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ADDENDUM TO "INTRODUCTION TO A-INFINITY ALGEBRAS AND MODULES" [HHA, V. 3 (2001) NO. 1, PP. 1-35]

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(*communicated by Lionel Schwartz*)

I thank J. Huebschmann for his detailed comments on the article [**15**]. With the help of his letter [**14**], I have compiled the following list of additions and corrections to be made in the respective sections of [**15**]:

1.2 History. An A-infinity structure may be described as a system of higher homotopies together with suitable coherence conditions. A basic observation, which has implicitly been exploited for long, is that A-infinity structures behave much better with respect to homotopy than strict (for example differential graded algebra) structures. Higher homotopies occurred in mathematics before A-infinity structures had been explicitly recognized, though. The system of $\cup_i$-products introduced by Steenrod [**19**] is an early example of higher homotopies. The $\cup_i$-products measure the failure from commutativity of the Alexander-Whitney map in a coherent fashion and prompted the development of s(trongly)h(omotopy)c(ommutative) structures as well as that of Steenrod operations. Massey products [**16**] may be seen as invariants of certain A-infinity structures. Homological perturbation theory (HPT) has nowadays become a standard tool to construct and handle A-infinity structures. The basic HPT-notion, that of contraction, was introduced in Section 12 of [**3**]. In that paper, Eilenberg and Mac Lane showed that the comparison between the reduced bar and W-constructions is a reduction, and they conjectured that it is a contraction. Using the perturbation lemma, in his Heidelberg diploma thesis (Diplomarbeit) supervised by J. Huebschmann, Wong has indeed verified this conjecture [**20**]. A geometric comparison between the bar and W-constructions has recently been obtained by Berger and Huebschmann in [**1**]. The notion of "recursive structure of triangular complexes"in Section 5 [**4**] is also an example of what was later identified as a perturbation. The "perturbation lemma" is lurking behind the formulas in Chapter II of Section 1 of [**18**] and seems to have first been made explicit by M. Barrat (unpublished). The first instance known to us where it appeared in print is [**2**]. A homological algebra and higher homotopies tradition was as well created by Berikashvili and his students in Georgia (at the time part of the USSR). More precise comments about the historical development until the mid eighties may be

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found in the article [**10**], and some specific comments about the Georgian tradition in [**12**].

In the early eighties, J. Huebschmann realized the relevance of A-infinity structures and homological perturbation theory (HPT) to homological algebra, as can be seen from the article [**5**]. In the articles [**6**], [**7**], [**8**], he used HPT to exploit A-infinity modules arising in group cohomology. The basic idea is this: Given a group extension $1 \to N \to G \to Q \to 1$, a free resolution $\mathcal{F}$ of the ground ring R in the category of N-modules, and a G-module M, the resulting differential graded G-module $Hom(\mathcal{F}, M)$ will *not* (in general) inherit a Q-module structure. However, it inherits an A-infinity Q-module structure, and the corresponding "A-infinity cohomology" yields the cohomology of G with values in M. Any A-infinity module structure admits a spectral sequence which is an invariant of the structure; in the case at hand the spectral sequence is that of the group extension. Starting from this idea, Huebschmann constructed free resolutions from which he was able to do explicit numerical calculations in group cohomology which until today still cannot be done by other methods. For example he obtained a complete description of the mod p cohomology algebra of any metacyclic group, including a description of the spectral sequence of the corresponding group extension: In particular, the spectral does not collapse from E_2. These results illustrate a typical phenomenon: Whenever a spectral sequence arises from a certain mathematical structure, there is, perhaps, a certain A-infinity module lurking behind, and the spectral sequence is an invariant thereof. The A-infinity structure is then somewhat finer than the spectral sequence itself, though. In this vein, A-infinity structures are lurking behind a number of other familiar structures in mathematics. One such example arises from complex manifolds where a certain A-infinity structure is hidden behind the Frölicher spectral sequence [**11, 13**].

The results of Huebschmann-Kadeishvili's [**10**] date from the end of the 80's. In this paper, a crucial role is played by the observation that the perturbation lemma is compatible with additional algebraic structure. This observation was used already in [**6**], [**7**], [**8**] to construct free resolutions, the additional structure being that of a module over the corresponding group.

3.2 Link with deformation theory. The article [**12**] gives a unified view on A-infinity structures, deformation theory, the classification of fibre spaces, the classification of rational homotopy types, gauge theory and Batalin-Vilkovisky techniques in cohomological physics.

3.3 Minimal models. The proofs of the minimality theorem 3.3 which are based on the perturbation lemma are no less explicit than the one of [**17**]. This becomes especially apparent in the clear presentation of the article [**10**], where the theorem is proved under much weaker hypotheses.

The minimality theorem (3.3) and the theorem on quasi-isomorphisms (3.7) have been known to many experts (especially in rational homotopy theory) since the early eighties. The references given in [**15**] are not exhaustive and are not meant as attributions of priority.

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