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Homotopy groups of generalized $E(2)$-local Moore spectra at the prime three

To the memory of the late Professor Masahiro Sugawara

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Abstract. Let $E(2)$ denote the Johnson-Wilson spectrum with homotopy groups $\pi_*(E(2)) = \mathbf{Z}_{(3)}[v_1, v_2, v_2^{-1}]$. Then the mod 3 Moore spectrum $V(0)$ satisfies $E(2)_*(V(0)) = E(2)_*/(3)$. We call a spectrum M generalized ($E(2)$-local) Moore spectrum if it satisfies $E(2)_*(M) = E(2)_*/(3) = E(2)_*(V(0))$ as an $E(2)_*E(2)$-comodule. We see that the Toda spectrum $\Sigma^{-21}V(1\frac{1}{2})$ is an example (*cf.* [10]) other than the Moore spectrum $V(0)$. Here we introduce other generalized Moore spectra and determine the homotopy groups of them.

1. Introduction

Let $\mathscr{S}_{(p)}$ denote the stable homotopy category of p-local spectra for a prime p, and $E(n) \in \mathscr{S}_{(p)}$ denote the Johnson-Wilson spectrum characterized by the homotopy groups $\pi_*(E(n)) = E(n)_* = v_n^{-1}\mathbf{Z}_{(p)}[v_1, v_2, \dots, v_n] \subset v_n^{-1}BP_* = v_n^{-1}\mathbf{Z}_{(p)}[v_1, v_2, \dots]$. Here, BP denotes the Brown-Peterson spectrum. We denote $L_n : \mathscr{S}_{(p)} \to \mathscr{S}_{(p)}$ as the Bousfield localization functor with respect to $E(n)$. We write $\mathscr{L}_n$ as the image of L_n. We call a spectrum $X \in \mathscr{L}_n$ invertible if there exists a spectrum Y such that $X \wedge Y = L_nS$ for the sphere spectrum S. Then Hovey and Sadofsky [6] showed that the collection of isomorphism classes of invertible spectra forms a group, which is called the Picard group of $\mathscr{L}_n$ and denoted by $\mathrm{Pic}(\mathscr{L}_n)$. We call an invertible spectrum X strict if $H\mathbf{Q}_0(X) = \mathbf{Q}$, and proper if X is strict and $X \not\simeq L_nS$. The strict invertible spectra define the subgroup $\mathrm{Pic}(\mathscr{L}_n)^0 \subset \mathrm{Pic}(\mathscr{L}_n)$. Hovey and Sadofsky also showed $\mathrm{Pic}(\mathscr{L}_n) = \mathrm{Pic}(\mathscr{L}_n)^0 \oplus \mathbf{Z}$, which means that an invertible spectrum is isomorphic to a suspension of a strict one.

In [6] and [10], it is shown that a spectrum $X \in \mathscr{L}_n$ is strict invertible if and only if $E(n)_*(X) = E(n)_* = E(n)_*(S)$ as an $E(n)_*E(n)$-comodule. We gener-

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alize this. We call $X \in \mathscr{L}_n$ a generalized ($E(n)$-local) k-th Smith-Toda spectrum if $E(n)_*(X) = E(n)_*/(p, v_1, \dots, v_k)$ as an $E(n)_*E(n)$-comodule. In particular, the point spectrum $*$ is a generalized k-th Smith-Toda spectrum for $k \geq n$, and the $E(n)$-localization of the k-th Smith-Toda spectrum $V(k)$ is a generalized one if $V(k)$ exists. Note that the 0-th Smith-Toda spectrum $V(0)$ is the mod p Moore spectrum, which is a cofiber of $p : S \to S$. So we call a generalized 0-th Smith-Toda spectrum a generalized ($E(n)$-local mod p) Moore spectrum, which exists for any $n \geq 0$ and any prime p. A generalized (-1)-st Smith-Toda spectrum X is an invertible spectrum as above. Let $\mathscr{V}_n(k)$ denote the collection of the isomorphism classes of generalized k-th Smith-Toda spectra in $\mathscr{L}_n$. In particular, $\mathscr{V}_n(-1) = \mathrm{Pic}(\mathscr{L}_n)^0$ and $\mathscr{V}_n(k) = \{*\}$ if $k \geq n$. Strickland shows that if X has a finitely generated $E(n)_*$-homology, then X is a small object in $\mathscr{L}_n$. (See Theorem 2.1.) This shows that $\mathscr{V}_n(k)$ is also a set. In [21] and [20], Yosimura, Yokotani and the second author showed the existence and uniqueness of a generalized Smith-Toda spectrum $L_n V(k)$ if $k < n$ and $n^2 + n < 2p$.

Proposition 1.1 ([21], [20], [6]). *If $n^2 + n < 2p$, then $\mathscr{V}_n(k) = \{L_n V(k)\}$. In particular, if $n^2 + n < 2p$, then each generalized* mod p *Moore spectrum is nothing but the $E(n)$-localization of the* mod p *Moore spectrum.*

In the same manner as the sphere spectrum acts on any spectrum, $\mathrm{Pic}(\mathscr{L}_n)^0$ acts on $\mathscr{V}_n(k)$ by $xv = [X \wedge V]$ for $x = [X] \in \mathrm{Pic}(\mathscr{L}_n)^0$ and $v = [V] \in \mathscr{V}_n(k)$. Since a generalized Smith-Toda spectrum V is small, the Spanier-Whitehead dual D defines an action on $\mathscr{V}_n(k)$ by $D_*(V) = \Sigma^{v(n)} DV = \Sigma^{v(n)} F(V, L_n S)$ with $D_* D_*(V) = V$. Here, $v(n) = \sum_{k=0}^{n} (2p^k - 1)$. Indeed, $E(n)_*(DV) = E(n)^*(V) = E(n)_*/(p, v_1, \dots, v_k)$. Note that $D_*(V(k)) = L_n V(k)$ for the Smith-Toda spectrum $V(k)$ if it exists. Let $\mathrm{Pic}(\mathscr{L}_n)^0\{V\}$ denote the orbit of V: $\{[V \wedge X] : X \in \mathrm{Pic}(\mathscr{L}_n)^0\}$. Hereafter, we write $V \in \mathscr{V}_n(k)$ for $[V] \in \mathscr{V}_n(k)$.

Conjecture A. For $V \in \mathscr{V}_n(k)$, $\mathrm{Pic}(\mathscr{L}_n)^0\{D_*(V)\} = \mathrm{Pic}(\mathscr{L}_n)^0\{V\}$.

This is true if $k = -1$.

Now we consider the generalized Moore spectra at the prime three. Then, Proposition 1.1 says that the generalized mod 3 Moore spectrum is $L_n V(0)$ if $n < 2$. So we work in $\mathscr{L}_2$. In [10], Kamiya and the second author constructed a proper invertible spectrum P that generates a summand $\mathbf{Z}/3 \subset \mathrm{Pic}(\mathscr{L}_2)^0$ and a monomorphism from $\mathrm{Pic}(\mathscr{L}_2)^0$ to the direct sum of the Adams-Novikov E_r-terms $\bigoplus_{r>1} E_r^{r,r-1}(S)$, which is isomorphic to $\mathbf{Z}/3 \oplus \mathbf{Z}/3$ by [19]. The invertible spectrum P is constructed by defining a map $f : P \to \Sigma^{-21} L_2 V(1\frac{1}{2})$ that induces the projection $f_* : E(2)_* \to E(2)_*/(3)$. There is a problem that asks whether or not there is another proper invertible spectrum

Q, which corresponds to the other summand of $E_5^{5,4}(S)$. Since each $E(2)_*$-homology sphere is an invertible spectrum by [10], $X \wedge V(0)$ for each invertible spectrum X is an example of generalized Moore spectra other than $L_2V(0)$. We have no idea whether or not an invertible spectrum X is an $E(2)$-localization of a finite spectrum, though X is a retract of $E(2)$-localization of a finite spectrum [6]. There is another generalized Moore spectrum $L_2V\left(1\frac{1}{2}\right)$ (*cf.* [10]) other than $L_2V(0)$, which is an $E(2)$-localization of a finite spectrum. Here $V\left(1\frac{1}{2}\right)$ is the Toda spectrum given in [22]. The construction is generalized as follows: It is shown the existence of a map $B'^{(i)} : \Sigma^{16i}S \to V(1)$ that induces $v_2^i : BP_* \to BP_*/(3, v_1)$ for $i = 0, 1, 5$ (*cf.* [12]). Since the order of $B'^{(i)}$ is three, $B'^{(i)}$ extends to $B^{(i)} : \Sigma^{16i}V(0) \to V(1)$. Let $\Sigma^{16i+5}V_i$ denote the cofiber of $B^{(i)}$. Then, $L_2V_i \in \mathscr{V}_2(0)$ for $i = 0, 1, 5$. Note that $V_0 = V(0)$ and $V_1 = \Sigma^{-21}V\left(1\frac{1}{2}\right)$.

We have another construction: Let X be an invertible spectrum and $\iota_X \in \pi_0(X)$ denote the element detected by $3g_X \in E_2^{0,0}(X) = \mathbf{Z}_{(3)}\{g_X\}$, and write WX as the cofiber of ι_X. In particular, $WL_2S^0 = L_2V(0)$. Note that W does not seem a good operation. Indeed, even though the Adams-Novikov differentials d_5 on the generators of $E_2^{0,0} = \mathbf{Z}/3$ for $WX \wedge X'$ and $W(X \wedge X')$ agree for any strict invertible spectra X and X', these spectra are not always homotopy equivalent. (If the Adams-Novikov differentials d_5 on the generators of $E_2^{0,0} = \mathbf{Z}_{(3)}$ for invertible spectra X and X' agree, then they are homotopy equivalent by [10].) For the proper invertible spectrum $P \in \mathrm{Pic}(\mathscr{L}_2)^0$, we have WP^j for $j \geq 0$. Here U^j for a spectrum U denotes the j-fold smash product of U for $j > 0$ and $U^0 = L_2S$. Note that $WP^2 = V_1 \wedge P$, since $P^2 \to P \xrightarrow{f} \Sigma^{-21}L_2V_1$ is a cofiber sequence [8]. It follows that $\mathrm{Pic}(\mathscr{L}_2)^0\{WP^2\} = \mathrm{Pic}(\mathscr{L}_2)^0\{L_2V_1\}$. If the other proper invertible spectrum Q exists, then we also have WQ, and $\mathscr{V}_2(0)$ contains the orbit $\mathrm{Pic}(\mathscr{L}_2)^0\{WQ, WQ^2\}$. Put

$$\mathscr{V}_2(0)^0 = \mathrm{Pic}(\mathscr{L}_2)^0\{L_2V_0, L_2V_1, L_2V_5, WP, (WQ, WQ^2 \text{ if they exist})\}.$$

Proposition 1.2. $\mathscr{V}_2(0)^0 \subset \mathscr{V}_2(0)$.

The size of $\mathrm{Pic}(\mathscr{L}_n)^0 = \mathscr{V}_n(-1)$ has an upper bound, but we have no idea about the size of $\mathscr{V}_n(k)$ for $k \geq 0$. Indeed, an generalized k-th Smith-Toda spectrum is not always a $V(k)$-module spectrum if $k \geq 0$.

Conjecture B. $\mathscr{V}_2(0)^0 = \mathscr{V}_2(0)$.

For the Spanier-Whitehead dual D, $D(X) = X^2$ for an invertible spectrum X (*cf.* [5], [10], Theorem 2.1), and we see that $D_*((WX) \wedge X) = WX \wedge X$, since the dual of the cofiber sequence $X \xrightarrow{\iota \wedge X} X^2 \longrightarrow (WX) \wedge X$ is $X \xrightarrow{D(\iota \wedge X)}$

$X^2 \to \Sigma D((WX) \wedge X)$. For V_i, we also see that $D_*(L_2V_i) = L_2V_i$. It follows that $D_*(V) \in \mathrm{Pic}(\mathscr{L}_2)^0\{V\}$ for $V \in \mathscr{V}_2(0)^0$.

Proposition 1.3. *If Conjecture B holds, then so does Conjecture A for* $k = 0$.

In this paper, we determine the homotopy groups of all spectra in $\mathscr{V}_2(0)^0$ other than WQ^i for $i = 1, 2$. The structure of the homotopy groups of L_2U for a spectrum U is more complicated than that of $\pi_*(L_{K(2)}U)$. Here $L_{K(2)}$ denotes the Bousfield localization functor with respect to the second Morava K-theory $K(2)$. So we recall [14] the chromatic spectra M_nU and N_nU for a spectrum U, which are defined inductively by

$$N_0U = U, \qquad M_nU = L_nN_nU \qquad \text{and the cofiber sequence}$$
$$N_nU \to M_nU \to N_{n+1}U.$$

Then the homotopy groups $\pi_*(L_{K(2)}U)$ are closely related with $\pi_*(M_2U)$, and the structure of $\pi_*(M_2U)$ is less complicated than that of $\pi_*(L_2U)$. Thus we consider $\pi_*(M_2U)$ instead of $\pi_*(L_2U)$. In [17], the homotopy groups $\pi_*(M_2V_0)$ for the mod 3 Moore spectrum $V_0 = V(0)$ are determined by use of the $E(2)$-based Adams spectral sequence. The E_2-term of it is a direct sum of modules A, B_h and B_t, and the homotopy groups are a direct sum of A, $\widetilde{B}_h$ and $\widetilde{B}_t$. Here, $\widetilde{B}_j$ denotes the permanent cycles of B_j. By use of the decomposition of the E_2-term, we obtain the homotopy groups in [8]:

$$\pi_*(M_2V_0 \wedge P^k) = A \oplus v_2^{9-3k}\widetilde{B}_h \oplus v_2^{9-3k}\widetilde{B}_t \qquad \text{and}$$
$$\pi_*(M_2V_1) = A \oplus v_2^3\widetilde{B}_h \oplus v_2^6\widetilde{B}_t.$$

Note that $v_2^9\widetilde{B}_j \cong \widetilde{B}_j$. These make us to predict the following theorem:

Theorem 1.4. $\pi_*(M_2V_1 \wedge P^k) = A \oplus v_2^{3-3k}\widetilde{B}_h \oplus v_2^{6-3k}\widetilde{B}_t$.

These results let us ask if there is a spectrum U_k such that $\pi_*(U_k) = A \oplus v_2^{6-3k}\widetilde{B}_h \oplus v_2^{3-3k}\widetilde{B}_t$. The elements of the orbit $\mathrm{Pic}(\mathscr{L}_2)^0\{WP\}$ give the affirmative answer.

Theorem 1.5. $\pi_*(M_2WP \wedge P^{k+1}) = A \oplus v_2^{6-3k}\widetilde{B}_h \oplus v_2^{3-3k}\widetilde{B}_t$.

For V_5, we have

Theorem 1.6. $\pi_*(M_2V_5 \wedge P^k) = A \oplus v_2^{3-3k}\widetilde{B}_h \oplus v_2^{9-3k}\widetilde{B}_t$.

Remark. $\pi_*(M_2V_5) = \pi_*(M_2WP \wedge P^2)$, while $L_2V_5 \not\simeq WP \wedge P^2$ by Lemma 5.3.

Remark. If Q exists, then the homotopy groups $\pi_*(M_2WQ)$ agree with none of above (*cf.* [9]).

This paper is organized as follows: In the next section, we include the result of Strickland. In section 3, we consider a decomposition of the $E(2)$-based Adams E_2-term, which plays a crucial role to determine the homotopy groups. Sections 4 and 5 are devoted to define the generalized Moore spectrum, and to show some properties of them, which show Proposition 1.2. In section 6, we prove Theorems 1.4 and 1.5. Theorem 1.6 is proved in section 7.

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2. Some results on generalized Smith-Toda spectra

Let BP and $E(n)$ denote the Brown-Peterson and the n-th Johnson-Wilson spectra, respectively, at a prime p. We write $\mathscr{L}_n$ as the stable homotopy category consisting of $E(n)$-local spectra. Then $BP_*BP = BP_*[t_1, t_2, \dots]$ has a structure of a Hopf algebroid over $BP_* = \pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2, \dots]$. Besides, it induces a Hopf algebroid structure on $E(n)_*E(n)$ over $E(n)_*$, since $E(n)_*E(n) = E(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(n)_*$ for $E(n)_* = v_n^{-1}\mathbf{Z}_{(p)}[v_1, v_2, \dots, v_n] \subset v_n^{-1}BP_*$.

We consider the $E(n)$-based Adams spectral sequence $E_r^{s,t}(U)$ for computing the homotopy groups $\pi_*(L_nU)$ of a spectrum U. The E_2-term is isomorphic to the Ext group

$$E_2^{s,t}(U) = \mathrm{Ext}^{s,t}_{E(n)_*E(n)}(E(n)_*, E(n)_*(U))$$

of $E(n)_*E(n)$-comodules.

The k-th Smith-Toda spectrum $V(k)$ is a spectrum with $BP_*(V(k)) = BP_*/(p, v_1, \dots, v_k)$. If $k < 4$, then the Smith-Toda spectrum exists if and only if $2k < p$ (*cf.* [23], [13]). We call a spectrum V a *generalized* ($E(n)$-*local*) k-*th Smith-Toda spectrum* if $E(n)_*(V) = E(n)_*/(p, v_1, \dots, v_k)$ as an $E(n)_*E(n)$-comodule. The E_2-term $E_2^{*,*}(V)$ of the $E(n)$-based Adams spectral sequence agrees with the E_2-term $E_2^*(V(k))$ if $V(k)$ exists. We write $\mathscr{V}_n(k)$ as the collection of isomorphism classes of generalized k-th Smith-Toda spectra. Then

$$\#(\mathscr{V}_n(k)) = 1 \qquad \text{if } n^2 + n < 2p$$

by [21] and [20].

Since $E(n)_*(V)$ is finitely generated, we see that $\mathscr{V}_n(k)$ is a set by [5, Th. 2.1.3] and a theorem of Strickland:

Theorem 2.1 (N. Strickland). *If $E(n)_*(X)$ is finitely generated, then X is small in the $E(n)$-local category.*

Proof. Let E be an S-algebra such that $E \simeq E(n)$, whose existence is certified in [2]. (A. Lazarev also has another argument.) Let $\mathscr{C}_X$ denote a full subcategory consisting of spectra Z such that the natural map

$$\bigoplus_i [X, Z \wedge Y_i] \to \left[X, Z \wedge \bigvee_i Y_i\right]$$

is an isomorphism for all families $\{Y_i\}$ of E-local spectra. Then, $\mathscr{C}_X$ is a thick subcategory. Note that X is small if $L_nS \in \mathscr{C}_X$. Since E_* is a ring of finite global dimension and $E_*(X)$ is finitely generated, $E \wedge X$ has a finite resolution by finitely generated free modules in the derived category $\mathscr{D}_E$ as in [1, Chap. IV], and hence $E \wedge X$ is small in $\mathscr{D}_E$. Since $\mathscr{D}_E(E \wedge X, E \wedge W \wedge Y) = [X, E \wedge W \wedge Y]$, every spectrum of the form $E \wedge W$ is in $\mathscr{C}_X$. Furthermore, since L_nS is E-nilpotent (*cf.* [15]), L_nS is in the thick subcategory generated by the spectra of the form $E \wedge W$. It follows that $L_nS \in \mathscr{C}_X$ as desired. □

3. A decomposition of $H^*M_1^1$

From this section on, we set the prime $p = 3$ and work in $\mathscr{L}_2$, the stable homotopy category of spectra localized with respect to the second Johnson-Wilson spectrum $E(2)$, whose coefficient ring is $E(2)_* = v_2^{-1}\mathbf{Z}_{(3)}[v_1, v_2] \subset v_2^{-1}BP_*$. Consider the mod 3 Moore spectrum $V(0)$ defined by the cofiber sequence

$$S \xrightarrow{3} S \xrightarrow{i} V(0) \xrightarrow{j} \Sigma S, \tag{3.1}$$

and the first Smith-Toda spectrum $V(1)$ defined by the cofiber sequence

$$\Sigma^4 V(0) \xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^5 V(0).$$

Here α denotes the Adams map.

Let $U(n)$ denote an n-th generalized Smith-Toda spectrum such that $E(2)_*(U(n))$ is isomorphic to $E(2)_*(V(n))$ as an $E(2)_*E(2)$-comodule, where $V(n)$ denotes the n-th Smith-Toda spectrum. Note that $U(n) = *$ if $n \geq 2$. For $n = 0$, we call a spectrum $U(0) \in \mathscr{L}_2$ a *generalized Moore spectrum*, and consider $M_2U(0)$ defined by the cofiber sequence

$$U(0) \to L_1U(0) \to \Sigma^{-1}M_2U(0)$$

(*cf.* [14]). Consider the $E(2)_*E(2)$-comodule M_2^0 and M_1^1 defined by $M_2^0 = E(2)_*/(3, v_1)$, which is also denoted by $K(2)_*$, and the short exact sequence

$$0 \to E(2)_*/(3) \to v_1^{-1}E(2)_*/(3) \to M_1^1 \to 0.$$

Then we have the short exact sequence

$$0 \longrightarrow M_2^0 \xrightarrow{1/v_1} M_1^1 \xrightarrow{v_1} M_1^1 \longrightarrow 0 \tag{3.2}$$

of $E(2)_*E(2)$-comodules. The E_2-terms of the $E(2)$-based Adams spectral sequences converging to $\pi_*(U(1))$ and $\pi_*(M_2U(0))$ are $H^*M_2^0$ and $H^*M_1^1$, respectively. Here H^*M for an $E(2)_*E(2)$-comodule M denotes $\mathrm{Ext}^*_{E(2)_*E(2)}(E(2)_*, M)$. The E_2-term $H^*M_2^0$ (resp. $H^*M_1^1$) is shown in [16] (resp. [17]) to be isomorphic to the tensor product of $\Lambda(\zeta_2)$ and the module M (resp. the direct sum of modules $(K(1)_*/k(1)_*) \otimes \Lambda(h_{10})$, $\bigoplus_{n\geq 0} F_n$ and $(F \oplus F^*) \otimes (\mathbf{Z}/3)[b_{10}]$). Here the modules M, F, F^* and F_n are given by

$$M = K(2)_*[b_{10}]\{1, h_{10}, h_{11}, \xi, \psi_0, \psi_1, b_{11}\xi\};$$

$$F = E(2,1)_*\{v_2^{\pm 1}/v_1, v_2h_{10}/v_1^2, v_2^2h_{11}/v_1^2, v_2^{\pm 1}b_{11}/v_1\},$$

$$F^* = E(2,1)_*\{\xi/v_1^2, v_2^{2\pm 1}\psi_0/v_1, v_2^{\pm 1}\psi_1/v_1, b_{11}\xi/v_1^2\} \qquad \text{and}$$

$$F_n = E(2, n+2)_*\{v_2^{\pm 3^{n+1}}/v_1^{4\times 3^n - 1}, v_2^{3^{n+1}}h_{10}/v_1^{6\times 3^n + 1},$$
$$v_2^{8\times 3^n}h_{10}/v_1^{10\times 3^n + 1}, v_2^{3^n(5\pm 3)+(3^n-1)/2}\xi/v_1^{4\times 3^n}\}$$

for

$$k(1)_* = (\mathbf{Z}/3)[v_1], \qquad E(2,n)_* = k(1)_*[v_2^{\pm 3^n}], \qquad K(1)_* = v_1^{-1}k(1)_* \qquad \text{and}$$
$$K(2)_* = (\mathbf{Z}/3)[v_2^{\pm 1}].$$

The element b_{10} acts on $(F \oplus F^*) \otimes (\mathbf{Z}/3)[b_{10}]$ freely. The action of b_{10} on F_n is seen as follows: Consider the exact sequence $H^sM_2^0 \to H^sM_1^1 \to H^sM_1^1 \xrightarrow{\delta} H^{s+1}M_2^0$ associated to (3.2) and suppose that $\delta(x) = y$ and $\delta(w) = yb_{10}$ for $x \in F_n$, $w \in F \oplus F^*$ and $y \neq 0 \in H^*M_2^0$. Then there exists an element $u \in H^*M_1^1$ such that $xb_{10} = w + v_1u$. Thus replacing w by $w + v_1u$, we have an isomorphism $(\mathbf{Z}/3)\{x\} \oplus (\mathbf{Z}/3)[b_{10}]\{w\} = (\mathbf{Z}/3)[b_{10}]\{x\}$. We also write $x = \overline{w}$. In this way, we compute the b_{10}-action in [8] and rewrite here the direct sum of the modules $\bigoplus_{n\geq 0} F_n$ and $(F \oplus F^*) \otimes (\mathbf{Z}/3)[b_{10}]$ as the direct sum of $\bigoplus_{n\geq 0} F_n'$ and $F' \otimes (\mathbf{Z}/3)[b_{10}]$. Here

$$F' = (\mathbf{Z}/3)_*\{v_2^{3(3t\pm1)}/v_1^3 = \overline{v_2^{9t\pm3-2}b_{11}/v_1},$$
$$v_2^{3^n(3t\pm1)}/v_1^{4\times3^{n-1}-1} = \overline{v_2^{3^{n-1}(9t\pm3-1)-1}b_{11}/v_1}\ \ (n>1),$$
$$v_2^{9t-2}b_{11}/v_1, v_2^{3^{n-1}(3t+1)-1}b_{11}/v_1, v_2^{3^{n-1}(9t-1)-1}b_{11}/v_1,$$
$$v_2^{3^n(3t+1)}h_{10}/v_1^{2\times3^n+1} = \overline{v_2^{3^{n+1}t+(3^n-1)/2}\psi_1/v_1}\ \ (n>0),$$
$$v_2^{3^n(9t+8)}h_{10}/v_1^{10\times3^n+1} = \overline{v_2^{3^{n+2}t+5\times3+(3^n-1)/2}\psi_1/v_1}\ \ (n\ge0), v_2^{3t-1}\psi_1/v_1,$$
$$v_2^{3^n(9t+5\pm3)+(3^n-1)/2}\xi/v_1^{4\times3^n} = \overline{v_2^{3^{n+1}(3t+1\pm1)+3(3^n-1)/2}b_{11}\xi/v_1^2}\ (n\ge0),$$
$$v_2^{3t}b_{11}\xi/v_1\}$$
$$\oplus E(2,1)_*\{v_2^{\pm1}/v_1, v_2h_{10}/v_1^2, v_2^2h_{11}/v_1^2, \xi/v_1^2, v_2^{2\pm1}\psi_0/v_1\} \qquad \text{and}$$
$$F_n' = E(2,n+2)_*\{v_2^{\pm3^{n+1}}/v_1^{4\times3^n-2}, v_2^{3^{n+1}}h_{10}/v_1^{6\times3^n},$$
$$v_2^{8\times3^n}h_{10}/v_1^{10\times3^n}, v_2^{3^n(5\pm3)+(3^n-1)/2}\xi/v_1^{4\times3^n-1}\}.$$

Put

(3.3) $$A = ((K(1)_*/k(1)_*)\otimes \Lambda(h_{10}) \oplus \textstyle\bigoplus_{n\ge0} F_n')\otimes \Lambda(\zeta_2) \qquad \text{and}$$
$$B = F'\otimes(\mathbf{Z}/3)[b_{10}]\otimes\Lambda(\zeta_2).$$

Then we have a decomposition of $H^*M_1^1$:

$$H^*M_1^1 = A\oplus B.$$

By observing the construction of modules, these modules satisfy the following:

(3.4)
1. $A \subset \bigoplus_{s=0}^3 H^sM_1^1$.
2. If $x\in B\cap H^sM_1^1$ for $s>5$, then $x = yb_{10}$ for some $y\in B$.
3. the generator b_{10} acts trivially on A and freely on B.

4. The generalized Moore spectrum WX for an invertible spectrum X at the prime three

We call a spectrum X *invertible* if there is a spectrum X' such that $X\wedge X' = L_2S$. An invertible spectrum X is called *strict* if $H\mathbf{Q}_0(X) = \mathbf{Q}$, and *proper* if $H\mathbf{Q}_0(X) = \mathbf{Q}$ and $X \not\simeq L_2S$. We denote by $\mathrm{Pic}(\mathscr{L}_2)^0$ the collection of isomorphism classes of strict invertible spectra, which is shown to be a group with multiplication defined by the smash product and with the unit L_2S. Note that every invertible spectrum is a suspension of a strict one (*cf.* [6]). It is shown in [6] and [10] that X is a strict invertible spectrum if and only if $E(2)_*(X) = E(2)_*$ as an $E(2)_*E(2)$-comodule. For a strict invertible spectrum X, the $E(2)$-based Adams E_2-term $E_2^*(X)$ is isomorphic to $E_2^*(S)$. In particular, $E_2^{0,0}(X) = E_2^{0,0}(S) = \mathbf{Z}_{(3)}$. Take the generator $g_X \in E_2^{0,0}(X)$. Then by [10], X is characterized by $d_5(g_X) \in E_2^{5,4}(X)$. For example, $d_5(g_X) = 0$ if and only if $X = L_2S$. If $d_5(g_X) \neq 0$, then g_X does not detect a homotopy element but $3g_X$ does. We denote $\imath_X$ as the homotopy element detected by $3g_X$, and write WX as the cofiber of $\imath_X$.

Let $V(1\frac{1}{2})$ denote the Toda spectrum defined in [22] by the cofiber sequence

$$\Sigma^{16}V(0) \xrightarrow{B} V(1) \to V(1\tfrac{1}{2}) \to \Sigma^{17}V(0),$$

in which the map B induces the homomorphism $v_2 : E(2)_*/(3) \to E(2)_*/(3, v_1)$, the multiplication by v_2. The element B is denoted by $[\beta i_1]$ in [22]. This is an example of generalized Moore spectra studied in the next section. In [10], a proper invertible spectrum P is constructed as well as the existence of a map $f : P \to \Sigma^{-21}L_2V(1\frac{1}{2})$ that realizes the projection $E(2)_* \to E(2)_*/(3)$. The spectrum P is characterized by the differential of the $E(2)$-based Adams spectral sequence as follows:

$$d_5(g_P) = v_2^{-2}h_{11}b_{10}^2 \in E_2^{5,4}(P) = E_2^{5,4}(S),$$

where the E_2-term $E_2^{5,4}(S)$ for $\pi_*(L_2S)$ is isomorphic to $(\mathbf{Z}/3)\{v_2^{-2}h_{11}b_{10}^2, v_2^{-1}\xi b_{10}\zeta_2\}$. Note then that $P^2 = P \wedge P$ is an invertible spectrum characterized by $d_5(g_{P^2}) = -v_2^{-2}h_{11}b_{10}^2 \in E_2^{5,4}(P^2) = E_2^{5,4}(S)$. Now the generalized Moore spectrum WP^k for $k = 1, 2$ fits in the cofiber sequence

$$S \xrightarrow{\imath_k} P^k \xrightarrow{i_k} WP^k \xrightarrow{j_k} \Sigma S, \tag{4.1}$$

where $\imath_k$ is the abbreviation of $\imath_{P^k}$.

Proposition 4.2. *WP and WP^2 are generalized Moore spectra, which are not $V(0)$-module spectra.*

Proof. By definition, the homotopy element $\imath_k \in \pi_0(P^k)$ for $k = 1, 2$ induces $(\imath_k)_* = 3 : E(2)_* = E(2)_*(S) \to E(2)_*(P^k) = E(2)_*$, and so $E(2)_*(WP^k)$ is isomorphic to $E(2)_*/(3)$ as an $E(2)_*E(2)$-comodule.

We show that $3id \neq 0 \in [WP, WP]_0$ for the identity map $id \in [WP, WP]_0$. Consider the diagram

$$\begin{array}{ccccccc} & & & & [WP, WP]_0 & & \\ & & & & \big\downarrow{\scriptstyle i_1^*} & & \\ [P, S]_0 & \xrightarrow{\imath_{1*}} & [P, P]_0 & \xrightarrow{i_{1*}} & [P, WP]_0 & \xrightarrow{j_{1*}} & [P, S]_{-1} \end{array}$$

The behavior of the map $\imath_{1*}:[P,S]_0\to[P,P]_0$ is observed by the one of $(\imath_1\wedge P^2)_*:[S,P^2]_0\to[S,S]_0$, since P is an invertible spectrum and P^2 is its inverse. The generator $\imath_2\in\mathbf{Z}\subset[S,P^2]_0$ is detected by $3g_{P^2}\in E_2^{0,0}(P^2)$ and $\imath_1$ induces also multiplication by 3 on the E_2-terms. Therefore, the induced map $(\imath_1\wedge P^2)_*$ assigns the element $3g_{P^2}$ to $9g_S$ on the E_2-terms, and so we have $(\imath_1\wedge P^2)_*(\imath_2)=9\in\pi_0(S)$ on the homotopy. It follows that the map i_1 generates $\mathbf{Z}/9\subset[P,WP]_0$. Therefore, $i_1^*(3id)=3i_1\neq0\in[P,WP]_0$, and $3id\neq0$. □

The results of [10] and [18] imply the following

Lemma 4.3. *If $U=WX$ for an invertible spectrum X, then* $d_5(g_U)\in(\mathbf{Z}/3)\{v_2^{-2}h_{11}b_{10}^2,v_2^{-1}\xi b_{10}\zeta_2\}\subset E_2^{5,4}(U)$.

Proof. By the definition of WX, there is an exact sequence

$$E_2^{5,4}(S)\xrightarrow{(\imath_X)_*=3}E_2^{5,4}(X)\xrightarrow{(i_X)_*}E_2^{5,4}(U)\xrightarrow{\delta}E_2^{6,4}(S)$$

associated to the cofiber sequence $S\xrightarrow{\imath_X}X\xrightarrow{i_X}U$. Since $E_2^{5,4}(X)=E_2^{5,4}(S)=(\mathbf{Z}/3)\{v_2^{-2}h_{11}b_{10}^2,v_2^{-1}\xi b_{10}\zeta_2\}$ by [18], $(i_X)_*$ is a monomorphism. Now the lemma follows from the naturality of the Adams differentials. □

5. The generalized Moore spectrum V_5 related to v_2^5

It is well known that there is a generator $v_2^i\in\pi_{16i}(V(1))$ for each $i=0,1,5$ ([12]), which are of order three. It follows that there exists a map $B^{(i)}:\Sigma^{16i}V(0)\to V(1)$ that induces $v_2^i:E(2)_*/(3)\to E(2)_*/(3,v_1)$. Note that $B^{(0)}=i_1$ and $B^{(1)}=\beta$, the Smith element. First we show that

Lemma 5.1. *The element $v_2^5\in\pi_{80}(V(1))$ does not extend to a self-map $v_2^5:\Sigma^{80}V(1)\to V(1)$.*

Proof. Suppose that such a self-map exists. Then we have a map $v_2^{10}:\Sigma^{160}V(1)\to V(1)$. Since there is an equivalence $v_2^9:\Sigma^{144}L_2V(1)\simeq L_2V(1)$ by [7], we obtain a self map $v_2:\Sigma^{16}L_2V(1)\to L_2V(1)$. This contradicts to the non-existence of the self map $v_2:\Sigma^{16}V(1)\to V(1)$, whose obstruction survives after localizing it with respect to $E(2)$. □

Now we write $\Sigma^{16i+5}V_i$ as the cofiber of $B^{(i)}$.

Proposition 5.2. *L_2V_5 is a generalized Moore spectrum.*

Proof. Consider the diagram

$$\begin{array}{ccccccc}
\Sigma^{84}V(0) & \xrightarrow{\alpha} & \Sigma^{80}V(0) & \xrightarrow{i_1} & \Sigma^{80}V(1) & \xrightarrow{j_1} & \Sigma^{85}V(0) \\
\downarrow & & \| & & \downarrow{\scriptstyle v_2^5} & & \downarrow \\
\Sigma^{-1}V_5 & \xrightarrow{j_{B^{(5)}}} & \Sigma^{80}V(0) & \xrightarrow{B^{(5)}} & V(1) & \xrightarrow{i_{B^{(5)}}} & \Sigma^{85}V_5.
\end{array}$$

Here the broken arrows exist after smashing with $E(2)$ so that the diagram commutes. Since v_2^5 is an isomorphism on the $E(2)_*$-homology of $V(1)$, we have an isomorphism $E(2)_*/(3) = E(2)_*(V(0)) = E(2)_*(V_5)$ by the Five Lemma. □

Lemma 5.3. *$L_2V_5 \not\simeq WX$ for any invertible spectrum X.*

Proof. Lemma 5.1 shows that $B^{(5)}\alpha \neq 0 \in \pi_{84}(L_2V(1))$. Write $x \neq 0 \in E_2^{*,*}(V(1))$ for an element that detects $B^{(5)}\alpha$. Then $x \in E_2^{4,88}(V(1))$ or $x \in E_2^{8,92}(V(1))$. By [16], we see that $E_6^{4,88}(V(1)) = (\mathbf{Z}/3)\{v_2^4 b_{10}h_{11}\zeta_2\}$ and $E_6^{8,92}(V(1)) = 0$. It follows that $x = \pm v_2^4 b_{10}h_{11}\zeta_2 \in E_2^{4,88}(V(1))$. Observe the cofiber sequence that defines V_5, and we see that $d_5(g_{V_5}) = \delta(x) = \pm v_1 v_2^{-3} b_{11} b_{10}\zeta_2 \in E_5^{5,4}(V_5)$. In fact, $\delta(v_2^4 b_{10}h_{11}\zeta_2) = [v_1^{-1}d(v_2^{-1}b_{10}h_{11}\zeta_2)] = v_1 v_2^{-3} b_{11} b_{10}\zeta_2$ read off from [17, Lemma 3.3]. Therefore, Lemma 4.3 shows that there is no invertible spectrum X such that $L_2V_5 = WX$. □

Proof of Proposition 1.2. Since $E(2)_*(X)$ for a strict invertible spectrum X is isomorphic to $E(2)_*$ as an $E(2)_*E(2)$-comodule, we see that $E(2)_*(U \wedge X) = E(2)_*(U) \otimes_{E(2)_*} E(2)_*(X) = E(2)_*(U)$ as an $E(2)_*E(2)$-comodule. So it suffices to show that WX and L_2V_i are generalized Moore spectra. For L_2V_0, it is trivial, and for L_2V_1, it is shown in [10]. L_2V_5 and WX are generalized Moore spectra by Propositions 5.2 and 4.2, respectively. □

6. The homotopy groups of generalized Moore spectra

We also work in $\mathscr{L}_2$ at the prime three. For a spectrum U, the spectra N_nU and M_nU are defined in [14] inductively by the cofiber sequence

$$N_nU \to M_nU \to N_{n+1}U,$$

setting $N_0U = U$ and $M_nU = L_nN_nU$. Then the $E(2)$-based Adams E_2-term for the homotopy groups $\pi_*(M_2U)$ of a generalized Moore spectrum U is isomorphic to

$$E_2^*(M_2U) = E_2^*(M_2V(0)) = H^*M_1^1,$$

where H^*M for an $E(2)_*E(2)$-comodule M denotes $\mathrm{Ext}^*_{E(2)_*E(2)}(E(2)_*, M)$, and the comodule M_1^1 is the cokernel of the localization map $E(2)_*/(3) \to$

$v_1^{-1}E(2)_*/(3)$. By observing the decomposition (3.3) of $E_2^*(M_2U) = H^*M_1^1$, we obtain

Lemma 6.1. *The $E(2)$-based Adams differentials d_5 and d_9 are trivial on $A \subset E_2^*(M_2U)$ for any generalized Moore spectrum U.*

Proof. Suppose that $d_r(x) = y$ for an element $x \in A$. Then $y \in B$ by (3.4) 1), since the filtration degree of y is greater than 4. Now apply b_{10}, and we have

$$0 = d_r(xb_{10}) = yb_{10} \in E_r^*(M_2U)$$

by (3.4) 3) and the naturality of the differential, since b_{10} detects the homotopy element $\beta_1 \in \pi_{10}(S)$. If $r = 5$, then b_{10} acts freely on B, and so $y = 0$.

If $r = 9$, then $yb_{10} \in E_2^*(M_2U)$ is zero or killed by an element u under the differential d_5. If $yb_{10} = 0$ in the E_2-term, then $y = 0$ by (3.4) 3). So we assume that $d_5(u) = yb_{10}$ for some $u \in E_2^*(M_2U)$. Since the filtration degree of y is greater than 8, the filtration degree of u is greater than 5, and so $u = wb_{10}$ for some $w \in B$ by (3.4) 2). It follows that $d_5(w) = y \bmod \operatorname{Ker} b_{10}$. Note that $\operatorname{Ker} b_{10} \subset B$. Since b_{10} acts freely on B in the $E_5(= E_2)$-term, we see that $\operatorname{Ker} b_{10} = 0$, which shows that $d_9(x) = 0$. □

Let P be the proper invertible spectrum constructed in [10], and consider the cofiber sequence $S \xrightarrow{\iota_k} P^k \xrightarrow{i_k} WP^k \xrightarrow{j_k} \Sigma S$ for $k = 0, 1, 2$, where we abbreviate ι_{P^k} by ι_k. Then it induces a long exact sequence

$$\pi_*(M_2S) \xrightarrow{\iota_{k*}} \pi_*(M_2P^k) \xrightarrow{i_{k*}} \pi_*(M_2WP^k) \xrightarrow{j_{k*}} \pi_{*-1}(M_2S) \xrightarrow{\iota_{k*}} \pi_{*-1}(M_2P^k).$$

The homotopy groups $\pi_*(M_2S)$ are computed by the $E(2)$-based Adams spectral sequence with E_2-term $H^*M_0^2$. Here M_0^2 is the comodule defined by the short exact sequences

$$0 \to E(2)_* \to 3^{-1}E(2)_* \to N_0^1 \to 0 \qquad \text{and} \qquad 0 \to N_0^1 \to v_1^{-1}N_0^1 \to M_0^2 \to 0$$

of $E(2)_*E(2)$-comodules, in which both of the inclusions are the localization maps. Since we have $H^*M_1^1 = A \oplus B$, the structure of the E_2-term $H^*M_0^2$ is described by using A and B, which is seen by observing the long exact sequence

$$0 \to H^0M_1^1 \xrightarrow{\phi_*} H^0M_0^2 \xrightarrow{3} H^0M_0^2 \xrightarrow{\delta} H^1M_1^1 \to \cdots \to H^sM_1^1 \xrightarrow{\phi_*} H^sM_0^2 \xrightarrow{3} H^sM_0^2 \xrightarrow{\delta} H^{s+1}M_1^1 \to \cdots$$

associated to the short exact sequence

$$0 \to M_1^1 \xrightarrow{\phi} M_0^2 \xrightarrow{3} M_0^2 \to 0. \tag{6.2}$$

In fact, let $\overline{M}$ for $M = A, B$ denote the module fitting in the exact sequence $M \xrightarrow{\phi} \overline{M} \xrightarrow{3} \overline{M} \xrightarrow{\delta} M$. It is shown in [18] that B is decomposed into B_h and B_t so that $\phi(B_h) = \bar{B}$ and $\delta(\bar{B}) = B_t$. Note that these are isomorphisms, that is, $B_h \cong \bar{B} \cong B_t$. Then, we obtain

$$H^* M_0^2 = \bar{A} \oplus \bar{B}$$

by [11, Remark 3.11]. The behaviors of the differentials d_5 and d_9 on the spectral sequence for $\pi_*(M_2 V(0))$ are studied in [17] as follows:

(6.3) 1. the differentials d_5 and d_9 act trivially on A.
2. the survivors of B in the E_{10}-term have the filtration degree less than 13.

Let $\widetilde{B_h}$ and $\widetilde{B_t}$ denote the submodules consisting of the survivors of B_h and B_t in E_{10}-term, respectively. The properties (6.3) 2) and (3.4) 1) show that the E_{10}-term has the horizontal vanishing line $s = 13$, which means that the E_{10}-term is the E_∞-term. Therefore, we obtain

$$\pi_*(M_2 V(0)) = A \oplus \widetilde{B_h} \oplus \widetilde{B_t}.$$

A similar properties hold for the spectral sequence for $\pi_*(M_2 S)$ [18]:

(6.4) 1. the differentials d_5 and d_9 act trivially on $\bar{A}$.
2. the survivors of $\bar{B}$ in the E_{10}-term have the filtration degree less than 13.

Therefore, the same argument as above shows

$$\pi_*(M_2 S) = \bar{A} \oplus \tilde{\bar{B}},$$

where $\tilde{\bar{B}}$ denotes the submodule consisting of the survivors of $\bar{B}$ in E_{10}-term.

Remark. In [18], the structure of $\bar{A}$ was left undetermined. It is determined in [19].

Under this notation, we determined in [8] the structure of the homotopy groups $\pi_*(P^k)$ for $k = 0, 1, 2$ as

$$\pi_*(M_2 P^k) = \bar{A} \oplus v_2^{9-3k} \tilde{\bar{B}}. \tag{6.5}$$

Note that $v_2^9 \tilde{\bar{B}} = \tilde{\bar{B}}$. Consider now the cofiber sequence

$$M_2 P^l \xrightarrow{\iota_k \wedge P^l} M_2 P^{k+l} \xrightarrow{i_k \wedge P^l} M_2 WP^k \wedge P^l \xrightarrow{j_k \wedge P^l} M_2 P^l \tag{6.6}$$

obtained by smashing P^l with the cofiber sequence (4.1). Then we compute the homotopy groups $\pi_*(M_2 WP^k \wedge P^l)$ by the structure (6.5) of $\pi_*(M_2 P^k)$.

Theorem 6.7. $\pi_*(M_2 WP^k \wedge P^l) = A \oplus v_2^{9-3l} \widetilde{B_h} \oplus v_2^{9-3(k+l)} \widetilde{B_t}$.

Proof. The cofiber sequence (6.6) induces a long exact sequence

$$H^sM_1^1 \xrightarrow{(j_k\wedge P^l)_*} H^sM_0^2 \xrightarrow{\ 3\ } H^sM_0^2 \xrightarrow{(i_k\wedge P^l)_*} H^{s+1}M_1^1$$

of the $E(2)$-based Adams E_2-terms. Since there are isomorphisms $(i_k\wedge P^l)_*(\bar{B}) = \delta(\bar{B}) = B_t$ and $(j_k\wedge P^l)_*(B_h) = \phi(B_h) = \bar{B}$ in the E_2-terms, we see that $v_2^{9-3(k+l)}\widetilde{B_t} \oplus v_2^{9-3l}\widetilde{B_h}$ is a summand of E_∞-term for $\pi_*(M_2WP^k\wedge P^l)$ by the naturality of the Adams differential. It also shows the same result as (6.3) 2). Therefore, the differentials d_r on A are trivial by Lemma 6.1. □

Proof of Theorem 1.4. In [8], we show that $P^2 \to P \xrightarrow{f} L_2V_1$ is a cofiber sequence for the proper invertible spectrum P. It follows that $V_1\wedge P = WP^2$, and $V_1 = WP^2\wedge P^2$. Thus the theorem follows from Theorem 6.7. □

Proof of Theorem 1.5. This is a corollary of Theorem 6.7. □

7. The homotopy groups $\pi_*(L_2V_5)$

Let $\Sigma^{85}V_5$ denote the cofiber of $B^{(5)}: \Sigma^{80}V(0) \to V(1)$ as above. By definition, we have the cofiber sequence

$$\Sigma^{84}L_2V_5 \xrightarrow{j_{B^{(5)}}} \Sigma^{80}L_2V(0) \xrightarrow{B^{(5)}} L_2V(1) \xrightarrow{i_{B^{(5)}}} \Sigma^{85}L_2V_5.$$

Then we obtain the cofiber sequence

$$L_2V(1) \xrightarrow{\phi_5} \Sigma^{84}M_2V_5 \xrightarrow{A_5} \Sigma^{80}M_2V(0) \xrightarrow{\partial_5} \Sigma L_2V(1)$$

by Verdier's axiom. This induces an exact sequence

$$(7.1)\quad \cdots \longrightarrow H^*M_2^0 \xrightarrow{v_2^{-5}/v_1} H^*M_1^1 \xrightarrow{v_1} H^*M_1^1 \xrightarrow{\delta_5} H^*M_2^0 \longrightarrow \cdots,$$

where $\delta_5 = v_2^5\delta$ for the connecting homomorphism δ associated to the short exact sequence (3.2).

We consider the Adams differentials on $B \subset E_2^*(M_2V_5) \cong H^*M_1^1$. For this sake, it suffices to know the behavior on the elements

$$v_2^{3t\pm1}/v_1,\qquad v_2^{3t+1}h_{10}/v_1^\varepsilon,\qquad v_2^{3t}\xi/v_1^\varepsilon,\qquad \text{and}\qquad v_2^{3t\pm1}\psi_1/v_1$$

for $t\in\mathbf{Z}$ and $\varepsilon = 1,2$. Indeed, using the multiplicative relations given in [16, Prop. 5.9], the behavior on the other elements is deduced by the actions of the homotopy elements $\beta_{6/3}\in\pi_{82}(S)$ and $\beta_1\in\pi_{10}(S)$, which are detected by $\widetilde{v_2^3b_{11}}\in E_2^{2,84}(S)$ and $b_{10}\in E_2^{2,12}(S)$, respectively. Here, $\widetilde{v_2^3b_{11}} \equiv v_2^3b_{11} \mod(3,v_1)$.

Lemma 7.2. *The behavior of the Adams differentials is given by:*

$$d_5(v_2^{3t+1}/v_1) = (1-t)v_2^{3t-1}h_{11}b_{10}^2/v_1,$$
$$d_5(v_2^{3t-1}/v_1) = 0,$$
$$d_9(v_2^{3t+1}h_{10}/v_1) = \pm t(t-1)v_2^{3t-2}b_{10}^5/v_1,$$
$$d_5(v_2^{3t+1}h_{10}/v_1^2) = tv_2^{3t-1}b_{10}^3/v_1 \quad \textit{up to sign},$$
$$d_9(v_2^{3t}\xi/v_1) = \pm(t^2-1)v_2^{3t-3}\psi_0 b_{10}^4/v_1,$$
$$d_5(v_2^{3t}\xi/v_1^2) = (1-t)v_2^{3t-2}\psi_0 b_{10}^2/v_1 \quad \textit{up to sign},$$
$$d_5(v_2^{3t+1}\psi_1/v_1) = -tv_2^{3t}\xi b_{10}^3/v_1 \quad \textit{and}$$
$$d_5(v_2^{3t-1}\psi_1/v_1) = 0.$$

Proof. First consider the elements in the image of v_2^{-5}/v_1. In the $E(2)_*$-based Adams spectral sequence for $\pi_*(L_2V(1))$,

(7.3)
$$d_5(v_2^j) = \begin{cases} 0 & j \equiv 0,1,5 \ (9) \\ -v_2^{j-2}h_{11}b_{10}^2 & j \equiv 3,4,8 \ (9) \\ v_2^{j-2}h_{11}b_{10}^2 & j \equiv 2,6,7 \ (9) \end{cases}$$

$$d_5(v_2^j\psi_1) = \begin{cases} 0 & j \equiv 2,6,7 \ (9) \\ -v_2^{j-1}\xi b_{10}^3 & j \equiv 0,1,5 \ (9) \\ v_2^{j-1}\xi b_{10}^3 & j \equiv 3,4,8 \ (9) \end{cases}$$

$$d_9(v_2^j h_{10}) = \begin{cases} v_2^{j-3}b_{10}^5 & j \equiv 3,4,8 \ (9) \\ 0 & \text{otherwise} \end{cases} \qquad d_9(v_2^j\xi) = \begin{cases} v_2^{j-3}\psi_0 b_{10}^4 & j \equiv 1,5,6 \ (9) \\ 0 & \text{otherwise} \end{cases}$$

by [16, Prop. 8.4, Prop. 9.10]. Note that the undetermined integer k in [16] is shown to be 1 in [4] (see also [3]). By the first two equations, we compute

$$d_5(v_2^{3t+1}/v_1) = (v_2^{-5}/v_1)_* d_5(v_2^{3t+6}) = (1-t)(v_2^{-5}/v_1)_*(v_2^{3t+4}h_{11}b_{10}^2),$$
$$d_5(v_2^{3t-1}/v_1) = (v_2^{-5}/v_1)_* d_5(v_2^{3t+4}) = -(t+1)(v_2^{-5}/v_1)_*(v_2^{3t+2}h_{11}b_{10}^2),$$
$$d_5(v_2^{3t+1}\psi_1/v_1) = (v_2^{-5}/v_1)_* d_5(v_2^{3t+6}\psi_1) = -t(v_2^{-5}/v_1)_*(v_2^{3t+5}\xi b_{10}^3) \quad \text{and}$$
$$d_5(v_2^{3t-1}\psi_1/v_1) = (v_2^{-5}/v_1)_* d_5(v_2^{3t+4}\psi_1) = (1-t)(v_2^{-5}/v_1)_*(v_2^{3t+3}\xi b_{10}^3).$$

Since $\delta_5(v_2^{3t\pm1}b_{10}^2/v_1) = \pm v_2^{3t\pm1+4}h_{11}b_{10}^2$ and $\delta_5(v_2^{3t\pm1}\psi_1 b_{10}^2/v_1) = \pm v_2^{3t\pm1+5}\xi b_{10}^3$ are the only relations related to our case by [17, Prop. 3.4], we see the first, the second, the seventh and the eighth equalities.

We also compute with (7.3) as

$$d_9(v_2^{3t+1}h_{10}/v_1) = (v_2^{-5}/v_1)_* d_9(v_2^{3t+6}h_{10}) = \pm t(t-1)(v_2^{-5}/v_1)_*(v_2^{3t+3}b_{10}^5) \quad \text{and}$$
$$d_9(v_2^{3t}\xi/v_1) = (v_2^{-5}/v_1)_* d_9(v_2^{3t+5}\xi) = \pm(t^2-1)(v_2^{-5}/v_1)_*(v_2^{3t+2}\psi_0 b_{10}^4).$$

Furthermore, $\delta_5(v_2^{3t+1}h_{10}b_{10}^4/v_1^2) = v_2^{3t+5}b_{10}^5$ and $\delta_5(v_2^{3t}\xi b_{10}^4/v_1^2) = -v_2^{3t+4}\psi_0 b_{10}^4$ are the only relations, and we see the third and fifth equalities.

We make a different argument to show the fourth and the sixth equalities. Since $d_5(v_2^{3t+1}h_{10}/v_1^2) \in E_2^{6,16(3t+1)}(L_2V_5) = (\mathbf{Z}/3)\{v_2^{3t-1}b_{10}^3/v_1\}$, we put $d_5(v_2^{3t+1}h_{10}/v_1^2) = kv_2^{3t-1}b_{10}^3/v_1$ for some $k \in \mathbf{Z}/3$. We see that $(v_2^{-5}/v_1)_*(kv_2^{3t+4}b_{10}^3) = kv_2^{3t-1}b_{10}^3/v_1$ and $d_5(v_2^{3t+4}b_{10}^3) = -(1+t)v_2^{3t+2}h_{11}b_{10}^5$. On the other hand, $\delta_5(d_9(v_2^{3t+1}h_{10}/v_1)) = t(t+1)\delta_5(v_2^{3t-2}b_{10}^5/v_1) = t(t+1)\cdot v_2^{3t+2}h_{11}b_{10}^5$. Therefore, $k(t+1) = t(t+1) \in \mathbf{Z}/3$ up to sign. It follows that $k = t \in \mathbf{Z}/3$ if $t \neq 2 \in \mathbf{Z}/3$. If $t = 2 \in \mathbf{Z}/3$, then $\partial_{5*}(v_2^{9s+7}h_{10}/v_1) = kv_2^{9s+10}b_{10}^3$. By applying b_{10}^3,

$$\begin{aligned} kv_2^{9s+10}b_{10}^6 &= \partial_{5*}(v_2^{9s+7}h_{10}b_{10}^3/v_1) = \partial_{5*}(d_5(v_2^{9s+7}b_{11}/v_1)) \\ &= d_9(\delta_5(v_2^{9s+7}b_{11}/v_1)) = d_9(v_2^{9s+13}h_{10}b_{10}) \\ &= v_2^{9s+10}b_{10}^6 \end{aligned}$$

up to sign by [17, Prop. 8.5], and $k \neq 0$. Thus the fourth equality follows. In the same manner as this, we obtain the sixth. In fact, $\delta_5(d_9(v_2^{3t}\xi/v_1)) = t(t-1)v_2^{3t}\xi b_{11}b_{10}^4 = ktv_2^{3t}\xi b_{11}b_{10}^4 = d_5(kv_2^{3t+3}\psi_0 b_{10}^2)$ if we put $d_5(v_2^{3t}\xi/v_1^2) = kv_2^{3t-2}\psi_0 b_{10}^2/v_1$ for some $k \in \mathbf{Z}/3$. Thus, $k = t-1$ if $t \neq 0$. If $t = 0$, we see that $k \neq 0$ by the equation $kv_2^{9s+3}\psi_0 b_{10}^5 = \partial_{5*}(d_5(v_2^{9s+1}\psi_1/v_1)) = d_9(\delta_{5*}(v_2^{9s+1}\psi_1/v_1)) = v_2^{9s+3}\psi_0 b_{10}^5$ up to sign. □

Proof of Theorem 1.6. For the homotopy groups $\pi_*(L_2V(0))$, the Adams differentials act as follows:

$$d_5(v_2^{3t+1}/v_1) = -tv_2^{3t-1}h_{11}b_{10}^2/v_1, \qquad d_9(v_2^{3t}\xi/v_1) = \pm t(t-1)v_2^{3t-3}\psi_0 b_{10}^4/v_1,$$
$$d_5(v_2^{3t-1}/v_1) = 0, \qquad d_5(v_2^{3t}\xi/v_1^2) = (1-t)v_2^{3t-2}\psi_0 b_{10}^2/v_1,$$
$$d_9(v_2^{3t+1}h_{10}/v_1) = \pm t(t+1)v_2^{3t-2}b_{10}^5/v_1, \quad d_5(v_2^{3t+1}\psi_1/v_1) = -(t+1)v_2^{3t}\xi b_{10}^3/v_1,$$
$$d_5(v_2^{3t+1}h_{10}/v_1^2) = tv_2^{3t-1}b_{10}^3/v_1, \qquad d_5(v_2^{3t-1}\psi_1/v_1) = 0.$$

We decompose B into eight summands, each of which is generated by the one of the above eight elements as $E(2)_*[\beta_1, \beta_{6/3}]/(\beta_{6/3}^2 - v_2^9\beta_1^2)$-modules. We name them F_h^0, F_t^0, F_h^1, F_t^1, F_h^{*0}, F_t^{*0}, F_h^{*1} and F_t^{*1}, respectively. Corresponding ones of L_2V_5, we name them G instead of F. Then Lemma 7.2 shows the isomorphisms

$$G_h^0 = v_2^3F_h^0, \qquad G_t^0 = F_t^0, \qquad G_h^1 = v_2^3F_h^1, \qquad G_t^1 = F_t^1,$$

$$G_h^{*0} = v_2^3 F_h^{*0}, \qquad G_t^{*0} = F_t^{*0}, \qquad G_h^{*1} = v_2^3 F_h^{*1} \qquad \text{and} \qquad G_t^{*1} = F_t^{*1}$$

as spectral sequences. Since $B = \bigoplus_{x=h,t,\varepsilon=0,1} G_x^{\varepsilon} \oplus G_x^{*\varepsilon}$, the E_{10}-term has the same horizontal vanishing line as the one for $\pi_*(L_2V(0))$. It follows that every element of A survives to the E_∞-term by Lemma 6.1. □

References

[1] A. D. Elmendorf, I. Kriz, M. A. Mandell and J. P. May, Rings, Modules, and Algebras in stable homotopy theory, Mathematical Surveys and Monographs **47**, Amer. Math. Soc., 1996.

[2] P. G. Goerss, Associative MU-algebra, preprint.

[3] P. Goerss, H.-W. Henn and M. Mahowald, The homotopy of $L_2V(1)$ for the prime 3, Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), Progr. Math., 215, Birkhäuser, Basel, (2004), 125–151.

[4] H.-W. Henn, Centralizers of elementary abelian p-subgroups and mod-p cohomology of profinite groups, Duke Math. J. **91** (1998), 933–941.

[5] M. Hovey, J. H. Palmieri and N. P. Strickland, Axiomatic stable homotopy theory, Memoirs A.M.S. **610** (1997).

[6] M. Hovey and H. Sadofsky, Invertible spectra in the $E(n)$-local stable homotopy category, J. London Math. Soc. **60** (1999), 284–302.

[7] I. Ichigi and K. Shimomura, $E(2)_*$-invertible spectra smashing with the Smith-Toda spectrum $V(1)$ at the prime 3, Proc. Amer. Math. Soc. **132** (2004), 3111–3119.

[8] I. Ichigi and K. Shimomura, The homotopy groups of $L_2V(1\frac{1}{2})$ and an invertible spectrum at the prime three, preprint.

[9] I. Ichigi and K. Shimomura, On the homotopy groups of an invertible spectrum in the $E(2)$-local category at the prime 3, JP Jour. Geometry & Topology **3** (2003), 257–268.

[10] Y. Kamiya and K. Shimomura, A relation between the Picard group of the $E(n)$-local homotopy category and $E(n)$-based Adams spectral sequence, the Proceedings of the Northwestern University Algebraic Topology Conference, March 2002, Contemp. Math. **346** (2004), 321–333.

[11] H. R. Miller, D. C. Ravenel, and W. S. Wilson, Periodic phenomena in Adams-Novikov spectral sequence, Ann. of Math. **106** (1977), 469–516.

[12] S. Oka, Note on the β-family in stable homotopy of spheres at the prime 3, Mem. Fac. Sci. Kyushu Univ. **35** (1981), 367–373.

[13] D. C. Ravenel, Complex cobordism and stable homotopy groups of spheres (Academic Press, 1986).

[14] D. C. Ravenel, Localization with respect to certain periodic homology theories, Amer. J. Math. **106** (1984), 415–446.

[15] D. C. Ravenel, Nilpotence and periodicity in Stable homotopy theory, Ann. of Math. Stud. **128**, Princeton University Press, Princeton, 1992.

[16] K. Shimomura, The homotopy groups of the L_2-localized Toda-Smith spectrum $V(1)$ at the prime 3, Trans. Amer. Math. Soc. **349** (1997), 1821–1850.

[17] K. Shimomura, The homotopy groups of the L_2-localized mod 3 Moore spectrum, J. Math. Soc. Japan, **52** (2000), 65–90.

[18] K. Shimomura, On the action of β_1 in the stable homotopy of spheres at the prime 3, Hiroshima Math. J. **30** (2000), 345–362.

[19] K. Shimomura and X. Wang, The homotopy groups $\pi_*(L_2S^0)$ at the prime 3, Topology **41** (2002), 1183–1198.
[20] K. Shimomura and M. Yokotani, Existence of the Greek letter elements in the stable homotopy groups of $E(n)_*$-localized spheres, Publ. RIMS, Kyoto Univ. **30** (1994), 139–150.
[21] K. Shimomura and Z. Yosimura, BP-Hopf module spectrum and BP_*-Adams spectral sequence, Publ. RIMS, Kyoto Univ. **21** (1986), 925–947.
[22] H. Toda, Algebra of stable homotopy of $\mathbf{Z}_p$-spaces and applications, J. Math. Kyoto Univ., **11** (1971), 197–251.
[23] H. Toda, On spectra realizing exterior parts of the Steenrod algebra. Topology **10** (1971), 53–66.

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