigg(\left(\int_\Omega (u^\varepsilon)^{p+1}(t)\right)^{\theta_1/(p+1)}\left(\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2\right)^{1-\theta_1/(p+1)} \\ &\quad + \left(\int_\Omega (u^\varepsilon)^{p+1}(t)\right)^{\theta_1/(p+1)}\left(\frac{2}{|\Omega|}\int_\Omega (u^\varepsilon)^{(k+p)/\alpha}(t)\right)^{\alpha(1-\theta_1/(p+1))}\Bigg). \quad (6.12)\end{aligned}$$

Set $B := 1 - \theta_1/(p+1)$ and remark that by equations (6.6) and (6.7) $B \in (0,1)$ and that B does not depend on p. Then

$$\frac{\theta_3}{(k+p)} = 1 - \frac{\theta_1}{p+1} = B$$

and

$$\theta_3 = B(k+p) = B(p+1) + \mu \tag{6.13}$$

with

$$\mu = (k-1)B.$$

Then equation (6.12) can be rewritten as

$$\int_\Omega (u^\varepsilon)^{m+p}(t) \le g_2\left(\left(\int_\Omega (u^\varepsilon)^{p+1}(t)\right)^{1-B}\left(\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2\right)^B \right.$$
$$\left. + \left(\int_\Omega (u^\varepsilon)^{p+1}(t)\right)^{1-B}\left(\frac{2}{|\Omega|}\int_\Omega (u^\varepsilon)^{(k+p)/\alpha}(t)\right)^{\alpha B}\right). \tag{6.14}$$

We set

$$\alpha = (k+p)\frac{B+\mu}{B(p+1)+\mu};$$

then in view of (6.13)

$$\begin{aligned}\left(\frac{2}{|\Omega|}\int_\Omega (u^\varepsilon)^{(k+p)/\alpha}(t)\right)^{\alpha B} &= \left(\frac{2}{|\Omega|}\int_\Omega (u^\varepsilon)^{(B(p+1)+\mu)/(B+\mu)}(t)\right)^{(k+p)((B+\mu)/(B(p+1)+\mu))B} \\ &= \left(\frac{2}{|\Omega|}\int_\Omega (u^\varepsilon)^{(B(p+1)+\mu)/(B+\mu)}(t)\right)^{B+\mu} \\ &= \left(\frac{2}{|\Omega|}\right)^{B+\mu}\left(\int_\Omega (u^\varepsilon)^{B(p+1)/(B+\mu)}(t)(u^\varepsilon)^{\mu/(B+\mu)}(t)\right)^{B+\mu}.\end{aligned}$$

Next we use that $B+\mu$ does not depend on p and apply Hölder's inequality to deduce that

$$\begin{aligned}\left(\frac{2}{|\Omega|}\int_\Omega (u^\varepsilon)^{(k+p)/\alpha}(t)\right)^{\alpha B} &= \left(\frac{2}{|\Omega|}\right)^{B+\mu}\left(\int_\Omega (u^\varepsilon)^{B(p+1)/(B+\mu)}(t)(u^\varepsilon)^{\mu/(B+\mu)}(t)\right)^{B+\mu} \\ &\le g_3(\Omega,k,m,q)\left(\int_\Omega (u^\varepsilon)^{p+1}(t)\right)^B\left(\int_\Omega u^\varepsilon(t)\right)^\mu,\end{aligned}$$

so that by Lemma 3.3 (iii)

$$\left(\frac{2}{|\Omega|}\int_\Omega (u^\varepsilon)^{(k+p)/\alpha}(t)\right)^{\alpha B} \le g_4(\Omega,M,k,m,q)\left(\int_\Omega (u^\varepsilon)^{p+1}(t)\right)^B. \tag{6.15}$$

Thus the second term of the right-hand-side of (6.14) can be estimate by

$$\left(\int_\Omega (u^\varepsilon)^{p+1}(t)\right)^{1-B}\left(\frac{2}{|\Omega|}\int_\Omega (u^\varepsilon)^{(k+p)/\alpha}(t)\right)^{\alpha B} \leq g_4 \int_\Omega (u^\varepsilon)^{p+1}(t). \tag{6.16}$$

Next we consider the first term on the right-hand-side of (6.14). We apply Young's inequality (Lemma 4.2 (i)) to obtain

$$\left(\int_\Omega (u^\varepsilon)^{p+1}(t)\right)^{1-B}\left(\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2\right)^{B} \leq \eta^{-B/(1-B)}\int_\Omega (u^\varepsilon)^{p+1}(t) + \eta\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2.$$

Setting $\eta = \dfrac{2kp}{Mg_2(k+p)^2}$ we have that

$$\begin{aligned}\left(\int_\Omega (u^\varepsilon)^{p+1}(t)\right)^{1-B}&\left(\int_\Omega |\nabla(u^\varepsilon)^{(k+p)/2}(t)|^2\right)^{B}\\ &\leq \left(Mg_2\frac{(k+p)^2}{2kp}\right)^{B/(1-B)}\int_\Omega (u^\varepsilon)^{p+1}(t)\\ &\quad + \frac{2kp}{Mg_2(k+p)^2}\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2.\end{aligned} \tag{6.17}$$

Next we substitute (6.17) and (6.16) into (6.14) to deduce that

$$\int_\Omega (u^\varepsilon)^{m+p}(t) \leq g_2\left(\left(Mg_2\frac{(k+p)^2}{2kp}\right)^{B/(1-B)}\int_\Omega (u^\varepsilon)^{p+1}(t) + \frac{2kp}{Mg_2(k+p)^2}\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2 + g_4\int_\Omega (u^\varepsilon)^{p+1}(t)\right).$$

Therefore we have that

$$\int_\Omega (u^\varepsilon)^{m+p}(t) \leq g_5(\Omega,k,m,q)(p+1)^{B/(1-B)}\int_\Omega (u^\varepsilon)^{p+1}(t) + \frac{2kp}{M(k+p)^2}\int_\Omega |\nabla\{(u^\varepsilon)^{(k+p)/2}\}(t)|^2,$$

which we substitute into (6.2) to deduce that

$$\frac{d}{dt}\int_{\Omega}(u^{\varepsilon})^{p+1}(t)+\frac{2kp(p+1)}{(k+p)^2}\int_{\Omega}|\nabla\{(u^{\varepsilon})^{(k+p)/2}\}(t)|^2$$
$$\leq Mg_5(p+1)^{1/(1-B)}\int_{\Omega}(u^{\varepsilon})^{p+1}(t),$$

which in turn implies the existence of positive constants c_1, c_2 only depending on Ω, k, m, M and q such that

$$\frac{d}{dt}\int_{\Omega}(u^{\varepsilon})^{p+1}(t)+c_1\int_{\Omega}|\nabla\{(u^{\varepsilon})^{(k+p)/2}\}(t)|^2\leq c_2(p+1)^{1/(1-B)}\int_{\Omega}(u^{\varepsilon})^{p+1}(t),$$

which completes the proof of (6.1). □

In order to complete the proof of Theorem 6.1, we recall a result due to Alikakos [Ali, Lemma 3.2] and Nakao [Nak, Lemma 3.1].

Theorem 6.3. *Let w be a (sufficiently smooth) function satisfying for all $t \in [0, T]$*

$$\frac{d}{dt}\int_{\Omega}|w|^{p+1}(t)+c_1\int_{\Omega}|\nabla\{w^{(k+p)/2}\}(t)|^2\leq c_2(1+p)^{\omega}\int_{\Omega}|w|^{p+1}(t)$$

for $p \geq 1$, where $\omega \geq 0$ is independent of p. Assume, moreover, that

$$\|w(t)\|_{L^{\infty}(0,T;L^{p}(\Omega))}\leq C_p$$

and that $\|w(0)\|_{L^{\infty}(\Omega)} \leq M$. Then there exists a constant C which depends only on C_p, M, Ω, c_1, c_2 and on ω such that

$$\sup_{t\in[0,T]}\|w(t)\|_{L^{\infty}(\Omega)}\leq C.$$

Proof of Theorem 6.1. By the lemmas 5.2 and 6.2, we can apply Theorem 6.3, which yields the result of Theorem 6.1. □

7. Existence and uniqueness of the solution of Problem (P)

In this section we prove that the sequence $(u^{\varepsilon}, v^{\varepsilon})$ tends to the unique solution of Problem (P) as $\varepsilon \to 0$. To that purpose, we will use a result due to DiBenedetto [DiB]. Since ε will tend to 0, we may assume that $\varepsilon < 1$. The main result of this section is given by

Theorem 7.1. *Problem (P) admits a unique weak solution (u, v). Moreover, the pair (u, v) is such that*

(i) $0 \leq u \leq C_0$ *and* $0 \leq v \leq M$ *in* $\Omega \times (0, \infty)$;

(ii) $\{u(t)\}_{\{t\geq 0\}}$ *and* $\{v(t)\}_{\{t\geq 0\}}$ *are equicontinuous;*

(iii) *Let* (u_1, v_1) *and* (u_2, v_2) *be two solutions of Problem* (P) *with initial functions* $(u_{1,0}, v_{1,0})$ *and* $(u_{2,0}, v_{2,0})$ *respectively; they satisfy the inequality*

$$\begin{aligned}\|u_1(.\,,t) - u_2(.\,,t)\|_{L^1(\Omega)} + \|v_1(.\,,t) - v_2(.\,,t)\|_{L^1(\Omega)}\\ \le Ce^{Ct}[\|u_{1,0} - u_{2,0}\|_{L^1(\Omega)} + \|v_{1,0} - v_{2,0}\|_{L^1(\Omega)}]\end{aligned}$$

for some positive constant C.

PROOF. We first prove the existence. We have that

$$\phi_\varepsilon(s) = (\varepsilon + s)^k - \varepsilon^k$$
$$\phi_\varepsilon'(s) = k(\varepsilon + s)^{k-1}$$
$$\phi_\varepsilon''(s) = k(k-1)(\varepsilon + s)^{k-2}.$$

so that ϕ_ε is a convex continuous bijection from $[0, C_0 + 1]$ into $[0, (\varepsilon + C_0 + 1)^k - \varepsilon^k]$ so that we may define $\beta_\varepsilon = \phi_\varepsilon^{-1}$. Moreover, we set $\beta(s) := s^{1/k}$. Since

$$[0, (C_0 + 1)^k] \subset [0, (\varepsilon + C_0 + 1)^k - \varepsilon^k]$$

and since $\phi_\varepsilon(u^\varepsilon) \in [0, (C_0 + 1)^k]$ for $0 < \varepsilon \le \varepsilon_0$ with ε_0 sufficiently small, we can fix the interval and restrict the definition of β_ε to the interval $[0, (C_0 + 1)^k]$. Then, we can easily see that β_ε' is a nonincreasing function (since ϕ_ε is a convex function) and that

$$0 < \frac{1}{k((C_0 + 1)^k + 1)^{(k-1)/k}} := \alpha_0 \le \beta_\varepsilon'(s) \le \beta'(s) \qquad \text{for all } s \in (0, (C_0 + 1)^k].$$

Next, we prove that $\beta_\varepsilon \to \beta$ uniformly on $[0, (C_0 + 1)^k]$. We remark that for each $s \in [0, C_0 + 1]$, the function $\varepsilon \mapsto \phi_\varepsilon(s)$ is nondecreasing. Then for each $s \in [0, (C_0 + 1)^k]$ the function $\varepsilon \mapsto \beta_\varepsilon(s)$ is nonincreasing so that as $\varepsilon \to 0$, $\{\beta_\varepsilon\}_{\varepsilon>0}$ is a nondecreasing sequence of continuous functions tending pointwise to the continuous function β on the compact set $[0, (C_0 + 1)^k]$. Therefore, we deduce from Dini's Theorem that

$$\beta_\varepsilon(s) \to s^{1/k} \qquad \text{uniformly on } [0, (C_0 + 1)^k] \text{ as } \varepsilon \to 0.$$

Setting $U^\varepsilon = \phi_\varepsilon(u^\varepsilon)$, we deduce that U^ε satisfies

$$\begin{cases} \dfrac{\partial}{\partial t}\beta_\varepsilon(U^\varepsilon) = \Delta U^\varepsilon + f_\varepsilon(U^\varepsilon, v^\varepsilon) & \text{for all } (x,t) \in Q_T,\\ \dfrac{\partial}{\partial \nu} U^\varepsilon = 0 & \text{for all } (x,t) \in \partial\Omega \times (0,T],\\ U^\varepsilon(x,0) = \phi_\varepsilon(u_0^\varepsilon(x)) & \text{for all } x \in \Omega, \end{cases}$$

with

$$f_\varepsilon(U^\varepsilon, v^\varepsilon) = v^\varepsilon \beta_\varepsilon(U^\varepsilon)^m - a(\beta_\varepsilon(U^\varepsilon), v^\varepsilon)\beta_\varepsilon(U^\varepsilon)^n - \varepsilon\beta_\varepsilon(U^\varepsilon)^{m+1}.$$

We remark that $f_\varepsilon(U^\varepsilon, v^\varepsilon)$ is bounded independently of ε. Multiplying the equation for U^ε by U^ε and integrating over Q_T gives

$$\int_0^T\int_\Omega \left\{\frac{\partial}{\partial t}\beta_\varepsilon(U^\varepsilon)\right\}U^\varepsilon + \int_0^T\int_\Omega |\nabla U^\varepsilon|^2 \le \phi_\varepsilon(C_0)\int_0^T\int_\Omega v^\varepsilon(u^\varepsilon)^m \le C_1(C_0+1)^k,$$

and, setting $F_\varepsilon(s) = \int_0^s \beta_\varepsilon'(v)v\,dv$ ($F_\varepsilon(s) \ge 0$ as $s \ge 0$), we have

$$\int_0^T\int_\Omega \left\{\frac{\partial}{\partial t}\beta_\varepsilon(U^\varepsilon)\right\}U^\varepsilon = \int_0^T\int_\Omega \frac{\partial}{\partial t}(F_\varepsilon(U^\varepsilon)) = \int_\Omega F_\varepsilon(U^\varepsilon(T)) - \int_\Omega F_\varepsilon(\phi_\varepsilon(u_0^\varepsilon))$$

and since $\beta_\varepsilon' \ge 0$

$$\begin{aligned}\int_\Omega F_\varepsilon(\phi_\varepsilon(u_0^\varepsilon)) &= \int_\Omega\int_0^{\phi_\varepsilon(u_0^\varepsilon)} \beta_\varepsilon'(v)v\,dv \\ &\le \int_\Omega \phi_\varepsilon(u_0^\varepsilon)\int_0^{\phi_\varepsilon(u_0^\varepsilon)} \beta_\varepsilon'(v)dv \\ &\le \int_\Omega \phi_\varepsilon(u_0^\varepsilon)u_0^\varepsilon \\ &\le C_0(C_0+1)^k|\Omega|.\end{aligned}$$

This proves that $\{U^\varepsilon\}$ is bounded in $L^2(0,T;H^1(\Omega))$ independently of ε and T. Next we remark that since $u_0^\varepsilon \to u_0$ uniformly on $\bar{\Omega}$, there exists a positive function ω such that $\omega(s) \to 0$ as $s \to 0$ and such that for all $0 < \varepsilon < \varepsilon_0$ we have that

$$|u_0^\varepsilon(x) - u_0^\varepsilon(x')| \le \omega(|x-x'|) \qquad \text{for all } x, x' \in \bar{\Omega}$$

$$\text{and thus } |u_0(x) - u_0(x')| \le \omega(|x-x'|) \qquad \text{for all } x, x' \in \bar{\Omega}.$$

Then, following DiBenedetto [DiB, Theorem 6.2 and Corollary], we deduce that $\{U^\varepsilon\}$ is equicontinuous in $\bar{Q}_T$ and thus precompact in $C(\bar{Q}_T)$. Similarly, setting $V^\varepsilon := \psi_\varepsilon(v^\varepsilon)$, we prove that $\{V^\varepsilon\}$ is precompact in $C(\bar{Q}_T)$. Thus there exist $\zeta, \xi \in C(\bar{Q}_T)$ and $\{U^{\varepsilon_j}\}, \{V^{\varepsilon_j}\}$ such that

$$\begin{cases} U^{\varepsilon_j} \to \zeta \\ V^{\varepsilon_j} \to \xi \end{cases}$$

uniformly in $C(\bar{Q}_T)$ as $\varepsilon_j \to 0$. Furthermore the difference

$$\begin{aligned}|u^{\varepsilon_j} - \zeta^{1/k}| &= |\beta_{\varepsilon_j}(U^{\varepsilon_j}) - \zeta^{1/k}| \\ &\le |\beta_{\varepsilon_j}(U^{\varepsilon_j}) - (U^{\varepsilon_j})^{1/k}| + |(U^{\varepsilon_j})^{1/k} - \zeta^{1/k}|,\end{aligned}$$

can be made arbitrary small since $\beta_\varepsilon(s) \to s^{1/k}$ uniformly on $[0,(C_0+1)^k]$ as $\varepsilon \to 0$. A similar inequality holds for $|v^{\varepsilon_j} - \xi^{1/l}|$. Setting $u = \zeta^{1/k}$ and $v = \xi^{1/l}$, we have proved that

$$\begin{cases} u^{\varepsilon_j} \to u \\ v^{\varepsilon_j} \to v \end{cases}$$

uniformly in $C(\bar{Q}_T)$ as $\varepsilon_j \to 0$.

Let $\varphi \in C^{2,1}(\bar{Q}_T)$ be such that $\dfrac{\partial}{\partial \nu}\varphi = 0$ for $(x,t) \in \partial\Omega \times [0,T]$. We have

$$\begin{aligned}\int_\Omega u^{\varepsilon_j}(t)\varphi(t) &= \int_\Omega u_0^{\varepsilon_j}\varphi(0) + \int_0^t\int_\Omega (U^{\varepsilon_j}\Delta\varphi + u^{\varepsilon_j}\varphi_t) \\ &\quad + \int_0^t\int_\Omega ((u^{\varepsilon_j})^m v^{\varepsilon_j} - a(u^{\varepsilon_j}, v^{\varepsilon_j})(u^{\varepsilon_j})^n - \varepsilon_j(u^{\varepsilon_j})^{m+1})\varphi\end{aligned}$$

and, as $j \to \infty$, we find

$$\int_\Omega u\varphi(t) = \int_\Omega u_0\varphi(0) + \int_0^t\int_\Omega (u^k\Delta\varphi + u\varphi_t + (u^m v - a(u,v)u^n)\varphi).$$

A computation similar as above shows that

$$\int_\Omega v(t)\varphi(t) = \int_\Omega v_0\varphi(0) + \int_0^t\int_\Omega (dv^l\Delta\varphi + v\varphi_t - u^m v\varphi).$$

Therefore (u,v) is a solution of Problem (P).

In view of Lemma 3.3 and Theorem 6.1, we deduce the bounds of Theorem 7.1 (i).

The equicontinuity of $\{u(t)\}_{t\ge 0}$ and $\{v(t)\}_{t\ge 0}$ follows from the proof of existence and the remark that the modulus of continuity of u and v do not depend on t, so that property (ii) holds.

The proof of the uniqueness of the solution of Problem (P) is completely similar to the uniqueness proof in the proof of Theorem 3.2, since the uniqueness result of Aronson, Crandall and Peletier [ACP, Corollary 11] holds for parabolic degenerate equations. □

Next we give some further properties of the solution pair (u,v) which will be useful for the study of the large time behavior in Section 8.

Theorem 7.2. (i) *For all* $q \geq (k+1)/2$, *the function*

$$t \mapsto \int_{\Omega} |\nabla u^q(t)|^2 \tag{7.1}$$

is in $L^1(0,\infty)$.

(ii) *For all* $q \geq (l+1)/2$ *the function*

$$t \mapsto \int_{\Omega} |\nabla v^q(t)|^2 \tag{7.2}$$

is in $L^1(0,\infty)$, *and for* $q \geq 1$

$$t \mapsto \int_{\Omega} v^q(t) \qquad \text{is a nonincreasing map.} \tag{7.3}$$

Proof. We recall that there exists a subsequence of $(u^\varepsilon, v^\varepsilon)$ which we still denote by $(u^\varepsilon, v^\varepsilon)$ such that

$$(u^\varepsilon, v^\varepsilon) \to (u, v) \qquad \text{in } C(\bar{Q}_T) \text{ as } \varepsilon \to 0.$$

(i) Let $p \geq 1$. Multiplying the equation for u^ε by $(u^\varepsilon)^p$ and integrating by parts gives

$$\frac{1}{p+1}\frac{d}{dt}\int_{\Omega}(u^\varepsilon)^{p+1} + \int_{\Omega} \nabla(u^\varepsilon)^k.\nabla(u^\varepsilon)^p \leq \int_{\Omega}(u^\varepsilon)^{m+p}v^\varepsilon \leq C_0^p \int_{\Omega}(u^\varepsilon)^m v^\varepsilon.$$

As in the proof of Lemma 5.1, we deduce that, also using Lemma 3.3 (ii),

$$\begin{aligned}\frac{1}{p+1}\int_{\Omega}(u^\varepsilon)^{p+1}(t) &+ \frac{4kp}{(k+p)^2}\int_0^t\int_{\Omega}|\nabla(u^\varepsilon)^{(k+p)/2}|^2 \\ &\leq C_0^p \int_0^t\int_{\Omega}(u^\varepsilon)^m v^\varepsilon + \frac{1}{p+1}\int_{\Omega}(u_0^\varepsilon)^p \leq C,\end{aligned} \tag{7.4}$$

where the constant C does not depend on t nor on ε. Set $q = (k+p)/2$. It follows from (7.4) that the sequence $\{\nabla(u^\varepsilon)^q\}$ is bounded in $L^2(\Omega \times (0,T))$. Thus there exists a subsequence $\nabla(u^{\varepsilon_j})^q$ and a function $U \in L^2(\Omega \times (0,T))$ such that

$$\nabla(u^{\varepsilon_j})^q \rightharpoonup U \qquad \text{weakly in } L^2(\Omega \times (0,T)).$$

Let φ be a smooth function satisfying $\dfrac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,T)$. The function u^{ε_j} satisfies the equality

$$\int_0^T\int_{\Omega}\nabla(u^{\varepsilon_j})^q\varphi = -\int_0^T\int_{\Omega}(u^{\varepsilon_j})^q\nabla\varphi,$$

in which we let $\varepsilon_j \to 0$ to deduce that

$$\int_0^T \int_\Omega U\varphi = -\int_0^T \int_\Omega u^q \nabla\varphi.$$

Therefore

$$U = \nabla u^q \qquad \text{in the sense of distributions.}$$

The function $f \mapsto \int_0^T \int_\Omega f^2$ is weakly lower semicontinuous so that

$$\int_0^T \int_\Omega |\nabla u^q|^2 \leq \liminf_{\varepsilon\to 0} \int_0^T \int_\Omega |\nabla (u^\varepsilon)^q|^2 \leq C,$$

where the constant C does not depend on T. This proves (i).

(ii) Let $p \geq 0$. Multiplying by $(v^\varepsilon)^p$ the partial differential equation for v^ε and integrating over Ω gives (since $v^\varepsilon > 0$)

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega (v^\varepsilon)^{p+1} + d\frac{4lp}{(l+p)^2}\int_\Omega |\nabla (v^\varepsilon)^{(l+p)/2}|^2 \leq 0 \tag{7.5}$$

and

$$\frac{1}{p+1}\int_\Omega (v^\varepsilon)^{p+1}(T) + d\frac{4lp}{(l+p)^2}\int_0^T \int_\Omega |\nabla (v^\varepsilon)^{(l+p)/2}|^2 \leq \frac{1}{p+1}\int_\Omega (v_0^\varepsilon)^{p+1}. \tag{7.6}$$

By equation (7.6) we conclude as in (i) that the function $v^{(l+p)/2}$ satisfies

$$\int_0^T \int_\Omega |\nabla v^{(l+p)/2}|^2 \leq \liminf_{\varepsilon\to 0} \int_0^T \int_\Omega |\nabla (v^\varepsilon)^{(l+p)/2}|^2 \leq C(v_0, l, d, p), \tag{7.7}$$

which completes the proof of (7.2). Integrating equation (7.5) over $(t, t+\tau)$ for $t > 0$ and $\tau > 0$ we have that

$$\int_\Omega (v^\varepsilon)^{p+1}(t+\tau) \leq \int_\Omega (v^\varepsilon)^{p+1}(t),$$

which in turn implies that, since $v^\varepsilon(t) \to v(t)$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$,

$$\int_\Omega v^{p+1}(t+\tau) \leq \int_\Omega v^{p+1}(t). \tag{7.8}$$

This completes the proof of Theorem 7.2 (ii). □

8. Large time behavior

Our first result is the following.

Lemma 8.1. *Let (u,v) be the weak solution of Problem (P) on $\Omega\times(0,\infty)$ and suppose that $a(u,v)$ satisfy Hypothesis H_a. Then*

(i) *if $a>0$, there exists a constant $v^\infty \in \left[0, \frac{1}{|\Omega|}\int_\Omega v_0\right]$ such that*

$$(u(t),v(t)) \to (0,v^\infty) \qquad \textit{uniformly in } \bar{\Omega} \textit{ as } t\to\infty;$$

(ii) *if $a=0$, then*

$$(u(t),v(t)) \to \left(\frac{1}{|\Omega|}\int_\Omega (u_0+v_0), 0\right) \qquad \textit{uniformly in } \bar{\Omega} \textit{ as } t\to\infty.$$

Proof. Since the functions u and v are bounded and since $\{u(t)\}_{t\geq 1}$, $\{v(t)\}_{t\geq 1}$ are equicontinuous (cf. Theorem 7.1 (ii)), it follows from Arzola-Ascoli's theorem that the sequences $\{u(t)\}_{t\geq 1}, \{v(t)\}_{t\geq 1}$ are precompact in $C(\bar{\Omega})$.

We first prove that there exists a nonnegative constant v^∞ such that

$$v(t)\to v^\infty \qquad \text{uniformly in } \bar{\Omega} \text{ as } t\to\infty. \tag{8.1}$$

Let $q\geq (l+1)/2$. It follows from Theorem 7.2 (ii) that, the function $t\mapsto \int_\Omega v^q(t)$ is nonincreasing and bounded from below so that it has a limit, say k_q, as $t\to\infty$. Then, Theorem 7.2 (ii) and the remark above imply that

$$\begin{cases} t\mapsto \int_\Omega |\nabla v^q|^2 & \text{is in } L^1(0,\infty),\\ \int_\Omega v^q(t)\to k_q & \text{as } t\to\infty,\\ \int_\Omega v^{2q}(t)\to k_{2q} & \text{as } t\to\infty. \end{cases}$$

Therefore

$$\int_\Omega \left|v^q(t)-\frac{1}{|\Omega|}k_q\right|^2 = \int_\Omega v^{2q}(t) - \frac{2}{|\Omega|}k_q\int_\Omega v^q(t) + \frac{1}{|\Omega|}k_q^2 \tag{8.2}$$

$$\to k_{2q} - \frac{1}{|\Omega|}k_q^2 \qquad \text{as } t\to\infty. \tag{8.3}$$

Moreover there exists a sequence $t_j\to\infty$ such that $\int_\Omega |\nabla v^q(t_j)|^2\to 0$. The sequence $v^q(t_j)$ satisfies

$$\begin{cases} \int_\Omega |\nabla v^q(t_j)|^2 \to 0 & \text{as } t_j\to\infty,\\ \int_\Omega v^q(t_j)\to k_q & \text{as } t_j\to\infty. \end{cases}$$

By Poincaré's inequality we have that

$$\left(\int_\Omega \left|v^q(t_j) - \frac{1}{|\Omega|}k_q\right|^2\right)^{1/2} \le \left(\int_\Omega \left|v^q(t_j) - \frac{1}{|\Omega|}\int_\Omega v^q(t_j)\right|^2\right)^{1/2}$$

$$+ \left(\int_\Omega \left|\frac{1}{|\Omega|}\int_\Omega v^q(t_j) - \frac{1}{|\Omega|}k_q\right|^2\right)^{1/2}$$

$$\le C(\Omega)\left(\int_\Omega |\nabla v^q(t_j)|^2\right)^{1/2}$$

$$+ |\Omega|^{1/2}\left|\frac{1}{|\Omega|}\int_\Omega v^q(t_j) - \frac{1}{|\Omega|}k_q\right|,$$

so that

$$\left(\int_\Omega \left|v^q(t_j) - \frac{1}{|\Omega|}k_q\right|^2\right)^{1/2} \to 0 \qquad \text{as } j \to \infty. \tag{8.4}$$

Comparing (8.3) and (8.4), we deduce that

$$k_{2q} = \frac{1}{|\Omega|}(k_q)^2.$$

In view of (8.3) we have proved that

$$\left\|v^q(t) - \frac{1}{|\Omega|}k_q\right\|_{L^2(\Omega)} \to 0 \qquad \text{as } t \to \infty,$$

and since $\{v^q(t)\}_{t\ge 1}$ is precompact in $C(\bar{\Omega})$, we deduce that

$$v^q(t) \to \frac{1}{|\Omega|}k_q \qquad \text{uniformly on } \bar{\Omega}.$$

Thus

$$v(t) \to v^\infty := \left(\frac{1}{|\Omega|}k_q\right)^{1/q} \qquad \text{uniformly on } \bar{\Omega}. \tag{8.5}$$

(i) Setting $\varphi = 1$ in the equations (3.1) and (3.2), we deduce that

$$\int_\Omega (u+v)(t) = \int_\Omega (u_0 + v_0) - \int_0^t \int_\Omega a(u,v)u^n. \tag{8.6}$$

This implies that the map $t \mapsto \int_\Omega (u+v)$ is nonincreasing and since $\int_\Omega (u+v)(t)$ is bounded from below, it has a limit as $t \to \infty$. Thus there exists u^∞ such that

$$\lim_{t\to\infty}\int_\Omega u(t) = u^\infty. \tag{8.7}$$

Moreover, equation (8.6) implies that

$$a(u,v)u^n \in L^1((0,\infty);L^1(\Omega)).$$

Since $a(u,v)$ is continuous on the compact set $[0,C_0]\times[0,M]$, it has a lower bound $a^- > 0$. It follows that

$$u^n \in L^1((0,\infty);L^1(\Omega)),$$

and therefore, by Hölder inequality, that

$$\left(\int_\Omega u\right)^n \in L^1(0,\infty).$$

Therefore there exists a sequence $t_j \to \infty$ as $j\to\infty$ such that

$$\int_\Omega u(t_j) \to 0 \qquad \text{as } j\to\infty, \tag{8.8}$$

so that $u_\infty = 0$. Finally we conclude that $u(t)$ converges to 0 uniformly in $\overline{\Omega}$ as $t\to\infty$.

(ii) We first suppose that $v_0 = 0$. In that case, if z is the solution of the porous medium equation

$$\begin{cases} z_t = \Delta z^k & \text{in } Q \\ \dfrac{\partial}{\partial \nu} z^k = 0 & \text{on } \partial\Omega\times(0,\infty) \\ z(x,0) = u_0(x) & \text{for all } x\in\Omega, \end{cases}$$

then the pair $(z(t),0)$ is the solution of Problem (P) with initial condition $(u_0,0)$. A result of Alikakos and Rostamian [AlRo] gives that

$$z(t) \to \frac{1}{|\Omega|}\int_\Omega u_0 \qquad \text{in } L^p(\Omega) \text{ for all } p\in[1,\infty) \text{ as } t\to\infty.$$

Thus since $\{z(t)\}_{t\geq 1}$ is precompact in $C(\overline{\Omega})$ and we have

$$z(t) \to \frac{1}{|\Omega|}\int_\Omega u_0 \qquad \text{in } C(\overline{\Omega}) \text{ as } t\to\infty.$$

Let (u,v) be the weak solution of Problem (P) with initial data (u_0,v_0), with $u_0 \neq 0$ and $v_0 \neq 0$, and z be as above. We can easily see that z is a lower solution for the equation satisfied by u so that

$$u \geq z \qquad \text{in } Q,$$

which in turn implies that there exists $\bar{t} \geq 0$ such that

$$u(t) \geq z(t) \geq c_3 := \frac{1}{2|\Omega|}\int_\Omega u_0 \qquad \text{for all } t \geq \bar{t}. \tag{8.9}$$

First we will prove that $v(t) \to 0$ uniformly on $\bar{\Omega}$ as $t \to \infty$. Since $u^m v$ is in $L^1((0,\infty); L^1(\Omega))$ (Lemma 3.3), and from

$$(c_3)^m v(t) \leq u^m(t)v(t)$$

for $t \geq \bar{t}$, we deduce that $v \in L^1((0,\infty); L^1(\Omega))$. Thus there is a sequence $\{t_j\}$ such that

$$\lim_{t_j\to\infty}\int_\Omega v(t_j) = 0, \tag{8.10}$$

which together with (8.5) implies that

$$v(t) \to 0 \qquad \text{uniformly in } \bar{\Omega} \text{ as } t \to 0.$$

Adding up the equations (3.1) and (3.2) and setting $\varphi = 1$ yields

$$\int_\Omega (u+v)(t) = \int_\Omega (u_0 + v_0)$$

and consequently

$$\lim_{t\to\infty}\int_\Omega u(t) = \int_\Omega (u_0 + v_0). \tag{8.11}$$

Next we prove that $u(t)$ tends to a constant uniformly in $\bar{\Omega}$ as $t \to \infty$. The method which we use here is similar to the proof of (8.1). First, we show that for each $q \geq (k+3)/2$, there exists a constant k_q such that $\lim_{t\to\infty}\int_\Omega u^q(t) = k_q$. Let $p \geq (k+1)/2$, $t \geq 0$ and $r > 0$; we take the duality product $((H^1)', H^1)$ of the equation for u by u^p and we deduce that

$$\int_t^{t+r}\langle u_t, u^p\rangle_{(H^1)',H^1} + \frac{4kp}{(k+p)^2}\int_t^{t+r}\int_\Omega |\nabla u^{(k+p)/2}|^2 = \int_t^{t+r}\int_\Omega u^{m+p}v.$$

The equality (as in [Tem, Chap. III, Lemma 1.2])

$$\int_t^{t+r}\langle u_t, u^p\rangle_{(H^1)',H^1} = \frac{1}{p+1}\int_\Omega u^{p+1}(t+r) - \frac{1}{p+1}\int_\Omega u^{p+1}(t), \tag{8.12}$$

implies that

$$\int_\Omega u^{p+1}(t+r) - \int_\Omega u^{p+1}(t)$$
$$= (p+1)\int_t^{t+r}\int_\Omega u^{m+p}v - \frac{4kp(p+1)}{(k+p)^2}\int_t^{t+r}\int_\Omega |\nabla u^{(k+p)/2}|^2. \qquad (8.13)$$

Since we have that, in view of Lemma 3.3 (ii) and Theorem 7.1 (i)

$$\int_0^\infty\int_\Omega u^{m+p}v \le (C_0)^p \int_0^\infty\int_\Omega u^m v \le (C_0)^p C_1,$$

and since, by Theorem 7.2,

$$\int_0^\infty\int_\Omega |\nabla u^{(k+p)/2}|^2 < \infty,$$

it follows from equation (8.13) that the sequence $\{\int_\Omega u^{p+1}(t)\}_{t\ge\bar{t}}$ is a Cauchy sequence. Therefore it has a limit as $t\to\infty$. Setting $q = p+1$, we have proved that there exists k_q such that

$$\lim_{t\to\infty}\int_\Omega u^q(t) = k_q \qquad \text{for all } q \ge (k+3)/2.$$

Then, in view of Theorem 7.2, we have proved that

$$\begin{cases} \int_\Omega |\nabla u^q|^2 \in L^1(0,\infty), \\ \lim_{t\to\infty}\int_\Omega u^q(t) = k_q, \\ \lim_{t\to\infty}\int_\Omega u^{2q}(t) = k_{2q}, \end{cases}$$

for all $q \ge (k+3)/2$. Now, as we have done for v in the beginning of this proof, we deduce that

$$u^q(t) \to \frac{1}{|\Omega|}k_q \qquad \text{in } C(\bar{\Omega}) \text{ as } t\to\infty.$$

Then, in view of equation (8.11), we have proved that

$$u(t) \to \frac{1}{|\Omega|}\int_\Omega (u_0+v_0) \qquad \text{in } C(\bar{\Omega}),$$

which completes the proof. □

Next we suppose that

$$u_0 \ne 0 \qquad \text{and} \qquad v_0 \ne 0, \qquad (8.14)$$

and that the function a is positive. We define two constants a^- and a^+ such that

$$0 < a^- \leq a(r,s) \leq a^+ \qquad \text{for all } (r,s) \in [0, C_0] \times [0, M],$$

and present some further characterization of the constant v^∞.

THEOREM 8.2. *The following results hold:*
(i) *If* $1 \leq m < n$ *then* $v^\infty = 0$*;*
(ii) *if* $1 \leq n \leq m$ *then* $v^\infty > 0$*;*
(iii) *if* $m = n$ *then* $v^\infty \leq a(0, v^\infty)$.

To begin with, we prove the following auxiliary result.

LEMMA 8.3. *The functions* $t \mapsto \int_\Omega u(t)$ *and* $t \mapsto \int_\Omega v(t)$ *are in* $C^1([0,\infty))$ *and*

$$\frac{d}{dt}\int_\Omega u(t) = \int_\Omega (u^m v - a(u,v)u^n)(t), \tag{8.15}$$

$$\frac{d}{dt}\int_\Omega v(t) = -\int_\Omega (u^m v)(t). \tag{8.16}$$

PROOF. Setting $\varphi = 1$ in the equality (3.1) one has

$$\int_\Omega u(t) = \int_\Omega u_0 + \int_0^t \int_\Omega (u^m v - a(u,v)u^n),$$

and the result follows from the fact that the function $t \mapsto \int_0^t \int_\Omega (u^m v - a(u,v)u^n)$ is in $C^1([0,\infty))$. The proof of (8.16) is similar. □

LEMMA 8.4. *We have that*

$$\int_\Omega u(t) > 0 \qquad \textit{and} \qquad \int_\Omega v(t) > 0 \qquad \textit{for all } t \geq 0.$$

PROOF. By equation (8.15) we have that

$$\frac{d}{dt}\int_\Omega u(t) = \int_\Omega (u^m v - a(u,v)u^n)(t)$$
$$\geq -a^+(C_0)^{n-1}\int_\Omega u(t),$$

which by Gronwall's Lemma and (8.14) implies that

$$\int_\Omega u(t) \geq \left(\int_\Omega u_0\right) \exp(-a^+ C_0^{n-1} t) > 0.$$

The proof for v is similar. □

PROOF OF THEOREM 8.2 (i). For the purpose of contradiction we suppose that $v^\infty > 0$. Then there exists $T \geq 0$ such that

$$0 \le a^+ u^{n-m}(t) \le v^\infty/4, \qquad 3v^\infty/4 \le v(t) \qquad \text{for all } t \ge T,$$

which in turn implies that

$$(v - a^+ u^{n-m})(t) \ge v^\infty/2.$$

Therefore, it follows from (8.15) that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u(t) &= \int_\Omega (u^m v - a(u,v)u^n)(t) = \int_\Omega (u^m(v - a(u,v)u^{n-m}))(t) \\ &\ge \int_\Omega (u^m(v - a^+ u^{n-m}))(t) \ge \frac{v^\infty}{2}\int_\Omega u^m(t) \\ &\ge 0.\end{aligned}$$

Thus in view of Lemma 8.4 we have that

$$\int_\Omega u(t) \ge \int_\Omega u(T) > 0 \qquad \text{for all } t \ge T.$$

This contradicts the fact that $u(t) \to 0$ as $t \to \infty$. We conclude that $v^\infty = 0$. □

Before proving Theorem 8.2 (ii), we suppose that $m \ge n$ and we set $f(x,t) := (u^{m-n}v - a(u,v))(x,t)$ and we consider the problem

$$(P_u) \quad \begin{cases} w_t = \Delta w^k + f(x,t)w^n & \text{in } \Omega \times (0,T), \\ \dfrac{\partial}{\partial \nu} w^k = 0 & \text{on } \partial\Omega \times (0,T), \\ w(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases}$$

Next we define solutions and upper and lower solutions of Problem P_u and present a comparison principle.

Definition 8.5. *We say that u is a solution of Problem* (P_u) *in* $\Omega \times [0,T^*]$, *if it satisfies:*

(i) $u \in C(\bar\Omega \times [0,T^*])$ *and* $0 \le u \le C$ *for a constant C;*

(ii) *For all* $\varphi \in C^{2,1}(\bar\Omega \times [O,T^*])$ *such that* $\varphi \ge 0$, $\dfrac{\partial\varphi}{\partial\nu} = 0$ *on* $\partial\Omega \times [0,T^*]$, *we have for all* $t \in [0,T^*]$

$$\int_\Omega u(t)\varphi(t) = \int_\Omega u_0\varphi(0) + \int_0^t\int_\Omega (u^k\Delta\varphi + u\varphi_t + f(x,t)u^n\varphi); \qquad (8.17)$$

We say that $\bar u$, *respectively* $\underline{u}$, *is a weak upper solution of Problem* (P_u), *respectively a weak lower solution, if it satisfies the property* (i)*, and the property* (ii) *with equality replaced by* $\geq$*, respectively* $\leq$*, in equation* (8.17).

We remark that if (u, v) is the unique weak solution of Problem (P), then u is also the unique weak solutions of Problem (P_u).

Then we have

Lemma 8.6 (Comparison Theorem). *If* $\bar{u}$ *is a upper solution and* $\underline{u}$ *a lower solution of Problem* (P_u) *in* $[0, T^*]$ *then*

$$\bar{u}(x,t) \geq \underline{u}(x,t) \qquad \textit{for all } (x,t) \in \Omega \times [O, T^*].$$

Proof. The proof goes as in [ACP, Theorem 12]. □

Now we prove the following lemma.

Lemma 8.7. *Suppose that* $n \leq m$ *and that there exists* $T > 0$ *such that* $u^{m-n}(t)v(t) \leq a^-/2$ *for all* $t \geq T$. *Then*
(i) *there exists a positive constant* C *such that for all* $t \geq T$

$$u(t) \leq \begin{cases} C_0 e^{-a^-(t-T)/2} & \textit{if } n = 1, \\ C(t - T + 1)^{-1/(n-1)} & \textit{if } n > 1; \end{cases}$$

(ii) $v^\infty > 0$.

Proof. (i) The condition $u^{m-n}(t)v(t) \leq \dfrac{a^-}{2}$ for all $t \geq T$ implies in particular that

$$f(x,t) = (u^{m-n}v - a(u,v))(x,t) \leq -a^-/2, \tag{8.18}$$

since by definition, we have $a(u,v)(x,t) \geq a^-$ for all $(x,t) \in \Omega \times (0,\infty)$. Then, for $t \geq T$, we define

$$\bar{u}(x,t) = \begin{cases} C_0 e^{-a^-(t-T)/2} & \text{if } n = 1, \\ C(t - T + 1)^{-1/(n-1)} & \text{if } n > 1; \end{cases} \tag{8.19}$$

where

$$C = \max\left(C_0, \left(\frac{2}{(n-1)a^-}\right)^{1/(n-1)}\right). \tag{8.20}$$

One can easily check that we have

$$\bar{u}_t \geq -a^- \bar{u}^n/2.$$

Now, let φ be as in Definition 8.5. We have that for all $t \geq T$

$$\int_\Omega \bar{u}(t)\varphi(t) - \int_\Omega \bar{u}(T)\varphi(T) - \int_T^t \int_\Omega (\bar{u}^k \Delta\varphi + \bar{u}\varphi_t + f(x,t)\bar{u}^n\varphi)$$
$$= \int_T^t \int_\Omega (\bar{u}_t\varphi + \nabla\bar{u}^k.\nabla\varphi - f(x,t)\bar{u}^n\varphi)$$
$$= \int_T^t \int_\Omega (\bar{u}_t - f(x,t)\bar{u}^n)\varphi$$
$$\geq \int_T^t \int_\Omega \left(-\frac{a^-}{2} - f(x,t)\right)\bar{u}^n\varphi \geq 0,$$

where the last inequality comes from inequality (8.18). Moreover we deduce from (8.19) and (8.20) that $\bar{u}(x,T) \geq u(x,T)$ for all $x \in \Omega$, so that $\bar{u}$ is a upper solution of Problem (P_u) in $[T,t]$ for all $t \geq T$, and it follows from Lemma 8.6 that $\bar{u}(x,t) \geq u(x,t)$ for all $(x,t) \in \Omega \times [T,\infty)$ which completes this part of the proof.

(ii) Next we prove that $v^\infty > 0$. We first consider the case that $n = 1$. The equality (8.16) and Theorem 8.7 (i) imply that for $t \geq T$

$$\frac{d}{dt}\int_\Omega v(t) \geq -\left(\int_\Omega v(t)\right)(C_0)^m e^{-a^- m(t-T)/2},$$

which in turn implies the inequality

$$\int_\Omega v(t) \geq \exp\left(\frac{2(C_0)^m}{a^- m}(e^{-a^- m(t-T)/2} - 1)\right)\int_\Omega v(T). \tag{8.21}$$

Letting $t \to \infty$ in (8.21) implies that

$$|\Omega| v^\infty = \lim_{t\to\infty}\int_\Omega v(t) \geq e^{-2(C_0)^m/a^- m}\int_\Omega v(T) > 0,$$

and therefore $v^\infty > 0$.

If $n > 1$ it follows from Theorem 8.7 (i) that

$$\frac{d}{dt}\int_\Omega v(t) \geq -\left(\int_\Omega v(t)\right)C^m(t - T + 1)^{-m/(n-1)},$$

so that finally

$$\int_\Omega v(t) \geq \exp\left(C^m \frac{n-1}{(m-n+1)}((t-T+1)^{-(m-n+1)/(n-1)} - 1)\right)\int_\Omega v(T). \tag{8.22}$$

Letting $t \to \infty$ in (8.22) yields

$$|\Omega| v^\infty = \lim_{t\to\infty} \int_\Omega v(t) \geq e^{-(n-1)/(C^m(m-n+1))} \int_\Omega v(T) > 0$$

so that $v^\infty > 0$. □

PROOF OF THEOREM 8.2 (ii). First we consider the case that $1 \leq n < m$. Since $u(t) \to 0$ as $t \to \infty$, let T be such that $u^{m-n}(t)v(t) \leq a^-/2$ for all $t \geq T$. Then we can apply Lemma 8.7 (ii) and conclude that $v^\infty > 0$.

Next we consider the case that $1 \leq n = m$; for the purpose of contradiction we suppose that $v^\infty = 0$. There exists $T > 0$ such that $u^{m-n}(t)v(t) \leq a^-/2$ for all $t \geq T$. Therefore we may apply Lemma 8.7 (ii) and conclude that $v^\infty > 0$. This contradicts the hypothesis that $v^\infty = 0$. Therefore $v^\infty > 0$. □

PROOF OF THEOREM 8.2 (iii). For the purpose of contradiction, we suppose that

$$v^\infty > a(0, v^\infty).$$

Since the function $(r,s) \mapsto s - a(r,s)$ is continuous, there exists $\eta \in (0, v^\infty)$ such that

$$s - a(r,s) \geq 0 \qquad \text{for all } (r,s) \in [0,\eta) \times (v^\infty - \eta, v^\infty + \eta).$$

Let T be such that

$$u(x,t) < \eta \qquad \text{and} \qquad |v(x,t) - v^\infty| < \eta \qquad \text{for all } (x,t) \in \overline{\Omega} \times [T,\infty).$$

Then for all $t \geq T$

$$v(x,t) - a(u,v)(x,t) \geq 0,$$

and

$$\begin{aligned} \frac{d}{dt}\int_\Omega u(t) &= \int_\Omega (u^m v - a(u,v)u^n)(t) \\ &= \int_\Omega (u^m(v - a(u,v)))(t) \\ &\geq 0, \end{aligned}$$

which by Lemma 8.4 implies that

$$\int_\Omega u(t) \geq \int_\Omega u(T) > 0.$$

This contradicts the fact that $u(t) \to 0$ as $t \to \infty$. Therefore $v^\infty \leq a(0, v^\infty)$. □

Appendix—Proof of Lemma 4.1

(i) We first consider the case that $N \geq 3$. For the purpose of contradiction we suppose that for all $\lambda > 0$ there exist $z \in H^1(\Omega)$ and $\alpha \geq 1$ such that

$$\lambda\left(\int_\Omega |\nabla z|^2 + \left(\frac{2}{|\Omega|}\int_\Omega |z|^{2/\alpha}\right)^\alpha\right) < \left(\int_\Omega |z|^{2^\star}\right)^{2/2^\star}. \tag{A.1}$$

We define a sequence $\{\lambda_n\}$ such that $\lambda_n \to \infty$ as $n \to \infty$. It follows from equation (A.1) that there exist $z_n \in H^1(\Omega)$ and $\alpha_n \geq 1$ such that

$$\lambda_n\left(\int_\Omega |\nabla z_n|^2 + \left(\frac{2}{|\Omega|}\int_\Omega |z_n|^{2/\alpha_n}\right)^{\alpha_n}\right) < \left(\int_\Omega |z_n|^{2^\star}\right)^{2/2^\star}, \tag{A.2}$$

which implies in particular that $z_n \neq 0$. We divide inequality (A.2) by $\|z_n\|^2_{L^{2^\star}(\Omega)}$ to obtain

$$\int_\Omega \left|\nabla \frac{z_n}{\|z_n\|_{L^{2^\star}(\Omega)}}\right|^2 + \left(\frac{2}{|\Omega|}\int_\Omega \left|\frac{z_n}{\|z_n\|_{L^{2^\star}(\Omega)}}\right|^{2/\alpha_n}\right)^{\alpha_n} < \frac{1}{\lambda_n}.$$

Setting

$$w_n = \frac{z_n}{\|z_n\|_{L^{2^\star}(\Omega)}}, \tag{A.3}$$

we deduce that

$$\int_\Omega |\nabla w_n|^2 + \left(\frac{2}{|\Omega|}\int_\Omega |w_n|^{2/\alpha_n}\right)^{\alpha_n} < \frac{1}{\lambda_n}. \tag{A.4}$$

It follows from (A.3) and (A.4) that

$$\begin{cases} \|w_n\|_{L^{2^\star}(\Omega)} = 1, \\ \|\nabla w_n\|_{L^2(\Omega)} \to 0 \quad \text{as } n \to \infty, \end{cases} \tag{A.5}$$

so that in particular there exists $w \in H^1(\Omega)$ such that as $n \to \infty$ $w_n \rightharpoonup w$ weakly in $H^1(\Omega)$ and $w_n \to w$ strongly in $L^2(\Omega)$ along a subsequence. It also follows from (A.5) and the weak lower semicontinuity of $z \mapsto \int_\Omega |\nabla z|^2$ that $\nabla w = 0$ in $L^2(\Omega)$. Thus there exists a constant l such that $w = l$ and $w_n \to l$ strongly in $H^1(\Omega)$. Since the embedding from $H^1(\Omega)$ into $L^{2^\star}(\Omega)$ [Bre, Corollary IX. 14] is continuous we have that

$$w_n \to l \qquad \text{strongly in } L^{2^\star}(\Omega) \text{ as } n \to \infty,$$

and therefore it follows from (A.5) that

$$|l| = |\Omega|^{-1/2^\star},$$

and we may suppose that $l > 0$. Furthermore we also deduce from (A.4) that

$$\left(\frac{2}{|\Omega|}\int_\Omega |w_n|^{2/\alpha_n} dx\right)^{\alpha_n} \to 0 \qquad \text{as } n \to \infty. \tag{A.6}$$

We consider two cases:

The case that $\{\alpha_n\}$ is bounded: Then there exists a subsequence of $\{\alpha_n\}$ which we denote again by $\{\alpha_n\}$ such that $\alpha_n \to \alpha \in [1, \infty)$ as $n \to \infty$. We have

$$\left|\int_\Omega |w_n|^{2/\alpha_n} - \int_\Omega l^{2/\alpha}\right| \le \left|\int_\Omega |w_n|^{2/\alpha_n} - \int_\Omega l^{2/\alpha_n}\right| + \left|\int_\Omega l^{2/\alpha_n} - \int_\Omega l^{2/\alpha}\right|. \tag{A.7}$$

Next we bound the first term on the right-hand-side of (A.7). Let $C > 0$ and $s \in [0, 1]$; the function $r \mapsto |r - C|^s - r^s + C^s$ is nonincreasing for $0 < r < C$ and nondecreasing for $r > C$ so that for all $r > 0$

$$|r^s - C^s| \le |r - C|^s,$$

which in turn implies that

$$\begin{aligned}
\left|\int_\Omega |w_n|^{2/\alpha_n} - \int_\Omega l^{2/\alpha_n}\right| &\le \int_\Omega \left|\, |w_n|^{2/\alpha_n} - l^{2/\alpha_n}\right| \\
&\le \int_\Omega |w_n^2 - l^2|^{1/\alpha_n} \\
&\le \left(\int_\Omega |w_n^2 - l^2|\right)^{1/\alpha_n} |\Omega|^{(\alpha_n-1)/\alpha_n} \\
&\le \left(\int_\Omega |(w_n - l)(w_n + l)|\right)^{1/\alpha_n} |\Omega|^{(\alpha_n-1)/\alpha_n} \\
&\le \left(\left(\int_\Omega (w_n - l)^2\right)^{1/2}\left(\int_\Omega (w_n + l)^2\right)^{1/2}\right)^{1/\alpha_n} |\Omega|^{(\alpha_n-1)/\alpha_n}.
\end{aligned}$$

Since as $n \to \infty$, we have that $w_n \to l$ in $L^2(\Omega)$, $w_n + l \to 2l$ in $L^2(\Omega)$ and in particular $w_n + l$ is bounded in $L^2(\Omega)$. Thus

$$\left|\int_\Omega |w_n|^{2/\alpha_n} - \int_\Omega l^{2/\alpha_n}\right| \le |\Omega|^{(\alpha_n-1)/\alpha_n}\left(\sup_n\left(\int_\Omega (w_n + l)^2\right)\right)^{1/2\alpha_n}\left(\int_\Omega (w_n - l)^2\right)^{1/2\alpha_n}. \tag{A.8}$$

Since the sequence $\{\alpha_n\}$ is bounded there exist two constants α_- and α_+ such that for all n we have $1 \le \alpha_- \le \alpha_n \le \alpha_+$; therefore inequality (A.8) gives

$$\begin{aligned}
\left|\int_\Omega |w_n|^{2/\alpha_n} - \int_\Omega l^{2/\alpha_n}\right| &\le (|\Omega| + 1)^{(\alpha_+-1)/\alpha_+}\left(\sup_n\left(\int_\Omega (w_n + l)^2\right) + 1\right)^{1/2\alpha_-} \\
&\quad \times \left(\int_\Omega (w_n - l)^2\right)^{1/2\alpha_+},
\end{aligned}$$

so that

$$\left| \int_\Omega |w_n|^{2/\alpha_n} - \int_\Omega l^{2/\alpha_n} \right| \to 0 \qquad \text{as } n \to \infty.$$

Moreover, we have

$$\left| \int_\Omega (l^{2/\alpha_n} - l^{2/\alpha}) \right| \to 0 \qquad \text{as } n \to \infty.$$

Thus we deduce from (A.7)

$$\left| \int_\Omega |w_n|^{2/\alpha_n} - \int_\Omega l^{2/\alpha} \right| \to 0 \qquad \text{as } n \to \infty. \tag{A.9}$$

Furthermore we have

$$\begin{aligned} &\left| \left(\frac{2}{|\Omega|} \int_\Omega |w_n|^{2/\alpha_n} \right)^{\alpha_n} - \left(\frac{2}{|\Omega|} \int_\Omega l^{2/\alpha} \right)^{\alpha} \right| \\ &\quad \le \left| \left(\frac{2}{|\Omega|} \int_\Omega |w_n|^{2/\alpha_n} \right)^{\alpha_n} - \left(\frac{2}{|\Omega|} \int_\Omega l^{2/\alpha} \right)^{\alpha_n} \right| \\ &\qquad + \left| \left(\frac{2}{|\Omega|} \int_\Omega l^{2/\alpha} \right)^{\alpha_n} - \left(\frac{2}{|\Omega|} \int_\Omega l^{2/\alpha} \right)^{\alpha} \right|. \end{aligned} \tag{A.10}$$

The second term of the right-hand-side of (A.10) tends to 0 as $n \to \infty$. Moreover, by (A.9) we can suppose that $\int_\Omega |w_n|^{2/\alpha_n} \in [0, \int_\Omega l^{2/\alpha} + 1]$. Since for $r, s \ge 0$, we have

$$\begin{aligned} |r^{\alpha_n} - s^{\alpha_n}| &\le \alpha_n (\max(r, s, 1))^{\alpha_n - 1} |r - s| \\ &\le \alpha_+ (\max(r, s, 1))^{\alpha_+ - 1} |r - s|, \end{aligned}$$

we deduce from (A.9) that

$$\left| \left(\frac{2}{|\Omega|} \int_\Omega |w_n|^{2/\alpha_n} \right)^{\alpha_n} - \left(\frac{2}{|\Omega|} \int_\Omega l^{2/\alpha} \right)^{\alpha_n} \right| \to 0 \qquad \text{as } n \to \infty,$$

which by (A.10) implies that

$$\left| \left(\frac{2}{|\Omega|} \int_\Omega |w_n|^{2/\alpha_n} \right)^{\alpha_n} - \left(\frac{2}{|\Omega|} \int_\Omega l^{2/\alpha} \right)^{\alpha} \right| \to 0 \qquad \text{as } n \to \infty. \tag{A.11}$$

Clearly (A.11) contradicts (A.6).

The case that $\{\alpha_n\}$ is unbounded: Then there exists a subsequence of $\{\alpha_n\}$ which we denote again by $\{\alpha_n\}$ such that $\alpha_n \to \infty$ as $n \to \infty$. Let μ be a positive number. We have

$$\mu \operatorname{mes}\{x, |w_n(x) - l| > \mu\} \le \int_\Omega |w_n - l| \to 0 \qquad \text{as } n \to \infty.$$

Note that $\int_\Omega |w_n - l| \le \mu^2$ for n large enough so that

$$\operatorname{mes}\{x, |w_n(x) - l| > \mu\} \le \mu,$$

for n large enough. Therefore

$$\operatorname{mes}\{x, |w_n(x) - l| \le \mu\} \ge |\Omega| - \mu. \tag{A.12}$$

Choosing $\mu < \dfrac{l}{2}$, we have in view of (A.12)

$$\begin{aligned}\int_\Omega |w_n|^{2/\alpha_n} &\ge \int_{\{x, |w_n(x)-l| \le \mu\}} |w_n|^{2/\alpha_n} \\ &\ge \int_{\{x, |w_n(x)-l| \le \mu\}} (l-\mu)^{2/\alpha_n} \\ &\ge \operatorname{mes}\{x, |w_n(x) - l| \le \mu\} \left(\frac{l}{2}\right)^{2/\alpha_n} \\ &\ge (|\Omega| - \mu)\left(\frac{l}{2}\right)^{2/\alpha_n}.\end{aligned}$$

Thus for all $\mu \in (0, l/2)$ we have

$$\liminf_{n\to\infty} \frac{2}{|\Omega|}\int_\Omega |w_n|^{2/\alpha_n} \ge 2\frac{|\Omega| - \mu}{|\Omega|} \lim_{n\to\infty}\left(\frac{l}{2}\right)^{2/\alpha_n} = 2\frac{|\Omega| - \mu}{|\Omega|},$$

and thus

$$\liminf_{n\to\infty} \frac{2}{|\Omega|}\int_\Omega |w_n|^{2/\alpha_n} \ge 2. \tag{A.13}$$

In turn (A.13) implies that

$$\liminf_{n\to\infty}\left(\frac{2}{|\Omega|}\int_\Omega |w_n|^{2/\alpha_n}\right)^{\alpha_n} = \infty$$

which contradicts (A.6). Therefore (A.4) is not satisfied so that finally (A.1) is not satisfied either, which completes the proof of (4.1).

(ii) In the case that $N = 1, 2$ one can prove the result as above, assuming for the purpose of contradiction the existence of sequences $\lambda_n \to \infty$, α_n and z_n such that

$$\lambda_n\left(\int_\Omega |\nabla z_n|^2 + \left(\frac{2}{|\Omega|}\int_\Omega |z_n|^{2/\alpha_n}\right)^{\alpha_n}\right) < \left(\int_\Omega |z_n|^q\right)^{2/q},$$

where $q \geq 1$. Setting $w_n = z_n \|z_n\|_{L^q(\Omega)}^{-1}$ we have

$$\begin{cases} \|\nabla w_n\|_{L^2(\Omega)} \to 0 \quad \text{as } n \to \infty, \\ \|w_n\|_{L^q(\Omega)} = 1 \end{cases} \tag{A.14}$$

and consequently we can suppose that

$$w_n \to l = \mathit{constant} \qquad \text{strongly in } H^1(\Omega)$$

along a subsequence. In view of (A.14) it follows that $|l| = |\Omega|^{-1/q}$ and we can suppose that $l > 0$. The proof is then similar to that of (i).

(iii) If $N = 1$, the proof is again similar. Assuming on the contrary the existence of sequences $\lambda_n \to \infty$, α_n and z_n such that

$$\lambda_n \left(\int_\Omega |\nabla z_n|^2 + \left(\frac{2}{|\Omega|} \int_\Omega |z_n|^{2/\alpha_n} \right)^{\alpha_n} \right) < (\sup z_n)^2. \tag{A.15}$$

Using the notation $w_n = z_n \|z_n\|_{L^\infty(\Omega)}^{-1}$ we have that

$$\begin{cases} \|\nabla w_n\|_{L^2(\Omega)} \to 0 \quad \text{as } n \to \infty, \\ \|w_n\|_{L^\infty(\Omega)} = 1 \end{cases} \tag{A.16}$$

and consequently, extracting a subsequence, we can suppose that

$$w_n \to l = \mathit{constant} \qquad \text{strongly in } H^1(\Omega)$$

where $|l| = 1$ (cf. (A.16)) and, we can suppose that $l > 0$. Then there exists $n_0 > 0$ such that $w_n \geq 1/2$ for all $n \geq n_0$. As in the case that $N \geq 3$, inequality (A.15) implies that

$$\left(\frac{2}{|\Omega|} \int_\Omega |w_n|^{2/\alpha_n} \right)^{\alpha_n} \to 0 \tag{A.17}$$

as $n \to \infty$. But we have that for $n \geq n_0$

$$\begin{aligned} \left(\frac{2}{|\Omega|} \int_\Omega |w_n|^{2/\alpha_n} \right)^{\alpha_n} &\geq \left(\frac{2}{|\Omega|} \int_\Omega \left(\frac{1}{2} \right)^{2/\alpha_n} \right)^{\alpha_n} \\ &\geq \left(2 \left(\frac{1}{2} \right)^{2/\alpha_n} \right)^{\alpha_n} \\ &\geq 2^{\alpha_n} \left(\frac{1}{2} \right)^2 \\ &\geq \frac{1}{2}, \end{aligned}$$

which contradicts (A.17). In turn (A.15) is not satisfied, which completes the proof of (4.3).

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References

[Ali] N. Alikakos, L^p bounds of solutions of reaction-diffusion equations, Comm. Partial Differential Equations **4** (1979), 827–868.

[AlRo] N. Alikakos & R. Rostamian, Large time behavior of solutions of Neumann boundary value problem for the porous medium equation, Indiana Univ. Math. Journal **30** (1981), 827–868.

[ACP] D. Aronson, M. G. Crandall & L. A. Peletier, Stabilization of solutions of a degenerate nonlinear diffusion problem, Nonlinear Anal. **6** (1982), 1001–1022.

[Bar] A. A. Barabanova, On the global existence of solutions of a reaction-diffusion equation with exponential nonlinearity, Proc. Amer. Math. Soc. **122** (1994), 827–831.

[Bre] H. Brezis, Analyse fonctionnelle Théorie et Applications, Masson, 1983.

[DiB] E. DiBenedetto, Continuity of weak solutions to a general porous medium equation, Indiana Univ. Math. J. **32** (1983), 83–118.

[GrSc] P. Gray & S. K. Scott, Sustained oscillations and other exotic patterns in isothermal reactions, J. Phys. Chem. **89** (1985), 22–32.

[HaKi] A. Haraux & M. Kirane, Estimations C^1 pour des problèmes paraboliques semi-linéaires, Ann. Fac. des Sci. Toulouse **5** (1983), 265–280.

[HaYo] A. Haraux & A. Youkana, On a result of K. Masuda concerning reaction-diffusion equations, Tôhoku Math. J. **40** (1988), 159–163.

[HiMiWe] D. Hilhorst, M. Mimura & R. Weidenfeld, Singular limit of a reaction-diffusion system with resource-consumer interaction, to appear in: Proceeding of the RIMS conference "Reaction-Diffusion Systems: Theory and Applications".

[Hos] H. Hoshino, On the convergence properties of global solutions for some reaction-diffusion systems under Neumann boundary conditions, Differential Integral Equations **9** (1996), 761–778.

[HoIl] Y. Hosono & B. Ilyas, Travelling waves for a simple diffusive epidemic model, Math. Models and Methods in Appl. Sciences **5** (1995), 935–966.

[Kan] J. I. Kanel, On global initial boundary-value problems for reaction-diffusion systems with balance conditions, Nonlinear Anal. **37** (1999), 971–995.

[Kit] S. Kitsunezaki, Interface dynamics for bacterial colony formation, J. Phys. Soc. Jpn., **66** (1997), 1544–1550.

[LSU] O. A. Ladyzenskaja, V. A. Solonnikov & N. N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, American Math. Society, 1968.

[Mas] K. Masuda, On the global existence and asymptotique behavior of solutions of reaction-diffusion equations, Hokkaido Math. J. **12** (1983), 360–370.

[MSM] M. Mimura, H. Sakaguchi & M. Matsushita, Reaction-Diffusion Modelling of Bacterial Colony Patterns, Physica A **282** (2000), 283–303.

[Nak] M. Nakao, L^p-estimates of solutions of some nonlinear degenerate diffusion equations, J. Math. Soc. Japan **37** (1985), 41–63.

[Nak2] M. Nakao, Existence, nonexistence and some asymptotic behaviour of global solutions of a nonlinear degenerate parabolic equation, Math. Rep. Kyushu Univ. **14** (1984), 1–21.

[Pel] L. A. Peletier, The porous media equation, in Applications of Nonlinear Analysis in the Physical Sciences (eds. H. Amann, N. Bazley and K. Kirchgaessner) Pitman (1981).

[ScSh] S. Scott & K. Showalter, Simple and complex reaction-diffusion fronts, in Chemical Waves and Patterns, (eds. R. Kapral and K. Showalter), Kluwer Academic Publ. (1995), 485–516.

[ShKa] N. Shigesada & K. Kawasaki, Biological Invasions: Theory and Practice, Oxford Series in Ecology and Evolution, Oxford Univ. Press 1997.

[Sma] D. R. Smart, Fixed points theorems, Cambridge University Press, 1974.

[Tem] R. Temam, Navier-Stokes Equations, North holland-publishing, 1977.

[Tur] A. Turing, The chemical bases of morphogenesis, Philos. Trans. R. Soc. Lond. B **237** (1952), 37–72.

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