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# SIMPLICIAL AND CROSSED LIE ALGEBRAS

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(*communicated by Ronald Brown*)

*Abstract*

In this paper we examine higher order Peiffer elements in simplicial Lie algebras and apply them to the Lie 2-crossed module and Lie crossed squares introduced by Ellis.

## Introduction

Simplicial Lie algebras arise in simplicial homotopical algebra, [**6**], [**13**], and their homotopy theory is of considerable interest. Kassel and Loday [**10**] (see also [**11**] and [**12**]) introduced crossed modules of Lie algebras as computational algebraic objects equivalent to simplicial Lie algebras with associated Moore complex of length 1. Following work of Conduché in a group-theoretic setting, Ellis [**9**] captured the algebraic structure of a Moore complex of length 2 in his definition of a 2-crossed module of Lie algebras. Within the homotopy theory of simplicial Lie algebras, analogues of Samelson and Whitehead products are given by sums over shuffles $(a; b)$ of Lie products of the form $[s_b x, s_a y]$. In this paper we explain the relationship of these shuffles to crossed modules and crossed 2-modules. More precisely, let **L** be a simplicial Lie algebra with Moore complex **NL**. Let $\partial$ denote the boundary homomorphism of the Moore complex. For $n > 1$ let $D_n$ be the ideal in $L_n$ generated by the degenerate elements. We show in Proposition 2.3 that if $L_n = D_n$, then

$$NL_n = I_n$$

where $I_n$ is an ideal in $L_n$ generated by certain shuffles. We use this equality to prove the following theorem (in which the face homomorphisms of the simplicial Lie algebra $L$ are denoted by $d_i$, and for $I \subseteq \{1, \ldots, n\}$ the intersection $\cap_{i \in I} \text{Ker} d_i$ is denoted by $K_I$).

**Theorem 1**. Let **L** be a simplicial Lie algebra.
(i) $L_2 = D_2$ then $\partial_2(NL_2) = [\text{Ker}d_0, \text{Ker}d_1]$.
(ii) $L_3 = D_3$ then

$$\partial_3(NL_3) = [K_{\{0,1\}}, K_{\{0,2\}}] + [K_{\{0,2\}}, K_{\{1,2\}}] + [K_{\{0,1\}}, K_{\{1,2\}}] + \sum_{I,J} [K_I, K_J].$$

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where $I \cup J = \{0,1,2\}$ , $I \cap J = \emptyset$.
(iii) If $L_n = D_n$ then

$$\partial_n(NL_n) \supseteq \sum_{I,J} [K_I, K_J]$$

where $I, J$ are nonempty subsets of $\{0, \ldots, n-1\}$ with $I \cup J = \{0, \ldots, n-1\}$ .
Part (i) of this theorem is an analogue of a group-theoretic result of Brown and Loday [**2**]

The paper is organised as follows. In Section 1 we recall some basics on simplicial Lie algebras and crossed modules. In Section 2 we prove Proposition 2.3. In Section 3 we prove Theorem 1. In Section 4 we use Theorem 1 to construct a functor from simplicial Lie algebras to 2-crossed modules; we also explain how Theorem 1 yields a functor from simplicial Lie algebras to the crossed $n$-cubes of Lie algebras introduced in [**8**].

## 1. Review of simplicial Lie algebras

All Lie algebras will be over a fixed commutative ring $k$.

A simplicial Lie algebra [**6**] $\mathbf{L}$ is a sequence of Lie algebras,

$$\mathbf{L} = \{L_0, L_1, \ldots L_n, \ldots\},$$

together with face and degeneracy maps

$$\begin{array}{llll} d_i = d_i^n : & L_n \to L_{n-1}, & 0 \leqslant i \leqslant n & (n \neq 0) \\ s_i = s_i^n : & L_n \to L_{n+1}, & 0 \leqslant i \leqslant n. & \end{array}$$

These maps are required to satisfied the simplicial identities

$$\begin{array}{lll} d_i d_j & = & d_{j-1} d_i \quad \text{for} \quad i < j \\ d_i s_j & = & \begin{cases} s_{j-1} d_i & \text{for} \quad i < j \\ \text{identity} & \text{for} \quad i = j, j+1 \\ s_j d_{i-1} & \text{for} \quad i > j+1 \end{cases} \\ s_i s_j & = & s_{j+1} s_i \quad \text{for} \quad i \leqslant j. \end{array}$$

It can be completely described as a functor $\mathbf{L}$: $\Delta^{op} \to \mathbf{LieAlg}_k$ where $\Delta$ is the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$ and increasing maps.

Elements $x \in L_n$ are called *n-dimensional simplices.* A simplex $x$ is called *degenerate* if $x = s_i(y)$ for some $y$.

A simplicial map $f : \mathbf{L} \to \mathbf{L}'$ is a family of homomorphisms $f_n : L_n \to L'_n$ commuting with the $d_i$ and $s_i$. We let **SLA** denote the category of simplicial Lie algebras.

An essential reference from our point of view is Carrasco's thesis, [**3**], where many of the basic techniques used here were developed systematically for the first time and the notion of hypercrossed complex was defined (although in a different context).

The following notation and terminology is derived from the analogous group theoretic case treated in [**3**], [**4**]. For the ordered set $[n] = \{0 < 1 < \ldots < n\}$, let $\alpha_i^n : [n+1] \to [n]$ be the increasing surjective map given by

$$\alpha_i^n(j) = \begin{cases} j & \text{if} \quad j \leqslant i \\ j-1 & \text{if} \quad j > i. \end{cases}$$

Let $S(n, n-r)$ be the set of all monotone increasing surjective maps from $[n]$ to $[n-r]$. This can be generated from the various $\alpha_i^n$ by composition. The composition of these generating maps is subject to the following rule $\alpha_j \alpha_i = \alpha_{i-1}\alpha_j,\ j < i$. This implies that every element $\alpha \in S(n, n-r)$ has a unique expression as $\alpha = \alpha_{i_1} \circ \alpha_{i_2} \circ \ldots \circ \alpha_{i_r}$ with $0 \leqslant i_1 < i_2 < \ldots < i_r \leqslant n-1$, where the indices $i_k$ are the elements of $[n]$ such that $\{i_1, \ldots, i_r\} = \{i : \alpha(i) = \alpha(i+1)\}$. We thus can identify $S(n, n-r)$ with the set $\{(i_r, \ldots, i_1)\ :\ 0 \leqslant i_1 < i_2 < \ldots < i_r \leqslant n-1\}$. In particular, the single element of $S(n, n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( ) denoted by $\emptyset_n$. Similarly the only element of $S(n, 0)$ is $(n-1, n-2, \ldots, 0)$. For all $n \geqslant 0$, let

$$S(n) = \bigcup_{0 \leqslant r \leqslant n} S(n, n-r).$$

We say that $\alpha = (i_r, \ldots, i_1) < \beta = (j_s, \ldots, j_1)$ in $S(n)$

$$\begin{array}{llll} \text{if} & i_1 = j_1, \ldots, i_k = j_k & \text{but} & i_{k+1} > j_{k+1} \quad (k \geqslant 0) \text{ or} \\ \text{if} & i_1 = j_1, \ldots, i_r = j_r & \text{and} & r < s. \end{array}$$

This makes $S(n)$ an ordered set. For instance, the orders in $S(2)$ and in $S(3)$ are respectively:

$$\begin{aligned} S(2) &= \{\emptyset_2 < (1) < (0) < (1,0)\}; \\ S(3) &= \{\emptyset_3 < (2) < (1) < (2,1) < (0) < (2,0) < (1,0) < (2,1,0)\}. \end{aligned}$$

We also define $\alpha \cap \beta$ as the set of indices which belong to both $\alpha$ and $\beta$.

**The Moore complex**

The Moore complex **NL** of a simplicial Lie algebra **L** is the complex

$$\mathbf{NL}: \qquad \cdots \to NL_n \xrightarrow{\partial_n} NL_{n-1} \to \cdots \to NL_1 \xrightarrow{\partial_1} NL_0 \xrightarrow{\partial_0} 0$$

where

$$NL_0 = L_0, \qquad NL_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i, \qquad \partial_n = d_n \text{ (restricted to } NL_n).$$

**Truncated Simplicial Lie Algebras**

By an *m-truncated Simplicial Lie Algebra*, we mean a collection of Lie algebras $\{L_0, \ldots, L_m\}$ and homomorphisms $d_i : L_n \to L_{n-1}$ for $0 \leqslant i \leqslant n,\ 0 \leqslant n \leqslant m$ and $s_i : L_n \to L_{n+1}$ for $0 \leqslant i \leqslant n,\ 0 \leqslant n \leqslant m-1$ which satisfy the simplicial identities.

**The Semidirect Decomposition of a Simplicial Lie Algebra**

The fundamental idea behind this can be found in Conduché [**5**]. A detailed investigation of it for the case of a simplicial group is given in Carrasco and Cegarra [**4**]. The algebra case of that structure is also given in Carrasco's thesis [**3**].

**Proposition 1.1.** *If* $\mathbf{L}$ *is a Simplicial Lie Algebra, then for any* $n \geqslant 0$

$$\begin{array}{rcl} L_n & \cong & (\ldots(NL_n \rtimes s_{n-1}NL_{n-1}) \rtimes \ldots \rtimes s_{n-2}\ldots s_0 NL_1)\rtimes \\ & & (\ldots(s_{n-2}NL_{n-1} \rtimes s_{n-1}s_{n-2}NL_{n-2}) \rtimes \ldots \rtimes s_{n-1}s_{n-2}\ldots s_0 NL_0). \end{array}$$

**Proof:** This is by repeated use of the following lemma. □

**Lemma 1.2.** *Let* $\mathbf{L}$ *be a Simplicial Lie Algebra. Then* $L_n$ *can be decomposed as a semidirect product:*

$$L_n \cong \mathrm{Ker} d_n^n \rtimes s_{n-1}^{n-1}(L_{n-1}).$$

**Proof:** The isomorphism is defined as follows:

$$\begin{array}{rccl} \theta: & L_n & \longrightarrow & \mathrm{Ker} d_n^n \rtimes s_{n-1}^{n-1}(L_{n-1}) \\ & l & \longmapsto & (l - s_{n-1}d_n l, s_{n-1}d_n l). \end{array}$$

□

The bracketting and the order of terms in this multiple semidirect product are generated by the sequence:

$$\begin{array}{rcl} L_1 & \cong & NL_1 \rtimes s_0 NL_0 \\ L_2 & \cong & (NL_2 \rtimes s_1 NL_1) \rtimes (s_0 NL_1 \rtimes s_1 s_0 NL_0) \\ L_3 & \cong & ((NL_3 \rtimes s_2 NL_2) \rtimes (s_1 NL_2 \rtimes s_2 s_1 NL_1))\rtimes \\ & & ((s_0 NL_2 \rtimes s_2 s_0 NL_1) \rtimes (s_1 s_0 NL_1 \rtimes s_2 s_1 s_0 NL_0)). \end{array}$$

and

$$\begin{array}{rcl} L_4 & \cong & (((NL_4 \rtimes s_3 NL_3) \rtimes (s_2 NL_3 \rtimes s_3 s_2 NL_2))\rtimes \\ & & ((s_1 NL_3 \rtimes s_3 s_1 NL_2) \rtimes (s_2 s_1 NL_2 \rtimes s_3 s_2 s_1 NL_1)))\rtimes \\ & & s_0(\text{decomposition of } L_3). \end{array}$$

Note that the term corresponding to $\alpha = (i_r, \ldots, i_1) \in S(n)$ is

$$s_\alpha(NL_{n-\#\alpha}) = s_{i_r \ldots i_1}(NL_{n-\#\alpha}) = s_{i_r} \ldots s_{i_1}(NL_{n-\#\alpha}),$$

where $\#\alpha = r$. Hence any element $x \in L_n$ can be written in the form

$$x = y + \sum_{\alpha \in S(n)} s_\alpha(x_\alpha) \qquad \text{with } y \in NL_n \text{ and } x_\alpha \in NL_{n-\#\alpha}.$$

**Crossed Modules of Lie Algebras**

The notion of crossed module of Lie algebras was defined by Kassel and Loday [**10**].

Let $M$ and $P$ be two Lie algebras. By an *action* of $P$ on $M$ we mean a $\mathbf{k}$-bilinear map $P \times M \to M$, $(p, m) \mapsto p \cdot m$ satisfying

$$\begin{array}{rcl} [p,p'] \cdot m & = & p \cdot (p' \cdot m) - p'(p \cdot m) \\ p \cdot [m,m'] & = & [p \cdot m, m'] + [m, p \cdot m'] \end{array}$$

for all $m, m' \in M, p, p' \in P$. For instance, if $P$ is a subalgebra of some Lie algebra $Q$ (including possibly the case $P = Q$), and if $M$ is an ideal in $Q$, then Lie multiplication in $Q$ yields an action of $P$ on $M$.

Suppose that $M$ and $N$ are Lie algebras with an action of $M$ on $N$ and action of $N$ on $M$. For any Lie algebra $Q$ we call a bilinear function $h : M \times N \to Q$ a *Lie pairing* [**7**] if

$$\begin{array}{rcl} h([m,m'],n) & = & h(m, m' \cdot n) - h(m', m \cdot n), \\ h(m,[n,n']) & = & h(n' \cdot m, n) - h(n \cdot m, n'), \\ h(n \cdot m, m' \cdot n') & = & -[h(m,n), h(m',n')], \end{array}$$

for all $m, m' \in M, n, n' \in N$. For example if $M$ and $N$ are both ideals of some Lie algebra then the function $M \times N \to M \cap N$, $(m, n) \mapsto [m, n]$ is a Lie pairing.

Recall from [**10**] the notion of a crossed module of Lie algebras. *A crossed module of Lie algebras* is a Lie homomorphism $\partial : M \to P$ together with an action of $P$ on $M$ such that ,

$$\text{CM1)} \quad \partial(p \cdot m) = [p, \partial m] \qquad \text{CM2)} \quad \partial m \cdot m' = [m, m']$$

for all $m, m' \in M, p \in P$.

The second condition (CM2) is called the *Peiffer identity.* A standard example of a crossed module is any ideal $I$ in $P$ giving an inclusion map $I \to P$, which is a crossed module. Conversely, given any crossed module $\partial : M \to P$, the image $I = \partial M$ of $M$ is an ideal in $P$.

## 2. Hypercrossed Complex Pairings and Boundaries in the Moore Complex

**Lemma 2.1.** *Let* $\boldsymbol{L}$ *be a simplicial Lie algebra and let* $\overline{NL}_n^{(r)} = \bigcap_{\substack{i=0 \\ i \neq r}}^{n} \mathrm{Ker} d_i$ *for* $0 \leqslant r \leqslant n$. *Then the mapping*

$$\varphi : NL_n \longrightarrow \overline{NL}_n^{(r)}$$

*in* $L_n$, *given by*

$$\varphi(l) = l - \sum_{k=0}^{n-r-1} (-1)^{k+1} s_{r+k} d_n l,$$

*is a* $\boldsymbol{k}$*-linear isomorphism.* □

This easily implies:

**Lemma 2.2.**

$$d_n(NL_n) = d_r(\overline{NL}_n^{(r)}).$$

□

**Proof of Theorem 1 (iii):** For any $J \subset [n-1], J \neq \emptyset$, let $r$ be the smallest element of $J$. If $r = 0$, then replace $J$ by $I$ and restart and if $0 \in I \cap J$, then re-define $r$ to be the smallest nonzero element of $J$. Otherwise continue.

Letting $l_0 \in \bigcap_{j \in J} \mathrm{Ker} d_j$ and $l_1 \in \bigcap_{i \in I} \mathrm{Ker} d_i$, one obtains

$$d_i[s_{r-1}l_0, s_r l_1] = 0 \text{ for } i \neq r$$

and hence $[s_{r-1}l_0, s_r l_1] \in \overline{NL}_n^{(r)}$. It follows that

$$[l_0, l_1] = d_r[s_{r-1}l_0, s_r l_1] \in d_r(\overline{NL}_n^{(r)}) = d_n NL_n \quad \text{by the previous lemma,}$$

and this implies

$$\left[\bigcap_{i \in I} \mathrm{Ker} d_i, \bigcap_{j \in J} \mathrm{Ker} d_j\right] \subseteq \partial_n NL_n.$$

**Hypercrossed complex pairings**

We recall from Carrasco [**3**] the construction of a family of $k$-linear morphisms. This was done there for associative algebras but adapts well to the Lie context. We define a set $P(n)$ consisting of pairs of elements $(\alpha, \beta)$ from $S(n)$ with $\alpha \cap \beta = \emptyset$, (for the definition of $\alpha \cap \beta$, see section 1), where $\alpha = (i_r, \ldots, i_1), \beta = (j_s, ..., j_1) \in S(n)$. The $k$-linear morphisms that we will need,

$$\{M_{\alpha,\beta} : NL_{n-\#\alpha} \times NL_{n-\#\beta} \longrightarrow NL_n : (\alpha, \beta) \in P(n), \quad n \geqslant 0\}$$

are given as composites by the diagrams

$$\begin{array}{ccc} NL_{n-\#\alpha} \times NL_{n-\#\beta} & \xrightarrow{M_{\alpha,\beta}} & NL_n \\ \Big\downarrow{\scriptstyle s_\alpha \times s_\beta} & & \Big\uparrow{\scriptstyle p} \\ L_n \times L_n & \xrightarrow[{[\ ,\ ]}]{} & L_n \end{array}$$

where

$$s_\alpha = s_{i_r} \ldots s_{i_1} : NL_{n-\#\alpha} \longrightarrow L_n \ , \quad s_\beta = s_{j_s} \ldots s_{j_1} : NL_{n-\#\beta} \longrightarrow L_n,$$

$p : L_n \to NL_n$ is defined by composite projections $p = p_{n-1} \ldots p_0$, where

$$p_j = 1 - s_j d_j \qquad \text{with} \quad j = 0, 1, \ldots n-1$$

and we denote the Lie bracket by $[\ ,\ ] : L_n \times L_n \to L_n$. Thus

$$\begin{aligned} M_{\alpha,\beta}(x_\alpha, y_\beta) &= p[\ ,\ ](s_\alpha \times s_\beta)(x_\alpha, y_\beta) \\ &= p([s_\alpha(x_\alpha), s_\beta(y_\beta)]) \\ &= (1 - s_{n-1}d_{n-1}) \ldots (1 - s_0 d_0)([s_\alpha(x_\alpha), s_\beta(y_\beta)]). \end{aligned}$$

We now define the ideal $I_n$ to be that generated by all elements of the form

$$M_{\alpha,\beta}(x_\alpha, y_\beta)$$

where $x_\alpha \in NL_{n-\#\alpha}$ and $y_\beta \in NL_{n-\#\beta}$ and for all $(\alpha,\beta) \in P(n)$.

Consider $M_{\alpha,\beta}(x_\alpha, y_\beta)$ and $M_{\beta,\alpha}(y_\beta, x_\alpha)$, here one uses $[s_\alpha(x_\alpha), s_\beta(y_\beta)]$, the other

$$[s_\beta(y_\beta), s_\alpha(x_\alpha)] = -[s_\alpha(x_\alpha), s_\beta(y_\beta)],$$

so the changing $\alpha$ and $\beta$ gives the only minus sign.

**Example** For $n = 2$, suppose $\alpha = (1)$, $\beta = (0)$ and $x, y \in NL_1 = \mathrm{Ker} d_0$. It follows that

$$\begin{aligned} M_{(1)(0)}(x,y) &= p_1p_0[s_1x, s_0y] \\ &= [s_1x, s_0y] - [s_1x, s_1y] \\ &= [s_1x, s_0y - s_1y] \end{aligned}$$

and these give the generator elements of the ideal $I_2$.

For $n = 3$, the linear morphisms are the following

$$\begin{array}{lll} M_{(1,0)(2)}, & M_{(2,0)(1)}, & M_{(2,1)(0)}, \\ M_{(2)(0)}, & M_{(2)(1)}, & M_{(1)(0)}. \end{array}$$

For all $x \in NL_1$, $y \in NL_2$, the corresponding generators of $I_3$ are:

$$\begin{aligned} M_{(1,0)(2)}(x,y) &= [s_1s_0x - s_2s_0x, s_2y], \\ M_{(2,0)(1)}(x,y) &= [s_2s_0x - s_2s_1x, s_1y - s_2y], \\ M_{(2,1)(0)}(x,y) &= [s_2s_1x, s_0y - s_1y + s_2y]; \end{aligned}$$

whilst for all $x, y \in NL_2$,

$$\begin{aligned} M_{(1)(0)}(x,y) &= [s_1x, s_0y - s_1y] + [s_2x, s_2y], \\ M_{(2)(0)}(x,y) &= [s_2x, s_0y], \\ M_{(2)(1)}(x,y) &= [s_2x, s_1y - s_2y]. \end{aligned}$$

**Proposition 2.3.** *Let $\boldsymbol{L}$ be a simplicial Lie algebra and $n > 0$, and $D_n$ the ideal in $L_n$ generated by degenerate elements. We suppose $L_n = D_n$, and let $I_n$ be ideal generated by elements of the form*

$$M_{\alpha,\beta}(x_\alpha, y_\beta) \qquad \text{with } (\alpha,\beta) \in P(n)$$

*where $x_\alpha \in NL_{n-\#\alpha}$, $y_\beta \in NL_{n-\#\beta}$. Then*

$$NL_n = I_n.$$

*As corollary we, of course, have that the image of $N_n$ is equal to the image of $NL_n$, i.e., $\partial_n(NL_n) = \partial_n(I_n)$.*

We omit its proof which can be obtained by changing slightly the corresponding results in [**1**]

## 3. Proof of first two parts of Theorem 1

**Proof of Theorem 1 (i)**: We know that any element $l_2$ of $L_2$ can be expressed in the form

$$l_2 = b + s_1y + s_0x + s_0u$$

with $b \in NL_2, x, y \in NL_1$ and $u \in s_0L_0$. We suppose $D_2 = L_2$. For $n = 1$, we take $\alpha = (1),\ \beta = (0)$ and $x, y \in NL_1 = \mathrm{Ker}d_0$. The ideal $I_2$ is generated by elements of the form

$$M_{(1)(0)}(x, y) = [s_1x, s_0y - s_1y].$$

The image of $I_2$ by $\partial_2$ is known to be $[[\mathrm{Ker}d_0, \mathrm{Ker}d_1]]$ by direct calculation. Indeed,

$$\begin{aligned} d_2[M_{(1)(0)}(x, y)] &= d_2[s_1x, s_0y - s_1y] \\ &= [x, s_0d_1y - y] \end{aligned}$$

where $x \in \mathrm{Ker}d_0$ and $[x, s_0d_1y - y] \in \mathrm{Ker}d_1$ and all elements of $\mathrm{Ker}d_1$ have this form due to Lemma 2.1.

As $\partial = \partial_1$ restricted to $NL_1$, this is precisely $d_2(M_{(1)(0)}(x, y))$. In other words the ideal $\partial I_2$ is the 'Peiffer ideal' of the precrossed module $\partial : NL_1 \to NL_0$, whose vanishing is equivalent to this being a crossed module. The description of $\partial I_2$ as $[\mathrm{Ker}d_0, \mathrm{Ker}d_1]$ gives that its vanishing in this situation is module-like behaviour since a module, $M$, is a Lie algebra with $[M, M] = 0$. Thus if $(\mathbf{NL}, \partial)$ yields a crossed module this fact will be reflected in the internal structure of $\mathbf{L}$ by the vanishing of $[\mathrm{Ker}d_0, \mathrm{Ker}d_1]$. Because the image of this $M_{(1)(0)}(x, y)$ is the Peiffer element determined by $x$ and $y$, we will call the $M_{\alpha,\beta}(x, y)$ in higher dimensions *higher order Peiffer elements* and will seek similar internal conditions for their vanishing.

We have seen that in all dimensions

$$\sum_{I,J}[K_I, K_J] \subseteq \partial_n(NL_n) = \partial I_n$$

and we will show shortly that this inclusion is an equality, not only in dimension 2 (as above), but also in dimension 3 and 4. The arguments are calculatory and do not generalise in an obvious way to higher dimensions although similar arguments can be used to get partial results there.

**Proof of Theorem 1 (ii)**: By Proposition 2.3, we know the generator elements of the ideal $I_3$ and $\partial_3(I_3) = \partial_3(NL_3)$. For each pair $\alpha, \beta \in S(3)$ with $\emptyset_3 < \alpha < \beta$ and $\alpha \cap \beta = \emptyset$, we take $x \in NL_{3-\#\alpha}, y \in NL_{3-\#\beta}$ and set $M_{\alpha,\beta}(x, y) = p_3p_2p_1\,[s_\alpha(x), s_\beta(y)]$ where $p_i(l) = l - s_id_i(l)$. This element is thus in $NL_3$. The valid pairs together with their corresponding pairing functions is given in the following table:

| | $\alpha$ | $\beta$ | $M_{\alpha,\beta}(x,y)$ |
|---|---|---|---|
| **1** | (1,0) | (2) | $[s_1s_0x - s_2s_0x, s_2y]$ |
| **2** | (2,0) | (1) | $[s_2s_0x - s_2s_1x, s_1y - s_2y]$ |
| **3** | (2,1) | (0) | $[s_2s_1x, s_0y - s_1y + s_2y]$ |
| **4** | (2) | (1) | $[s_1x, s_0y - s_1y] + [s_2x, s_2y]$ |
| **5** | (2) | (0) | $[s_2x, s_0y]$ |
| **6** | (1) | (0) | $[s_2x, s_1y - s_2y]$ |

The explanation of this table is the following:

$\partial_3 M_{\alpha,\beta}(x,y)$ is in $[K_I, K_J]$ in the simple cases corresponding to the first 4 rows. In row 5, $\partial_3 M_{(2)(0)}(x,y) \in [K_{\{0,1\}}, K_{\{1,2\}}] + [K_{\{0,1\}}, K_{\{0,2\}}]$ and similarly in row 6, the higher Peiffer element is in the sum of the indicated $[K_I, K_J]$. To illustrate the sort of argument used we look at the case of $\alpha = (1,0)$ and $\beta = (2)$, i.e. row 1. For $x \in NL_1$ and $y \in NL_2$,

$$\begin{aligned} d_3[M_{(1,0)(2)}(x,y)] &= d_3[s_1s_0x - s_2s_0x, s_2y] \\ &= [s_1s_0d_1x - s_0x, y] \end{aligned}$$

and so

$$d_3[M_{(1,0)(2)}(x,y)] = [s_1s_0d_1x - s_0x, y] \in [\mathrm{Ker}d_2, \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1].$$

We have denoted $[\mathrm{Ker}d_2, \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1]$ by $[K_{\{2\}}, K_{\{0,1\}}]$ where $I = \{2\}$ and $J = \{0,1\}$. Rows 2, 3 and 4 are similar. For Row 5, $\alpha = (2)$, $\beta = (0)$ with $x, y \in NL_2 = \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1$,

$$\begin{aligned} d_3[M_{(2)(0)}(x,y)] &= d_3[s_2x, s_0y] \\ &= [x, s_0d_2y]. \end{aligned}$$

We can assume, for $x, y \in NL_2$,

$$x \in \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1 \quad \text{and} \quad y + s_0d_2y - s_1d_2y \in \mathrm{Ker}d_1 \cap \mathrm{Ker}d_2$$

and, multiplying them together,

$$\begin{aligned} [x, y + s_0d_2y - s_1d_2y] &= [x, y - s_1d_2y] + [x, s_0d_2y] \\ &= d_3[M_{(2)(1)}(x,y)] + d_3[M_{(2)(0)}(x,y)] \end{aligned}$$

and so

$$\begin{aligned} d_3[M_{(2)(0)}(x,y)] &\in [K_{\{0,1\}}, K_{\{1,2\}}] + d_3[M_{(2)(1)}(x,y)] \\ &\subseteq [K_{\{0,1\}}, K_{\{1,2\}}] + [K_{\{0,1\}}, K_{\{0,2\}}]. \end{aligned}$$

For Row 6, for $\alpha = (1)$, $\beta = (0)$ and $x, y \in NL_2 = \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1$,

$$\begin{aligned} d_3[M_{(1)(0)}(x,y)] &= d_3\big([s_1x, s_0y - s_1y] + [s_2x, s_2y]\big) \\ &= [s_1d_2x, s_0d_2y - s_1d_2x] + [x, y]. \end{aligned}$$

We can take the following elements

$$(s_0d_2y - s_1d_2y + y) \in \mathrm{Ker}d_1 \cap \mathrm{Ker}d_2 \quad \text{and} \quad (s_1d_2x - x) \in \mathrm{Ker}d_0 \cap \mathrm{Ker}d_2.$$

When we multiply them together, we get

$$\begin{aligned}[s_0d_2y - s_1d_2y + y, s_1d_2x - x] &= [s_0d_2y, s_1d_2x] - [s_1d_2y, s_1d_2x] + [y, x]\\ &\quad -[x, s_0d_2y] + [x, (s_1d_2y - y)]\\ &\quad +[y, (s_1d_2x - x)]\\ &= d_3[M_{(1)(0)}(x, y)] - d_3[M_{(2)(0)}(x, y)] +\\ &\quad d_3[M_{(2)(1)}(x, y) + M_{(2)(1)}(y, x)]\end{aligned}$$

and hence

$$d_3[M_{(1)(0)}(x, y)] \in [K_{\{0,2\}}, K_{\{1,2\}}] + [K_{\{0,1\}}, K_{\{1,2\}}] + [K_{\{0,1\}}, K_{\{0,2\}}].$$

So we have shown

$$\partial_3 I_3 \subseteq \sum_{I,J}[K_I, K_J] + [K_{\{0,1\}}, K_{\{0,2\}}] + [K_{\{0,2\}}, K_{\{1,2\}}] + [K_{\{0,1\}}, K_{\{1,2\}}].$$

The opposite inclusion can be verified by using proposition 2.3. Therefore

$$\begin{aligned}\partial_3(NL_3) = \ & [\mathrm{Ker}d_2, (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1)] + [\mathrm{Ker}d_1, (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2)] +\\ & [\mathrm{Ker}d_0, (\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2)] + [(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1), (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2)] +\\ & [(\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2), (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2)] + [(\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2), (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1)]\end{aligned}$$

This completes the proof of Theorem 1 (ii).

## 4. Application to 2-Crossed Modules and Crossed Squares of Lie Algebras

The following definition is due to Ellis [**9**].

**Definition 4.1.** *A 2-crossed module of Lie algebras consists of a complex of Lie algebras*

$$M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0$$

*with $\partial_2, \partial_1$ morphisms of Lie algebras, where the algebra $M_0$ acts on itself by Lie bracket $M_0$ acts on $M_1$ and $M_2$ such that*

$$M_2 \xrightarrow{\partial_2} M_1$$

*is a crossed module in which $M_1$ acts on $M_2$ via $M_0$, further, there is a $M_0$-bilinear function giving*

$$\{\ \ ,\ \ \} : M_1 \times M_1 \longrightarrow M_2,$$

*called a Peiffer lifting, which satisfies the following axioms:*

$$\begin{array}{lrcl}
PL1: & \partial_2\{y_0,y_1\} & = & [y_0,y_1]-y_0\cdot\partial_1(y_1),\\
PL2: & \{\partial_2(x_1),\partial_2(x_2)\} & = & [x_1,x_2],\\
PL3: & \{\partial_2(x),y\} & = & y\cdot x-\partial_1(y)\cdot x,\\
PL4: & \{y,\partial_2(x)\} & = & y\cdot x,\\
PL5: & \{y_0,y_1\}\cdot z & = & \{y_0\cdot z,y_1\}+\{y_0,y_1\cdot z\},\\
PL6: & \{[y_0,y_1],y_2\} & = & \{y_0,y_2\}\cdot\partial_1 y_1+\{y_1,[y_0,y_2]\}\\
 & & & -\{y_1,y_2\}\cdot\partial_1 y_0-\{y_0,[y_1,y_2]\},\\
PL7: & \{y_0,[y_1,y_2]\} & = & \{y_0,y_1\}\cdot\partial_1 y_2-\{y_0,y_2\}\cdot\partial_1 y_1\\
 & & & +\{y_1,[y_0,y_2]-y_0\cdot\partial_1 y_2\}-\{y_2,[y_0,y_1]-y_0\cdot\partial_1 y_1\},
\end{array}$$

*for all* $x,x_1,x_2\in M_2,\ y,\ y_0,\ y_1,\ y_2\in M_1$ *and* $z\in M_0$.

We denote such a 2-crossed module of algebras by $\{M_2,\ M_1,\ M_0,\ \partial_2,\ \partial_1\}$.

The following result is an analogous result of the commutative algebra version, cf. [**1**].

**Proposition 4.2.** *Let* $\boldsymbol{L}$ *be a simplicial Lie algebra with the Moore complex* $\boldsymbol{NL}$*. Then the complex of Lie algebras*

$$NL_2/\partial_3(NL_3\cap D_3)\xrightarrow{\overline{\partial}_2}NL_0$$

*is a 2-crossed module of Lie algebras, where the Peiffer map is defined as follows:*

$$\begin{array}{rcl}
\{\ \ ,\ \ \}: NL_1\times NL_1 & \longrightarrow & NL_2/\partial_3(NL_3\cap D_3)\\
\\
(y_0,y_1) & \longmapsto & \overline{[s_1y_0,s_1y_1-s_0y_1]}.
\end{array}$$

*Here the right hand side denotes a coset in* $NL_2/\partial_3(NL_3\cap D_3)$ *represented by the corresponding element in* $NL_2$.

**Proof:** We will show that all axioms of a 2-crossed module are verified. It is readily checked that the morphism $\overline{\partial}_2 : NL_2/\partial_3(NL_3\cap D_3)\to NL_1$ is a crossed module. (In the following calculations we display the elements omitting the overlines.)

PL1:

$$\begin{array}{rcl}
\overline{\partial}_2\{y_0,y_1\} & = & \partial_2[s_1y_0,s_1y_1-s_0y_1]\\
 & = & [y_0,y_1]-y_0\cdot\partial_1 y_1.
\end{array}$$

PL2: From $\partial_3(M_{(1)(0)}(x_1,x_2))=[s_1d_2(x_1),s_0d_2(x_2)-s_1d_2(x_2)]+[x_1,x_2]$, one obtains

$$\begin{array}{rcl}
\{\overline{\partial}_2(x_1),\overline{\partial}_2(x_2)\} & = & [s_1d_2x_1,s_1d_2x_2-s_0d_2x_2]\\
 & \equiv & [x_1,x_2]\mod \partial_3(NL_3\cap D_3).
\end{array}$$

PL3:

$$\{\overline{\partial}_2(x),y\}\ =\ [s_1\partial_2 x,s_1y-s_0y],$$

but

$$\partial_3(M_{(2,0)(1)}(y,x)) = [s_0y - s_1y, s_1d_2x] - [s_0y - s_1y, x] \in \partial_3(NL_3 \cap D_3)$$

and

$$\partial_3(M_{(1,0)(2)}(y,x)) = [s_1s_0d_1y - s_0y, x] \in \partial_3(NE_3 \cap D_3),$$

so then

$$\begin{array}{rcll} \{\overline{\partial}_2(x), y\} & \equiv & [s_1(y), x] - [s_0(y), x] & \mod \partial_3(NL_3 \cap D_3) \\ & = & y \cdot x - \partial_1(y) \cdot x & \text{by the definition of the action,} \end{array}$$

PL4: since $\partial_3(M_{(2,1)(0)}(y,x)) = [s_1y, s_0d_2x - s_1d_2x] + [s_1(y), x]$,

$$\begin{array}{rcl} \{y, \overline{\partial}_2(x)\} & = & [s_1y, s_1\partial_2x - s_0\partial_2x] \\ & \equiv & [s_1(y), x] \quad \mod \partial_3(NL_3 \cap D_3) \\ & = & y \cdot x \qquad \text{by the definition of the action.} \end{array}$$

PL5: By the definition of the action, we get

$$\{y_0, y_1\} \cdot z = \{y_0 \cdot z, y_1\} + \{y_0, y_1 \cdot z\}$$

with $x, x_1, x_2 \in NL_2/\partial_3(NL_3 \cap D_3)$, $y, y_0, y_1, y_2 \in NL_1$ and $z \in NL_0$. Verification of axioms PL6 and PL7 are omitted as they are routine. This completes the proof of the proposition. □

This only used the higher dimension Peiffer elements. A result in terms of $[K_I, K_J]$ vanishing can also be given:

**Proposition 4.3.** *If in a simplicial Lie algebra* $\boldsymbol{L}$*, one has* $[K_I, K_J] = 0$ *in dimension 2 for the following cases:* $I \cup J = [2], I \cap J = \emptyset; I = \{0,1\}, J = \{0,2\}$ *or* $I = \{1,2\}$*; and* $I = \{0,2\}, J = \{1,2\}$ *then*

$$NL_2 \longrightarrow NL_1 \longrightarrow NL_0$$

*can be given the structure of a 2-crossed module.*

Another application of higher order Peiffer elements is a Lie crossed square. First we recall from [**8**] the notion of crossed $n$-cubes of Lie algebras.

*A crossed n-cube of Lie algebras* is a family of Lie algebras, $M_A$ for $A \subseteq < n >= \{1, ..., n\}$ together with homomorphisms $\mu_i : M_A \to M_{A-\{i\}}$ for $i \in < n >$ and for $A, B \subseteq < n >$, functions

$$h : M_A \times M_B \longrightarrow M_{A \cup B}$$

such that for all $\lambda \in \Lambda,\ a, a' \in M_A,\ b, b' \in M_B,\ c \in M_C,\ i, j \in < n >$ and $A \subseteq B$

$$
\begin{array}{lrcl}
1) & \mu_i a & = & a \quad \text{if } i \notin A \\
2) & \mu_i \mu_j a & = & \mu_j \mu_i a \\
3) & \mu_i h(a,b) & = & h(\mu_i a, \mu_i b) \\
4) & h(a,b) & = & h(\mu_i a, b) = h(a, \mu_i b) \quad \text{if } i \in A \cap B \\
5) & h(a,a') & = & [a,a'] \\
6) & h(a,a) & = & 0 \\
7) & h(a+a',b) & = & h(a,b) + h(a',b) \\
8) & h(a,b+b') & = & h(a,b) + h(a,b') \\
9) & \lambda h(a,b) & = & h(\lambda a, b) = h(a, \lambda b) \\
10) & h(h(a,b),c) & + & h(h(b,c),a) + h(h(c,a),b) = 0.
\end{array}
$$

*A morphism of crossed n-cubes* is defined in the obvious way. We thus denote a category of crossed $n$-cubes by $\mathbf{Crs^n}$.

Theorem 1 (iii) can be used to verify that the following construction in a functor from simplicial Lie algebras to crossed $n$-cubes.

For a simplicial Lie algebra $\mathbf{L}$ and a given $n$, we write $\mathbf{M}(\mathbf{L},\ n)$ for the crossed $n$-cube, arising from the functor

$$\mathbf{M}(-,\mathbf{n}) : \mathbf{SLA} \longrightarrow \mathbf{Crs^n}.$$

Then the crossed $n$-cube $\mathbf{M}(\mathbf{L},n)$ is determined by:

(i) for $A \subseteq < n >$,

$$\mathbf{M}(\mathbf{L},n)_A = \frac{\bigcap_{j\in A} \mathrm{Ker} d^n_{j-1}}{d^{n+1}_{n+1}(\mathrm{Ker} d^{n+1}_0 \cap \{\bigcap_{j\in A} \mathrm{Ker} d^{n+1}_j\})};$$

(ii) the inclusion

$$\bigcap_{j\in A} \mathrm{Ker} d^n_{j-1} \longrightarrow \bigcap_{j\in A-\{i\}} \mathrm{Ker} d^n_{j-1}$$

induces the morphism

$$\mu_i : \mathbf{M}(\mathbf{L},n)_A \longrightarrow \mathbf{M}(\mathbf{L},n)_{A-\{i\}};$$

(iii) the functions, for $A, B \subseteq < n >$,

$$h : \mathbf{M}(\mathbf{L},n)_A \times \mathbf{M}(\mathbf{L},n)_B \longrightarrow \mathbf{M}(\mathbf{L},n)_{A\cup B}$$

given by

$$h(\bar{x},\bar{y}) = \overline{[x,y]},$$

where an element of $\mathbf{M}(\mathbf{L},n)_A$ is denoted by $\bar{x}$ with $x \in \bigcap_{j\in A} \mathrm{Ker} d^n_{j-1}$.

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