HIROSHIMA MATH. J.
33 (2003), 137–156

The structure of the general chromatic E_1-term $\mathrm{Ext}^0_{\Gamma(2)}(BP_*, M_2^1)$ at the prime 2

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(Received March 22, 2002)
(Revised August 20, 2002)

ABSTRACT. Ravenel [8] has introduced p-local spectra $T(m)$ for $m \geq 0$. The Adams-Novikov E_2-term converging to $\pi_*(T(m))$ is isomorphic to

$$\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*, BP_*),$$

where $\Gamma(m+1) = BP_*[t_{m+1}, t_{m+2}, \dots]$, and thus we may follow the chromatic method introduced in [4] to compute the E_2-term. One of the crucial point is to determine the Ext groups $\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*, M_s^n)$. In particular $\mathrm{Ext}^0_{\Gamma(m+1)}(BP_*, M_2^1)$ has already been known except for $p = 2$ and $m = 1$. In this paper we will give the explicit description of the last unknown case.

1. Introduction

The homotopy groups of Ravenel spectrum $T(m)$ give information on the homotopy groups of spheres using "the method of infinite descent", which was the main subject of [8] Chapter 7. Its BP-homology group is given by $BP_*(T(m)) \cong BP_*[t_1, \dots, t_m]$. The Adams-Novikov E_2-term for $T(m)$ is

$$\mathrm{Ext}^*_{BP_*(BP)}(BP_*, BP_*(T(m))),$$

which is isomorphic to $\mathrm{Ext}_{\Gamma(m+1)}(BP_*, BP_*)$ by the change-of-rings isomorphism. So this object is computable using the chromatic spectral sequence introduced in [4]. Define comodules M_m^n by

$$M_m^n = v_{m+n}^{-1} BP_*/(p, \dots, v_{m-1}, v_m^\infty, \dots, v_{m+n-1}^\infty)$$

as usual. Then the chromatic E_1-term is

$$E_1^{s,t} = \mathrm{Ext}^t_{\Gamma(m+1)}(BP_*, M_0^s).$$

2000 *Mathematics Subject Classification*. 55Q10, 55Q51.

Key words and phrases. Adams-Novikov spectral sequence, Chromatic spectral sequence, Ravenel spectrum.

We can determine the structure of this Ext group beginning with the s-th Morava stabilizer algebra $\mathrm{Ext}_{\Gamma(m+1)}(BP_*, M_s^0)$ by Bockstein spectral sequences

$$\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*, M_{s-n}^n) \Rightarrow \mathrm{Ext}^*_{\Gamma(m+1)}(BP_*, M_{s-n-1}^{n+1}).$$

Recently, these Ext groups have been researched by the first author, Ravenel, Shimomura and their coworkers. Notice that Shimomura denotes our $\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*, M_{s-n}^n)$ by $\mathrm{Ext}^*_{BP_*BP}(BP_*, M_{s-n}^n[m])$, and he has determined the complete structure of $\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*, M_{s-1}^1)$ in [9] for $m \geq s^2 - s - 1$.

Moreover, $\mathrm{Ext}^0_{\Gamma(m+1)}(BP_*, M_2^1)$ is known for various p and m. In particular, it is determined by

Ichigi-Nakai-Ravenel [2]	for $p = 2$ and $m \geq 3$ or $p \geq 3$ and $m \geq 2$,
Ichigi-Shimomura [3]	for $p = 3$ and $m = 1$,
Mitsui-Shimomura [5]	for $p \geq 5$ and $m = 1$,
Ichigi [1]	for $p = 2$ and $m = 2$.

The purpose of this paper is to determine the structure of $\mathrm{Ext}^0_{\Gamma(m+1)}(BP_*, M_2^1)$ in case that $p = 2$ and $m = 1$, which had been the last unsolved case. We will define integers $\hat{a}(k)$ in (4.2) and elements $\hat{x}_k$ $(k \geq 0)$ inductively on k by

$$\hat{x}_0 = v_4$$
$$\text{and} \qquad \hat{x}_k = \hat{x}_{k-1}^2 + \hat{y}_k \quad \text{for } k \geq 1,$$

where each $\hat{y}_k$ is v_2-multiple and defined in (4.4). We will see that $\hat{x}_k/v_2^{\hat{a}(k)}$ is in $\mathrm{Ext}^0_{\Gamma(2)}(BP_*, M_2^1)$ and that the image of $\hat{x}_k/v_2^{\hat{a}(k)}$ under the connecting homomorphism $\delta : \mathrm{Ext}^0_{\Gamma(2)}(BP_*, M_2^1) \to \mathrm{Ext}^1_{\Gamma(2)}(BP_*, M_3^0)$ is nontrivial and cohomologous to the image of $v_4^{2^k}/v_2^{\hat{a}(k)}$.

Denote $\mathbf{Z}/(p)[v_2^{\pm 1}, v_3]$ by $\hat{K}(2)_*$ and $\mathbf{Z}/(p)[v_2, v_3]$ by $\hat{k}(2)_*$ respectively. Then our main theorem is

Theorem 1.1. *Assume that $p = 2$. Then, as a $v_3^{-1}\hat{k}(2)_*$-module, $\mathrm{Ext}^0_{\Gamma(2)}(BP_*, M_2^1)$ is the direct sum of*

(i) *the cyclic $\mathbf{Z}/(2)[v_2, v_3^{\pm 1}]$-modules isomorphic to $\mathbf{Z}/(2)[v_2, v_3^{\pm 1}]/(v_2^{\hat{a}(k)})$ generated by $\hat{x}_k^s/v_2^{\hat{a}(k)}$ for $k \geq 0$ and $2 \nmid s > 0$; and*

(ii) *$v_3^{-1}\hat{K}(2)_*/\hat{k}(2)_*$, generated by $1/v_2^j$ for $j \geq 1$.*

Although Ichigi-Shimomura [3] has shown that $\hat{x}_k$ for $p = 3$ are the same as those for $p > 3$ [5], our result shows that $p = 2$ case differs from the odd p cases.

In §2 we review some basic facts about Brown-Peterson theory (cobar complex, Bockstein spectral sequence and Morava stabilizer algebra). In §3 we list up formulas for the right unit η_R on Hazewinkel generators v_n and elements $\hat{w}_4$ and $\hat{w}_5$ given in (3.2). In §4 we construct key elements $\hat{x}_k$ and compute the first cobar differential $d = \eta_R - \eta_L$ on $\hat{x}_k$. The proof of Theorem 1.1 is completed in §5.

We wish to thank Ippei Ichigi for reading carefully a draft of this paper and checking some calculations. We also would like to thank the referee for his helpful advice.

2. Bockstein spectral sequence

Hereafter we will abbreviate $\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*, M)$ to $\mathrm{Ext}^*_{\Gamma(m+1)}(M)$ for simplicity.

It is well known that $\mathrm{Ext}^*_{\Gamma(m+1)}(M)$ can be computed as cohomology groups of the cobar complex

$$0 \longrightarrow M \xrightarrow{d_0} C^1_{\Gamma(m+1)}(M) \xrightarrow{d_1} C^2_{\Gamma(m+1)}(M) \xrightarrow{d_2} \cdots \xrightarrow{d_{k-1}} C^k_{\Gamma(m+1)}(M) \xrightarrow{d_k} \cdots,$$

where $C^n_{\Gamma(m+1)}(M) = \Gamma(m+1)^{\otimes n} \otimes M$ (n-fold tensor product). The differentials of this complex are defined using the right unit η_R and the coproduct Δ of Hopf algebroid $(BP_*, \Gamma(m+1))$. In particular we have $d_0 = \eta_R - \eta_L$ and $\mathrm{Ext}^0_{\Gamma(m+1)}(M) = \ker d_0$.

We will determine the structure of $\mathrm{Ext}^0_{\Gamma(m+1)}(M_2^1)$ for $p = 2$ and $m = 1$ using Bockstein spectral sequence. In fact, the following lemma plays a fundamental role.

Lemma 2.1 ([4] Remark 3.11). *Assume that there exists a $v_3^{-1}\hat{k}(2)_*$-submodule B^t of $\mathrm{Ext}^t_{\Gamma(2)}(M_2^1)$ for each $t < N$, such that the following sequence is exact:*

$$0 \longrightarrow \mathrm{Ext}^0_{\Gamma(2)}(M_3^0) \xrightarrow{1/v_2} B^0 \xrightarrow{v_2} B^0 \xrightarrow{\delta} \mathrm{Ext}^1_{\Gamma(2)}(M_3^0) \xrightarrow{1/v_2} \cdots$$

$$\cdots \xrightarrow{1/v_2} B^{N-1} \xrightarrow{v_2} B^{N-1} \xrightarrow{\delta} \mathrm{Ext}^N_{\Gamma(2)}(M_3^0),$$

where δ is the restriction of the coboundary map

$$\delta : \mathrm{Ext}^t_{\Gamma(2)}(M_2^1) \to \mathrm{Ext}^{t+1}_{\Gamma(2)}(M_3^0).$$

Then the inclusion map $i_t : B^t \to \mathrm{Ext}^t_{\Gamma(2)}(M_2^1)$ is an isomorphism between $\hat{k}(2)_*$-modules for each $t < N$.

In order to apply this lemma we will construct a module B^0 which satisfies the above condition. Because B^0 has a submodule isomorphic to $\mathrm{Ext}^0_{\Gamma(2)}(M_3^0)$, it is a natural way to construct B^0 by extending $\mathrm{Ext}^0_{\Gamma(2)}(M_3^0)$.

The following generalization of Morava-Landweber theorem is straightforward.

Lemma 2.2 (cf. [8] Proposition 7.1.7). *For any prime p, we have*

$$\mathrm{Ext}^0_{\Gamma(2)}(M_3^0) \cong K(3)_*[v_4] = \mathbf{Z}/(p)[v_3^{\pm 1}, v_4]. \tag{2.3}$$

This Ext group is the starting point to construct B^0. Notice that for $x/v_2^i \in B^0$ there is an element $x' = x + (v_2^i\text{-multiples})$ such that $x'/v_2^{i+1} \in B^0$ if $\delta(x/v_2^i) = 0$. In this sense, an element of B^0 is divided by v_2 and we obtain a new element in B^0 if its δ image is zero.

We will choose elements $\hat{x}_k$ $(k \geq 0)$ each of which is $v_4^{2^k}$ plus v_2-multiples in (4.3) and (4.4), and denote the minimal exponent of v_2 by $\hat{a}(k)$ (4.2) such that $\delta(\hat{x}_k/v_2^{\hat{a}(k)}) \neq 0$. Then the following lemma is standard.

Lemma 2.4. *We may define B^0 in Lemma 2.1 by*

$$v_3^{-1}\hat{k}(2)_*\{\hat{x}_k^s/v_2^{\hat{a}(k)} : k \geq 0 \text{ and } p \nmid s > 0\} \oplus v_3^{-1}\hat{K}(2)_*/\hat{k}(2)_*,$$

if the set

$$\{\delta(\hat{x}_k^s/v_2^{\hat{a}(k)}) : k \geq 0 \text{ and } p \nmid s > 0\} \subset \mathrm{Ext}^1_{\Gamma(2)}(M_3^0) \tag{2.5}$$

is linearly independent over $\mathbf{Z}/(p)[v_3^{\pm 1}]$, where δ is the coboundary map in Lemma 2.1.

In order to check the condition (2.5) we have to know the first cohomology $\mathrm{Ext}^1_{\Gamma(2)}(M_3^0)$, which has fortunately been obtained in [7].

Proposition 2.6 ([7] Theorem 1.1).

$$\mathrm{Ext}^1_{\Gamma(2)}(M_3^0) \cong \hat{K}(3)_*\{\hat{h}_{1,0}, \hat{h}_{1,1}, \hat{h}_{1,2}, \hat{h}_{2,0}, \hat{h}_{2,1}, \hat{h}_{2,2}, \rho_3\}, \tag{2.7}$$

where each $\hat{h}_{i,j}$ is the class corresponding to $t_{i+1}^{p^j}$ and ρ_3 is a suitable element with degree 0.

By this proposition the basis of the vector space $\mathrm{Ext}^1_{\Gamma(2)}(M_3^0)$ is described explicitly, so that it is easy to confirm whether the set (2.5) is linearly independent or not.

3. Preliminary calculations

Here we list up some formulas which we will use in §4. By the formula (1.1) and (1.3) in [4], we can deduce the formulas of $\eta_R(v_i)$.

Lemma 3.1. *The right unit*

$$\eta_R : v_3^{-1}BP_* \to v_3^{-1}BP_* \otimes_{BP_*} \Gamma(2)$$

on Hazewinkel generators v_i are expressed as

$$\eta_R(v_3) \equiv v_3 \qquad \bmod(2, v_1),$$

$$\eta_R(v_4) \equiv v_4 + v_2 t_2^4 + v_2^4 t_2 \qquad \bmod(2, v_1),$$

$$\eta_R(v_5) \equiv v_5 + v_3^4 t_2 + v_3 t_2^8 + v_2^8 t_3 + v_2 t_3^4 \qquad \bmod(2, v_1),$$

$$\eta_R(v_6) \equiv v_6 + v_4^4 t_2 + v_4 t_2^{16} + v_3^8 t_3 + v_3 t_3^8 + v_2 t_4^4 \qquad \bmod(2, v_1, v_2^2),$$

$$\eta_R(v_7) \equiv v_7 + v_5^4 t_2 + v_5 t_2^{32} + v_4^8 t_3 + v_4 t_3^{16} + v_3^{16} t_4 + v_3^{16} t_2^5 + v_3^4 t_2^{33} + v_3 t_4^8 \qquad \bmod(2, v_1, v_2).$$

Define elements $\hat{w}_i \in v_3^{-1} BP_*$ $(i = 4, 5)$ by

$$\hat{w}_4 = v_3^{-1} v_5 \quad \text{and} \quad \hat{w}_5 = v_3^{-1}(v_6 + v_4 \hat{w}_4^2). \tag{3.2}$$

Lemma 3.3. *The differentials*

$$d = \eta_R - \eta_L : v_3^{-1} BP_* \to v_3^{-1} BP_* \otimes_{BP_*} \Gamma(2)$$

on $\hat{w}_i$ are expressed as

$$d(\hat{w}_4) \equiv v_3^3 t_2 + v_2^8 v_3^{-1} t_3 + v_2 v_3^{-1} t_3^4 + t_2^8 \qquad \bmod(2, v_1),$$

$$d(\hat{w}_5) \equiv t_3^8 + v_3^7 t_3 + v_3^5 v_4 t_2^2 + v_3^{-1} v_4^4 t_2 + v_2(v_3^{-1} t_2^{20} + v_3^{-1} t_4^4 + v_3^5 t_2^6 + v_3^{-3} v_5^2 t_2^4) \qquad \bmod(2, v_1, v_2^2).$$

Proof. $d(\hat{w}_4)$ is straightforward by Lemma 3.1. For $d(\hat{w}_5)$, we observe that

$$d(v_3^{-1} v_6) \equiv t_3^8 + v_3^7 t_3 + v_3^{-1} v_4 t_2^{16} + v_3^{-1} v_4^4 t_2 + v_2 v_3^{-1} t_4^4,$$

$$d(v_3^{-1} v_4 \hat{w}_4^2) \equiv v_3^{-1} v_4 t_2^{16} + v_3^5 v_4 t_2^2 + v_2(v_3^{-1} t_2^{20} + v_3^5 t_2^6 + v_3^{-3} v_5^2 t_2^4)$$

modulo $(2, v_1, v_2^2)$. Summing these two congruences, we have the desired formula.

By this lemma we have $d(\hat{w}_4) \equiv v_3^3 t_2 + t_2^8$ and $d(\hat{w}_5) \equiv t_3^8 + v_3^7 t_3 + v_3^5 v_4 t_2^2 + v_3^{-1} v_4^4 t_2$ modulo $(2, v_1, v_2)$. These show that

$$t_2 = v_3^{-3} t_2^8 \quad \text{and} \quad t_3 = v_3^{-7} t_3^8 + v_3^{-2} v_4 t_2^2 + v_3^{-8} v_4^4 t_2$$

in $\mathrm{Ext}^1_{\Gamma(2)}(BP_*, M_3^0)$. So we may replace $\hat{h}_{1,i}$ with $\hat{h}_{1,i+3}$ and $\hat{h}_{2,i}$ with $\hat{h}_{2,i+3}$ in (2.7). Therefore Proposition 2.7 implies

COROLLARY 3.4.

$$\mathrm{Ext}^1_{\Gamma(2)}(M_3^0) \cong \hat{K}(3)_*\{\hat{h}_{1,2}, \hat{h}_{1,3}, \hat{h}_{1,4}, \hat{h}_{2,4}, \hat{h}_{2,5}, \hat{h}_{2,6}, \rho_3\}.$$

We will use this $\hat{K}(3)_*$-basis rather than the one of (2.7) because it would allow us to make the construction of $\hat{x}_k$ easy.

4. The elements $\hat{x}_k$ and its δ-image

The elements $\hat{x}_k$ are constructed by adding some v_2-multiples to $v_4^{2^k} \in \mathrm{Ext}^0_{\Gamma(2)}(M_3^0)$. In other words, they satisfy the equality

$$v_4^{2^k s}/v_2 = \hat{x}_k^s/v_2$$

in $\mathrm{Ext}^0_{\Gamma(2)}(M_2^1)$. In this section we will make the full description of $\hat{x}_k$ and compute the first cobar differential

$$d = \eta_R - \eta_L : v_3^{-1}BP_* \to v_3^{-1}BP_* \otimes_{BP_*} \Gamma(2)$$

on $\hat{x}_k$ in Lemma 4.5.

From now on, we set $v_3 = 1$ for simplicity because v_3 is a unit in $v_3^{-1}BP_*$. Define elements ϕ_i $(1 \le i \le 6)$ by

$$\phi_1 = v_2\hat{w}_4^4 + v_4, \qquad \phi_2 = v_2^2\hat{w}_4 + v_4^2, \qquad \phi_3 = v_2\hat{w}_5^4 + v_4^{17},$$
$$\phi_4 = v_7^{16} + \hat{w}_4^{80}, \qquad \phi_5 = v_2^2\hat{w}_4^8 + v_4^2 \qquad \text{and} \qquad \phi_6 = v_2^4\hat{w}_4^2 + v_4^4.$$

Using Lemma 3.1 and Lemma 3.3 we can compute $d(\phi_i)$ easily.

LEMMA 4.1. *The differentials on ϕ_i are expressed as*

$$d(\phi_1) \equiv v_2t_2^{32} + v_2^4t_2 + v_2^5t_3^{16} \qquad \mathrm{mod}(2, v_1, v_2^{12}),$$
$$d(\phi_2) \equiv v_2^2t_2 + v_2^3t_3^4 + v_2^8t_2^2 \qquad \mathrm{mod}(2, v_1, v_2^{10}),$$
$$d(\phi_3) \equiv v_2(t_3^4 + t_3^{32} + v_4^4t_2^8) + v_2^4v_4^{16}t_2,$$
$$+ v_2^5(t_2^{24} + t_2^{80} + t_4^{16} + v_5^8t_2^{16}) \qquad \mathrm{mod}(2, v_1, v_2^8),$$
$$d(\phi_4) \equiv t_4^{16} + t_4^{128} + v_4^{16}t_3^{256} + v_4^{128}t_3^{16},$$
$$+ t_2^{192} + t_2^{640} + v_5^{16}t_2^{64} + v_5^{64}t_2^{128} \qquad \mathrm{mod}(2, v_1, v_2^3),$$
$$d(\phi_5) \equiv v_2^2t_2^{64} \qquad \mathrm{mod}(2, v_1, v_2^5),$$
$$d(\phi_6) \equiv v_2^4t_2^2 \qquad \mathrm{mod}(2, v_1, v_2^6).$$

Define integers $\hat{a}(k)$ $(k \geq 0)$ inductively on k by

$$\hat{a}(k) = \begin{cases} 2^k & (0 \leq k \leq 2), \\ 3 \cdot 2^{k-1} & (3 \leq k \leq 4), \\ 50 & (k = 5), \\ 103 & (k = 6), \\ 207 & (k = 7), \\ 49 \cdot 2^{k-5} + \hat{a}(k-4) & (k \geq 8). \end{cases} \tag{4.2}$$

Define elements $\hat{x}_k \in v_3^{-1}BP_*$ $(k \geq 0)$ inductively on k by

$$\hat{x}_k = \hat{x}_{k-1}^2 + \hat{y}_k, \tag{4.3}$$

where

$$\hat{y}_k = \begin{cases} 0 & \text{for } 0 \leq k \leq 2, \\ v_2^7\phi_1 + v_2^9\phi_2 + v_2^{11}\phi_3 + v_2^{10}v_4^4\hat{x}_1 + v_2^{13}v_4^{16}\phi_2 & \\ \qquad + v_2^{15}v_4^{16}\phi_3 + v_2^{14}v_4^{20}\hat{x}_1 & \text{for } k = 3, \\ v_2^{27}v_4^4\phi_1 & \text{for } k = 4, \\ v_2^{24}\hat{x}_4 + v_2^{36}\hat{x}_3 + v_2^{44}v_4^{64}\hat{x}_2 + v_2^{47}v_4^{16}\phi_1 + v_2^{48}\hat{w}_5^{16} & \text{for } k = 5, \\ v_2^{98}v_4^{32}\hat{x}_2 + v_2^{99}v_4^4\phi_1 + v_2^{101}v_4^4\phi_2 + v_2^{102}v_4^{32}\hat{w}_4^2 & \text{for } k = 6, \\ v_2^{205}v_4^2\phi_1 & \text{for } k = 7, \\ v_2^{364}\hat{x}_5 + v_2^{412}v_4^{192}\hat{x}_2 + v_2^{413}(v_4^{16}\phi_2 + \phi_6) & \\ \qquad + v_2^{414}(v_4^{20}\hat{x}_1 + \phi_5 + v_4^{160}\phi_5) + v_2^{415}(v_4^{16}\phi_3 + v_4^{256}\phi_1) & \\ \qquad + v_2^{416}(\phi_4 + v_4^{16}\hat{w}_5^{32} + v_4^{128}\hat{w}_5^{16} + v_4^{128}\hat{w}_5^{32}) & \text{for } k = 8, \\ v_2^{49\cdot 2^{k-5}}\hat{x}_{k-4}(\hat{x}_{k-4} + \hat{x}_{k-5}^2) & \text{for } k \geq 9. \end{cases} \tag{4.4}$$

Then we have

Lemma 4.5. *Modulo $(2, v_1, v_2^{1+\hat{a}(k)})$, the differentials on $\hat{x}_k$ are expressed as*

$$d(\hat{x}_k) \equiv \begin{cases} v_2^{2^k}t_2^{2^{k+2}} & \text{for } 0 \leq k \leq 2, \\ v_2^{3\cdot 2^{k-1}}(t_3^{2^{k+1}} + t_3^{2^{k+2}}) & \text{for } 3 \leq k \leq 4, \\ v_2^{50}v_4^2t_2^{16} & \text{for } k = 5, \\ v_2^{103}v_4t_2^{16} & \text{for } k = 6, \\ v_2^{207}v_4t_2^{8} & \text{for } k = 7, \\ v_2^{49\cdot 2^{k-5}}v_4^{2^{k-4}}d(\hat{x}_{k-4}) & \text{for } k \geq 8. \end{cases}$$

PROOF. For $0 \le k \le 2$, it directly follows from (3.1). For $k = 3$, we have

$$
\begin{aligned}
d(v_2^7\phi_1) &\equiv v_2^8 t_2^{32} + v_2^{11} t_2 + v_2^{12} t_3^{16}, \\
d(v_2^9\phi_2) &\equiv v_2^{11} t_2 + v_2^{12} t_3^4 + v_2^{17} t_2^2, \\
d(v_2^{11}\phi_3) &\equiv v_2^{12}(t_3^4 + t_3^{32} + v_4^4 t_2^8) + v_2^{15} v_4^{16} t_2 + v_2^{16}(t_2^{24} + t_2^{80} + t_4^{16} + v_5^8 t_2^{16}), \\
d(v_2^{10} v_4^4 \hat{x}_1) &\equiv v_2^{12} v_4^4 t_2^8 + v_2^{14} v_4^2 t_2^{16} + v_2^{16} t_2^{24} + v_2^{18} v_4^4 t_2^2, \\
d(v_2^{13} v_4^{16}\phi_2) &\equiv v_2^{15} v_4^{16} t_2 + v_2^{16} v_4^{16} t_3^4, \\
d(v_2^{15} v_4^{16}\phi_3) &\equiv v_2^{16} v_4^{16}(t_3^4 + t_3^{32} + v_4^4 t_2^8), \\
d(v_2^{14} v_4^{20} \hat{x}_1) &\equiv v_2^{16} v_4^{20} t_2^8 + v_2^{18} v_4^{18} t_2^{16}
\end{aligned}
$$

modulo $(2, v_1, v_2^{19})$. Summing these congruences, we obtain

$$
\begin{aligned}
d(\hat{y}_3) \equiv{} & v_2^8 t_2^{32} + v_2^{12}(t_3^{16} + t_3^{32}) + v_2^{14} v_4^2 t_2^{16} + v_2^{16}(t_2^{80} + t_4^{16} + v_4^{16} t_3^{32} + v_5^8 t_2^{16}) \\
& + v_2^{17} t_2^2 + v_2^{18}(v_4^4 t_2^2 + v_4^{18} t_2^{16})
\end{aligned}
$$

and

$$
\begin{aligned}
d(\hat{x}_3) \equiv{} & v_2^{12}(t_3^{16} + t_3^{32}) + v_2^{14} v_4^2 t_2^{16} + v_2^{16}(t_2^{80} + t_4^{16} + v_4^{16} t_3^{32} + v_5^8 t_2^{16}) \\
& + v_2^{17} t_2^2 + v_2^{18}(v_4^4 t_2^2 + v_4^{18} t_2^{16})
\end{aligned}
$$

modulo $(2, v_1, v_2^{19})$. For $k = 4$, we have

$$
\begin{aligned}
d(\hat{y}_4) &= d(v_2^{27} v_4^4 \phi_1) \\
&\equiv v_2^{28} v_4^4 t_2^{32}
\end{aligned}
$$

and

$$
d(\hat{x}_4) \equiv v_2^{24}(t_3^{32} + t_3^{64})
$$

modulo $(2, v_1, v_2^{31})$. For $k = 5$, we have

$$
\begin{aligned}
d(v_2^{48} \hat{w}_5^{16}) &\equiv v_2^{48}(t_3^{16} + t_3^{128} + v_4^{16} t_2^{32} + v_4^{64} t_2^{16}), \\
d(v_2^{47} v_4^{16} \phi_1) &\equiv v_2^{48} v_4^{16}(t_2^{32} + v_2^3 t_2 + v_2^4 t_3^{16}), \\
d(v_2^{44} v_4^{64} \hat{x}_2) &\equiv v_2^{48} v_4^{64} t_2^{16}, \\
d(v_2^{36} \hat{x}_3) &\equiv v_2^{48}(t_3^{16} + t_3^{32}) + v_2^{50} v_4^2 t_2^{16} + v_2^{52}(t_2^{80} + t_4^{16} + v_4^{16} t_3^{32} + v_5^8 t_2^{16}) \\
&\quad + v_2^{53} t_2^2 + v_2^{54}(v_4^4 t_2^2 + v_4^{18} t_2^{16}), \\
d(v_2^{24} \hat{x}_4) &\equiv v_2^{48}(t_3^{32} + t_3^{64})
\end{aligned}
$$

modulo $(2, v_1, v_2^{55})$. Summing these congruences, we obtain

$$d(\hat{y}_5) \equiv v_2^{48}(t_3^{64} + t_3^{128}) + v_2^{50}v_4^2t_2^{16} + v_2^{51}v_4^{16}t_2 + v_2^{52}(t_2^{80} + t_4^{16} + v_4^{16}t_3^{16} + v_4^{16}t_3^{32} + v_5^8t_2^{16}) + v_2^{53}t_2^2 + v_2^{54}(v_4^4t_2^2 + v_4^{18}t_2^{16})$$

and

$$d(\hat{x}_5) \equiv v_2^{50}v_4^2t_2^{16} + v_2^{51}v_4^{16}t_2 + v_2^{52}(t_2^{80} + t_4^{16} + v_4^{16}t_3^{16} + v_4^{16}t_3^{32} + v_5^8t_2^{16}) + v_2^{53}t_2^2 + v_2^{54}(v_4^4t_2^2 + v_4^{18}t_2^{16})$$

modulo $(2, v_1, v_2^{55})$. For $k = 6$, we have

$$\begin{aligned}
d(v_2^{99}v_4^4\phi_1) &\equiv v_2^{99}\{d(v_4^4)\phi_1 + \eta_R(v_4^4)d(\phi_1)\} \\
&\equiv v_2^{100}v_4^4t_2^{32} + v_2^{103}(v_4t_2^{16} + v_4^4t_2) + v_2^{104}(t_2^{48} + v_4^4t_3^{16} + v_5^4t_2^{16}), \\
d(v_2^{102}v_4^{32}\hat{w}_4^2) &\equiv v_2^{102}(v_4^{32}t_2^2 + v_4^{32}t_2^{16}) + v_2^{104}v_4^{32}t_3^8, \\
d(v_2^{98}v_4^{32}\hat{x}_2) &\equiv v_2^{102}v_4^{32}t_2^{16}, \\
d(v_2^{101}v_4^4\phi_2) &\equiv v_2^{103}v_4^4t_2 + v_2^{104}v_4^4t_3^4 + v_2^{105}v_4^2t_2^{16}
\end{aligned}$$

modulo $(2, v_1, v_2^{106})$. Summing these congruences, we obtain

$$d(\hat{y}_6) \equiv v_2^{100}v_4^4t_2^{32} + v_2^{102}v_4^{32}t_2^2 + v_2^{103}v_4t_2^{16} + v_2^{104}(t_2^{48} + v_4^4t_3^4 + v_4^4t_3^{16} + v_4^{32}t_3^8 + v_5^4t_2^{16}) + v_2^{105}v_4^2t_2^{16}$$

and

$$d(\hat{x}_6) \equiv v_2^{103}v_4t_2^{16} + v_2^{104}(t_2^{48} + t_2^{160} + t_4^{32} + v_4^4t_3^4 + v_4^4t_3^{16} + v_4^{32}t_3^8 + v_4^{32}t_3^{32} + v_4^{32}t_3^{64} + v_5^4t_2^{16} + v_5^{16}t_2^{32}) + v_2^{105}v_4^2t_2^{16}$$

modulo $(2, v_1, v_2^{106})$. For $k = 7$, we have

$$\begin{aligned}
d(\hat{y}_7) &= d(v_2^{205}v_4^2\phi_1) \\
&= v_2^{205}\{d(v_4^2)\phi_1 + \eta_R(v_4^2)d(\phi_1)\} \\
&\equiv v_2^{206}v_4^2t_2^{32} + v_2^{207}v_4t_2^8 + v_2^{208}(t_2^{40} + v_5^4t_2^8) + v_2^{209}v_4^2t_2
\end{aligned}$$

and

$$d(\hat{x}_7) \equiv v_2^{207}v_4t_2^8 + v_2^{208}(t_2^{40} + t_2^{96} + t_2^{320} + t_4^{64} + v_4^8t_3^8 + v_4^8t_3^{32} + v_4^{64}t_3^{16} + v_4^{64}t_3^{64} + v_4^{64}t_3^{128} + v_5^4t_2^8 + v_5^8t_2^{32} + v_5^{32}t_2^{64}) + v_2^{209}v_4^2t_2$$

modulo $(2, v_1, v_2^{210})$. For $k = 8$, we have

$$
\begin{aligned}
d(v_2^{364}\hat{x}_5) &\equiv v_2^{414}v_4^2t_2^{16} + v_2^{415}v_4^{16}t_2 + v_2^{416}(t_2^{80} + t_4^{16} + v_4^{16}t_3^{16} + v_4^{16}t_3^{32} + v_5^8t_2^{16}) \\
&\quad + v_2^{417}t_2^2 + v_2^{418}(v_4^4t_2^2 + v_4^{18}t_2^{16}), \\
d(v_2^{413}v_4^{16}\phi_2) &\equiv v_2^{415}v_4^{16}t_2 + v_2^{416}v_4^{16}t_3^4, \\
d(v_2^{415}v_4^{16}\phi_3) &\equiv v_2^{416}(v_4^{16}t_3^4 + v_4^{16}t_3^{32} + v_4^{20}t_2^8), \\
d(v_2^{414}v_4^{20}\hat{x}_1) &\equiv v_2^{416}v_4^{20}t_2^8 + v_2^{418}v_4^{18}t_2^{16}, \\
d(v_2^{416}\phi_4) &\equiv v_2^{416}(t_4^{16} + t_4^{128} + v_4^{16}t_3^{256} + v_4^{128}t_3^{16} + t_2^{192} + t_2^{640} + v_5^{16}t_2^{64} + v_5^{64}t_2^{128}), \\
d(v_2^{416}v_4^{16}\hat{w}_5^{32}) &\equiv v_2^{416}(v_4^{16}t_3^{32} + v_4^{16}t_3^{256} + v_4^{48}t_2^{64} + v_4^{144}t_2^{32}), \\
d(v_2^{414}\phi_5) &\equiv v_2^{416}v_4^{48}t_2^{64}, \\
d(v_2^{416}v_4^{128}\hat{w}_5^{16}) &\equiv v_2^{416}(v_4^{128}t_3^{16} + v_4^{128}t_3^{128} + v_4^{144}t_2^{32} + v_4^{192}t_2^{16}), \\
d(v_2^{412}v_4^{192}\hat{x}_2) &\equiv v_2^{416}v_4^{192}t_2^{16}, \\
d(v_2^{416}v_4^{128}\hat{w}_5^{32}) &\equiv v_2^{416}(v_4^{128}t_3^{32} + v_4^{128}t_3^{256} + v_4^{160}t_2^{64} + v_4^{256}t_2^{32}), \\
d(v_2^{414}v_4^{160}\phi_5) &\equiv v_2^{416}v_4^{160}t_2^{64}, \\
d(v_2^{415}v_4^{256}\phi_1) &\equiv v_2^{416}v_4^{256}t_2^{32}, \\
d(v_2^{413}\phi_6) &\equiv v_2^{417}t_2^2
\end{aligned}
$$

modulo $(2, v_1, v_2^{419})$. Summing these congruences, we obtain

$$
\begin{aligned}
d(\hat{y}_8) &\equiv v_2^{414}v_4^2t_2^{16} + v_2^{418}v_4^4t_2^2 \\
&\quad + v_2^{416}(t_2^{80} + t_2^{192} + t_2^{640} + t_4^{128} + v_4^{16}t_3^{16} + v_4^{16}t_3^{32} \\
&\qquad + v_4^{128}t_3^{32} + v_4^{128}t_3^{128} + v_4^{128}t_3^{256} + v_5^8t_2^{16} + v_5^{16}t_2^{64} + v_5^{64}t_2^{128})
\end{aligned}
$$

and
$$
\begin{aligned}
d(\hat{x}_8) &\equiv v_2^{416}(v_4^{16}t_3^{32} + v_4^{16}t_3^{64}) \\
&\equiv v_2^{416}v_4^{16}d(\hat{x}_4)
\end{aligned}
$$

modulo $(2, v_1, v_2^{419})$.

For $k \geq 9$, we prove the formula by induction. Assume that the congruence

$$
d(\hat{x}_{k-1}) \equiv v_2^{49\cdot 2^{k-6}}\hat{x}_{k-5}d(\hat{x}_{k-5}) \quad \mathrm{mod}(v_2^{3+\hat{a}(k-1)}) \tag{4.6}
$$

is satisfied. Notice that (4.2) may be rewritten as

$$\hat{a}(k) = 2\hat{a}(k-1) + \begin{cases} 2 & \text{for } k \equiv 0 \ (4), \\ 2 & \text{for } k \equiv 1 \ (4), \\ 3 & \text{for } k \equiv 2 \ (4), \\ 1 & \text{for } k \equiv 3 \ (4). \end{cases}$$

This suggests that we should compute $d(\hat{x}_k)$ modulo $(v_2^{3+\hat{a}(k)})$ rather than modulo $(v_2^{1+\hat{a}(k)})$. Denote $\hat{x}_k + \hat{x}_{k-1}^2$ by $\hat{z}_k$. By definition, $\hat{z}_k$ is related to $\hat{z}_{k-4}$ by

$$\hat{z}_k = v_2^{\hat{a}(k)-\hat{a}(k-4)}\hat{x}_{k-4}\hat{z}_{k-4}.$$

Then $d(\hat{z}_k)$ is computed as

$$\begin{aligned} d(\hat{z}_k) &= v_2^{\hat{a}(k)-\hat{a}(k-4)} d(\hat{x}_{k-4}\hat{z}_{k-4}) \\ &= v_2^{\hat{a}(k)-\hat{a}(k-4)}(d(\hat{x}_{k-4})\hat{z}_{k-4} + \eta_R(\hat{x}_{k-4})d(\hat{z}_{k-4})). \end{aligned} \tag{4.7}$$

Define integers $n(k)$ by

$$n(k) = \begin{cases} 24 & \text{for } k = 9, \\ 98 & \text{for } k = 10, \\ 205 & \text{for } k = 11, \\ 364 & \text{for } k = 12, \\ \hat{a}(k-4) - \hat{a}(k-8) & \text{for } k \geq 13. \end{cases}$$

By definition, $\hat{z}_{k-4}$ is divisible by $v_2^{n(k-4)}$ so that we observe that

$$\begin{aligned} v_2^{\hat{a}(k)-\hat{a}(k-4)} \cdot d(\hat{x}_{k-4})\hat{z}_{k-4} &= v_2^{\hat{a}(k)-\hat{a}(k-4)} \cdot d(\hat{x}_{k-4}) v_2^{n(k)} z \\ &= v_2^{\hat{a}(k)-\hat{a}(k-4)+n(k)} z \cdot d(\hat{x}_{k-4}) \end{aligned}$$

for some z. Because $d(\hat{x}_{k-4})$ is trivial modulo $(v_2^{\hat{a}(k-4)})$ and the inequality

$$\hat{a}(k) + n(k) \geq \hat{a}(k) + 3$$

is satisfied, we can ignore the first term of (4.7) modulo $(v_2^{3+\hat{a}(k)})$. We can also apply the similar statement for the second term. Thus we have

$$d(\hat{z}_k) \equiv v_2^{\hat{a}(k)-\hat{a}(k-4)}\hat{x}_{k-4} d(\hat{z}_{k-4}) \qquad \mathrm{mod}(v_2^{\hat{a}(k)+3}). \tag{4.8}$$

On the other hand, by assumption (4.6) we have

$$\begin{aligned} d(\hat{x}_{k-1}^2) &\equiv v_2^{49 \cdot 2^{k-5}} \hat{x}_{k-5}^2 d(\hat{x}_{k-5}^2) \\ &\equiv v_2^{49 \cdot 2^{k-5}} \hat{x}_{k-4} d(\hat{x}_{k-5}^2). \end{aligned} \tag{4.9}$$

Summing (4.8) and (4.9), we obtain the desired formula.

5. Proof of the Theorem 1.1

Define integers $\hat{c}(k)$ $(k \geq 0)$ by

(5.1)
$$\hat{c}(k) = \begin{cases} 0 & \text{for } 0 \le k \le 4, \\ 2 & \text{for } k = 5, \\ 1 & \text{for } 6 \le k \le 7, \\ 2^{k-4} + \hat{c}(k-4) & \text{for } k \ge 8. \end{cases}$$

This is the exponent of v_4 in $d(\hat{x}_k)$ and thus Lemma 4.5 is rewritten as

(5.2)
$$d(\hat{x}_k) \equiv v_2^{\hat{a}(k)} v_4^{\hat{c}(k)} \begin{cases} t_2^{2^{k+2}} & \text{for } 0 \le k \le 2, \\ (t_3^{16} + t_3^{32}) & \text{for } k = 3, \\ (t_3^{32} + t_3^{64}) & \text{for } k \equiv 0 \mod(4), \\ t_2^{16} & \text{for } k \equiv 1 \text{ and } 2 \mod(4), \\ t_2^{8} & \text{for } k \equiv 3 \mod(4). \end{cases}$$

modulo $(v_2^{1+\hat{a}(k)})$. By (5.2) and the multiplicative property of η_R, we obtain

(5.3)
$$\delta(\hat{x}_k^s / v_2^{\hat{a}(k)}) = v_4^{2^k(s-1)+\hat{c}(k)} \begin{cases} \hat{h}_{1,k+2} & \text{for } 0 \le k \le 2, \\ (\hat{h}_{2,4} + \hat{h}_{2,5}) & \text{for } k = 3, \\ (\hat{h}_{2,5} + \hat{h}_{2,6}) & \text{for } k \equiv 0 \mod(4), \\ \hat{h}_{1,4} & \text{for } k \equiv 1 \text{ and } 2 \mod(4), \\ \hat{h}_{1,3} & \text{for } k \equiv 3 \mod(4). \end{cases}$$

Because of the condition (2.5), it suffices to show that these elements are linearly independent over $\mathbf{Z}/(2)$ (recall that we set $v_3 = 1$).

For $k \ge 8$, set $k = k_0 + 4k_1$ with $4 \le k_0 \le 7$ and $k_1 \ge 1$. Then (5.1) may be rewritten as

$$\hat{c}(k) = \begin{cases} 2^4(16^{k_1-1} + 16^{k_1-2} + \cdots + 16^2 + 16 + 1) & \text{for } k \equiv 0 \mod(4), \\ 2^5(16^{k_1-1} + 16^{k_1-2} + \cdots + 16^2 + 16 + 1) + 2 & \text{for } k \equiv 1 \mod(4), \\ 2^6(16^{k_1-1} + 16^{k_1-2} + \cdots + 16^2 + 16 + 1) + 1 & \text{for } k \equiv 2 \mod(4), \\ 2^7(16^{k_1-1} + 16^{k_1-2} + \cdots + 16^2 + 16 + 1) + 1 & \text{for } k \equiv 3 \mod(4). \end{cases}$$

Denote $2^k(s-1) + \hat{c}(k)$ (the exponent of v_4 in $\delta(\hat{x}_k^s / v_2^{\hat{a}(k)})$) by $D(k,s)$. The next is the table which classified $D(k,s)$ according to the classes $\hat{h}_{1,j}$ $(2 \le j \le 4)$ or $\hat{h}_{2,j} + \hat{h}_{2,j+1}$ $(j = 4,5)$ (see (5.3)).

class	exponent of v_4
$\hat{h}_{1,2}$	$D(0,s)$
$\hat{h}_{1,3}$	$D(1,s)$, $D(7,s)$, $D(7+4k_1,s)$
$\hat{h}_{1,4}$	$D(2,s)$, $D(5,s)$, $D(6,s)$, $D(5+4k_1,s)$, $D(6+4k_1,s)$
$\hat{h}_{2,4} + \hat{h}_{2,5}$	$D(3,s)$
$\hat{h}_{2,5} + \hat{h}_{2,6}$	$D(4,s)$, $D(4+4k_1,s)$

In order to confirm that the set of elements (5.3) is linearly independent, it is enough to check that two $D(k,s)$ belonging to the same class are different each other. For example, $D(k,s)$ corresponding to $\hat{h}_{1,3}$ are

$$\begin{cases} D(1,s) = 2(s-1), \\ D(7,s) = 2^7(s-1)+1, \\ D(7+4k_1,s) = 2^{7+4k_1}(s-1) + 2^7(16^{k_1-1} + 16^{k_1-2} + \cdots + 16^2 + 16 + 1) + 1. \end{cases}$$

$D(1,s)$ is clearly different from other cases because only $D(1,s)$ is even. Moreover, we see that

$$D(7,s) \equiv 1 \qquad \mathrm{mod}(2^8),$$

$$\text{but} \qquad D(7+4k_1,s) \equiv 2^7+1 \qquad \mathrm{mod}(2^8).$$

because $s-1$ is even, so $D(7,s) \neq D(7+4k_1,s)$.

We also have to confirm that two integers $D(7+4\ell_1,s_1)$ and $D(7+4\ell_2,s_2)$ are different each other whenever $(\ell_1,s_1) \neq (\ell_2,s_2)$. Assume that

$$D(7+4\ell_1,s_1) = D(7+4\ell_2,s_2)$$

with $\ell_1 < \ell_2$. Then we see that

$$\begin{aligned} 2^{7+4\ell_1}(s_1-1) - 2^{7+4\ell_2}(s_2-1) &= 2^7(16^{\ell_2-1} + 16^{\ell_2-2} + \cdots + 16^2 + 16 + 1) \\ &\quad - 2^7(16^{\ell_1-1} + 16^{\ell_1-2} + \cdots + 16^2 + 16 + 1) \\ &= 2^7(16^{\ell_2-1} + 16^{\ell_2-2} + \cdots + 16^{\ell_1}) \\ &= 2^7 \cdot 16^{\ell_1}(16^{\ell_2-\ell_1-1} + 16^{\ell_2-\ell_1-2} + \cdots + 16 + 1) \\ &= 2^{7+4\ell_1}(16^{\ell_2-\ell_1-1} + 16^{\ell_2-\ell_1-2} + \cdots + 16 + 1). \end{aligned}$$

Dividing both sides by $2^{7+4\ell_1}$, we have

$$s_1 - 1 - 2^{4(\ell_2-\ell_1)}(s_2-1) = 16^{\ell_2-\ell_1-1} + 16^{\ell_2-\ell_1-2} + \cdots + 16 + 1.$$

Observe that the left hand side is even (because s_1-1 is even) while the right hand side is odd in this equality. This is a contradiction and we can conclude that $D(7+4\ell_1,s_1) \neq D(7+4\ell_2,s_2)$.

Similar statements are satisfied in other cases, too. Consequently, (5.3) is a linearly independent set over $\mathbf{Z}/(2)$ and thus B^0 is isomorphic to our target $\mathrm{Ext}^0_{\Gamma(2)}(M_2^1)$.

Appendix A. A beginner's guide to the calculation by Mathematica program

Here we exhibit some Mathematica programs which would be useful for those who are working in *BP* theory. The programming is not so difficult, but there are few guides for programming in terms of Brown-Peterson theory or related topics. *That is why we think that it might be better to give explanation for the program which we used to obtain the results in this paper.*

Our steps to obtain the results in this paper were as follows: First we used Mathematica to obtain the exact definition of $\hat{a}(k)$ (4.2) and $\hat{y}_k$ (4.4). Next we confirmed it both by Mathematica and by hand.

Because it is possible to make some mistakes in programming, it is not good to depend only on calculating by computer, we think. However, it is of benefit to reduce the amount of our computational jobs. (We usually waste almost of our time to get exact definition of $\hat{x}_k$ and $\hat{a}(k)$!) So we checked our results not only by computer but also by hand. In fact, the first author used Mathematica to obtain the results of [6] and [2] in a similar way.

The first author studied how to program using Mathematica from some programs by D. C. Ravenel. We thank him so much for giving us his useful Mathematica programs and permission to exhibit some of them here.

A.1. Definition of some functions. We must specify a prime number at first. For example, if we set $p = 2$, then we write

```
Clear[p]; p = 2;
```

Denote ℓ_i (generators of $BP_* \otimes \mathbf{Q}$) by `l[i]` and Hazewinkel generators v_i by `v[i]` as usual. Because of formulas

$$p\ell_1 = v_1$$

$$\text{and} \qquad p\ell_i = v_i + \sum_{k=1}^{i-1} v_{i-k}^{p^k} \ell_k \quad \text{for } i \geq 2$$

we can express `l[i]` using `v[i]` as

```
Clear[l, v, t];
l[0] = 1; l[1] = v[1]/p; t[0] = 1;
l[i_] := Expand[
        (v[i] + Sum[v[i - k]^(p^k)*l[k], {k, 1, i - 1}])/p
              ]; i >= 2;
```

The following program (originally due to Ravenel) is designed in order to define the algebra structure of the right unit $\eta_R : BP_* \to BP_*(BP)$.

(A.1)
```
Clear[RU, RRU]
RU[x_ + y_] := RU[x] + RU[y];
RU[x_*y_] := RU[x]*RU[y];
RU[x_/y_] := RU[x]/RU[y];
```

```
RU[x_^i_Integer] := RU[x]^i;
RU[x_Rational] := x;
RU[x_Integer] := x;
RU[x_Rational*y_] := x*RU[y];
RU[x_Integer*y_] := x*RU[y];
RRU[x_] := Expand[RU[x] - x];
```

Here the symbol `RU` means the right unit η_R and `RRU` means the reduced right unit $\eta_R - \eta_L$. Recall that $\eta_R(v_i)$ is given by the recursive formulas

$$\eta_R(v_1) = p\eta_R(\ell_1)$$

$$\text{and} \qquad \eta_R(v_i) = p\eta_R(\ell_i) - \sum_{k=1}^{i-1} \eta_R(v_{i-k}^{p^k})\eta_R(\ell_k).$$

If we denote $\eta_R(\ell_i)$ by `RUonl[i]`, then the above formulas are rewritten as

(A.2)
```
Clear[RUonl]
RUonl[i_] := Sum[l[k]*t[i - k]^(p^k), {k, 0, i}];
RU[v[1]] = Expand[p*RUonl[1]];
RU[v[i_]] := Expand[p*RUonl[i]
           - Sum[RU[v[i - k]]^(p^k)*RUonl[k],
           {k, 1, i - 1}]];
```

Under these preparations, we can make the program calculate $\eta_R(v_i)$. For example, input as

```
In[1] := RU[v[1]]
         RU[v[2]]
         RU[v[3]]
```

Then the corresponding outputs are

```
Out[1] := 2t[1] + v[1]
Out[2] := -4t[1]^3 + 2t[2] - 5t[1]^2v[1] - 3t[1]v[1]^2 +v[2]
Out[3] := -16t[1]^7 - 4t[1]t[2]^2 + 2t[3] - 56t[1]^6v[1]
          - 4t[1]^3t[2]v[1] - t[2]^2v[1] - 85t[1]^5v[1]^2
          - 2t[1]^2t[2]v[1]^2 - 70t[1]^4v[1]^3 - 2t[1]t[2]v[1]^3
          - 36t[1]^3v[1]^4 - t[2]v[1]^4 - 11t[1]^2v[1]^5
          - 2t[1]v[1]^6 + t[1]^4v[2] - 4t[1]t[2]v[2]
          - 2t[1]^3v[1]v[2] - 2t[2]v[1]v[2] - t[1]^2v[1]^2v[2]
          - t[1]v[1]^3v[2] - t[1]v[2]^2 + v[3]
```

A.2. Programs for mod p calculation. Here we introduce some programs which we actually used to obtain the results in this paper.

If there are so many processes in running programs, then a computer would need very long time (or stop). So, whenever we do programming, we must make our best effort at designing programs so as to reduce the size of computation.

When we do calculations modulo p, then we put the following program in front of (A.1):

```
Clear[PM, PR, SP]
PM[x_] := PolynomialMod[x, p];
PR[x_, e_] := PolynomialRemainder[x, v[2]^e, v[2]];
SP[x_Plus, k_] := (#^(p^k)) & /@ x
SP[x_Times, k_] := (#^(p^k)) & /@ x
SP[x_Integer, k_] := x
SP[x_, k_] := x^(p^k)
```

and change the first line of (A.1) into

```
Clear[RU, RRU, padic, pdigits]
padic[i_] := IntegerDigits[i, p];
pdigits[i_] := Length[padic[i]];
```

the fifth line of (A.1) into

```
RU[x_^i_Integer] := Product[
                    SP[RU[x]^(padic[i][[pdigits[i] - k]]), k],
                    {k, 0, pdigits[i] - 1}];
```

and the fourth line of (A.2) into

```
RU[v[i_]] := PM[p*RUonl[i]
          - Sum[RU[v[i - k]]^(p^k)*RUonl[k],
          {k, 1, i - 1}]];
```

In the above program `PM[`(polynomial)`]` means the reduced polynomial modulo (p), `PR[`(polynomial), e`]` means the reduced polynomial modulo (v_2^e), and `SP[`$\sum_i x_i, k$`]` (each x_i is a monomial) means $\sum_i x_i^{p^k}$.

In this paper we considered the right unit $M_2^1 \to M_2^1 \otimes BP_*(BP)/(t_1)$, so we set

```
t[1] = 0; v[1] = 0;
```

Moreover, because M_2^1 is v_3-local, we may also set

```
v[3] = 1;
```

For $d(\hat{x}_k)$ $(0 \le k \le 2)$, input data of $\hat{x}_k$ as

```
Clear[a, x]
a[0] = 1; a[1] = p; a[2] = p^2;
x[0] = v[4]; x[1] = x[0]^p; x[2] = x[1]^p;

Do[
  Print[""]
    Print["d(x[", i, "]) is computed as follows:"]
    Print[" a[", i, "]=", a[i]]
    Print[" x[", i, "]=", x[i]]
    Print[" d(x[", i, "])=", PM[RRU[x[i]]],
           " mod (", p, ",v[1])"]
    Print[" d(x[", i, "])=", PR[PM[RRU[x[i]]], 1 + a[i]],
           " mod (", p, ",v[1],", v[2]^(1 + a[i]), ")"]
  , {i, 0, 2}
  ]
```

Then the corresponding outputs are

```
d on x[0] is computed as follows:
 a[0] = 1
 x[0] = v[4]
 d(x[0]) = t[2]^4 v[2] + t[2]v[2]^4      mod (2,v[1])
 d(x[0]) = t[2]^4 v[2]      mod (2,v[1],v[2]^2)

d on x[1] is computed as follows:
 a[1] = 2
 x[1] = v[4]^2
 d(x[1]) = t[2]^8 v[2]^2 + t[2]^2 v[2]^8      mod (2,v[1])
 d(x[1]) = t[2]^8 v[2]^2      mod (2,v[1],v[2]^3)

d on x[2] is computed as follows:
 a[2] = 4
 x[2] = v[4]^4
 d(x[2]) = t[2]^16 v[2]^4 + t[2]^4 v[2]^16      mod (2,v[1])
 d(x[2]) = t[2]^16 v[2]^4      mod (2,v[1],v[2]^5)
```

For $d(\hat{x}_k)$ $(k \ge 3)$, programming becomes more complicated because there are many monomials in $\hat{x}_k - \hat{x}_{k-1}^2$. The next program is designed in order to make outputs easy to see.

```
Do[Dx[j] = PM[RRU[x[j]]], {j, 0, 2}];

Result[k_] := (
      DA[k, i_] = RRU[A[k, i]];
      Dy[k] =
        Collect[PM[Sum[PR[PM[DA[k, i]], aa[k]],
               {i, ElementNum[k]}]], v[2]];
      Dx[k] = Collect[PM[SP[Dx[k - 1], 1] + Dy[k]], v[2]];
      RU[x[k]] = Dx[k] + x[k];

      Print["If we set "]
        Print[" a[", k, "]=", a[k]]
        Print[" aa[", k, "]=", aa[k]]
        Print["then we have "]
        Do[Print[" d(A[", k",", i, "])=", PR[PM[DA[k, i]],
                 aa[k]], " mod(", v[2]^(aa[k]), "),"],
                 {i, ElementNum[k]}]
        Print["Summing these congruences, we have "]
        Print[" d(y[", k, "])=", Dy[k],
              " mod(", v[2]^(aa[k]), "),"]
        Print["Consequently, we obtain "]
        Print[" d(x[", k, "])=", PR[Dx[k], aa[k]],
              " mod(", v[2]^(aa[k]), ")"]
        Print[" d(x[", k, "])=", PR[Dx[k], 1 + a[k]],
              " mod(", v[2]^(1 + a[k]), ")"]
        );
```

Here the symbols `a[k]`, `x[k]`, `Dx[k]` and `Dy[k]` mean $\hat{a}(k)$, $\hat{x}_k$, $d(\hat{x}_k)$ and $d(\hat{y}_k)$ respectively, and `aa[k]` is a larger integer than `a[k]`. Notice that we actually computed $d(\hat{x}_k)$ modulo $(v_2^{\mathtt{aa[k]}})$ in the proof of Lemma 4.5.

Each `A[k,i]` consists of elements added to $\hat{x}_{k-1}$, i.e.,

$$\hat{y}_k = \sum_i \mathtt{A[k,i]}$$

and `DA[k,i]` is $d(\mathtt{A[k,i]})$. `ElementNum[k]` is the number of such `A[k,i]` (so i runs from 1 to this number in the above sum).

The next program defines $\hat{w}_i$ $(i = 4, 5)$ in (3.3):

```
Clear[w];
w[4] = v[5]; w[5] = v[6] + v[4]*v[5]^2;
```

Under these preparations, we are ready to compute $d(\hat{x}_k)$ for $k \geq 3$. For example, input data of elements of $\hat{x}_3 - \hat{x}_2^2$ as

```
Clear[k, A]
k = 3;

A[k, 1] = v[2]^8*w[4]^4;
A[k, 2] = v[2]^7*x[0];
A[k, 3] = v[2]^(11)*w[4];
A[k, 4] = v[2]^9*x[1];
A[k, 5] = v[2]^(12)*SP[w[5], 2];
A[k, 6] = v[2]^(11)*v[4]^(16)*x[0];
A[k, 7] = v[2]^(10)*v[4]^4*x[1];
A[k, 8] = v[2]^(15)*v[4]^(16)*w[4];
A[k, 9] = v[2]^(13)*v[4]^(16)*x[1];
A[k, 10] = v[2]^(16)*v[4]^(16)*SP[w[5], 2];
A[k, 11] = v[2]^(15)*v[4]^(32)*x[0];
A[k, 12] = v[2]^(14)*v[4]^(20)*x[1];

ElementNum[k] = 12; a[k] = 12; aa[k] = 19;

Result[k]
```

Then the corresponding outputs show the same results as described in the proof of Lemma 4.5. We have got the results on $d(\hat{x}_k)$ for higher k in similar ways. We believe that interested readers can follow $k \geq 4$ cases, referring to the above-mentioned programs.

To obtain $d(\hat{x}_k)$ we needed about 1.733 second for $k = 3$, 0.233 second for $k = 4$, 1.233 second for $k = 5$, 0.817 second for $k = 6$, 0.4 second for $k = 7$, 11.967 second for $k = 8$, and so on, with Mathematica Ver. 4.1 and 667 MHz PowerBook G4 with Mac OS Ver. 9.2.2.

Of course, we think that the programs exhibited here might be naive and that some professional persons can make better programs. We will appreciate it if the reader could show us a better way.

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