GENERALIZED STEENROD HOMOLOGY THEORIES ARE STRONG SHAPE INVARIANT

PETER MROZIK

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Abstract

It is shown that a reduced homology theory on the category of pointed compact metric spaces is strong shape invariant if and only if its homology functors h_n satisfy the *quotient exactness axiom*, which means that for each pointed compact metric pair (X, A, a_0) the natural sequence $h_n(A, a_0) \to h_n(X, a_0) \to h_n(X/A, *)$ is exact. As a consequence, all generalized Steenrod homology theories are strong shape invariant.

1. Introduction

The concept of a generalized Steenrod homology theory for compact metric spaces was introduced in [13] by Kaminker and Schochet. Their purely axiomatic approach distills the essential properties which are common to classical (ordinary) Steenrod homology and to Brown-Douglas-Fillmore homology ϵ_* whose origin lies in the theory of operator algebras (see, e.g., [3]).

Steenrod homology theories are reduced theories defined for single spaces. In [13] the authors dealt with the unpointed case. In the subsequent paper [14], Kahn, Kaminker and Schochet switched to the pointed case which is the more natural setting for reduced homology and on which we shall focus.

By a *reduced pointed homology system* on the category $\mathbf{CM_0}$ of pointed compact metric spaces we denote any system $(h_n, \sigma_n)_{n \in \mathbb{Z}}$ of covariant homotopy invariant functors $h_n \colon \mathbf{CM_0} \to \mathbf{Ab}$ (= category of abelian groups) and of natural isomorphisms $\sigma_n \colon h_n \to h_{n+1} \circ \mathbf{S}$, where $\mathbf{S}$ denotes the reduced suspension functor on $\mathbf{CM_0}$. A reduced pointed homology system is called *exact* if it satisfies the following *exactness axiom*: For each pointed compact metric pair (X, A, a_0) the natural sequence

$$h_n(A, a_0) \xrightarrow{i_*} h_n(X, a_0) \xrightarrow{j_*} h_n(X \cup C(A, a_0), *)$$

is exact for all $n \in \mathbb{Z}$.

Here, $(X \cup C(A, a_0), *)$ denotes the pointed adjunction space obtained by attaching the reduced cone $C(A, a_0)$ to X via the inclusion $A \hookrightarrow C(A, a_0)$.

In the literature, exact reduced pointed homology systems are customarily denoted as reduced pointed *homology theories* (see, e.g., [**5, 19**]), and we shall adopt this notation here.

We now come to a "stronger" variant of exactness. A reduced pointed homology system $(h_n, \sigma_n)_{n \in \mathbb{Z}}$ on $\mathbf{CM_0}$ is called *quotient exact* if it satisfies the following *quotient exactness axiom*: For each pointed compact metric pair (X, A, a_0) the natural sequence

$$h_n(A, a_0) \xrightarrow{i_*} h_n(X, a_0) \xrightarrow{p_*} h_n(X/A, *)$$

is exact for all $n \in \mathbb{Z}$.

Each quotient exact reduced pointed homology system yields a commutative diagram

$$
\begin{array}{ccccc}
h_n(A, a_0) & \xrightarrow{\ i_*\ } & h_n(X, a_0) & \xrightarrow{\ p_*\ } & h_n(X/A, *) \\
 & & \downarrow{\scriptstyle j_*} & & \| \\
h_n(C(A, a_0), *) & \xrightarrow{\ i'_*\ } & h_n(X \cup C(A, a_0), *) & \xrightarrow{\ p'_*\ } & h_n(X/A, *)
\end{array}
$$

with exact rows. Since $h_n(C(A, a_0), *) = h_n(*, *) = 0$, p'_* is a monomorphism which implies that the exactness axiom is satisfied. That is, each *quotient exact reduced pointed homology system* is a *reduced pointed homology theory*. This legitimizes the use of the phrase *homology theory* in the following definition (cf. [**14**]).

A *Steenrod homology theory* on $\mathbf{CM_0}$ is a reduced pointed homology system $(h_n, \sigma_n)_{n \in \mathbb{Z}}$ on $\mathbf{CM_0}$, which is quotient exact and, in addition, satisfies the following *cluster axiom*: For each sequence $(X_i, x_i)_{i \in \mathbb{N}}$ of pointed compacta, the projections $p_k \colon \mathbf{Cl}_{i=1}^{\infty}(X_i, x_i) \to (X_k, x_k)$ defined on the cluster $\mathbf{Cl}_{i=1}^{\infty}(X_i, x_i)$ induce isomorphisms

$$\pi_n \colon h_n(\mathbf{Cl}_{i=1}^{\infty}(X_i, x_i)) \to \prod_{k=1}^{\infty} h_n(X_k, x_k)$$

for all $n \in \mathbb{Z}$.

An abundance of Steenrod homology theories is constructed in [**14**] by associating to any CW-spectrum $\underline{E}$ a Steenrod homology theory ${}^s\underline{E}_*$, which is an extension of the standard reduced pointed homology theory $\underline{E}_*$ defined on the category $\mathbf{CW_0^{fin}}$ of finite pointed CW-complexes (see, e.g., [**19**, Section 8]). Therefore each reduced pointed homology theory h_* on $\mathbf{CW_0^{fin}}$ admits a *Steenrod extension* ${}^sh_*$ to $\mathbf{CM_0}$ because h_* can be represented by a CW-spectrum.

In [**6**, Section 8.2] Edwards and Hastings give an alternative construction of a Steenrod extension ${}^sh_*$ of a reduced pointed homology theory h_* on $\mathbf{CW_0^{fin}}$. Their approach brings together Steenrod extensions and strong shape theory by factorizing ${}^sh_*$ over the Vietoris functor $V \colon \mathbf{CM_0} \to \mathbf{Ho}(\mathbf{pro} - \mathbf{SS})$. Since V is strong shape invariant, the Steenrod extensions defined in [**6**] are also strong shape invariant.

For a deeper understanding of Steenrod extensions and their relation to strong shape theory, see Bauer's papers [**1, 2**].

It is worth mentioning that ordinary Steenrod homology theory is also known to be strong shape invariant. This is excellently explained in Ferry's paper [**7**].

The object of this paper is to show that *all* Steenrod homology theories are strong shape invariant. This will be derived directly from the axioms by continuing the approach of [**18**], where we showed that, among unreduced homology theories on the category of compact metric pairs, the strong shape invariant theories are precisely those satisfying the strong excision axiom. Our main result is the following.

Strong Shape Invariance Theorem (Theorem 3.5): The strong shape invariant reduced pointed homology theories on $\mathbf{CM_0}$ are precisely the quotient exact reduced pointed homology systems. In particular, all Steenrod homology theories on $\mathbf{CM_0}$ are strong shape invariant.

The proof of the Strong Shape Invariance Theorem is worth sketching here because of its exceptional clarity.

The reason why exactness plus strong shape invariance imply quotient exactness is that quotient maps collapsing a contractible subspace to a point are strong shape equivalences. This yields an isomorphism $q_* \colon h_n(X \cup C(A, a_0), *) \to h_n(X/A, *)$, which results in quotient exactness.

The main ingredient in the proof of the converse is a representation of the pointed strong shape category of pointed compacta $\mathbf{SSh_0}$ as a quotient category: In Section 7 we show that the strong shape functor $S \colon \mathbf{CM_0} \to \mathbf{SSh_0}$ localizes $\mathbf{CM_0}$ at the class of *pointed cylinder base embeddings*. A characteristic feature of pointed cylinder base embeddings $i \colon (A, a_0) \hookrightarrow (X, a_0)$ is that they are *inclusions such that the quotient space $(X/A, *)$ is pointed contractible* (see Proposition 7.4). But now there is a really remarkable coincidence: Any quotient exact reduced pointed homology system on $\mathbf{CM_0}$ takes such inclusions to isomorphisms which proves its strong shape invariance.

Although in this paper we concentrate on reduced *pointed* homology theories, we also treat reduced *unpointed* homology theories and obtain an analogue of the Strong Shape Invariance Theorem (Theorem 4.3). Moreover, in Section 6 we show that:

- strong shape invariant reduced pointed homology theories,
- strong shape invariant reduced unpointed homology theories, and
- strong shape invariant unreduced homology theories

are equivalent which means that they can be transformed into each other without losing information.

In particular, we see that the strong excision axiom in the unreduced case and the quotient exactness axiom in the reduced case are equivalent characterizations of the same phenomenon — namely strong shape invariance — which may possibly lead to a deeper understanding of the general context of these axioms.

It is also interesting that strong shape invariant homology theories occur in the entourage of the Novikov conjecture. To understand why ordinary Steenrod homology theory arises in studies of the Novikov Conjecture, the reader should consult [**7**]. In the most general form, strong shape invariant homology theories appear in Higson's and Roe's paper [**11**] on the coarse Baum-Connes conjecture. In [**11**], the authors consider *coarsenings* of generalized homology theories M_*. In Proposition 4.3, which states that the coarsening map $c \colon M_*(\mathcal{O}Y) \to MX_*(\mathcal{O}Y)$ is an isomorphism for any finite-dimensional compact metric space Y, they have to assume that the homology theory M_* satisfies the strong excision axiom, i.e., is strong shape invariant.

2.   Reduced homology theories: Pointed case

Reduced homology theories can be defined for pointed and unpointed spaces. We concentrate on the pointed case since this is the "natural setting" for reduced homology. The unpointed case will be addressed in short form in Section 4.

In the sequel we shall need the four functors:

- $\mathrm{Abs}\colon \mathbf{CM_0^2} \to \mathbf{CM_0}, \mathrm{Abs}(X, A, a_0) = (X, a_0), \mathrm{Abs}(f) = f,$
- $\mathrm{Res}\colon \mathbf{CM_0^2} \to \mathbf{CM_0}, \mathrm{Res}(X, A, a_0) = (A, a_0), \mathrm{Res}(f) = f\mid_A,$
- $\Gamma\colon \mathbf{CM_0^2} \to \mathbf{CM_0}, \Gamma(X, A, a_0) = (X \cup C(A, a_0), *), \Gamma(f) = f \cup C(f\mid_A),$
- $Q\colon \mathbf{CM_0^2} \to \mathbf{CM_0}, Q(X, A, a_0) = (X/A, *), Q(f) = $ induced quotient map.

The above functors come along with four natural transformations:

- Inclusion transformation $i\colon \mathrm{Res} \to \mathrm{Abs}, i^{(X,A,a_0)}\colon (A, a_0) \to (X, a_0),$
- Inclusion transformation $j\colon \mathrm{Abs} \to \Gamma, j^{(X,A,a_0)}\colon (X, a_0) \to (X \cup C(A, a_0), *),$
- Quotient transformation $p\colon \mathrm{Abs} \to Q, p^{(X,A,a_0)}\colon (X, a_0) \to (X/A, *),$
- Quotient transformation $q\colon \Gamma \to Q, q^{(X,A,a_0)}\colon (X \cup C(A, a_0), *) \to (X/A, *) = ((X \cup C(A, a_0))/C(A, a_0), *).$

In the following definitions let $h\colon \mathbf{CM_0} \to \mathbf{Ab}$ be an arbitrary covariant functor.

Definition 2.1 (Exactness Axiom). h satisfies the *exactness axiom* if for all pointed pairs $(X, A, a_0) \in \mathbf{CM_0^2}$ the sequence

$$
\begin{array}{ccccc}
h(A, a_0) & \xrightarrow{\ i_*\ } & h(X, a_0) & \xrightarrow{\ j_*\ } & h(X \cup C(A, a_0), *) \\
\| & & \| & & \| \\
h(\mathrm{Res}(X, A, a_0)) & & h(\mathrm{Abs}(X, A, a_0)) & & h(\Gamma(X, A, a_0))
\end{array}
$$

is exact.

Definition 2.2 (Quotient Axiom). h satisfies the *quotient exactness axiom* if for all pointed pairs $(X, A, a_0) \in \mathbf{CM_0^2}$ the sequence

$$
\begin{array}{ccccc}
h(A, a_0) & \xrightarrow{\ i_*\ } & h(X, a_0) & \xrightarrow{\ p_*\ } & h(X/A, *) \\
\| & & \| & & \| \\
h(\mathrm{Res}(X, A, a_0)) & & h(\mathrm{Abs}(X, A, a_0)) & & h(Q(X, A, a_0))
\end{array}
$$

is exact.

Definition 2.3 (Cone Collapsing Axiom). h satisfies the *cone collapsing axiom* if for all pointed pairs $(X, A, a_0) \in \mathbf{CM_0^2}$ the quotient map $q\colon (X \cup C(A, a_0), *) \to (X/A, *)$ induces an isomorphism

$$
\begin{array}{ccc}
h(X \cup C(A, a_0), *) & \xrightarrow{\ q_*\ } & h(X/A, *) \\
\| & & \| \\
h(\Gamma(X, A, a_0)) & & h(Q(X, A, a_0)).
\end{array}
$$

The following result (which is essentially due to [**3**]) relates the above concepts.

Theorem 2.4. *Let h be homotopy invariant. Then the following are equivalent:*

(1) *h satisfies the quotient exactness axiom.*

(2) *h satisfies the cone collapsing axiom and the exactness axiom.*

Proof of (1) $\Rightarrow$ (2). The cone collapsing axiom follows from the proofs of [**3**, Lemmas 2.16 and 2.17]. These proofs deal with $h = \text{Ext}$ in the unpointed case. The only properties of Ext used in [**3**] are homotopy invariance and quotient exactness so that the proofs apply to our situation. Replacing unreduced mapping cylinders, mapping cones, cones and suspensions by their reduced versions, we see that if $(X, A, a_0) \in \mathbf{CM_0^2}$ is a pointed pair such that (A, a_0) is pointed contractible, then $p_* \colon h(X, a_0) \to h(X/A, *)$ is an isomorphism. The cone collapsing axiom is just a special case of this result.

The exactness axiom is satisfied because $p_* = q_* \circ j_*$.

Proof of (2) $\Rightarrow$ (1). This is obvious since $p_* = q_* \circ j_*$. $\qquad\qquad\square$

Definition 2.5.

1. A *reduced pointed homology system* on $\mathbf{CM_0}$ is a collection of homotopy invariant covariant functors $h_n \colon \mathbf{CM_0} \to \mathbf{Ab}$ and natural isomorphisms $\sigma_n \colon h_n \to h_{n+1} \circ \mathbf{S}$, $n \in \mathbb{Z}$.

2. A *reduced pointed homology theory* is a reduced pointed homology system such that all homology functors h_n satisfy the exactness axiom.

3. A *quotient exact reduced pointed homology system* is a reduced pointed homology system such that all homology functors h_n satisfy the quotient exactness axiom.

From Theorem 2.4 we derive

Corollary 2.6. *On $\mathbf{CM_0}$ the quotient exact reduced pointed homology systems coincide with the reduced pointed homology theories which satisfy the cone collapsing axiom (i.e., for which all homology functors h_n satisfy the cone collapsing axiom).*

One can give an alternative definition of a reduced pointed homology theory as a collection of homotopy invariant covariant functors $h_n \colon \mathbf{CM_0} \to \mathbf{Ab}$ and natural transformations $\partial_n \colon h_{n+1} \circ \Gamma \to h_n \circ \text{Res}$ satisfying the *long exactness axiom*, which means that for each $(X, A, a_0) \in \mathbf{CM_0^2}$ there is a long exact sequence

$$\cdots \xrightarrow{\partial_n} h_n(A, a_0) \xrightarrow{i_*} h_n(X, a_0) \xrightarrow{j_*} h_n(X \cup C(A, a_0), *)$$

$$\xrightarrow{\partial_{n-1}} h_{n-1}(A, a_0) \xrightarrow{i_*} \cdots .$$

To verify the equivalence of both definitions we need the natural transformation

$$\xi \colon \Gamma \to \mathbf{S} \circ \text{Res}$$

given by the maps

$$\xi^{(X, A, a_0)} \colon (X \cup C(A, a_0), *) \xrightarrow{\pi} ((X \cup C(A, a_0))/X, *)$$

$$= (C(A, a_0)/A, *) \xrightarrow{h} \mathbf{S}(A, a_0),$$

where π is the quotient map and h is the obvious homeomorphism. Note that the

inclusion $X \hookrightarrow X \cup C(A, a_0)$ is a cofibration. Hence $\xi^{(X,A,a_0)}$ is a pointed homotopy equivalence if (X, a_0) is pointed contractible.

Given a reduced pointed homology theory in the sense of the original definition, we define $\partial_n = \partial(\sigma_n)$ as the composition

$$h_{n+1}(X \cup C(A, a_0), *) \xrightarrow{\xi_*} h_{n+1}(\mathbf{S}(A, a_0)) \xrightarrow{\sigma_n^{-1}} h_n(A, a_0).$$

The proof of long exactness can be found in [19, Section 7.33]. Observe that [19] gives a different definition of $\partial(\sigma_n)$ by interposing the automorphism $\nu_* \colon h_{n+1}(\mathbf{S}(A, a_0)) \to h_{n+1}(\mathbf{S}(A, a_0))$ induced by the homotopy inverse ν on the h-cogroup $\mathbf{S}(A, a_0)$. That is, $\partial(\sigma_n)$ as defined here is nothing else but $\partial(\nu_* \circ \sigma_n)$ as defined in [19]. But $(h_n, \nu_* \circ \sigma_n)$ is also a reduced pointed homology theory so we get the desired exactness result.

Conversely, given a reduced pointed homology theory in the alternative sense, we can construct natural isomorphisms $\sigma_n = \sigma(\partial_n)$ as follows: The long exact sequence of the pointed pair $(C(A, a_0), A, a_0)$ yields isomorphisms

$$h_{n+1}(\Gamma(C(A, a_0), A, a_0) \xrightarrow{\partial_n} h_n(A, a_0)$$

since all $h_n(C(A, a_0), *) = 0$. We can therefore define

$$\sigma_n \colon h_n(A, a_0) \xrightarrow{\partial_n^{-1}} h_{n+1}(\Gamma(C(A, a_0), A, a_0)) \xrightarrow{\xi_*} h_{n+1}(\mathbf{S}(A, a_0)).$$

In the above situation ξ is a pointed homotopy equivalence because $C(A, a_0)$ is pointed contractible. Therefore σ_n is a *natural isomorphism.*

The assignments $\sigma_n \mapsto \partial(\sigma_n)$ and $\partial_n \mapsto \sigma(\partial_n)$ are inverse to each other; i.e., we have $\sigma(\partial(\sigma_n)) = \sigma_n$ and $\partial(\sigma(\partial_n)) = \partial_n$.

To verify the first equation observe that $\xi_* = \xi_*^{(C(A,a_0),A,a_0)}$ is an isomorphism whence

$$\begin{aligned}
\sigma(\partial(\sigma_n)) &= \xi_*^{(C(A,a_0),A,a_0)} \circ \partial(\sigma_n)^{-1} \\
&= \xi_*^{(C(A,a_0),A,a_0)} \circ ((\sigma_n)^{-1} \circ \xi_*^{(C(A,a_0),A,a_0)})^{-1} \\
&= \sigma_n.
\end{aligned}$$

To check the second equation, let us consider the inclusion maps $\mu \colon (X, A, a_0) \to (C(X, a_0), A, a_0)$ and $\nu \colon (C(A, a_0), A, a_0) \to (C(X, a_0), A, a_0)$. The following diagram commutes because ξ and ∂_n are natural transformations:

This implies

$$\begin{aligned}
\partial(\sigma(\partial_n)) &= \sigma(\partial_n)^{-1} \circ \xi_*^{(X,A,a_0)} = (\xi_*^{(C(A,a_0),A,a_0)} \circ \partial_n^{-1})^{-1} \circ \xi_*^{(X,A,a_0)} \\
&= \partial_n \circ (\xi_*^{(C(A,a_0),A,a_0)})^{-1} \circ \xi_*^{(X,A,a_0)} \\
&= \partial_n \circ \Gamma(\nu)_* \circ (\xi_*^{(C(A,a_0),A,a_0)})^{-1} \circ \xi_*^{(X,A,a_0)} \\
&= \partial_n \circ (\xi_*^{(C(X,a_0),A,a_0)})^{-1} \circ \xi_*^{(X,A,a_0)} \\
&= \partial_n \circ \Gamma(\mu)_* = \partial_n.
\end{aligned}$$

Let us finally consider yet another concept of exactness.

Definition 2.7. A *reduced pointed Q-homology theory* on $\mathbf{CM_0}$ is a collection of homotopy invariant covariant functors $h_n \colon \mathbf{CM_0} \to \mathbf{Ab}$ and natural transformations $\Delta_n \colon h_{n+1} \circ Q \to h_n \circ \mathrm{Res}$, $n \in \mathbb{Z}$, such that for each $(X, A, a_0) \in \mathbf{CM_0^2}$ there is a long exact sequence

$$\cdots \xrightarrow{\Delta_n} h_n(A, a_0) \xrightarrow{i_*} h_n(X, a_0) \xrightarrow{p_*} h_n(X/A, *) \xrightarrow{\Delta_{n-1}} h_{n-1}(A, a_0) \xrightarrow{i_*} \cdots .$$

Lemma 2.8. *Each reduced pointed Q-homology theory* (h_n, Δ_n) *satisfies the cone collapsing axiom (i.e., all functors* h_n *satisfy the cone collapsing axiom).*

Proof. The long exact sequence of the pointed pair $(X \cup C(A, a_0), C(A, a_0), *)$ shows that all

$$q_* \colon h_n(X \cup C(A, a_0), *) \to h_n(X/A, *)$$

are isomorphisms. $\qquad\square$

Our above alternative definition of a reduced pointed homology theory immediately yields the following corollary which is closely related to [**3**, Theorem 2.19].

Corollary 2.9. *On* $\mathbf{CM_0}$ *the reduced pointed Q-homology theories* (h_n, Δ_n) *can be identified with the reduced pointed homology theories* (h_n, ∂_n) *which satisfy the cone collapsing axiom. The identification is given by the assignment*

$$\Delta_n \mapsto \partial_n = \Delta_n \circ q_*.$$

3. Reduced pointed homology theories on $\mathbf{CM_0}$ and strong shape invariance

Definition 3.1 (Strong Shape Invariance Axiom)**.** A reduced pointed homology system (h_n, σ_n) on $\mathbf{CM_0}$ is *strong shape invariant* if each pointed strong shape equivalence $f \colon (X, x_0) \to (Y, y_0)$ between pointed compacta induces isomorphisms

$$f_* \colon h_n(X, x_0) \to h_n(Y, y_0)$$

for all $n \in \mathbb{Z}$.

Remark 3.2. The strong shape functor $S \colon \mathbf{CM_0} \to \mathbf{SSh_0}$ localizes $\mathbf{CM_0}$ at the class of pointed strong shape equivalences (see Section 7). As a consequence, for each reduced pointed homology system (h_n, σ_n) on $\mathbf{CM_0}$, strong shape invariance is equivalent to the existence of covariant functors $h'_n \colon \mathbf{SSh_0} \to \mathbf{Ab}$ such that $h_n = h'_n \circ S$ for all $n \in \mathbb{Z}$. These functors h'_n are of course uniquely determined.

Theorem 3.3. *Let (h_n, σ_n) be a reduced pointed homology theory on $\mathbf{CM_0}$. Then the following are equivalent:*

(1) *(h_n, σ_n) is strong shape invariant.*

(2) *(h_n, σ_n) satisfies the cone collapsing axiom.*

Proof of (1) $\Rightarrow$ (2). The quotient map $q \colon (X \cup C(A, a_0), *) \to (X/A, *)$ is a pointed strong shape equivalence because $C(A, a_0)$ is contractible see, e.g., [**9**, Theorem 2 and Corollary 3]).

Proof of (2) $\Rightarrow$ (1). Let $i \colon (A, a_0) \to (X, x_0)$ be a pointed cylinder base embedding (cf. Section 7). We shall show that i induces isomorphisms $i_* \colon h_n(A, a_0) \to h_n(X, x_0)$ for all $n \in \mathbb{Z}$.

It is no restriction to assume that $A \subset X$ and $a_0 = x_0$ (otherwise decompose i into a homeomorphism and the genuine inclusion from $i(A, a_0)$ into (X, x_0) which is again a pointed cylinder base embedding). Consider the long exact sequence of the pointed pair (X, A, a_0)

$$\cdots \to h_{n+1}(X \cup C(A, a_0), *) \xrightarrow{\partial_n} h_n(A, a_0) \xrightarrow{i_*} h_n(X, a_0)$$
$$\xrightarrow{j_*} h_n(X \cup C(A, a_0), *) \to \cdots .$$

By assumption, all $q_* \colon h_k(X \cup C(A, a_0), *) \to h_k(X/A, *)$ are isomorphisms. Since $(X/A, *)$ is pointed contractible (cf. Proposition 7.4), $h_k(X/A, *) = 0$.

Thus all $h_k(X \cup C(A, a_0), *) = 0$, which implies that i_* is an isomorphism for all $n \in \mathbb{Z}$.

We have now shown that the h_n take pointed cylinder base embeddings to isomorphisms. By Theorem 7.7 there exist $h'_n \colon \mathbf{SSh_0} \to \mathbf{Ab}$ such that $h_n = h'_n \circ S$. This implies that the h_n take pointed strong shape equivalences to isomorphisms. $\square$

Remark 3.4. The implication (1) $\Rightarrow$ (2) holds true for any *reduced pointed homology system* (h_n, σ_n). However, the exactness axiom is essential for the proof of (2) $\Rightarrow$ (1), which means that this implication holds true only when (h_n, σ_n) is a *reduced pointed homology theory*. Moreover, in the proof (2) $\Rightarrow$ (1) it would be sufficient to assume that the $q_* \colon h_k(X \cup C(A, a_0), *) \to h_k(X/A, *)$ are monomorphisms.

Theorem 3.5. *Let (h_n, σ_n) be a reduced pointed homology system on $\mathbf{CM_0}$. Then the following are equivalent:*

(1) *(h_n, σ_n) is quotient exact.*

(2) *(h_n, σ_n) is exact and strong shape invariant, i.e., is a strong shape invariant reduced pointed homology theory.*

In other words: On $\mathbf{CM_0}$ the quotient exact reduced pointed homology systems coincide with the strong shape invariant reduced pointed homology theories.

Proof of (1) $\Rightarrow$ (2) By Theorem 2.4, (h_n, σ_n) is exact and satisfies the cone collapsing axiom. Theorem 3.3 then yields strong shape invariance.

Proof. Proof of (2) $\Rightarrow$ (1) By Theorem 3.3, (h_n, σ_n) satisfies the cone collapsing axiom. Theorem 2.4 yields quotient exactness. $\square$

Corollary 3.6. *Each Steenrod homology theory on* $\mathbf{CM_0}$ *is a strong shape invariant reduced pointed homology theory.*

Corollary 3.7. *On* $\mathbf{CM_0}$ *the reduced pointed Q-homology theories* (h_n, Δ_n) *can be identified with the strong shape invariant reduced pointed homology theories* (h_n, ∂_n). *The identification is given by the assignment*

$$\Delta_n \mapsto \partial_n = \Delta_n \circ q_*.$$

4. Reduced homology theories: unpointed case

Reduced theories can also be defined in the unpointed setting. We shall be short since everything is almost identical with the pointed case. In the unpointed case the categories $\mathbf{CM_0}$ (resp. $\mathbf{CM_0^2}$) have to be replaced by their unpointed counterparts $\mathbf{CM_{ne}}$ (resp. $\mathbf{CM_{ne}^2}$) whose objects are all nonempty compact metric spaces X (resp. all compact metric pairs (X, A) such that $A \neq \emptyset$).

The definitions of the *Exactness Axiom*, the *Quotient Exactness Axiom*, the *Cone Collapsing Axiom* and the *Strong Shape Invariance Axiom* in the unpointed setting are omitted here since they are completely analogous as in the pointed case (of course, the reduced cone appearing in the Exactness Axiom has to be replaced by the unreduced cone).

Definition 4.1.

1. A *reduced unpointed homology system* on $\mathbf{CM_{ne}}$ is a collection of homotopy invariant covariant functors $h_n \colon \mathbf{CM_{ne}} \to \mathbf{Ab}$ and natural isomorphisms

$$\sigma_n \colon h_n \to h_{n+1} \circ \mathbf{S}, n \in \mathbb{Z}$$

   (where $\mathbf{S} \colon \mathbf{CM_{ne}} \to \mathbf{CM_{ne}}$ is the unreduced suspension functor).

2. A *reduced unpointed homology theory* is a reduced unpointed homology system such that all homology functors h_n satisfy the exactness axiom.

3. A *quotient exact reduced unpointed homology system* is a reduced unpointed homology system such that all homology functors h_n satisfy the quotient exactness axiom.

Our definition of a reduced unpointed homology theory is essentially that given in [**12, 15**] and can therefore be regarded as the "standard" definition.

As in the pointed case there is yet another concept of exactness (see, e.g., [**10**] and [**13**, Proposition 2.5]).

Definition 4.2. A *reduced unpointed Q-homology theory* on $\mathbf{CM_{ne}}$ is a collection of homotopy invariant covariant functors $h_n \colon \mathbf{CM_{ne}} \to \mathbf{Ab}$ and natural transformations $\Delta_n \colon h_{n+1} \circ Q \to h_n \circ \mathrm{Res}, n \in \mathbb{Z}$, satisfying the axiom of *Q-exactness*, which says that for each $(X, A) \in \mathbf{CM_{ne}^2}$ there is a long exact sequence

$$\cdots \xrightarrow{\Delta_n} h_n(A) \xrightarrow{i_*} h_n(X) \xrightarrow{p_*} h_n(X/A) \xrightarrow{\Delta_{n-1}} h_{n-1}(A) \xrightarrow{i_*} \cdots.$$

Let us now state the main results. The proofs are omitted because they can be copied from the pointed case (using [**17**] instead of Section 7).

Theorem 4.3. *On* $\mathbf{CM_{ne}}$ *the following coincide:*

- *Quotient exact reduced unpointed homology systems,*
- *Reduced unpointed homology theories which satisfy the cone collapsing axiom,*
- *Strong shape invariant reduced unpointed homology theories.*

The concept of a Steenrod homology theory was originally introduced in [**13**] for the unpointed case, that is on $\mathbf{CM_{ne}}$ (see also [**6**]).

Theorem 4.4. *Each Steenrod homology theory on* $\mathbf{CM_{ne}}$ *is a strong shape invariant reduced unpointed homology theory.*

Theorem 4.5. *On* $\mathbf{CM_{ne}}$ *the reduced unpointed Q-homology theories* (h_n, Δ_n) *can be identified with the strong shape invariant reduced unpointed homology theories* (h_n, ∂_n). *The identification is given by the assignment*

$$\Delta_n \mapsto \partial_n = \Delta_n \circ q_*.$$

5. Strong shape invariance for unreduced homology theories on $\mathbf{CM^2}$ revisited

In [**18**] we have shown that strong shape invariance and strong excision are equivalent for unreduced homology theories on $\mathbf{CM^2}$. We shall slightly improve this result. In the following definitions, let (H_n, ∂_n) be an unreduced homology theory on $\mathbf{CM^2}$.

Definition 5.1 (Cone Collapsing Axiom)**.** (H_n, ∂_n) satisfies the *cone collapsing axiom* if for all $(X, A) \in \mathbf{CM^2}$ the quotient map $p \colon (X \cup CA, CA) \to (X/A, *)$ induces isomorphisms

$$p_* \colon H_n(X \cup CA, CA) \to H_n(X/A, *)$$

for all $n \in \mathbb{Z}$.

Remark 5.2. (H_n, ∂_n) satisfies the cone collapsing axiom if and only if it satisfies the *cone collapsing axiom in the absolute case*, which means that $p \colon X \cup CA \to X/A$ induces isomorphisms

$$p_* \colon H_n(X \cup CA) \to H_n(X/A)$$

for all $n \in \mathbb{Z}$.

Proof. Let $(X, A) \in \mathbf{CM^2}$. Consider the infinite commutative diagram consisting of the long exact sequences of the pairs $(X \cup CA, CA)$ and $(X/A, *)$ and the homomorphisms p_* connecting both sequences. Since all $p_* \colon H_n(CA) \to H_n(*)$ are isomorphisms, the five lemma yields the assertion. $\square$

Definition 5.3 (Strong Excision Axiom)**.** (H_n, ∂_n) satisfies the *strong excision axiom* if, for all $(X, A) \in \mathbf{CM^2}$, the quotient map $p \colon (X, A) \to (X/A, *)$ induces isomorphisms

$$p_* \colon H_n(X, A) \to H_n(X/A, *)$$

for all $n \in \mathbb{Z}$.

Remark 5.4. In case $A = \emptyset$, we define $(X/\emptyset, *) = (X^+, +)$ and let

$$p \colon (X, \emptyset) \to (X/\emptyset, *)$$

denote inclusion. Then p is an excision map (excising the one-point subspace $+$) and therefore induces isomorphisms p_* in homology.

Definition 5.5. An unreduced homology theory (H_n, ∂_n) on $\mathbf{CM^2}$ is *strong shape invariant in the absolute case* if each strong shape equivalence $f \colon X \to Y$ between compacta induces isomorphisms

$$f_* \colon H_n(X) \to H_n(Y)$$

for all $n \in \mathbb{Z}$.

Theorem 5.6. *Let (H_n, ∂_n) be an unreduced homology theory on $\mathbf{CM^2}$. Then the following are equivalent:*

(1) (H_n, ∂_n) *is strong shape invariant (cf. [**18**]).*

(2) (H_n, ∂_n) *is strong shape invariant in the absolute case.*

(3) (H_n, ∂_n) *satisfies the cone collapsing axiom.*

(4) (H_n, ∂_n) *satisfies the strong excision axiom.*

Proof of $(1) \Rightarrow (2)$. This is obvious.

Proof of $(2) \Rightarrow (3)$. This follows from Remark 5.2 and the fact that $p \colon X \cup CA \to X/A$ is a strong shape equivalence (cf. [**9**]).

Proof of $(3) \Rightarrow (4)$. This follows from the ordinary excision axiom as shown in the proof of [**18**, Theorem 2.3].

Proof of $(4) \Rightarrow (1)$. This is covered by [**18**, Theorem 2.3]. $\qquad\square$

6. The equivalence of unreduced and reduced strong shape invariant homology theories

It is commonly understood that unreduced and reduced homology theories are equivalent. Unfortunately, it is not true in this general form. In fact, one has to make *additional assumptions* to establish such an equivalence.

Concerning the equivalence of unreduced and reduced *pointed* homology theories (which constitute the "natural" framework for reduced homology), the reader should consult [**19**, Section 7]. The price paid in [**19**] is to require that homology theories satisfy the weak homotopy axiom and that all pointed spaces under consideration have a non-degenerate basepoint.

The situation is better for reduced *unpointed* homology theories. Kelly [**15**] has shown that unreduced homology theories which satisfy the dimension axiom and a slightly weakened form of the excision axiom are equivalent to reduced unpointed homology theories which satisfy a variant of the dimension axiom and a slightly modified form of the exactness axiom (in which $X \cup CA$ is replaced by the space $X \cup Ap$ got from X by "joining the subset A to a point p"). For compact metric

spaces Kelly's modified axioms agree with the standard axioms and it is, moreover, easy to verify that the dimension axiom can be omitted without affecting the proof of the equivalence of unreduced and reduced theories. In the compact metric case we may therefore restate Kelly's results as follows.

Theorem 6.1. *Unreduced homology theories on* $\mathbf{CM^2}$ *are equivalent to reduced unpointed homology theories on* $\mathbf{CM_{ne}}$.

For an unreduced theory (H_n, ∂_n), the associated reduced theory $(\overline{H}_n, \sigma(\partial_n))$ is given by

$$\overline{H}_n(X) = H_{n+1}(CX, X) \approx \ker(c_* \colon H_n(X, \emptyset) \to H_n(*, \emptyset)),$$

$$\overline{H}_n(f) = H_n(Cf).$$

For a reduced unpointed theory (h_n, σ_n), the associated unreduced theory $(\hat{h}_n, \partial(\sigma_n))$ is given by

$$\hat{h}_n(X, A) = h_n(\Gamma(X, A)),$$

$$\hat{h}_n(f) = h_n(\Gamma(f)).$$

The formulae for $\sigma(\partial_n)$ and $\partial(\sigma_n)$ have been omitted because they are irrelevant for our purposes. Note that in case $A = \emptyset$ we set $\Gamma(X, \emptyset) = X^+$.

The purpose of this section is to show that in the realm of compact metric spaces the concepts of

- strong shape invariant unreduced homology theories,
- strong shape invariant reduced unpointed homology theories and
- strong shape invariant reduced pointed homology theories

are *completely equivalent*.

The first part is obtained as an easy deduction of Theorem 6.1.

Theorem 6.2. *Under the equivalence of Theorem 6.1, strong shape invariant unreduced homology theories on* $\mathbf{CM^2}$ *correspond to strong shape invariant reduced unpointed homology theories on* $\mathbf{CM_{ne}}$.

In other words: Unreduced homology theories on $\mathbf{CM^2}$ which satisfy the strong excision axiom correspond to quotient exact reduced unpointed homology theories on $\mathbf{CM_{ne}}$.

Proof. Assume that (H_n, ∂_n) is strong shape invariant. Consider a strong shape equivalence $f \colon X \to Y$ between nonempty compacta. Then $C(f) \colon (CX, X) \to (CY, Y)$ is a strong shape equivalence of pairs; thus $\overline{H}_n(f)$ is an isomorphism. This shows that $(\overline{H}_n, \sigma(\partial_n))$ is strong shape invariant.

Assume that (h_n, σ_n) is strong shape invariant. Consider a strong shape equivalence $f \colon X \to Y$ between compacta. Then also $\Gamma(f) = f^+ \colon X^+ \to Y^+$ is a strong shape equivalence; hence $\hat{h}_n(f)$ is an isomorphism. This means that $(\hat{h}_n, \partial(\sigma_n))$ is strong shape invariant in the absolute case and therefore strong shape invariant. $\square$

In the reduced pointed case we cannot establish an analogue of Theorem 6.1 but confine ourselves to

Theorem 6.3. *Strong shape invariant unreduced homology theories on* $\mathbf{CM^2}$ *are equivalent to strong shape invariant reduced pointed homology theories on* $\mathbf{CM_0}$.

In other words: Unreduced homology theories on $\mathbf{CM^2}$*, which satisfy the strong excision axiom, are equivalent to quotient exact reduced pointed homology theories on* $\mathbf{CM_0}$.

Proof. We establish an equivalence between unreduced homology theories on $\mathbf{CM^2}$ which satisfy the strong excision axiom and reduced pointed Q-homology theories on $\mathbf{CM_0}$. By Corollary 3.7 this proves the theorem.

Given an unreduced homology theory (H_n, ∂_n) on $\mathbf{CM^2}$, which satisfies the strong excision axiom, we define reduced homology functors $\overline{H}_n \colon \mathbf{CM_0} \to \mathbf{Ab}$ by

$$\overline{H}_n(X, x_0) = H_n(X, \{x_0\}),$$
$$\overline{H}_n(f) = H_n(f).$$

For any pointed pair $(X, A, a_0) \in \mathbf{CM_0^2}$ we have a commutative diagram

$$
\begin{array}{ccccccc}
H_{n+1}(X,\{a_0\}) & \longrightarrow & H_{n+1}(X,A) & \xrightarrow{\partial_n} & H_n(A,\{a_0\}) & \longrightarrow & H_n(X,\{a_0\}) \\
\| & & {\scriptstyle p_*}\downarrow{\scriptstyle\approx} & & \| & & \| \\
& & H_{n+1}(X/A,\{*\}) & & & & \\
\| & & \| & & \| & & \| \\
\overline{H}_{n+1}(X,a_0) & \xrightarrow{p_*} & \overline{H}_{n+1}(X/A,*) & \xrightarrow{\overline{\partial}_n} & \overline{H}_n(A,a_0) & \longrightarrow & \overline{H}_n(X,a_0),
\end{array}
$$

where the top row is the long exact sequence of the triple $(X, A, \{a_0\})$, the vertical arrow p_* is the strong excision isomorphism and $\overline{\partial}_n$ is *defined* to make the middle square commutative.

It is then clear that $(\overline{H}_n, \overline{\partial}_n)$ is a reduced pointed Q-homology theory on $\mathbf{CM_0}$.

Given a reduced pointed Q-homology theory (h_n, Δ_n) on $\mathbf{CM_0}$, we define unreduced homology functors $\hat{h}_n \colon \mathbf{CM^2} \to \mathbf{Ab}$ by

$$\hat{h}_n(X, A) = h_n(X/A, *),$$
$$\hat{h}_n(f) = h_n(\overline{f}).$$

For any pair $(X, A) \in \mathbf{CM^2}$ we have a commutative diagram

$$
\begin{array}{ccccccc}
h_{n+1}(X^+,+) & \longrightarrow & h_{n+1}(X^+/A^+,*) & \xrightarrow{\Delta_n} & h_n(A^+,+) & \longrightarrow & h_n(X^+,+) \\
\| & & \| & & \| & & \| \\
& & h_{n+1}(X/A,*) & & & & \\
\| & & \| & & \| & & \| \\
\hat{h}_{n+1}(X,\emptyset) & \longrightarrow & \hat{h}_{n+1}(X,A) & \xrightarrow{\hat{\Delta}_n} & \hat{h}_n(A,\emptyset) & \longrightarrow & \hat{h}_n(X,\emptyset),
\end{array}
$$

where the top row is the long exact sequence of the pointed pair $(X^+, A^+, +)$ and $\hat{\Delta}_n$ is *defined* to make the middle square commutative.

It is then clear that $(\hat{h}_n, \hat{\Delta}_n)$ is an unreduced homology theory on $\mathbf{CM^2}$ which by construction satisfies the strong excision axiom (since $(X/A)/\{*\} = X/A$).

It is easy to see that the unreduced homology theories $(\hat{\bar{H}}_n, \hat{\bar{\partial}}_n)$ and (H_n, ∂_n) are naturally isomorphic: Observe that

$$\hat{\bar{H}}_n(X, A) = H_n(X/A, \{*\})$$

and verify that the strong excision isomorphism $p_* \colon H_n \to \hat{\bar{H}}_n$ is an isomorphism of unreduced homology theories.

It is also easy to see that the reduced pointed Q-homology theories $(\bar{\hat{h}}_n, \bar{\hat{\Delta}}_n)$ and (h_n, Δ_n) are naturally isomorphic: Observe that

$$\bar{\hat{h}}_n(X, x_0) = h_n(X/\{x_0\}, *)$$

and verify that the pointed homeomorphisms $q^{(X, x_0)} \colon (X, x_0) \to (X/\{x_0\}, *)$ induce an isomorphism $q_* \colon h_n \to \bar{\hat{h}}_n$ of reduced pointed Q-homology theories. $\square$

Remark 6.4. It follows from Theorems 6.2 and 6.3 that strong shape invariant reduced pointed homology theories on $\mathbf{CM_0}$ are equivalent to strong shape invariant reduced unpointed homology theories on $\mathbf{CM_{ne}}$. Using the fact that X is a strong deformation retract of $X \cup C\{x_0\}$, one easily verifies the following: Given a reduced unpointed homology theory (h_n, σ_n) on $\mathbf{CM_{ne}}$, an equivalent reduced pointed homology theory $(\tilde{h}_n, \tilde{\sigma}_n)$ on $\mathbf{CM_0}$ is obtained by

$$\tilde{h}_n(X, x_0) = h_n(X), \tilde{h}_n(f) = h_n(f),$$

i.e., by forgetting the basepoints.

Corollary 6.5. *Let (h_n, σ_n) be a quotient exact reduced pointed homology system on $\mathbf{CM_0}$. If two pointed maps $f, g \colon (X, x_0) \to (Y, y_0)$ between pointed compacta induce the same unpointed strong shape morphism, then they induce the same homomorphism $f_* = g_* \colon h_n(X, x_0) \to h_n(Y, y_0)$. This applies, in particular, if f, g are freely homotopic.*

7. Pointed cylinder base embeddings

It is well-known that *in the absolute case* the strong shape functor $S \colon \mathbf{CM} \to \mathbf{SSh}$ localizes $\mathbf{CM}$ at the class of strong shape equivalences see, e.g., [**16**, Section 10.7]). In [**18**] we have shown that this generalizes to compact metric pairs. In this section we treat the pointed case.

The pointed strong shape category $\mathbf{SSh_0}$ can be regarded as a full subcategory of the strong shape category of compact metric pairs $\mathbf{SSh^2}$. This follows from the "standard" constructions of $\mathbf{SSh^2}$ and $\mathbf{SSh_0}$ as presented in [**16**]. The construction of $\mathbf{SSh^2}$ is based on resolutions of pairs, the construction of $\mathbf{SSh_0}$ on pointed resolutions which occur as a *special case* of resolutions of pairs. See also [**9**] for an explicit reference.

We can therefore apply the results of [**18**]. We recall that a *cylinder base embedding of pairs* is an embedding of compact metric pairs $i \colon (B, B_0) \to (C, C_0)$ such

that $C_0 \backslash i(B_0) \subset C \backslash i(B)$ and $(C \backslash i(B), C_0 \backslash i(B_0)) \approx (Z, Z_0) \times (0, 1]$ for some compact metric pair (Z, Z_0).

Definition 7.1. A *cylinder base embedding for pointed compacta* is a cylinder base embedding of pairs $i \colon (B, B_0) \to (C, C_0)$ such that $B_0 = \{b_0\}$ for some point $b_0 \in B$ and $(C \backslash i(B), C_0 \backslash i(B_0)) \approx (Z, z_0) \times (0, 1]$ for some pointed compactum (Z, z_0).

By [**18**, Theorem A.1] each cylinder base embedding for pointed compacta is a strong shape equivalence of pairs of compacta. Observe that $C_0 \backslash \{i(b_0)\} \approx \{z_0\} \times (0, 1]$. Thus C_0 is a homeomorphic copy of the unit interval I. Note also that $i(B) \cap C_0 = \{i(b_0)\}$.

Definition 7.2. A *collapse* is a map $p \colon (C, C_0) \to (D, d_0)$ from a compact metric pair (C, C_0) to a pointed metric space (D, d_0) such that p maps $C \backslash C_0$ homeomorphically onto $D \backslash \{d_0\}$.

For each compact metric pair (C, C_0), the quotient map $q \colon (C, C_0) \to (C/C_0, [C_0])$ is a collapse. For any collapse $p \colon (C, C_0) \to (D, d_0)$ there is a unique homeomorphism $\kappa \colon (C/C_0, [C_0]) \to (D, d_0)$ such that $\kappa \circ q = p$.

Definition 7.3. A map of pointed compacta $j \colon (B, b_0) \to (D, d_0)$ is called a *pointed cylinder base embedding* if it can be decomposed as $j = p \circ i$, where $i \colon (B, b_0) \to (C, C_0)$ is a cylinder base embedding for pointed compacta and $p \colon (C, C_0) \to (D, d_0)$ is a collapse.

Note that j is in fact an *embedding* because $i(B) \cap C_0 = \{i(b_0)\}$. Also observe that the collapse p occurring in the definition of a pointed cylinder base embedding is a strong shape equivalence of pairs since C_0 is a homeomorphic copy of the unit interval (see [**9**]).

Proposition 7.4. *Let* $j \colon (B, b_0) \to (D, d_0)$ *be a pointed cylinder base embedding. Then the quotient space* $(D/j(B), [j(B)])$ *obtained from* (D, d_0) *by collapsing its base* $j(B)$ *to a point is pointed contractible.*

Proof. Let $i \colon (B, b_0) \to (C, C_0)$ be as in Definition 7.3. Let (Z, z_0) be a pointed compactum such that $(C \backslash i(B), C_0 \backslash i(b_0)) \approx (Z, z_0) \times (0, 1]$. Then $(D/j(B), [j(B)]) \approx C(Z, z_0) =$ reduced cone of (Z, z_0). $\square$

Proposition 7.5. *Each pointed cylinder base embedding is a pointed strong shape equivalence of pointed compacta.*

Proof. By definition, each pointed cylinder base embedding is the composition of two strong shape equivalences of pairs of compacta and therefore is itself a strong shape equivalence of pairs of compacta. Since $\mathbf{SSh_0}$ is a full subcategory of $\mathbf{SSh^2}$, we are done. $\square$

Proposition 7.6. *Let* $\Phi \colon (X, x_0) \to (Y, y_0)$ *be a pointed strong shape morphism between pointed compacta. Then there exist a pointed cylinder base embedding* $j \colon (Y, y_0) \to (D, d_0)$ *and a pointed map* $f \colon (X, x_0) \to (D, d_0)$ *such that* $\Phi = S(j)^{-1} \circ S(f)$.

Proof. By [**18**], Φ has a mapping cylinder in the category $\mathbf{SSh^2}$. This means that there exist a cylinder base embedding for pointed compacta $i: (Y, y_0) \to (C, C_0)$ and a map $g: (X, x_0) \to (C, C_0)$ such that $\Phi = S(i)^{-1} \circ S(g)$ in $\mathbf{SSh^2}$. Let $p: (C, C_0) \to (D, d_0)$ be a collapse. Then $j = p \circ i$ is a pointed cylinder base embedding. Letting $f = p \circ g$, we see that $\Phi = S(j)^{-1} \circ S(f)$ in $\mathbf{SSh_0}$. $\square$

Let Σ_{CBE_0} be the class of pointed cylinder base embeddings of pointed compacta.

Theorem 7.7. $S: \mathbf{CM_0} \to \mathbf{SSh_0}$ *localizes* $\mathbf{CM_0}$ *at* Σ_{CBE_0}. *A fortiori S localizes* $\mathbf{CM_0}$ *at pointed strong shape equivalences of pointed compacta.*

To prepare the proof we recall the concept of a *calculus of left fractions*. Let $\mathbf{C}$ be a category and Σ a set of morphisms in $\mathbf{C}$. Σ *admits a calculus of left fractions* if the following conditions are satisfied:

(LF0) For each object X in $\mathbf{C}$ there is a *set* $\Sigma_X \subset \Sigma$ such that for each morphism $s: X \to Y$ of Σ there exists an isomorphism $u: Y \to Y'$ in $\mathbf{C}$ such that $u \circ s \in \Sigma_X$.

(LF1) Σ contains all identity morphisms in $\mathbf{C}$.

(LF2) If $s: X \to Y$ and $t: Y \to Z$ are in Σ, then also $t \circ s$ is in Σ.

(LF3) Each pair of morphisms $X' \xleftarrow{s} X \xrightarrow{u} Y$ with s in Σ can be completed to a commutative square

$$
\begin{array}{ccc}
X & \xrightarrow{\;u\;} & Y \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\
X' & \xrightarrow[\;u'\;]{} & Y'
\end{array}
$$

such that s' is in Σ.

(LF4) Let $u, v: X \to Y$ be morphisms in $\mathbf{C}$. If there exists a morphism $s: X' \to X$ in Σ with $u \circ s = v \circ s$, then there exists a morphism $t: Y \to Y'$ in Σ with $t \circ u = t \circ v$.

If Σ admits a calculus of left fractions, then we can construct the *category of left fractions* $\mathbf{C}/\Sigma$. It has the same objects as $\mathbf{C}$ and the morphisms from X to Y are equivalence classes of pairs of morphisms $(f: X \to Z, s: Y \to Z)$ in $\mathbf{C}$ with $s \in \Sigma$ (for details see [**8**]). The equivalence class of (f, s) is denoted by $f \backslash s$. A canonical quotient functor $Q: \mathbf{C} \to \mathbf{C}/\Sigma$ is defined by $Q(X) = X$ for the objects and $Q(f) = f \backslash \operatorname{id}$ for the morphisms. This functor Q *localizes* $\mathbf{C}$ at the class Σ.

We need a slight modification of the concept of a calculus of left fractions.

Definition 7.8. Let $\mathbf{C}$ be a category and Σ a set of morphisms in $\mathbf{C}$. We say that Σ *admits a weak calculus of left fractions* if conditions (LF0), (LF3) and (LF4) are satisfied.

Definition 7.9. Let Σ admit a weak calculus of fractions. The *closure* of Σ is defined to be the class $\overline{\Sigma}$ of all identities in $\mathbf{C}$ and all finite compositions $s_n \circ \cdots \circ s_1$ of elements $s_1, \ldots, s_n \in \Sigma$ (with arbitrary length n).

Proposition 7.10. *Let Σ admit a weak calculus of fractions. Then the closure $\overline{\Sigma}$ admits a calculus of left fractions. The quotient functor $Q\colon \mathbf{C} \to \mathbf{C}\backslash\overline{\Sigma}$ localizes $\mathbf{C}$ at Σ.*

Proof. This is a straightforward exercise. □

Let $\mathbf{HCM_0}$ be the homotopy category of $\mathbf{CM_0}$ and let $H\Sigma_{CBE_0}$ be the set of homotopy classes of maps in Σ_{CBE_0}.

Theorem 7.11. *$H\Sigma_{CBE_0}$ admits a weak calculus of left fractions in $\mathbf{HCM_0}$.*

Proof of (LF0). (LF0) is obvious since each pointed compactum has a homeomorphic copy in the Hilbert cube.

Proof of (LF3). Let $(X', x_0') \xleftarrow{j} (X, x_0) \xrightarrow{u} (Y, y_0)$ be a pair of morphisms in $\mathbf{CM_0}$ with $j \in \Sigma_{CBE_0}$. We have $j = p \circ i$, where $i\colon (X, x_0) \to (C, C_0)$ is a cylinder base embedding for pointed compacta, and $p\colon (C, C_0) \to (X', x_0')$ is the collapse associated to i. Consider the pushout diagram

$$
\begin{array}{ccc}
(X, x_0) & \xrightarrow{\ u\ } & (Y, y_0) \\
\ \downarrow{\scriptstyle i} & & \ \downarrow{\scriptstyle i'} \\
(C, C_0) & \xrightarrow{\ u'\ } & (Y', Y_0').
\end{array}
$$

It is easy to see that i' is a a cylinder base embedding for pointed compacta and that u' maps C_0 homeomorphically onto Y_0'.

Let $p'\colon (Y', Y_0') \to (Y'', y_0'')$ be the collapse associated to i'. Then u' induces a map $u''\colon (X', x_0') \to (Y'', y_0'')$ such that the diagram

$$
\begin{array}{ccc}
(C, C_0) & \xrightarrow{\ u'\ } & (Y', y_0') \\
\ \downarrow{\scriptstyle p} & & \ \downarrow{\scriptstyle p'} \\
(X', x_0') & \xrightarrow[\ u''\ ]{} & (Y'', y_0'')
\end{array}
$$

commutes. Since $j' = p' \circ i'\colon (Y, y_0) \to (Y'', y_0'')$ is a pointed cylinder base embedding, we are done.

Proof of (LF4). Let f_0 and $f_1\colon (X, x_0) \to (Y, y_0)$ be two pointed maps and let $j\colon (X', x_0') \to (X, x_0)$ be a pointed cylinder base embedding such that $f_0 \circ j \simeq f_1 \circ j$ ($\simeq$ denotes *basepoint preserving homotopy*). By definition, there exist a cylinder base embedding for pointed compacta $i\colon (X', x_0') \to (C, C_0)$ and a collapse $p\colon (C, C_0) \to (X, x_0)$ such that $p \circ i = j$.

There is a homotopy $H\colon (j(X'), x_0) \times I \to (Y, y_0)$ such that $H_0 = f_0\mid_{j(X')}$ and $H_1 = f_1\mid_{j(X')}$. We can extend H to a map

$$
H'\colon (X \times \{0,1\} \cup j(X') \times I, x_0 \times I) \to (Y, y_0)
$$

such that $H_0' = f_0$ and $H_1' = f_1$.

Let

$$\iota\colon (X \times \{0,1\} \cup j(X') \times I, x_0 \times I) \to (X, x_0) \times I$$

denote inclusion and let

$$(D, d_0) = ((X \times \{0,1\} \cup j(X') \times I)/x_0 \times I, [x_0 \times I])$$

denote the pointed quotient space obtained by contracting $x_0 \times I$ to a point. We have quotient maps

$$q\colon (X \times \{0,1\} \cup j(X') \times I, x_0 \times I) \to (D, d_0)$$

and

$$q'\colon (X, x_0) \times I \to (X \times I/x_0 \times I, [x_0 \times I]).$$

H' induces a pointed map

$$H''\colon (D, d_0) \to (Y, y_0)$$

such that $H'' \circ q = H'$ and ι induces an embedding

$$\iota'\colon (D, d_0) \to (X \times I/x_0 \times I, [x_0 \times I])$$

of pointed compacta such that $\iota' \circ q = q' \circ \iota$. We shall show that ι' is a pointed cylinder embedding. Let $R = C \times \{0,1\} \cup i(X') \times I$ and $R_0 = C_0 \times \{0,1\} \cup i(x_0') \times I$. The map $p \times \mathrm{id}_I\colon (C, C_0) \times I \to (X, x_0) \times I$ restricts to a map

$$\bar{p}\colon (R, R_0) \to (X \times \{0,1\} \cup j(X') \times I, x_0 \times I).$$

Let

$$r\colon (R, R_0) \to (R/R_0, [R_0])$$

denote the quotient map. It is easy to see that $q \circ \bar{p}$ induces a map

$$\hat{p}\colon (R/R_0, [R_0]) \to (D, d_0)$$

such that $\hat{p} \circ r = q \circ \bar{p}$. Since p is a collapse, $\hat{p}$ is a homeomorphism.

We shall first show that the inclusion

$$\iota''\colon (R, R_0) \to (C \times I, C_0 \times I)$$

is a cylinder base embedding of compact metric pairs.

Since i is a cylinder base embedding for pointed compacta, there exist a pointed compactum (Z, z_0) and a homeomorphism

$$h\colon (Z, z_0) \times (0,1] \to (C\backslash i(X'), C_0\backslash i(x_0')).$$

Let

$$h_0\colon \{z_0\} \times (0,1] \to C_0\backslash i(x_0')$$

denote the restriction of h. Clearly h_0 extends to a homeomorphism

$$k_0\colon \{z_0\} \times I \to C_0.$$

Define a homeomorphism h' by

$$h' = h \times \mathrm{id}_{(0,1)}\colon (Z, z_0) \times (0,1] \times (0,1) \to (C\backslash i(X'), C_0\backslash i(x_0')) \times (0,1)$$
$$= (C \times I\backslash R, C_0 \times I\backslash R_0)$$

and let

$$h'_0\colon \{z_0\} \times (0,1] \times (0,1) \to C_0 \times I \backslash R_0$$

denote the restriction of h'. By construction, the homeomorphism

$$k'_0 = k_0 \times \mathrm{id}_I \colon \{z_0\} \times I \times I \to C_0 \times I$$

is an extension of h'_0.

Choose a homeomorphism $\lambda\colon I \times I \to I \times I$ such that $\lambda(I \times (0,1]) = (0,1] \times (0,1)$. We obtain a homeomorphism

$$h'' = h' \circ (\mathrm{id}_Z \times \lambda)\colon (Z,z_0) \times I \times (0,1] \to (C \times I \backslash R, C_0 \times I \backslash R_0)$$

which shows that ι'' is a cylinder base embedding of compact metric pairs.

Next let

$$h''_0\colon \{z_0\} \times I \times (0,1] \to C_0 \times I \backslash R_0$$

denote the restriction of h''. Again by construction, the homeomorphism

$$k''_0 = k'_0 \circ (\mathrm{id}_{z_0} \times \lambda)\colon \{z_0\} \times I \times I \to C_0 \times I$$

is an extension of h''_0.

The fibres $F_t = k''_0(\{z_0\} \times I \times \{t\})$, $t \in I$, are homeomorphic copies of the unit interval I. We have $F_0 = R_0$ and $F_t \cap R = \emptyset$ for $t \in (0,1]$. Define an equivalence relation $\sim$ on $C \times I$ by

$$\xi_1 \sim \xi_2 \text{ if and only if } \xi_1 = \xi_2 \text{ or there exists } t \in I \text{ such that } \xi_1, \xi_2 \in F_t.$$

$\sim$ restricts to an equivalence relation on C_0. Define

$$(E, E_0) = (C \times I/\sim, C_0 \times I/\sim).$$

(E, E_0) is a compact metric pair such that E_0 is a homeomorphic copy of I. Let $\pi\colon (C \times I, C_0 \times I) \to (E, E_0)$ be the quotient map. It contracts each fibre $F_t \subset C_0$ to a point and compresses the square $C_0 \times I$ to E_0. Since $\pi(R_0) = [F_0] \in [E_0]$, the restriction of π to (R, R_0) induces a map

$$\pi'\colon (R/R_0, [R_0]) \to (E, E_0)$$

such that $\pi \circ \iota'' = \pi' \circ r$. Since $F_t \cap R = \emptyset$ for $t \in (0,1]$, π' is an embedding.

By construction, $h''\colon (Z,z_0) \times I \times (0,1] \to (C \times I \backslash R, C_0 \times I \backslash R_0)$ maps the fibres $F'_t = \{z_0\} \times I \times \{t\}$ homeomorphically onto

$$h''(F'_t) = h''_0(F'_t) = k''_0(F'_t) = F_t, t \in (0,1].$$

Hence h'' induces a homeomorphism

$$h'''\colon (Z/\{z_0\} \times I, [\{z_0\} \times I]) \times (0,1] \to (E \backslash \pi'(R/R_0), E_0 \backslash \pi'([R_0])).$$

This means that π' is a cylinder base embedding for pointed compacta.

Consider the map

$$q' \circ (p \times \mathrm{id}_I)\colon (C \times I, C_0 \times I) \to (X \times I/x_0 \times I, [x_0 \times I]).$$

It is a collapse which contracts the square $C_0 \times I$ to the point $[x_0 \times I]$. Since it maps points which are equivalent with respect to $\sim$ to the same point in $X \times I/x_0 \times I$, it

induces a map

$$s \colon (E, E_0) \to (X \times I/x_0 \times I, [x_0 \times I]).$$

It is clear by construction that s is a collapse. Therefore

$$J = s \circ \pi' \colon (R/R_0, [R_0]) \to (X \times I/x_0 \times I, [x_0 \times I])$$

is a pointed cylinder base embedding. It is easy to verify that $J = \iota' \circ \hat{p}$. Since $\hat{p}$ is a homeomorphism, we conclude that ι' is a pointed cylinder base embedding as we claimed above.

As in the proof of (LF3) there exists a commutative diagram

$$
\begin{array}{ccc}
(D, d_0) & \xrightarrow{\;\;H''\;\;} & (Y, y_0) \\
\iota' \downarrow & & \downarrow j' \\
(X \times I/x_0 \times I, [x_0 \times I]) & \xrightarrow[\;u\;]{} & (Y', y_0'),
\end{array}
$$

where j' is a pointed cylinder base embedding. We therefore obtain a commutative diagram

$$
\begin{array}{ccc}
((X \times \{0,1\} \cup j(X') \times I, x_0 \times I) & \xrightarrow{\;\;H'\;\;} & (Y, y_0) \\
\iota \downarrow & & \downarrow j' \\
(X \times I, x_0 \times I) & \xrightarrow[\;q' \circ u\;]{} & (Y', y_0').
\end{array}
$$

The pointed homotopy $G = q' \circ u$ satisfies $G_0 = j' \circ H_0' = \iota \circ f_0$ and $G_1 = j' \circ f_1$ which proves (LF4). □

We now prove Theorem 7.7.

Let us first consider any functor $\Theta \colon \mathbf{CM_0} \to \mathbf{K}$ taking Σ_{CBE_0} to isomorphisms in the category $\mathbf{K}$. Then Θ is homotopy invariant, i.e., factorizes as $\Theta = H\Theta \circ H$ with $H\Theta \colon \mathbf{HCM_0} \to \mathbf{K}$. This comes from the fact that the inclusions $i_k \colon (X, x_0) \to (X, x_0) \times I/\{x_0\} \times I, i_k(x) = [x, k], k = 0, 1$ are pointed cylinder base embeddings and therefore induce isomorphisms $\Theta(i_k)$. Hence $\Theta(i_0) = \Theta(i_1)$ since $\Theta(p)$, where $p \colon (X, x_0) \times I/\{x_0\} \times I \to (X, x_0)$ denotes the projection map, is a common right inverse for both. This implies that homotopic maps have the same image under Θ.

It is therefore sufficient to show that the functor $HS \colon \mathbf{HCM_0} \to \mathbf{SSh_0}$ localizes $\mathbf{HCM_0}$ at $H\Sigma_{CBE_0}$.

Consider the closure $\overline{H\Sigma_{CBE_0}}$ of $H\Sigma_{CBE_0}$. Let $\mathbf{HCM_0}/\overline{H\Sigma_{CBE_0}}$ be the category of left fractions and let $\Lambda \colon \mathbf{HCM_0} \to \mathbf{HCM_0}/\overline{H\Sigma_{CBE_0}}$ be the quotient functor which localizes $\mathbf{HCM_0}$ at $H\Sigma_{CBE_0}$ (cf. Proposition 7.10).

By Proposition 7.5, HS takes $H\Sigma_{CBE_0}$ to isomorphisms. Hence HS also takes the closure $\overline{H\Sigma_{CBE_0}}$ to isomorphisms. Therefore there exists a unique functor

$$\Omega \colon \mathbf{HCM_0}/\overline{H\Sigma_{CBE_0}} \to \mathbf{SSh_0}$$

such that $HS = \Omega \circ \Lambda$. On the morphisms, $\Omega([f]\backslash[s]) = S([s])^{-1} \circ S([f])$. We shall show that Ω is a category isomorphism; this will prove Theorem 7.7.

By Proposition 7.6, Ω is a full functor.We shall show that Ω is faithful. Using the fact that $\overline{H\Sigma_{CBE_0}}$ admits a calculus of left fractions, it suffices to show the following (see, e.g., [**4**, Lemma 1.19]):

For any pointed homotopy classes $[f_0], [f_1] \colon (X, x_0) \to (Y, y_0)$ such that $HS([f_0]) = HS([f_1])$, there exists $[s] \in \overline{H\Sigma_{CBE_0}}$ such that $[s] \circ [f_0] = [s] \circ [f_1]$.

We shall show that we can even find an $[s] \in H\Sigma_{CBE_0}$ with this property. This will be done using mapping cylinders of approaching maps (see [**17**]). We may assume that X and Y are contained in the Hilbert cube Q. Then we can identify pointed strong shape morphisms $(X, x_0) \to (Y, y_0)$ with approaching homotopy classes of approaching maps $(Q, X, x_0) \Rightarrow (Q, Y, y_0)$.

Choose approaching extensions

$$\varphi_k \colon (Q, X, x_0) \Rightarrow (Q, Y, y_0) \quad \text{of} \quad f_k \colon (X, x_0) \to (Y, y_0).$$

By assumption, there exists an approaching homotopy $\phi \colon (Q, X, x_0) \times I \Rightarrow (Q, Y, y_0)$ connecting φ_0 and φ_1. We identify the mapping cylinders $C^2(\varphi_k) = C^2(f_k)$ with the left and right face of the mapping cylinder $C^2(\phi)$ (as sets, $C^2(g_k) = (X, x_0) \times \{k\} \times (0, 1] + (Y, y_0) \subset (X, x_0) \times I \times (0, 1] + (Y, y_0) = C^2(\phi)$). We choose a homeomorphism

$$h \colon (X, x_0) \times I \times (0, 1] \to C^2(\phi) \backslash (Y, y_0).$$

Let $\alpha_k \colon C^2(f_k) \to C^2(\phi)$ be the left and right face embeddings, let $i_{(X, x_0)}^{(k)} \colon (X, x_0) \to C^2(f_k)$ be the top embeddings (identifying (X, x_0) with $h((X, x_0) \times \{k\} \times \{1\})$) and let $j_{(Y, y_0)}^{(k)} \colon (Y, y_0) \to C^2(f_k)$ as well as $j_{(Y, y_0)} \colon (Y, y_0) \to C^2(\phi)$ be the base embeddings. We then have

$$j_{(Y, y_0)} \circ f_0 = \alpha_0 \circ j_{(Y, y_0)}^{(0)} \circ f_0 \simeq \alpha_0 \circ i_{(X, x_0)}^{(0)}$$
$$\simeq \alpha_1 \circ i_{(X, x_0)}^{(1)}$$
$$\simeq \alpha_1 \circ j_{(Y, y_0)}^{(1)} \circ f_1 = j_{(Y, y_0)} \circ f_1.$$

Here, $\simeq$ denotes *homotopy of pairs*.

We have $C^2(\phi) = (C(\phi), C(\phi_0))$, where $\phi_0 \colon (Q, x_0) \times I \Rightarrow (Q, y_0)$ is the restriction of ϕ. Let

$$q \colon C^2(\phi) \to (C(\phi)/C(\phi_0), [C(\phi_0)])$$

be the quotient map. Defining $s = q \circ j_{(Y, y_0)}$, we see that $s \circ f_0$ and $s \circ f_1$ are pointed homotopic. We shall show that s is a pointed cylinder base embedding.

Define an equivalence relation $\sim$ on $C(\phi)$ by

$\xi_1 \sim \xi_2$ if and only if $\xi_1 = \xi_2$ or there exists $t \in (0, 1]$

such that $\xi_1, \xi_2 \in h(x_0 \times I \times \{t\})$.

$\sim$ restricts to an equivalence relation on $C(\phi_0)$. Let

$$r \colon C^2(\phi) \to (C(\phi)/\sim, C(\phi_0)/\sim)$$

denote the quotient map. Then $i = r \circ j_{(Y, y_0)}$ is a cylinder base embedding for pointed

compacta since h induces a homeomorphism

$$((C(\phi)/\sim)\backslash i(Y), (C(\phi_0)/\sim)\backslash i(y_0)) \approx (X \times I/x_0 \times I, [x_0 \times I]) \times (0,1].$$

It is clear that q maps points which are equivalent with respect to $\sim$ to the same point in $X \times I/x_0 \times I$. Therefore q induces a map

$$p\colon ((C(\phi)/\sim)\backslash i(Y), (C(\phi_0)/\sim)\backslash i(y_0)) \to (C(\phi)/C(\phi_0), [C(\phi_0)])$$

such that $p \circ r = q$. This map obviously is a collapse.

We have now shown that $s = q \circ j_{(Y,y_0)} = p \circ r \circ j_{(Y,y_0)} = p \circ i$ is a pointed cylinder base embedding which completes the proof. The *a fortiori* part of the theorem holds because pointed strong shape equivalences are precisely the morphisms which are taken to isomorphisms by S.

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Peter Mrozik   pmrozik@gmx.de

Berliner Str. 11, 61194 Niddatal, Germany