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The Morel–Voevodsky localization theorem in spectral algebraic geometry

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We prove an analogue of the Morel–Voevodsky localization theorem over spectral algebraic spaces. As a corollary we deduce a "derived nilpotent-invariance" result which, informally speaking, says that A^1–homotopy-invariance kills all higher homotopy groups of a connective commutative ring spectrum.

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1 Introduction

Let R be a connective $\mathcal{E}_\infty$–ring spectrum, and denote by $\mathrm{CAlg}^{\text{ét}}_R$ the ∞–category of étale $\mathcal{E}_\infty$–algebras over R. The starting point for this paper is the following fundamental result of J Lurie, which says that the small étale topos of R is equivalent to the small étale topos of $\pi_0(R)$ (see [12, Theorem 7.5.0.6] and [13, Remark B.6.2.7]):

Theorem 1.0.1 (Lurie) *For any connective $\mathcal{E}_\infty$–ring R, let $\pi_0(R)$ denote its 0–truncation (viewed as a discrete $\mathcal{E}_\infty$–ring). Then restriction along the canonical functor $\mathrm{CAlg}^{\text{ét}}_R \to \mathrm{CAlg}^{\text{ét}}_{\pi_0(R)}$ induces an equivalence from the ∞–category of étale sheaves of spaces $\mathrm{CAlg}^{\text{ét}}_{\pi_0(R)} \to \mathrm{Spc}$ to the ∞–category of étale sheaves of spaces $\mathrm{CAlg}^{\text{ét}}_R \to \mathrm{Spc}$.*

Theorem 1.0.1 can be viewed as a special case of the following result (see [13, Proposition 3.1.4.1]):

Theorem 1.0.2 (Lurie) *Let $R \to R'$ be a homomorphism of connective $\mathcal{E}_\infty$–rings that is surjective on π_0. Then restriction along the canonical functor $\mathrm{CAlg}^{\text{ét}}_R \to \mathrm{CAlg}^{\text{ét}}_{R'}$ defines a fully faithful embedding of the ∞–category of étale sheaves $\mathrm{CAlg}^{\text{ét}}_{R'} \to \mathrm{Spc}$ into the ∞–category of étale sheaves $\mathrm{CAlg}^{\text{ét}}_R \to \mathrm{Spc}$. Moreover, a sheaf $\mathcal{F}$ belongs to the essential image if and only if its restriction to the complement of the closed subset $\mathrm{Spec}(R') \subseteq \mathrm{Spec}(R)$ is (weakly) contractible.*

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Theorems 1.0.1 and 1.0.2 are particular to small sites: for example, they do not hold for sheaves on the big site $\mathrm{CAlg}_R^{\mathrm{sm}}$ of *smooth* R–algebras. Our main objective in this paper is to show that, if we restrict to sheaves that are $\boldsymbol{A}^1$*–homotopy-invariant*, then these results do have analogues on the big sites (and we can even replace the étale topology by the coarser Nisnevich topology). To be precise, we have (see Corollary 3.2.9 and Theorem 3.2.4):

Theorem A *For any connective $\mathcal{E}_\infty$–ring R, restriction along the canonical functor $\mathrm{CAlg}_R^{\mathrm{sm}} \to \mathrm{CAlg}_{\pi_0(R)}^{\mathrm{sm}}$ induces an equivalence from the ∞–category of $\boldsymbol{A}^1$–homotopy-invariant Nisnevich sheaves $\mathrm{CAlg}_{\pi_0(R)}^{\mathrm{sm}} \to \mathrm{Spc}$ to the ∞–category of $\boldsymbol{A}^1$–homotopy-invariant Nisnevich sheaves $\mathrm{CAlg}_R^{\mathrm{sm}} \to \mathrm{Spc}$.*

Theorem B *Let $i\colon Z \to S$ be a closed immersion of quasicompact quasiseparated spectral algebraic spaces, with quasicompact open complement $j\colon U \hookrightarrow S$. Denote by $\mathrm{Sm}_{/S}$ and $\mathrm{Sm}_{/Z}$ the ∞–categories of smooth spectral algebraic spaces over S and Z, respectively. Then the direct image functor i_* defines a fully faithful embedding of the ∞–category of $\boldsymbol{A}^1$–invariant Nisnevich sheaves on $\mathrm{Sm}_{/Z}$ into the ∞–category of $\boldsymbol{A}^1$–invariant Nisnevich sheaves on $\mathrm{Sm}_{/S}$. Moreover, an object $\mathcal{F}$ belongs to the essential image if and only if its inverse image $j^*(\mathcal{F})$ is (weakly) contractible.*

Theorem B can also be viewed as an analogue of Kashiwara's lemma in D–modules (as generalized by Gaitsgory and Rozenblyum [5] to the setting of spectral algebraic geometry over fields of characteristic zero). It is essentially a reformulation of our main result, an analogue of the localization theorem of Morel and Voevodsky [14, Theorem 3.2.21] in the setting of spectral algebraic geometry. By analogy with loc. cit., we define a *motivic space* over a spectral algebraic space S as an $\boldsymbol{A}^1$–invariant Nisnevich sheaf of spaces on $\mathrm{Sm}_{/S}$. Then we have (see Theorem 3.2.2):

Theorem C (localization) *Let $i\colon Z \to S$ be a closed immersion of quasicompact quasiseparated spectral algebraic spaces, with quasicompact open complement $j\colon U \hookrightarrow S$. Let $j_\sharp$ denote the "extension by zero" functor, left adjoint to j^*. Then, for any motivic space over S, there is a cocartesian square*

$$\begin{tikzcd} j_\sharp j^*(\mathcal{F}) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ j_\sharp(\mathrm{pt}_U) \arrow[r] & i_* i^*(\mathcal{F}) \end{tikzcd}$$

of motivic spaces over S.

1.1 Outline

In order to make sense of Theorem A, we need to define the notions of *smoothness* and of $\boldsymbol{A}^1$*–homotopy-invariance* in the world of $\mathcal{E}_\infty$–ring spectra. There are two natural ways to define smoothness for a homomorphism of connective $\mathcal{E}_\infty$–rings $A \to B$:

- One can require that B is flat as an A–module, and that the induced homomorphism of ordinary commutative rings $\pi_0(A) \to \pi_0(B)$ is smooth.
- One can require that B is locally of finite presentation as an A–algebra, and that the relative cotangent complex $L_{B/A}$ is a finitely generated projective B–module.

There are also two candidate "affine lines" over a connective $\mathcal{E}_\infty$–ring R:

- The "flat affine line" $\boldsymbol{A}^1_{\flat,\mathrm{Spec}(R)}$ (Remark 2.1.5), whose $\mathcal{E}_\infty$–ring of functions is the polynomial $\mathcal{E}_\infty$–algebra $R[T] = R \otimes \Sigma^\infty_+(\boldsymbol{N})$. This affine line is smooth in the first sense, and is compatible with the affine line in classical algebraic geometry. That is, when R is discrete, $\boldsymbol{A}^1_{\flat,\mathrm{Spec}(R)}$ is the classical affine line over R.
- The "spectral affine line" $\boldsymbol{A}^1_{\mathrm{Spec}(R)}$ (Example 2.1.2), whose $\mathcal{E}_\infty$–ring of functions is the free $\mathcal{E}_\infty$–algebra $R\{T\}$ on one generator T (in degree zero). This spectral algebraic space is smooth in the second sense, and represents the functor sending an R–algebra A to its underlying space $\Omega^\infty(A)$.

In this paper we work with the second definition of smoothness, and with the "intrinsic" spectral affine line $\boldsymbol{A}^1_{\mathrm{Spec}(R)}$. We review the appropriate definitions in detail in Section 2. In the setting of derived algebraic geometry (formed out of simplicial commutative rings), the two affine lines collapse into one, so that the resulting $\boldsymbol{A}^1$–homotopy theory is a much simpler version of the theory developed here (see the author's PhD thesis [9]). When R is of characteristic zero (an $\mathcal{E}_\infty$-$\boldsymbol{Q}$–algebra), the theory of spectral algebraic geometry over R is equivalent to derived algebraic geometry over R, and the $\boldsymbol{A}^1$–homotopy theory constructed here recovers the construction of loc. cit. Over a general R, we have two different affine lines and two a priori different versions of $\boldsymbol{A}^1$–homotopy theory (see Warning 2.4.7).

In Section 3 we turn to our main results, which are all centred around the functor i_* of direct image along a closed immersion i. We begin by proving that i_* commutes with almost all colimits (Theorem 3.1.1). We then state the localization theorem (Theorem C above) as Theorem 3.2.2, postponing its proof to Section 4. We first explain how it implies Theorem B (Theorem 3.2.4) and Theorem A (Corollary 3.2.9). As another application, we then proceed to develop part of the formalism of Grothendieck's six operations: the proper base change and projection formulas in the case of closed immersions (Propositions 3.3.2 and 3.4.2) and a smooth-closed base change formula (Proposition 3.5.2).

Finally, Section 4 is dedicated to the proof of Theorem C. Aside from generalizing the theorem of Morel and Voevodsky [14] to the spectral setting, our statement also differs in a couple other (mutually orthogonal) ways:

- We do not impose noetherian hypotheses. For this reason, we give a proof of Theorem C that avoids the use of "points" and therefore applies to sheaves satisfying *Čech descent*, as opposed to the (a priori) stronger condition of hyperdescent (see Remark 2.2.10). An alternative approach to removing noetherian hypotheses is to use continuity arguments to reduce to the noetherian case, as described in the classical setting in Hoyois [7, Appendix C].
- We generalize the result from (spectral) schemes to (spectral) algebraic spaces. The key point is that every quasicompact quasiseparated algebraic space is Nisnevich-locally affine (see Knutson [10, Chapter II, Theorem 6.4]). To be precise, one needs a little more than this: see the proof of Proposition 2.2.13. Repeating the proof of Theorem C in the setting of classical algebraic geometry, one can similarly generalize the statement of [14, Theorem 3.2.21] to algebraic spaces.

Our proof follows the same general strategy as the original proof of Morel and Voevodsky, but differs in some details. Let $i\colon Z \hookrightarrow S$ be a closed immersion as in the statement. The first step is to use Proposition 2.2.13 and Theorem 3.1.1 to reduce to the case of (the motivic localization of) a sheaf represented by a smooth spectral algebraic space X over the base S. Then, given a partially defined section $t\colon Z \hookrightarrow X$ over S, we have to show that a certain presheaf $\mathrm{h}_S(X,t)$ is motivically contractible. We achieve this in a few steps:

- Nisnevich-locally on X, we can lift the partially defined section $t\colon Z \hookrightarrow X$ to a section $s\colon S \hookrightarrow X$ (Lemma 4.2.4).
- Nisnevich-locally on X, the section s can be approximated by the zero section of a trivial vector bundle on S, up to some étale morphism that induces an isomorphism over S (Lemma 4.2.3). Moreover, the construction $\mathrm{h}_S(X,t)$ is invariant under such approximations (Lemma 4.2.6).

- If X is a vector bundle over S (and t is the restriction of the zero section), then $\mathrm{h}_S(X,t)$ is $\boldsymbol{A}^1$–contractible (Lemma 4.2.5).

1.2 Notation and conventions

We will use the language of ∞–categories freely throughout the text. Our main references are Lurie [11; 12]. The ∞–category of spaces will be denoted by Spc, and a morphism in an ∞–category will be called an *isomorphism* if it is invertible (= an *equivalence* in the language of [11]).

The term *spectral algebraic space* will mean a quasicompact quasiseparated spectral algebraic space as defined in [13]. An affine spectral scheme is a spectral algebraic space of the form $\mathrm{Spec}(R)$, where R is a connective $\mathcal{E}_\infty$–ring (see eg [12]). Any spectral algebraic space S admits a finite Nisnevich covering by affine spectral schemes [13, Example 3.7.1.5]; it is a (quasicompact quasiseparated) spectral *scheme* in the sense of [13] if and only if it moreover admits a *Zariski* covering by finitely many affines. It is a (quasicompact, quasiseparated) *classical* algebraic space if and only if it admits a Nisnevich covering by finitely many *classical* affines (of the form $\mathrm{Spec}(R)$ with R discrete). Given a spectral algebraic space S, we write S_{cl} for its *underlying classical algebraic space*, so that $\mathrm{Spec}(R)_{\mathrm{cl}} = \mathrm{Spec}(\pi_0(R))$ for any connective $\mathcal{E}_\infty$–ring R.

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2 Motivic spaces

2.1 Sm–fibred spaces

Definition 2.1.1 Let $f\colon X \to S$ be a morphism of spectral algebraic spaces. We say that f is *smooth* if it is of finite presentation and the relative cotangent complex $\mathcal{L}_{X/S}$ is locally free of finite rank. If moreover the cotangent complex vanishes, then we say that f is *étale*.

Example 2.1.2 Let $\boldsymbol{S}$ denote the sphere spectrum, and $\boldsymbol{S}\{T_1,\dots,T_n\}$ the free $\mathcal{E}_\infty$–algebra on n generators (in degree zero). Given a spectral algebraic space S, consider for any $n \geqslant 0$ the *n–dimensional spectral affine space*

$$\boldsymbol{A}^n_S = S \times \mathrm{Spec}(\boldsymbol{S}\{T_1, \ldots, T_n\}).$$

Then the projection $\boldsymbol{A}^n_S \to S$ has cotangent complex free of rank n, and is smooth. More generally, if $\mathcal{E}$ is a locally free sheaf of finite rank on S, then the associated vector bundle $\pi\colon \mathrm{Spec}_S(\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{E})) \to S$ has relative cotangent complex $\pi^*(\mathcal{E})$, and is again smooth.

Remark 2.1.3 Nisnevich-locally on X, any smooth morphism of spectral algebraic spaces $f\colon X \to S$ can be factored through an étale morphism $X \to \boldsymbol{A}^n_S$ and the projection $\boldsymbol{A}^n_S \to S$ (see [13, Proposition 11.2.2.1]).

Warning 2.1.4 Unlike in classical algebraic geometry, smooth morphisms in spectral algebraic geometry are generally not flat: étale morphisms are flat, but $\boldsymbol{A}^n_S \to S$ is flat if and only if $n = 0$ or S is of characteristic zero. In particular, if S is classical, the spectral algebraic space $\boldsymbol{A}^n_S$ is classical if and only if $n = 0$ or S is of characteristic zero (see Warning 2.4.7).

Remark 2.1.5 There is a variant of the construction $\boldsymbol{A}^n_S$ that is flat over S (but usually not smooth). Namely, let $\boldsymbol{S}[T_1, \ldots, T_n]$ denote the polynomial $\mathcal{E}_\infty$–algebra on n generators over $\boldsymbol{S}$ (in degree zero); this is by definition the suspension spectrum $\Sigma^\infty_+(\boldsymbol{N}^n)$, where the (additive) commutative monoid $\boldsymbol{N}^n$ is viewed as a discrete $\mathcal{E}_\infty$–space. If we set

$$\boldsymbol{A}^n_{\flat,S} = S \times \mathrm{Spec}(\boldsymbol{S}[T_1, \ldots, T_n]),$$

then the projection $\boldsymbol{A}^n_{\flat,S} \to S$ is flat.

Definition 2.1.6 Let S be a spectral algebraic space. An Sm–*fibred space* over S, or simply a *fibred space* over S, is a presheaf of spaces on the ∞–category $\mathrm{Sm}_{/S}$ of smooth spectral algebraic spaces over S. We write $\mathrm{Spc}(S)$ for the ∞–category of Sm–fibred spaces over S, and denote the Yoneda embedding by $X \mapsto \mathrm{h}_S(X)$.

2.2 Nisnevich descent

In this subsection we discuss the property of Nisnevich descent for an Sm–fibred space. One very pleasant feature of the Nisnevich topology, compared to the étale topology, is that the sheaf condition can be described using finite limits (see Theorem 2.2.6).

Definition 2.2.1 Let S be a spectral algebraic space and $X \in \mathrm{Sm}_{/S}$. A *Nisnevich square* over X is a cartesian square of spectral algebraic spaces

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd} \tag{2-1}$$

where j is an open immersion, p is étale and there exists a closed immersion $Z \hookrightarrow X$ complementary to j such that the induced morphism $p^{-1}(Z) \to Z$ is invertible.

Definition 2.2.2 Let $\mathcal{F} \in \mathrm{Spc}(S)$ be a fibred space over S. We say that $\mathcal{F}$ satisfies *Nisnevich excision* if it is reduced, ie the space $\Gamma(\varnothing, \mathcal{F})$ is contractible, and for any Nisnevich square over X of the form (2-1), the induced square of spaces

$$\begin{tikzcd} \Gamma(X,\mathcal{F}) \arrow[r, "j^*"] \arrow[d, "p^*"] & \Gamma(U,\mathcal{F}) \arrow[d] \\ \Gamma(V,\mathcal{F}) \arrow[r] & \Gamma(W,\mathcal{F}) \end{tikzcd}$$

is cartesian.

Remark 2.2.3 Being defined by finite limits, the property of Nisnevich excision is stable under filtered colimits and small limits in $\mathrm{Spc}(S)$.

Definition 2.2.4 Let $\mathcal{F} \in \mathrm{Spc}(S)$ be a fibred space over S. We say that $\mathcal{F}$ satisfies *Nisnevich descent* if it is reduced and, for any Nisnevich square (2-1), the canonical morphism of spaces

$$\Gamma(X,\mathcal{F}) \to \mathrm{Tot}\big(\Gamma(\check{\mathrm{C}}(\widetilde{X}/X)_\bullet, \mathcal{F})\big)$$

is invertible, where $\widetilde{X} = U \sqcup V$, the simplicial object $\check{\mathrm{C}}(\widetilde{X}/X)_\bullet$ is the Čech nerve of the morphism $\widetilde{X} \to X$, and "Tot" denotes totalization of a cosimplicial diagram.

Construction 2.2.5 Consider the Grothendieck pretopology on $\mathrm{Sm}_{/S}$ generated by the following covering families: (a) the empty family, covering the empty scheme $\varnothing$; (b) for any $X \in \mathrm{Sm}_{/S}$ and for any Nisnevich square over X of the form (2-1), the family $\{U \to X,\ V \to X\}$, covering X. We call the associated Grothendieck topology the *Nisnevich topology*. Then $\mathcal{F} \in \mathrm{Spc}(S)$ satisfies Nisnevich descent in the sense of Definition 2.2.4 if and only if it is a sheaf with respect to the Nisnevich topology (in the sense of [11]).

Theorem 2.2.6 *Let S be a spectral algebraic space and $\mathcal{F}$ a Sm–fibred space over S. Then $\mathcal{F}$ satisfies Nisnevich excision if and only if it satisfies Nisnevich descent.*

Theorem 2.2.6 follows from a general result of Voevodsky [17, Corollary 5.10] (see also [2, Theorem 3.2.5]).

Theorem 2.2.7 (Voevodsky) *Let $\boldsymbol{C}$ be an ∞–category admitting fibred products. Let $\mathcal{E}$ be a set of cartesian squares which is closed under isomorphism and satisfies the following properties:*

(a) *The set $\mathcal{E}$ is closed under base change. More precisely, suppose that Q is a cartesian square in $\boldsymbol{C}$ of the form*

$$\begin{tikzcd} Q(1,1) \arrow[r] \arrow[d] & Q(0,1) \arrow[d] \\ Q(1,0) \arrow[r] & Q(0,0) \end{tikzcd} \tag{2-2}$$

that belongs to $\mathcal{E}$. Then its base change along any morphism $c \to Q(0,0)$ in $\boldsymbol{C}$ also belongs to $\mathcal{E}$.

(b) *For every square Q in $\mathcal{E}$ of the form (2-2), the lower horizontal arrow $Q(1,0) \to Q(0,0)$ is a monomorphism* (ie its diagonal $\Delta\colon Q(1,0) \to Q(1,1) \times_{Q(0,0)} Q(1,0)$ is invertible).

(c) *For every square Q in $\mathcal{E}$ of the form (2-2), the right-hand vertical arrow $Q(0,1) \to Q(0,0)$ is k–truncated for some $k \geqslant 0$.*

(d) *For every square Q in $\mathcal{E}$ of the form (2-2), the induced square*

$$\begin{tikzcd} Q(1,1) \arrow[r] \arrow[d, "\Delta"] & Q(0,1) \arrow[d, "\Delta"] \\ Q(1,1) \times_{Q(1,0)} Q(1,1) \arrow[r] & Q(0,1) \times_{Q(0,0)} Q(0,1) \end{tikzcd}$$

also belongs to $\mathcal{E}$.

Then, for any presheaf $\mathcal{F}\colon (\boldsymbol{C})^{\mathrm{op}} \to \mathrm{Spc}$, the following two conditions are equivalent:

(i) *The presheaf $\mathcal{F}$ sends every square in $\mathcal{E}$ to a cartesian square of spaces.*

(ii) *For any square $Q \in \mathcal{E}$, write $\check{\mathrm{C}}(Q)_\bullet$ for the Čech nerve of the morphism $Q(1,0) \sqcup Q(0,1) \to Q(0,0)$. Then the canonical map of spaces*

$$\mathcal{F}(Q(0,0)) \to \mathrm{Tot}(\mathcal{F}(\check{\mathrm{C}}(Q)_\bullet))$$

is invertible.

Notation 2.2.8 Given a square $Q \in \mathcal{E}$ of the form (2-2), it will be useful to generalize the notation as follows: for each pair of integers $i, j \geqslant 0$, let $Q(i, j)$ denote the object

$$Q(i,j) := Q(1,0)^{\times i} \times_{Q(0,0)} Q(0,1)^{\times j}$$

in $\boldsymbol{C}$, where $Q(1,0)^{\times i}$ denotes the i–fold fibred product of $Q(1,0)$ with itself over $Q(0,0)$ (and similarly for $Q(0,1)^{\times j}$).

Proof of Theorem 2.2.7 The proof is essentially the same as in the case where $\boldsymbol{C}$ is a 1–category, but we reproduce it here for the reader's convenience. Given a square $Q \in \mathcal{E}$, let K_Q denote the colimit of the diagram $\mathrm{h}(Q(0,1)) \leftarrow \mathrm{h}(Q(1,1)) \rightarrow \mathrm{h}(Q(1,0))$ (formed in the ∞–category of presheaves), and let $\mathcal{K}_{\mathcal{E}}$ denote the set of canonical morphisms $k_Q \colon K_Q \to \mathrm{h}(Q(0,0))$ for all $Q \in \mathcal{E}$. Note that a presheaf $\mathcal{F}$ satisfies condition (i) if and only if it is $\mathcal{K}_{\mathcal{E}}$–local. Similarly, let C_Q denote the geometric realization of the Čech nerve $\check{\mathrm{C}}(Q)_\bullet$, and $\mathcal{C}_{\mathcal{E}}$ the set of canonical morphisms $c_Q \colon \check{\mathrm{C}}(Q)_\bullet \to \mathrm{h}(Q(0,0))$ for all $Q \in \mathcal{E}$. Then a presheaf $\mathcal{F}$ satisfies condition (ii) if and only if it is $\mathcal{C}_{\mathcal{E}}$–local. For any $Q \in \mathcal{E}$ as in (2-2), form the cartesian square of presheaves

$$\begin{tikzcd} K_Q \times_{\mathrm{h}(Q(0,0))} C_Q \arrow[r, "p(Q)"] \arrow[d, "q(Q)"] & K_Q \arrow[d, "k_Q"] \\ C_Q \arrow[r, "c_Q"] & \mathrm{h}(Q(0,0)) \end{tikzcd} \tag{2-3}$$

We will show that (1) the morphism $p(Q)$ is invertible, and (2) $q(Q)$ is both a $\mathcal{K}_{\mathcal{E}}$–local equivalence and a $\mathcal{C}_{\mathcal{E}}$–local equivalence. Since any class of local equivalences is strongly saturated [11, Lemma 5.5.4.11] and in particular satisfies the two-of-three property, it will follow that the classes of $\mathcal{K}_{\mathcal{E}}$–local and $\mathcal{C}_{\mathcal{E}}$–local equivalences coincide, and therefore that conditions (i) and (ii) are equivalent.

For (1), it suffices by universality of colimits to show that c_Q becomes invertible after base change along any of the morphisms $Q(0,1) \to Q(1,1)$, $Q(1,0) \to Q(0,0)$ or $Q(1,1) \to Q(0,0)$. Since the morphism $Q(0,1) \sqcup Q(1,0) \to Q(0,0)$ splits after any of these base changes, it follows that the augmented simplicial object $\check{\mathrm{C}}(Q)_\bullet \to \mathrm{h}(Q(1,1))$ also becomes split after any of these base changes.

For (2), write Q_{ij} for the base change of the square Q along $Q(i,j) \to Q(0,0)$ for $i, j \geqslant 0$. By universality of colimits, it will suffice to show that each $k_{Q_{ij}}$ is a $\mathcal{K}_{\mathcal{E}}$–local and $\mathcal{C}_{\mathcal{E}}$–local equivalence (for $i + j \geqslant 1$). The former claim follows from assumption (a). For $i \geqslant 1$, assumption (b) implies that the lower horizontal arrow in the square Q_{ij} is invertible; in this case it is clear that the morphism $k_{Q_{ij}}$ is invertible.

Therefore we may set $i = 0$ and consider the squares Q_{0j} for $j \geqslant 1$; it will suffice to show that $k_{Q_{ij}}$ is invertible for sufficiently large j. Note that in the commutative diagram

$$\begin{tikzcd}
Q(1,j) \arrow[r] \arrow[d, "\Delta"] & Q(0,j) \arrow[d, "\Delta"] \\
Q(1,j+1) \arrow[r] \arrow[d] & Q(0,j+1) \arrow[d] \\
Q(1,j) \arrow[r] & Q(0,j)
\end{tikzcd}$$

both squares are cartesian, the vertical composites are identities and the lower square is canonically identified with Q_{0j}. Since the class of $\mathcal{C}_{\mathcal{E}}$–local equivalences is closed under retracts and cobase change, it will suffice to show that $k_{Q'}$ is a $\mathcal{C}_{\mathcal{E}}$–local equivalence, where Q' denotes the upper square. Note that by assumptions (a) and (d), the square Q' belongs to $\mathcal{E}$. By assumption (b), its lower horizontal arrow is a monomorphism, and by (c) its right-hand vertical arrow is $(k-1)$–truncated (where k is such that $Q(0,1) \to Q(0,0)$ is k–truncated). Therefore we may replace Q by Q' and assume that the vertical arrow $Q(1,0) \to Q(0,0)$ is $(k-1)$–truncated. Repeating the above argument recursively we eventually reduce to the case where both horizontal and vertical legs of the square Q are (-1)–truncated (= monomorphisms). For such Q, observe that in each of the squares Q_{ij} $(i + j \geqslant 1)$, one of the legs is invertible. Then it is obvious that $k_{Q_{ij}}$ is invertible, so that $q(Q)$ is invertible. Then the square (2-3) shows that k_Q is a $\mathcal{C}_{\mathcal{E}}$–local equivalence. □

Proof of Theorem 2.2.6 Apply Theorem 2.2.7 to the set of Nisnevich squares. It is easy to see that the assumptions hold (recall that étale morphisms are 0–truncated). □

Remark 2.2.9 From [13, Theorem 3.7.5.1] and Theorem 2.2.6 it follows that the topology defined in Construction 2.2.5 coincides with Lurie's version of the Nisnevich topology constructed in [13, Section 3.7.4].

Remark 2.2.10 In our definition of the Nisnevich descent property we consider only Čech covers as opposed to arbitrary hypercovers (see [11, Section 6.5.4]). If S is noetherian and of finite dimension, then there is no difference [13, Corollary 3.7.7.3].

Construction 2.2.11 Let $\mathrm{Spc}_{\mathrm{Nis}}(S)$ denote the full subcategory of $\mathrm{Spc}(S)$ spanned by Nisnevich excisive fibred spaces. By Theorem 2.2.6 this is an exact left localization, and we denote the localization functor by $\mathcal{F} \mapsto \mathrm{L}_{\mathrm{Nis}}(\mathcal{F})$. We say that a morphism in

$\mathrm{Spc}(S)$ is a *Nisnevich-local equivalence* if it induces an isomorphism after Nisnevich localization.

Example 2.2.12 Given a spectral algebraic space X, let $\mathrm{K}(X)$ denote the nonconnective algebraic K–theory spectrum of the stable ∞–category of perfect complexes on X. Then the assignment $X \mapsto \Omega^\infty \mathrm{K}(X)$, viewed as an Sm–fibred space over a spectral algebraic space S, is Nisnevich excisive. This follows from compact generation of quasicoherent sheaves on spectral algebraic spaces[1] as in [13, Theorem 9.6.1.1 and Corollary 9.6.3.2; 1, Proposition 6.9 and Theorem 6.11] (see eg [4, Proposition A.13]).

Proposition 2.2.13 *For any spectral algebraic space S, the ∞–category $\mathrm{Spc}_{\mathrm{Nis}}(S)$ is generated under sifted colimits by objects of the form $\mathrm{h}_S(X)$, where $X \in \mathrm{Sm}_{/S}$ is affine and $X \to S$ factors through an étale morphism to a spectral affine space $\boldsymbol{A}^n_S$ for some $n \geqslant 0$.*

Proof Say that $X \in \mathrm{Sm}_{/S}$ is *good* if it admits an étale S–morphism to $\boldsymbol{A}^n_S$ for some $n \geqslant 0$. Let $\boldsymbol{C}$ denote the full subcategory of $\mathrm{Spc}_{\mathrm{Nis}}(S)$ generated under sifted colimits by representables of the form $\mathrm{h}_S(X)$ with X affine and good. From [11, Lemma 5.5.8.14] it follows that $\mathrm{Spc}_{\mathrm{Nis}}(S)$ is generated under sifted colimits by the representables, so it will suffice to show that every representable space $\mathrm{h}_S(X)$ with $X \in \mathrm{Sm}_{/S}$ belongs to $\boldsymbol{C}$. Using Remark 2.1.3 we may write $\mathrm{h}_S(X)$ as the colimit of a simplicial diagram where each term is good (namely, take the Čech nerve of a suitable Nisnevich covering family of X). We may assume therefore that X is good.

Choose a *scallop decomposition*[2] $\varnothing = U_0 \hookrightarrow U_1 \hookrightarrow \cdots \hookrightarrow U_n = X$ [13, Theorem 3.5.2.1], ie a sequence of open immersions such that for each $1 \leqslant i \leqslant n$ there exists an affine V_i fitting in a commutative square

$$\begin{tikzcd} W_i \arrow[r, hook] \arrow[d] & V_i \arrow[d] \\ U_{i-1} \arrow[r, hook] & U_i \end{tikzcd}$$

which is cartesian and cocartesian (in the ∞–category of spectral algebraic spaces), with $V_i \to U_i$ étale.

For each i the morphisms $U_{i-1} \hookrightarrow U_i$ and $V_i \to U_i$ generate a Nisnevich covering, so $\mathrm{h}_S(U_i)$ is the colimit of the Čech nerve of $U_{i-1} \sqcup V_i \to U_i$. Moreover, every term

[1] Recall that for us, all spectral algebraic spaces are implicitly quasicompact and quasiseparated.

[2] If X admits a Zariski covering by affines, then one can take $V_i \to U_i$ to be open immersions, with $U_i = V_1 \cup V_2 \cup \cdots \cup V_i$.

of this simplicial diagram is separated, because $U_{i-1} \times_{U_i} V_i = W_i$ is separated (being an open subspace in V_i). By induction we may therefore assume that X is separated and good. For this case, choose again a scallop decomposition as above. Since each U_i is now separated and good, the cartesian squares

$$\begin{tikzcd} W_i = U_{i-1} \times_{U_i} V_i \arrow[r] \arrow[d] & U_{i-1} \times V_i \arrow[d] \\ U_i \arrow[r, "\Delta"] & U_i \times U_i \end{tikzcd}$$

show that the W_i are *affine*. The Čech nerve of $U_{i-1} \sqcup V_i \to U_i$ is therefore a simplicial diagram with all terms affine and good, so we conclude by induction. □

Proposition 2.2.14 *Let S be a spectral algebraic space. Denote by $\mathrm{Sm}^{\mathrm{sch}}_{/S}$ and $\mathrm{Sm}^{\mathrm{aff}}_{/S}$ the full subcategories of $\mathrm{Sm}_{/S}$ spanned by smooth spectral schemes and affine smooth spectral schemes, respectively, over S.*

(i) *If S is a spectral scheme, then restriction along the inclusion $\mathrm{Sm}^{\mathrm{sch}}_{/S} \hookrightarrow \mathrm{Sm}_{/S}$ induces an equivalence on ∞–categories of Nisnevich sheaves. In particular, every Nisnevich sheaf on $\mathrm{Sm}_{/S}$ is a right Kan extension of its restriction to $\mathrm{Sm}^{\mathrm{sch}}_{/S}$.*

(ii) *If S is affine, then restriction along the inclusion $\mathrm{Sm}^{\mathrm{aff}}_{/S} \hookrightarrow \mathrm{Sm}_{/S}$ induces an equivalence on ∞–categories of Nisnevich sheaves. In particular, every Nisnevich sheaf on $\mathrm{Sm}_{/S}$ is a right Kan extension of its restriction to $\mathrm{Sm}^{\mathrm{aff}}_{/S}$.*

Proof For claim (i), let ι denote the inclusion functor and ι^* the functor of restriction of presheaves along ι. This admits fully faithful left and right adjoints $\iota_!$ and ι_*, given respectively by left and right Kan extension. Since ι is topologically continuous (preserves Nisnevich covering families), the functor ι^* preserves Nisnevich sheaves. It is also topologically cocontinuous (see before Definition 3.1.5), so ι_* preserves Nisnevich sheaves. Therefore at the level of Nisnevich sheaves the functor ι^* is left adjoint to ι_* and right adjoint to $\mathrm{L}_{\mathrm{Nis}}\,\iota_!$. It follows formally that $\mathrm{L}_{\mathrm{Nis}}\,\iota_!$ is also fully faithful. Since its essential image is generated under colimits by objects of the form $\mathrm{L}_{\mathrm{Nis}}\,\iota_!(\mathrm{h}_S(X)) \simeq \mathrm{L}_{\mathrm{Nis}}\,\mathrm{h}_S(X) \simeq \mathrm{h}_S(X)$ for $X \in \mathrm{Sm}^{\mathrm{sch}}_{/S}$, it follows from Proposition 2.2.13 that it is essentially surjective. The proof of claim (ii) is the same. □

2.3 A^1–homotopy-invariance

Definition 2.3.1 Let S be a spectral algebraic space and $\mathcal{F} \in \mathrm{Spc}(S)$ a fibred space over S. We say that $\mathcal{F} \in \mathrm{Spc}(S)$ satisfies *A^1–homotopy-invariance* if for every

$X \in \mathrm{Sm}_{/S}$, the canonical map of spaces

$$p^*\colon \Gamma(X, \mathcal{F}) \to \Gamma(\boldsymbol{A}^1_X, \mathcal{F})$$

is invertible, where $p\colon \boldsymbol{A}^1_X \to X$ is the projection of the spectral affine line over X (Example 2.1.2). Let $\mathrm{Spc}_{\boldsymbol{A}^1}(S)$ denote the full subcategory of $\mathrm{Spc}(S)$ spanned by $\boldsymbol{A}^1$–homotopy-invariant fibred spaces.

Remark 2.3.2 The full subcategory $\mathrm{Spc}_{\boldsymbol{A}^1}(S) \subset \mathrm{Spc}(S)$ is stable under small colimits and limits.

Definition 2.3.3 Note that $\mathrm{Spc}_{\boldsymbol{A}^1}(S)$ can be described as the (accessible) left localization of $\mathrm{Spc}(S)$ at the set of canonical projections $\boldsymbol{A}^1_X \to X$ for $X \in \mathrm{Sm}_{/S}$. Since this set is essentially small, there is a localization functor $\mathcal{F} \mapsto \mathrm{L}_{\boldsymbol{A}^1}(\mathcal{F})$. We say that a morphism in $\mathrm{Spc}(S)$ is an $\boldsymbol{A}^1$*–local equivalence* if it induces an isomorphism in $\mathrm{Spc}_{\boldsymbol{A}^1}(S)$ after $\boldsymbol{A}^1$–localization.

Example 2.3.4 The Sm–fibred space $X \mapsto \Omega^\infty \mathrm{K}(X)$ of Example 2.2.12 is rarely $\boldsymbol{A}^1$–homotopy-invariant, and its $\boldsymbol{A}^1$–localization no longer satisfies Nisnevich descent. There is however a variant of K–theory, studied in [3], which is both $\boldsymbol{A}^1$–invariant and Nisnevich excisive.

Remark 2.3.5 The fact that $\boldsymbol{A}^1$–projections are stable under base change implies that the $\boldsymbol{A}^1$–localization functor commutes with finite products, and in fact admits the following description: for any fibred space $\mathcal{F} \in \mathrm{Spc}(S)$, the space of sections over any $X \in \mathrm{Sm}_{/S}$ is computed by a sifted colimit:

$$\Gamma(X, \mathrm{L}_{\boldsymbol{A}^1}(\mathcal{F})) \simeq \varinjlim_{(Y \to X) \in (\boldsymbol{A}_X)^{\mathrm{op}}} \Gamma(Y, \mathcal{F}), \tag{2-4}$$

where $\boldsymbol{A}_X$ is the full subcategory of $\mathrm{Sm}_{/X}$ spanned by composites of $\boldsymbol{A}^1$–projections. Moreover, the ∞–category $\mathrm{Spc}_{\boldsymbol{A}^1}(S)$ has universality of colimits. See Proposition 3.4 of [8].

Definition 2.3.6 Let $\boldsymbol{S}$ denote the sphere spectrum and $\boldsymbol{S}\{T\}$ the free $\mathcal{E}_\infty$–algebra on one generator T (in degree zero). The two morphisms $\boldsymbol{S}\{T\} \to \boldsymbol{S}$ sending T to 0 and 1, respectively, induce, for any $X \in \mathrm{Sm}_{/S}$, two sections i_0 and i_1 of the projection $p\colon \boldsymbol{A}^1_X \to X$. Given two morphisms $\varphi_0, \varphi_1\colon \mathcal{F} \rightrightarrows \mathcal{G}$ in $\mathrm{Spc}(S)$, an *elementary* $\boldsymbol{A}^1$*–homotopy* from φ_0 to φ_1 is a morphism $\mathrm{h}_S(\boldsymbol{A}^1_S) \times \mathcal{F} \to \mathcal{G}$ whose restrictions to

$h_S(S) \times \mathcal{F} = \mathcal{F}$ along i_0 and i_1 are isomorphic to φ_0 and φ_1, respectively. We say that φ_0 and φ_1 are *$\boldsymbol{A}^1$–homotopic* if there exists a sequence of elementary $\boldsymbol{A}^1$–homotopies connecting them; in this case the induced morphisms $\mathrm{L}_{\boldsymbol{A}^1}(\mathcal{F}) \rightrightarrows \mathrm{L}_{\boldsymbol{A}^1}(\mathcal{G})$ coincide. A morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ in $\mathrm{Spc}(S)$ is called a *strict $\boldsymbol{A}^1$–homotopy equivalence* if there exists a morphism $\psi\colon \mathcal{G} \to \mathcal{F}$ such that the composites $\varphi \circ \psi$ and $\psi \circ \varphi$ are $\boldsymbol{A}^1$–homotopic to the identities. Any strict $\boldsymbol{A}^1$–homotopy equivalence is an $\boldsymbol{A}^1$–local equivalence.

2.4 Motivic spaces

Definition 2.4.1 A *motivic space* over S is an Sm–fibred space $\mathcal{F} \in \mathrm{Spc}(S)$ that is Nisnevich-local and $\boldsymbol{A}^1$–local. We write $\boldsymbol{H}(S)$ for the full subcategory of $\mathrm{Spc}(S)$ spanned by motivic spaces. This is an accessible left localization, and we write $\mathcal{F} \mapsto \mathrm{L}_{\mathrm{mot}}(\mathcal{F})$ for the localization functor. We say that a morphism in $\mathrm{Spc}(S)$ is a *motivic equivalence* if it induces an isomorphism in $\boldsymbol{H}(S)$ after application of $\mathrm{L}_{\mathrm{mot}}$. We write $\mathrm{M}_S(X) := \mathrm{L}_{\mathrm{mot}}\, h_S(X)$ for the motivic space represented by an object $X \in \mathrm{Sm}_{/S}$.

Remark 2.4.2 The ∞–category $\boldsymbol{H}(S)$ has universality of colimits (since $\mathrm{Spc}_{\mathrm{Nis}}(S)$ and $\mathrm{Spc}_{\boldsymbol{A}^1}(S)$ do). Similarly, the functor $\mathrm{L}_{\mathrm{mot}}$ commutes with finite products (since $\mathrm{L}_{\mathrm{Nis}}$ and $\mathrm{L}_{\boldsymbol{A}^1}$ do). By adjunction it follows that $\boldsymbol{H}(S)$ is cartesian closed: for any $\mathcal{F} \in \boldsymbol{H}(S)$ and $\mathcal{G} \in \mathrm{Spc}(S)$, the internal hom object $\underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{F}) \in \mathrm{Spc}(S)$ is a motivic space.

Remark 2.4.3 Since the conditions of Nisnevich and $\boldsymbol{A}^1$–locality are each stable under filtered colimits (Remarks 2.2.3 and 2.3.2), it follows that motivic localization can be described as the transfinite composite

$$\mathrm{L}_{\mathrm{mot}}(\mathcal{F}) = \varinjlim_{n \geqslant 0} (\mathrm{L}_{\boldsymbol{A}^1} \circ \mathrm{L}_{\mathrm{Nis}})^{\circ n}(\mathcal{F}) \tag{2-5}$$

for any $\mathcal{F} \in \mathrm{Spc}(S)$.

By Proposition 2.2.13 we get:

Proposition 2.4.4 *The ∞–category $\boldsymbol{H}(S)$ is generated under sifted colimits by objects of the form $\mathrm{M}_S(X)$, where $X \in \mathrm{Sm}_{/S}$ is affine and $X \to S$ factors through an étale morphism to a spectral affine space $\boldsymbol{A}^n_S$ for some $n \geqslant 0$.*

Corollary 2.4.5 *Let S be a spectral algebraic space. Consider the full subcategories $\mathrm{Sm}^{\mathrm{sch}}_{/S}$ and $\mathrm{Sm}^{\mathrm{aff}}_{/S}$ of $\mathrm{Sm}_{/S}$ as defined in Proposition 2.2.14. Then:*

(i) *If S is a spectral scheme, then restriction along the inclusion $\mathrm{Sm}^{\mathrm{sch}}_{/S} \hookrightarrow \mathrm{Sm}_{/S}$ induces an equivalence on ∞–categories of $\boldsymbol{A}^1$–homotopy-invariant Nisnevich sheaves.*

(ii) *If S is an affine spectral scheme, then restriction along the inclusion $\mathrm{Sm}^{\mathrm{aff}}_{/S} \hookrightarrow \mathrm{Sm}_{/S}$ induces an equivalence on ∞–categories of $\boldsymbol{A}^1$–homotopy-invariant Nisnevich sheaves.*

Proof Let ι denote either inclusion functor. The claim will follow by repeating the proof of Proposition 2.2.14 and using the following assertion:

($*$) The restriction functor ι^* preserves $\boldsymbol{A}^1$–invariant spaces, as does its right adjoint ι_*.

The first claim follows immediately from the fact that ι preserves $\boldsymbol{A}^1$–projections (so that the left Kan extension functor $\iota_!$ preserves $\boldsymbol{A}^1$–local equivalences). The second claim is equivalent by adjunction to the assertion that ι^* preserves $\boldsymbol{A}^1$–local equivalences. For this it will suffice to show that, for any $X \in \mathrm{Sm}_{/S}$, the canonical morphism

$$\iota^* \mathrm{h}_S(X \times \boldsymbol{A}^1) \to \iota^* \mathrm{h}_S(X)$$

is an $\boldsymbol{A}^1$–local equivalence. By universality of colimits it suffices to show that, for any $Y \in \mathrm{Sm}_{/S}$ and any morphism $\varphi\colon \mathrm{h}_S(Y) \to \iota^* \mathrm{h}_S(X)$, the base change

$$\iota^* \mathrm{h}_S(X \times \boldsymbol{A}^1) \times_{\iota^* \mathrm{h}_S(X)} \mathrm{h}_S(Y) \to \mathrm{h}_S(Y)$$

is an $\boldsymbol{A}^1$–local equivalence. Since φ factors as $\mathrm{h}_S(Y) \to \iota^*\iota_! \mathrm{h}_S(Y) \simeq \iota^* \mathrm{h}_S(Y) \to \iota^* \mathrm{h}_S(X)$, the morphism in question is a base change of the morphism

$$\iota^* \mathrm{h}_S(X \times \boldsymbol{A}^1) \times_{\iota^* \mathrm{h}_S(X)} \iota^* \mathrm{h}_S(Y) \to \iota^* \mathrm{h}_S(Y).$$

Since ι^* commutes with limits, this is identified with the canonical morphism

$$\mathrm{h}_S(Y \times \boldsymbol{A}^1) \to \mathrm{h}_S(Y).$$

This is obviously an $\boldsymbol{A}^1$–local equivalence, so the claim follows. □

Let S be a classical scheme. Then there is a parallel variant $\boldsymbol{H}_{\mathrm{cl}}(S)$ of the construction $\boldsymbol{H}(S)$ using the site of smooth *classical* schemes, imposing Nisnevich descent and homotopy-invariance with respect to the *classical* affine line; see eg [7, Appendix C] (where it is denoted by $\boldsymbol{H}(S)$). This agrees with the original construction of Morel and Voevodsky [14] when the latter is defined (that is, when S is noetherian and of finite dimension). If we assume that S is of characteristic zero, then it also agrees with the spectral version $\boldsymbol{H}(S)$:

Proposition 2.4.6 *Let S be a classical scheme of characteristic zero. Then there is a canonical equivalence of ∞–categories*

$$\boldsymbol{H}(S) \simeq \boldsymbol{H}_{\mathrm{cl}}(S).$$

Proof This follows from Corollary 2.4.5 and the fact that, if S is of characteristic zero, there is a canonical equivalence between $\mathrm{Sm}^{\mathrm{sch}}_{/S}$ and the category of smooth classical S–schemes, which preserves and detects Nisnevich covers, and sends the spectral affine line to the classical one. □

Warning 2.4.7 The characteristic zero hypothesis in Proposition 2.4.6 is necessary in the proof: For example, if $S = \mathrm{Spec}(\boldsymbol{F}_p)$, then the site $\mathrm{Sm}^{\mathrm{sch}}_{/S}$ is not equivalent to the site of smooth classical S–schemes. Indeed the classical affine line $\mathrm{Spec}(\boldsymbol{F}_p[T])$ is not smooth over S when viewed as a spectral scheme. On the other hand, the site $\mathrm{Sm}^{\mathrm{sch}}_{/S}$ contains objects like the spectral affine line $\boldsymbol{A}^1_S = \mathrm{Spec}(\boldsymbol{F}_p\{T\})$, which is not a classical scheme. See [16, Proposition 2.4.1.5].

We conclude this subsection by introducing the pointed variant of $\boldsymbol{H}(S)$:

Construction 2.4.8 Given a spectral algebraic space S, write $\boldsymbol{H}(S)_\bullet$ for the ∞–category of pointed objects in $\boldsymbol{H}(S)$. The forgetful functor $\boldsymbol{H}(S)_\bullet \to \boldsymbol{H}(S)$ admits a left adjoint $\mathcal{F} \mapsto \mathcal{F}_+$ which freely adjoins a basepoint. The ∞–category $\boldsymbol{H}(S)_\bullet$ is equivalent to the full subcategory of the ∞–category $\mathrm{Spc}(S)_\bullet$ of fibred pointed spaces $\mathcal{F}$ whose underlying fibred space is motivic. It is an accessible left localization of $\mathrm{Spc}(S)_\bullet$ and the localization functor satisfies $\mathrm{L}_{\mathrm{mot}}(\mathcal{F}_+) \simeq \mathrm{L}_{\mathrm{mot}}(\mathcal{F})_+$ for any $\mathcal{F} \in \boldsymbol{H}(S)$. The cartesian monoidal structure on $\boldsymbol{H}(S)$ extends uniquely to a symmetric monoidal structure on $\boldsymbol{H}(S)_\bullet$ with the property that the functor $\mathcal{F} \mapsto \mathcal{F}_+$ is symmetric monoidal [15, Corollary 2.32]. We write $\wedge$ for the monoidal product; the monoidal unit is the object $(\mathrm{pt}_S)_+ \in \boldsymbol{H}(S)_\bullet$. It follows from Proposition 2.4.4 that $\boldsymbol{H}(S)_\bullet$ is generated under sifted colimits by objects of the form $\mathrm{M}_S(X)_+$ for all affine $X \in \mathrm{Sm}_{/S}$ which admit an étale S–morphism to $\boldsymbol{A}^n_S$ for some $n \geqslant 0$.

2.5 Functoriality

We now discuss the basic functorialities that the system of categories $\boldsymbol{H}(S)$ enjoys as S varies. For any morphism $f\colon T \to S$, we will define a pair of adjoint functors

$(f_{\boldsymbol{H}}^*, f_*^{\boldsymbol{H}})$. If f happens to be smooth, there will be a further adjunction $(f_\sharp^{\boldsymbol{H}}, f_{\boldsymbol{H}}^*)$. When there is no risk of confusion we will usually omit the decorations ${}^{\boldsymbol{H}}$ and ${}_{\boldsymbol{H}}$.

Construction 2.5.1 Let $f\colon T \to S$ be a morphism of spectral algebraic spaces. Restriction along the base change functor $\mathrm{Sm}_{/S} \to \mathrm{Sm}_{/T}$ defines a canonical functor

$$f_*^{\mathrm{Spc}}\colon \mathrm{Spc}(T) \to \mathrm{Spc}(S).$$

It admits a left adjoint f^*_{Spc} which is uniquely characterized by commutativity with small colimits and the formula $f^*_{\mathrm{Spc}}(\mathrm{h}_S(X)) \simeq \mathrm{h}_T(X \times_S T)$ for $X \in \mathrm{Sm}_{/S}$.

Construction 2.5.2 The base change functor $\mathrm{Sm}_{/S} \to \mathrm{Sm}_{/T}$ preserves Nisnevich covering families and $\boldsymbol{A}^1$–projections, so the functor f^*_{Spc} preserves motivic equivalences. By adjunction its right adjoint f_*^{Spc} preserves motivic spaces and induces a functor ("direct image")

$$f_*^{\boldsymbol{H}}\colon \boldsymbol{H}(T) \to \boldsymbol{H}(S).$$

This is right adjoint to the functor $f^*_{\boldsymbol{H}} = \mathrm{L}_{\mathrm{mot}}\, f^*_{\mathrm{Spc}}$ ("inverse image"), characterized uniquely by commutativity with colimits and the formula $f^*_{\boldsymbol{H}}(\mathrm{M}_S(X)) \simeq \mathrm{M}_T(X \times_S T)$ for $X \in \mathrm{Sm}_{/S}$.

Remark 2.5.3 By Remark 2.4.2 it follows that $f^*_{\boldsymbol{H}}$ commutes with finite products.

When the morphism f is smooth, the inverse image functor $f^*_{\boldsymbol{H}}$ also admits a *left* adjoint $f_\sharp$. When f is an open immersion, this is the functor of "extension by zero". More generally, when f is étale, it is the functor of "compactly support direct image".

Construction 2.5.4 Let $p\colon T \to S$ be a smooth morphism of spectral algebraic spaces. Then the base change functor admits a right adjoint $\mathrm{Sm}_{/T} \to \mathrm{Sm}_{/S}$, the forgetful functor $(X \to T) \mapsto (X \to T \xrightarrow{p} S)$. It follows that the functor p^*_{Spc} coincides with restriction along the forgetful functor, and admits a left adjoint

$$p_\sharp^{\mathrm{Spc}}\colon \mathrm{Spc}(T) \to \mathrm{Spc}(S),$$

which is uniquely characterized by commutativity with colimits and the formula $p_\sharp^{\mathrm{Spc}}(\mathrm{h}_T(X)) \simeq \mathrm{h}_S(X)$ for $X \in \mathrm{Sm}_{/T}$.

Construction 2.5.5 Since the forgetful functor $\mathrm{Sm}_{/T} \to \mathrm{Sm}_{/S}$ preserves Nisnevich covering families and $\boldsymbol{A}^1$–projections, it follows that $p_\sharp^{\mathrm{Spc}}$ preserves motivic equivalences. In particular, its right adjoint p^*_{Spc} preserves motivic spaces, and induces a functor

$$p^*_{\boldsymbol{H}}\colon \boldsymbol{H}(S) \to \boldsymbol{H}(T).$$

This is right adjoint to the functor $p^{\boldsymbol{H}}_\sharp(\mathcal{F}) \simeq \mathrm{L}_{\mathrm{mot}}(p^{\mathrm{Spc}}_\sharp(\mathcal{F}))$, characterized by commutativity with colimits and the formula $p^{\boldsymbol{H}}_\sharp(\mathrm{M}_T(X)) \simeq \mathrm{M}_S(X)$ for $X \in \mathrm{Sm}_{/T}$.

Remark 2.5.6 The functor $p^{\boldsymbol{H}}_\sharp$ commutes with binary products if and only if p is a monomorphism, hence if and only if p is an open immersion.

Proposition 2.5.7 (Nisnevich separation) *Let $(p_\alpha\colon S_\alpha \to S)_\alpha$ be a Nisnevich covering family of spectral algebraic spaces. Then the functors $(p_\alpha)^*\colon \boldsymbol{H}(S) \to \boldsymbol{H}(S_\alpha)$ form a conservative family.*

Proof Let $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism in $\boldsymbol{H}(S)$ and assume that $(p_\alpha)^*(\varphi)$ is invertible for each α. It suffices to show that the map $\Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_2)$ is invertible for all $X \in \mathrm{Sm}_{/S}$. Passing to the Čech nerve of the covering family, we may assume that $X \in \mathrm{Sm}_{/S_\alpha}$ for some α. Then the claim follows from the assumption because we have, by adjunction, natural isomorphisms $\Gamma(X_\alpha, \mathcal{F}_i) \simeq \Gamma(X, (p_\alpha)_\sharp(p_\alpha)^*\mathcal{F}_i)$ for each i. □

Construction 2.5.8 Suppose we have a cartesian square

$$\begin{tikzcd} T' \arrow[r, "f'"] \arrow[d, "p'"] & S' \arrow[d, "p"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

of spectral algebraic spaces. If p is smooth, then the square

$$\begin{tikzcd} \boldsymbol{H}(S') \arrow[r, "(f')^*"] \arrow[d, "p_\sharp"] & \boldsymbol{H}(T') \arrow[d, "(p')_\sharp"] \\ \boldsymbol{H}(S) \arrow[r, "f^*"] & \boldsymbol{H}(T) \end{tikzcd}$$

commutes up to the canonical natural transformation

$$(p')_\sharp(f')^* \xrightarrow{\text{unit}} (p')_\sharp(f')^* p^* p_\sharp \simeq (p')_\sharp(p')^* f^* p_\sharp \xrightarrow{\text{counit}} f^* p_\sharp.$$

Proposition 2.5.9 (smooth base change formula) *Suppose given a cartesian square of spectral algebraic spaces as above, with p smooth. Then there are canonical invertible natural transformations*

$$(p')^{\boldsymbol{H}}_\sharp (f')^*_{\boldsymbol{H}} \to f^*_{\boldsymbol{H}} p^{\boldsymbol{H}}_\sharp, \quad p^*_{\boldsymbol{H}} f^{\boldsymbol{H}}_* \to (f')^{\boldsymbol{H}}_* (p')^*_{\boldsymbol{H}}$$

of functors $\boldsymbol{H}(S') \to \boldsymbol{H}(T)$ and $\boldsymbol{H}(T) \to \boldsymbol{H}(S')$, respectively.

Proof The second transformation is obtained from the first by passage to right adjoints. For the first, note that each of the functors involved commutes with colimits. Therefore by Proposition 2.4.4 it suffices to show that the transformation induces an isomorphism after evaluation at any object $\mathrm{M}_{S'}(X')$ with $X' \in \mathrm{Sm}_{/S'}$, which is obvious. $\square$

Corollary 2.5.10 *Let $j\colon U \hookrightarrow S$ be an open immersion of spectral algebraic spaces. Then the functors $j_\sharp^{\boldsymbol{H}}$ and $j_*^{\boldsymbol{H}}$ are fully faithful.*

Proof Applying Proposition 2.5.9 to the square

$$\begin{tikzcd} U \arrow[r, equal] \arrow[d, equal] & U \arrow[d, hook, "j"] \\ U \arrow[r, hook, "j"] & S \end{tikzcd}$$

which is cartesian because j is a monomorphism, we deduce that the natural transformations $\mathrm{id} \to j^* j_\sharp$ and $j^* j_* \to \mathrm{id}$ are invertible. $\square$

Corollary 2.5.11 *Let $i\colon Z \hookrightarrow S$ be a closed immersion of spectral algebraic spaces with quasicompact open complement $j\colon U \hookrightarrow S$. Then the natural transformations*

$$\varnothing_Z \to i_{\boldsymbol{H}}^* j_\sharp^{\boldsymbol{H}}, \quad j_{\boldsymbol{H}}^* i_*^{\boldsymbol{H}} \to \mathrm{pt}_U$$

are invertible.

Proof Apply Proposition 2.5.9 to the cartesian square

$$\begin{tikzcd} \varnothing \arrow[r, hook] \arrow[d, hook] & Z \arrow[d, hook, "i"] \\ U \arrow[r, hook, "j"] & S \end{tikzcd}$$

$\square$

Construction 2.5.12 Let $p\colon S' \to S$ be a smooth morphism of spectral algebraic spaces. Given motivic spaces $\mathcal{F}' \in \boldsymbol{H}(S')$, $\mathcal{F} \in \boldsymbol{H}(S)$ and $\mathcal{G} \in \boldsymbol{H}(S)$, we get a morphism

$$\mathcal{F}' \times_{p_{\boldsymbol{H}}^*(\mathcal{G})} p_{\boldsymbol{H}}^*(\mathcal{F}) \xrightarrow{\text{unit}} p_{\boldsymbol{H}}^* p_\sharp^{\boldsymbol{H}}(\mathcal{F}') \times_{p_{\boldsymbol{H}}^*(\mathcal{G})} p_{\boldsymbol{H}}^*(\mathcal{F}) \simeq p_{\boldsymbol{H}}^*(p_\sharp^{\boldsymbol{H}}(\mathcal{F}') \times_{\mathcal{G}} \mathcal{F}),$$

which corresponds by adjunction to a canonical morphism

$$p_\sharp^{\boldsymbol{H}}(\mathcal{F}' \times_{p_{\boldsymbol{H}}^*(\mathcal{G})} p_{\boldsymbol{H}}^*(\mathcal{F})) \to p_\sharp^{\boldsymbol{H}}(\mathcal{F}') \times_{\mathcal{G}} \mathcal{F}.$$

Proposition 2.5.13 (smooth projection formula) *Let $p\colon S' \to S$ be a smooth morphism of spectral algebraic spaces. Given motivic spaces $\mathcal{F}' \in \boldsymbol{H}(S')$, $\mathcal{F} \in \boldsymbol{H}(S)$ and $\mathcal{G} \in \boldsymbol{H}(S)$, we have canonical bifunctorial isomorphisms*

$$p_\sharp^{\boldsymbol{H}}(\mathcal{F}' \times_{p_{\boldsymbol{H}}^*(\mathcal{G})} p_{\boldsymbol{H}}^*(\mathcal{F})) \xrightarrow{\sim} p_\sharp^{\boldsymbol{H}}(\mathcal{F}') \times_{\mathcal{G}} \mathcal{F},$$
$$p_{\boldsymbol{H}}^*(\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})) \xrightarrow{\sim} \underline{\mathrm{Hom}}(p_{\boldsymbol{H}}^*(\mathcal{F}), p_{\boldsymbol{H}}^*(\mathcal{G})),$$
$$\underline{\mathrm{Hom}}(p_\sharp^{\boldsymbol{H}}(\mathcal{F}'), \mathcal{F}) \xrightarrow{\sim} p_*^{\boldsymbol{H}}\,\underline{\mathrm{Hom}}(\mathcal{F}', p_{\boldsymbol{H}}^*(\mathcal{F})).$$

The functorialities under discussion extend freely to the setting of pointed motivic spaces:

Construction 2.5.14 Given a spectral algebraic space S, let $\boldsymbol{H}(S)_\bullet$ denote the ∞–category of pointed motivic spaces over S (Construction 2.4.8). For any morphism $f\colon T \to S$, the direct image functor $f_*^{\boldsymbol{H}}$ preserves terminal objects and therefore induces a functor $f_*^{\boldsymbol{H}_\bullet}$ (which commutes with passage to underlying motivic spaces). Its left adjoint $f_{\boldsymbol{H}_\bullet}^*$ is uniquely characterized by commutativity with sifted colimits and the formula $f_{\boldsymbol{H}_\bullet}^*(\mathcal{F}_+) \simeq f_{\boldsymbol{H}}^*(\mathcal{F})_+$ for any $\mathcal{F} \in \boldsymbol{H}(S)$. Similarly, for any smooth morphism $p\colon T \to S$, there is a functor $p_\sharp^{\boldsymbol{H}_\bullet}$ left adjoint to $p_{\boldsymbol{H}_\bullet}^*$ which is uniquely characterized by commutativity with sifted colimits and the formula $p_\sharp^{\boldsymbol{H}_\bullet}(\mathcal{F}_+) \simeq p_\sharp^{\boldsymbol{H}}(\mathcal{F})_+$ for any $\mathcal{F} \in \boldsymbol{H}(T)$. The smooth base change and projection formulas (Propositions 2.5.9 and 2.5.13) have obvious pointed analogues that we leave to the reader to formulate.

3 Results

3.1 Exactness of i_*

Our first goal is to prove the following:

Theorem 3.1.1 *Let $i\colon Z \hookrightarrow S$ be a closed immersion of spectral algebraic spaces. Then the direct image functor $i_*^{\boldsymbol{H}}\colon \boldsymbol{H}(Z) \to \boldsymbol{H}(S)$ commutes with contractible colimits.*

In other words, $i_*^{\boldsymbol{H}}$ commutes with colimits of diagrams indexed by contractible[3] ∞–categories. At the level of pointed spaces, we get:

[3] An essentially small ∞–category is called *contractible* if the ∞–groupoid obtained by formally adjoining inverses to all morphisms is (weakly) contractible.

Corollary 3.1.2 *The direct image functor $i_*^{\boldsymbol{H}_\bullet}\colon \boldsymbol{H}(Z)_\bullet \to \boldsymbol{H}(S)_\bullet$ commutes with small colimits.*

Proof It suffices to show that $i_*^{\boldsymbol{H}_\bullet}$ commutes with contractible colimits and preserves the initial object.[4] The former condition follows directly from the unpointed statement (Theorem 3.1.1) and the fact that $i_*^{\boldsymbol{H}_\bullet}$ preserves the initial object (= terminal object) is obvious. □

Remark 3.1.3 It follows from Corollary 3.1.2 that the functor $i_*^{\boldsymbol{H}_\bullet}$ admits a right adjoint $i^!_{\boldsymbol{H}_\bullet}$, called the *exceptional inverse image* functor. A concrete description of the functor $i^!_{\boldsymbol{H}_\bullet}$ can be given using the localization theorem; see Remark 3.2.5.

The geometric input into the proof of Theorem 3.1.1 is a spectral version of [6, Proposition 18.1.1]:

Proposition 3.1.4 *Let $i\colon Z \hookrightarrow S$ be a closed immersion of spectral algebraic spaces. For any smooth* (*resp. étale*) *morphism $q\colon Y \to Z$, there exists, Nisnevich-locally on Y, a smooth* (*resp. étale*) *morphism $p\colon X \to S$, and a cartesian square*

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p} \\ Z & \hookrightarrow & S \end{array}$$

Proof The smooth case follows from the étale case by factoring $q\colon Y \to Z$ through an étale morphism to an affine space $\boldsymbol{A}^n_Z$ (which can always be done Nisnevich-locally). Therefore it will suffice to show the étale case. The question being Nisnevich-local, we may assume that S, Z and Y are affine; let $S = \operatorname{Spec}(R)$, $Z = \operatorname{Spec}(R')$ and $Y = \operatorname{Spec}(A')$. The étale R'–algebra A' induces an étale $\pi_0(R')$–algebra $A' \otimes_{R'} \pi_0(R') \simeq \pi_0(A')$. Note that it will suffice to demonstrate the claim Zariski-locally on $\pi_0(A')$ (since any Zariski covering of $\pi_0(A')$ lifts uniquely to a Zariski covering of A'). By [6, Proposition 18.1.1] the $\pi_0(R')$–algebra $\pi_0(A')$ lifts, Zariski-locally on $\pi_0(A')$, to an étale $\pi_0(R)$–algebra A_0. By [12, Theorem 7.5.0.6], the latter lifts uniquely to an étale R–algebra A. □

[4] Let D be a diagram indexed on an ∞–category $\boldsymbol{I}$. Then the initial object defines a cone over D, which we may view as another diagram D' with the same colimit but which is indexed on a *contractible* ∞–category.

In order to deduce Theorem 3.1.1 from Proposition 3.1.4, we will need to make a small topos-theoretic digression. Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be essentially small ∞–categories, equipped with Grothendieck topologies $\tau_{\boldsymbol{C}}$ and $\tau_{\boldsymbol{D}}$, respectively. Recall that a functor $u\colon \boldsymbol{C} \to \boldsymbol{D}$ is *topologically cocontinuous* if the following condition is satisfied:

(COC) For every $\tau_{\boldsymbol{D}}$–covering sieve $R' \hookrightarrow h(u(c))$, the sieve $R \hookrightarrow h(c)$, generated by morphisms $c' \to c$ such that $h(u(c')) \to h(u(c))$ factors through R', is $\tau_{\boldsymbol{C}}$–covering.

Definition 3.1.5 Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be essentially small ∞–categories, equipped with Grothendieck topologies $\tau_{\boldsymbol{C}}$ and $\tau_{\boldsymbol{D}}$, respectively. Assume that $\boldsymbol{D}$ admits an initial object $\varnothing_{\boldsymbol{D}}$. A functor $u\colon \boldsymbol{C} \to \boldsymbol{D}$ is *topologically quasicocontinuous* if it satisfies the following condition:

(COC′) For every $\tau_{\boldsymbol{D}}$–covering sieve $R' \hookrightarrow h(u(c))$, the sieve $R \hookrightarrow h(c)$, generated by morphisms $c' \to c$ such that either $u(c')$ is initial or $h(u(c')) \to h(u(c))$ factors through $R' \hookrightarrow h(u(c))$, is $\tau_{\boldsymbol{C}}$–covering.

Let $\mathrm{PSh}(\boldsymbol{D})$ denote the ∞–category of presheaves of spaces on $\boldsymbol{C}$, $\mathrm{Sh}_{\tau_{\boldsymbol{D}}}(\boldsymbol{D})$ the full subcategory of $\mathrm{PSh}(\boldsymbol{D})$ spanned by $\tau_{\boldsymbol{D}}$–sheaves. We write $\mathcal{F} \mapsto \mathrm{L}_{\tau_{\boldsymbol{D}}}(\mathcal{F})$ for the (left-exact) localization functor.

Lemma 3.1.6 *With notation as in Definition 3.1.5, let $u\colon \boldsymbol{C} \to \boldsymbol{D}$ be a topologically quasicocontinuous functor. Assume that the initial object $\varnothing_{\boldsymbol{D}}$ is **strict** in the sense that for any object $d \in \boldsymbol{D}$, any morphism $d \to \varnothing_{\boldsymbol{D}}$ is invertible. Assume also that, for any object $d \in \boldsymbol{D}$, the sieve $\varnothing_{\mathrm{PSh}(\boldsymbol{D})} \hookrightarrow h(d)$ is $\tau_{\boldsymbol{D}}$–covering if and only if d is initial* (where $\varnothing_{\mathrm{PSh}(\boldsymbol{D})}$ denotes the initial object of $\mathrm{PSh}(\boldsymbol{D})$)*. Then we have:*

(i) *The restriction functor $u^*\colon \mathrm{PSh}(\boldsymbol{D}) \to \mathrm{PSh}(\boldsymbol{C})$ sends $\tau_{\boldsymbol{D}}$–local equivalences between reduced*[5] *presheaves to $\tau_{\boldsymbol{C}}$–local equivalences.*

(ii) *The functor $\mathrm{Sh}_{\tau_{\boldsymbol{D}}}(\boldsymbol{D}) \to \mathrm{Sh}_{\tau_{\boldsymbol{C}}}(\boldsymbol{C})$, given by the assignment $\mathcal{F} \mapsto \mathrm{L}_{\tau_{\boldsymbol{C}}}(u^*(\mathcal{F}))$, commutes with contractible colimits.*

Proof Let $\mathrm{PSh}_{\mathrm{red}}(\boldsymbol{D})$ the full subcategory of $\mathrm{PSh}(\boldsymbol{D})$ spanned by reduced presheaves. This is a left localization, and the localization functor $\mathcal{F} \mapsto \mathrm{L}_{\mathrm{red}}(\mathcal{F})$ has the effect of forcing $\mathrm{L}_{\mathrm{red}}(\mathcal{F})(\varnothing) \simeq \mathrm{pt}$ (while $\mathcal{F}(d) \to \mathrm{L}_{\mathrm{red}}(\mathcal{F})(d)$ is an isomorphism whenever d is not initial). Note also that the ∞–category $\mathrm{PSh}_{\mathrm{red}}(\boldsymbol{D})$ is the free completion of $\boldsymbol{D}$ by contractible colimits.

[5] We say that a presheaf $\mathcal{F}$ is reduced if it sends the initial object to a contractible space.

Let $\mathcal{A}$ denote the set of morphisms in $\mathrm{PSh}(\boldsymbol{D})$ containing all isomorphisms, the canonical morphism $e\colon \varnothing_{\mathrm{PSh}(\boldsymbol{D})} \hookrightarrow h(\varnothing_{\boldsymbol{D}})$ and all $\tau_{\boldsymbol{D}}$–covering sieves $R' \hookrightarrow h(d)$ of a noninitial object $d \in \boldsymbol{D}$. Then the set of $\tau_{\boldsymbol{D}}$–local equivalences in $\mathrm{PSh}(\boldsymbol{D})$ is the closure of $\mathcal{A}$ under small colimits, cobase change and the two-of-three property. It follows that the set of $\tau_{\boldsymbol{D}}$–local equivalences in $\mathrm{PSh}_{\mathrm{red}}(\boldsymbol{D})$ is the closure of the set $\mathrm{L}_{\mathrm{red}}(\mathcal{A})$ under *contractible* colimits, cobase change and the two-of-three property. Therefore, for the first claim it will suffice to show that u^* sends every morphism in $\mathrm{L}_{\mathrm{red}}(\mathcal{A})$ to a $\tau_{\boldsymbol{C}}$–local equivalence (since the subcategory $\mathrm{PSh}_{\mathrm{red}}(\boldsymbol{D})$ is closed under contractible colimits). This is clear for the morphism $\mathrm{L}_{\mathrm{red}}(e)$, as it is already invertible.

Now let $s\colon R' \hookrightarrow h(d)$ be a $\tau_{\boldsymbol{D}}$–covering sieve of a noninitial object $d \in \boldsymbol{D}$. Note that we have $\mathrm{L}_{\mathrm{red}}(s) = s$, as $h(d)$ is reduced and hence so is R' (since s is a monomorphism and R' is by assumption nonempty). Thus we need to show that $u^*(s)$ is a $\tau_{\boldsymbol{C}}$–local equivalence. By universality of colimits it is sufficient to show that, for every object c of $\boldsymbol{C}$ and every morphism $\varphi\colon h(c) \to u^*h(d)$, the base change

$$u^*R' \times_{u^*h(d)} h(c) \hookrightarrow h(c)$$

is a $\tau_{\boldsymbol{C}}$–covering sieve. By adjunction, φ factors through the unit $h(c) \to u^*u_!h(c) = u^*h(u(c))$ and the morphism $u^*(\varphi^\flat)\colon u^*h(u(c)) \to u^*h(d)$, where $\varphi^\flat$ is the left transpose of φ. The base change of φ by $u^*h(u(c)) \to u^*h(d)$ is identified, since u^* commutes with limits, with the canonical morphism

$$u^*\big(R' \times_{h(d)} h(u(c))\big) \to u^*h(u(c)).$$

Since the sieve $R' \times_{h(d)} h(u(c)) \hookrightarrow h(u(c))$ is $\tau_{\boldsymbol{D}}$–covering, as the base change of a $\tau_{\boldsymbol{D}}$–covering sieve, the conclusion now follows from the condition (COC′).

The second assertion is a formal consequence of the first, using the fact that every $\tau_{\boldsymbol{D}}$–sheaf is reduced (by assumption). □

Lemma 3.1.7 *Let $i\colon Z \hookrightarrow S$ be a closed immersion of spectral algebraic spaces. Then the base change functor $\mathrm{Sm}_{/S} \to \mathrm{Sm}_{/Z}$ is topologically quasicocontinuous* (*with respect to the Nisnevich topology*).

Proof Unravelling the definition, this amounts to the following assertion:

(∗) For any $X \in \mathrm{Sm}_{/S}$ and any Nisnevich covering sieve R' of $X \times_S Z$, let $R \hookrightarrow \mathrm{h}_S(X)$ denote the sieve generated by morphisms $X' \to X$ such that either (i) the empty sieve on $X' \times_S Z$ is Nisnevich-covering, or (ii) $X' \times_S Z \to X \times_S Z$ factors through R'. Then $R \hookrightarrow \mathrm{h}_S(X)$ is Nisnevich-covering.

This follows directly from Proposition 3.1.4, which says that étale morphisms can be lifted (Nisnevich-locally) along i. □

Proof of Theorem 3.1.1 This follows from Lemmas 3.1.6 and 3.1.7. □

3.2 The localization theorem

We now state the main result of this paper, and explain some of its immediate consequences.

Construction 3.2.1 Let $i\colon Z \hookrightarrow S$ be a closed immersion of spectral algebraic spaces with quasicompact open complement $j\colon U \hookrightarrow S$. Given a motivic space $\mathcal{F} \in \boldsymbol{H}(S)$, consider the tautologically commuting square

$$\begin{tikzcd} j_\sharp^{\boldsymbol{H}} j_{\boldsymbol{H}}^*(\mathcal{F}) \arrow[r, "\text{counit}"] \arrow[d, "\text{unit}"] & \mathcal{F} \arrow[d, "\text{unit}"] \\ j_\sharp^{\boldsymbol{H}} j_{\boldsymbol{H}}^* i_*^{\boldsymbol{H}} i_{\boldsymbol{H}}^*(\mathcal{F}) \arrow[r, "\text{counit}"] & i_*^{\boldsymbol{H}} i_{\boldsymbol{H}}^*(\mathcal{F}) \end{tikzcd}$$

Up to the canonical identification $j_{\boldsymbol{H}}^* i_*^{\boldsymbol{H}} \simeq \mathrm{pt}_U$ (Corollary 2.5.11), this square is identified with a canonical commutative square

$$\begin{tikzcd} j_\sharp^{\boldsymbol{H}} j_{\boldsymbol{H}}^*(\mathcal{F}) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ j_\sharp^{\boldsymbol{H}}(\mathrm{pt}_U) \arrow[r] & i_*^{\boldsymbol{H}} i_{\boldsymbol{H}}^*(\mathcal{F}) \end{tikzcd} \tag{3-1}$$

that we call the *localization square* associated to i.

Theorem 3.2.2 (localization) *Let $i\colon Z \hookrightarrow S$ be a closed immersion of spectral algebraic spaces with quasicompact open complement $j\colon U \hookrightarrow S$. Then, for every motivic space $\mathcal{F} \in \boldsymbol{H}(S)$, the localization square (3-1) is cocartesian.*

The proof of Theorem 3.2.2 will be carried out in Section 4. Here we record a few of its consequences.

Corollary 3.2.3 *Let $i\colon Z \hookrightarrow S$ be a closed immersion of spectral algebraic spaces with quasicompact open complement $j\colon U \hookrightarrow S$. For any pointed motivic space $\mathcal{F} \in \boldsymbol{H}(S)_\bullet$, there is a cofibre sequence*

$$j_\sharp j^*(\mathcal{F}) \to \mathcal{F} \to i_* i^*(\mathcal{F}) \tag{3-2}$$

and a fibre sequence

$$i_* i^!(\mathcal{F}) \to \mathcal{F} \to j_* j^*(\mathcal{F}) \tag{3-3}$$

in $\boldsymbol{H}(S)_\bullet$.

Proof The claim is that the commutative square

$$\begin{tikzcd} j_\sharp^{\boldsymbol{H}_\bullet} j^*_{\boldsymbol{H}_\bullet}(\mathcal{F}) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathrm{pt}_S \arrow[r] & i_*^{\boldsymbol{H}_\bullet} i^*_{\boldsymbol{H}_\bullet}(\mathcal{F}) \end{tikzcd}$$

is cocartesian in $\boldsymbol{H}(S)_\bullet$. Since the forgetful functor $\boldsymbol{H}(S)_\bullet \to \boldsymbol{H}(S)$ reflects contractible colimits, it will suffice to show that the induced square of underlying motivic spaces

$$\begin{tikzcd} j_\sharp^{\boldsymbol{H}} j^*_{\boldsymbol{H}}(\mathcal{F}) \sqcup_{j_\sharp^{\boldsymbol{H}} j^*_{\boldsymbol{H}}(\mathrm{pt}_S)} \mathrm{pt}_S \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathrm{pt}_S \arrow[r] & i_*^{\boldsymbol{H}} i^*_{\boldsymbol{H}}(\mathcal{F}) \end{tikzcd}$$

is cocartesian. By Theorem 3.2.2, the composite square

$$\begin{tikzcd} j_\sharp^{\boldsymbol{H}} j^*_{\boldsymbol{H}}\mathcal{F} \arrow[r] \arrow[d] & (j_\sharp^{\boldsymbol{H}} j^*_{\boldsymbol{H}}\mathcal{F}) \sqcup_{j_\sharp^{\boldsymbol{H}} j^*_{\boldsymbol{H}}(\mathrm{pt}_S)} \mathrm{pt}_S \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ j_\sharp^{\boldsymbol{H}} j^*_{\boldsymbol{H}}(\mathrm{pt}_S) \arrow[r] & \mathrm{pt}_S \arrow[r] & i_*^{\boldsymbol{H}} i^*_{\boldsymbol{H}}\mathcal{F} \end{tikzcd}$$

is cocartesian. Since the left-hand square is evidently cocartesian, it follows that the right-hand square is also cocartesian. □

The following reformulation of the localization theorem is an analogue of Kashiwara's lemma in the setting of D–modules:

Theorem 3.2.4 *Let $i\colon Z \hookrightarrow S$ be a closed immersion of spectral algebraic spaces with quasicompact open complement. Then the direct image functor $i_*^{\boldsymbol{H}}$ is fully faithful, and its essential image is spanned by objects $\mathcal{F} \in \boldsymbol{H}(S)$ whose restriction $j^*_{\boldsymbol{H}}(\mathcal{F}) \in \boldsymbol{H}(U)$ is contractible.*

Proof First we show that i_* is fully faithful. An application of Theorem 3.2.2 to the motivic space $i_*(\mathcal{F}) \in \boldsymbol{H}(S)$ shows that the counit induces an isomorphism $i_* i^* i_* \to i_*$. By a standard argument it therefore suffices to show that i_* is conservative. For this let $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism in $\boldsymbol{H}(Z)$ such that $i_*(\varphi)$ is invertible. To show that φ is invertible, it will suffice to show that

$$\Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_2)$$

is invertible for each $X \in \mathrm{Sm}_{/Z}$. By Proposition 3.1.4, we may assume that X is the base change of some $Y \in \mathrm{Sm}_{/S}$. Then we have natural isomorphisms $\Gamma(X, \mathcal{F}_k) \simeq \Gamma(Y, i_*(\mathcal{F}_k))$ for each k, so the claim follows.

Next we identify the essential image of i_*. Suppose that $\mathcal{F} \in \boldsymbol{H}(S)$ with $j^*(\mathcal{F})$ contractible. Then Theorem 3.2.2 yields that the unit map $\mathcal{F} \to i_* i^*(\mathcal{F})$ is invertible, so that $\mathcal{F}$ belongs to the essential image of i_*. The other inclusion follows from Corollary 2.5.11. □

Remark 3.2.5 We can use the localization theorem to give a concrete description of the abstractly defined functor $i^!_{\boldsymbol{H}_\bullet}$. Namely, it is given by

$$i^!_{\boldsymbol{H}_\bullet}(\mathcal{F}) \simeq \mathrm{Fib}(i^*(\mathcal{F}) \to i^* j_* j^*(\mathcal{F}))$$

for any $\mathcal{F} \in \boldsymbol{H}(Z)_\bullet$.

Corollary 3.2.6 (nilpotent-invariance) *Let $i\colon S_0 \hookrightarrow S$ be a closed immersion of spectral algebraic spaces. Suppose that i induces an isomorphism $(S_0)_{\mathrm{cl,red}} \simeq S_{\mathrm{cl,red}}$ of reduced classical algebraic spaces. Then the functors $i^*_{\boldsymbol{H}}$ and $i^{\boldsymbol{H}}_*$ are mutually inverse equivalences.*

Proof Since the complement of i is empty, this follows from Theorem 3.2.4. □

Corollary 3.2.7 *Let S be a spectral algebraic space, and let $i\colon S_{\mathrm{cl}} \hookrightarrow S$ denote the inclusion of its underlying classical algebraic space* (viewed as a discrete spectral algebraic space)*. Then the functors $i^*_{\boldsymbol{H}}$ and $i^{\boldsymbol{H}}_*$ define mutually inverse equivalences $\boldsymbol{H}(S) \simeq \boldsymbol{H}(S_{\mathrm{cl}})$.*

Warning 3.2.8 Corollary 3.2.7 does not assert that $\boldsymbol{H}(S)$ is equivalent to the classical motivic homotopy category over S_{cl}, denoted by $\boldsymbol{H}_{\mathrm{cl}}(S)$ in Proposition 2.4.6.

Corollary 3.2.9 *Let R be a connective $\mathcal{E}_\infty$–ring and consider the affine spectral scheme $S = \operatorname{Spec}(R)$. Denote by $i\colon S_{\mathrm{cl}} \to S$ the inclusion of the underlying classical scheme. Then the functor $i_*^{\boldsymbol{H}}$ induces an equivalence from the ∞–category of $\boldsymbol{A}^1$–homotopy-invariant Nisnevich sheaves on $\mathrm{Sm}^{\mathrm{aff}}_{/S_{\mathrm{cl}}}$ to the ∞–category of $\boldsymbol{A}^1$–homotopy-invariant Nisnevich sheaves of spaces on $\mathrm{Sm}^{\mathrm{aff}}_{/S}$.*

Proof Combine Corollary 3.2.7 with Corollary 2.4.5. □

3.3 Closed base change formula

Construction 3.3.1 Consider a cartesian square

$$\begin{tikzcd} Y \arrow[r, hook, "k"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Z \arrow[r, hook, "i"] & S \end{tikzcd}$$

of spectral algebraic spaces, where i and k are closed immersions with quasicompact open complements. Then the square

$$\begin{tikzcd} \boldsymbol{H}(Z) \arrow[r, "i_*"] \arrow[d, "g^*"] & \boldsymbol{H}(S) \arrow[d, "f^*"] \\ \boldsymbol{H}(Y) \arrow[r, "k_*"] & \boldsymbol{H}(X) \end{tikzcd}$$

commutes up to a natural transformation

$$f^* i_* \xrightarrow{\text{unit}} k_* k^* f^* i_* \simeq k_* g^* i^* i_* \xrightarrow{\text{counit}} k_* g^*.$$

Proposition 3.3.2 *Suppose given a cartesian square of spectral algebraic spaces as above, with i and k closed immersions with quasicompact open complements. Then the canonical natural transformation*

$$f^*_{\boldsymbol{H}} i_*^{\boldsymbol{H}} \to k_*^{\boldsymbol{H}} g^*_{\boldsymbol{H}}$$

is invertible.

Proof Since i_* is fully faithful (Theorem 3.2.4) it suffices to show that the natural transformation

$$f^* i_* i^* \to k_* g^* i^*$$

is invertible. This follows by a straightforward application of the localization theorem (Theorem 3.2.2) for the closed immersions i and k, respectively, using the smooth base change formula (Proposition 2.5.9) for the open complements. □

Recall that in the pointed setting, the functor $i_*^{\boldsymbol{H}_\bullet}\colon \boldsymbol{H}(Z)_\bullet \to \boldsymbol{H}(S)_\bullet$ admits a right adjoint $i^!_{\boldsymbol{H}_\bullet}$ (Corollary 3.1.2).

Corollary 3.3.3 *Suppose given a cartesian square of spectral algebraic spaces as above, with i and k closed immersions with quasicompact open complements. Then the natural transformations*

$$k_*^{\boldsymbol{H}_\bullet} g^*_{\boldsymbol{H}_\bullet} \to f^*_{\boldsymbol{H}_\bullet} i_*^{\boldsymbol{H}_\bullet}, \quad i^!_{\boldsymbol{H}_\bullet} f_*^{\boldsymbol{H}_\bullet} \to g_*^{\boldsymbol{H}_\bullet} k^!_{\boldsymbol{H}_\bullet}$$

are invertible.

Proof The first follows immediately from the unpointed version (Proposition 3.3.2), and the second follows from the first by passing to right adjoints. □

3.4 Closed projection formula

Construction 3.4.1 Let $i\colon Z \hookrightarrow S$ be a closed immersion with quasicompact open complement. Given pointed motivic spaces $\mathcal{F}' \in \boldsymbol{H}(Z)_\bullet$ and $\mathcal{F} \in \boldsymbol{H}(S)_\bullet$, we get a morphism

$$i^*_{\boldsymbol{H}_\bullet}(i_*^{\boldsymbol{H}_\bullet}(\mathcal{F}') \wedge \mathcal{F}) \simeq i^*_{\boldsymbol{H}_\bullet} i_*^{\boldsymbol{H}_\bullet}(\mathcal{F}') \wedge i^*_{\boldsymbol{H}_\bullet}(\mathcal{F}) \xrightarrow{\text{counit}} \mathcal{F}' \wedge i^*_{\boldsymbol{H}_\bullet}(\mathcal{F}),$$

which corresponds by adjunction to a canonical morphism

$$i_*^{\boldsymbol{H}_\bullet}(\mathcal{F}') \wedge \mathcal{F} \to i_*^{\boldsymbol{H}_\bullet}(\mathcal{F}' \wedge i^*_{\boldsymbol{H}_\bullet}(\mathcal{F})).$$

Proposition 3.4.2 *Let $i\colon Z \hookrightarrow S$ be a closed immersion with quasicompact open complement. Given pointed motivic spaces $\mathcal{F}' \in \boldsymbol{H}(Z)_\bullet$, $\mathcal{F} \in \boldsymbol{H}(S)_\bullet$, and $\mathcal{G} \in \boldsymbol{H}(S)_\bullet$, there are canonical bifunctorial isomorphisms*

$$i_*^{\boldsymbol{H}_\bullet}(\mathcal{F}') \wedge \mathcal{F} \to i_*^{\boldsymbol{H}_\bullet}(\mathcal{F}' \wedge i^*_{\boldsymbol{H}_\bullet}(\mathcal{F})), \quad i^!_{\boldsymbol{H}_\bullet} \underline{\mathrm{Hom}}_S(\mathcal{G}, \mathcal{F}) \to \underline{\mathrm{Hom}}_Z(i^*_{\boldsymbol{H}_\bullet}(\mathcal{G}), i^!_{\boldsymbol{H}_\bullet}(\mathcal{F})).$$

Proof The second statement follows from the first by passing to right adjoints. For the first, it suffices by fully faithfulness of i_* (Theorem 3.2.4) to show that the morphism

$$i_*(i^*\mathcal{F}) \wedge \mathcal{G} \to i_*(i^*\mathcal{F} \wedge i^*\mathcal{G})$$

is invertible for all pointed motivic spaces $\mathcal{F}, \mathcal{G} \in \boldsymbol{H}(S)_\bullet$. This follows from the localization theorem (Corollary 3.2.3), using the smooth projection formula (Proposition 2.5.13) for the open complement $j\colon U \hookrightarrow S$. □

3.5 Smooth-closed base change formula

Construction 3.5.1 Consider a cartesian square of spectral algebraic spaces

$$\begin{tikzcd} Y \arrow[r, hook, "k"] \arrow[d, "q"] & X \arrow[d, "p"] \\ Z \arrow[r, hook, "i"] & S \end{tikzcd}$$

where i and k are closed immersions with quasicompact open complements, and p and q are smooth. Then by Proposition 2.5.9 it follows that the square

$$\begin{tikzcd} \boldsymbol{H}(Y)_\bullet \arrow[r, "k_*"] \arrow[d, "q_\sharp"] & \boldsymbol{H}(X)_\bullet \arrow[d, "p_\sharp"] \\ \boldsymbol{H}(Z)_\bullet \arrow[r, "i_*"] & \boldsymbol{H}(S)_\bullet \end{tikzcd}$$

commutes up to a natural transformation

$$p_\sharp k_* \xrightarrow{\text{unit}} i_* i^* p_\sharp k_* \simeq i_* q_\sharp k^* k_* \xrightarrow{\text{counit}} i_* q_\sharp.$$

Proposition 3.5.2 *Suppose given a cartesian square of spectral algebraic spaces as above, where i and k are closed immersions with quasicompact open complements, and p and q are smooth. Then the canonical natural transformations*

$$p_\sharp^{\boldsymbol{H}_\bullet} k_*^{\boldsymbol{H}_\bullet} \to i_*^{\boldsymbol{H}_\bullet} q_\sharp^{\boldsymbol{H}_\bullet}, \quad q^*_{\boldsymbol{H}_\bullet} i^!_{\boldsymbol{H}_\bullet} \to k^!_{\boldsymbol{H}_\bullet} p^*_{\boldsymbol{H}_\bullet}$$

are invertible.

Proof The second transformation is obtained from the first by passing to right adjoints. Since the direct image functor k_* is fully faithful (Theorem 3.2.4), it suffices to show that the transformation $p_\sharp k_* k^* \to i_* q_\sharp k^*$, obtained by precomposition with k^*, is invertible. This follows directly from the localization theorem (Corollary 3.2.3) and the smooth base change formula (Proposition 2.5.9). □

4 Proof of the localization theorem

This section is dedicated to the proof of our main result, Theorem 3.2.2. For the duration of the section, we fix a closed immersion of spectral algebraic spaces $i: Z \hookrightarrow S$ with quasicompact open complement $j: U \hookrightarrow S$. Given $X \in \mathrm{Sm}_{/S}$, we will use the notation $X_U := X \times_S U \in \mathrm{Sm}_{/U}$ and $X_Z := X \times_S Z \in \mathrm{Sm}_{/Z}$.

4.1 The space of Z–trivialized maps

In this subsection we formulate Proposition 4.1.6, which aside from Theorem 3.1.1 is the main input that goes into the proof of the localization theorem; Section 4.2 will be dedicated to its proof.

Construction 4.1.1 Let $X \in \mathrm{Sm}_{/S}$ and denote by $\mathrm{h}^Z_S(X) \in \mathrm{Spc}(S)$ the fibred space

$$\mathrm{h}^Z_S(X) := \mathrm{h}_S(X) \sqcup_{\mathrm{h}_S(X_U)} \mathrm{h}_S(U).$$

Note that for $Y \in \mathrm{Sm}_{/S}$, the space $\Gamma(Y, \mathrm{h}_S(U))$ is either empty or contractible depending on whether $Y_Z \in \mathrm{Sm}_{/Z}$ is empty. It follows that the space of sections $\Gamma(Y, \mathrm{h}^Z_S(X))$ is contractible when Y_Z is empty, and otherwise is given by the mapping space $\mathrm{Maps}_S(Y, X)$.

Remark 4.1.2 There is a canonical identification $i^*_{\mathrm{Spc}}(\mathrm{h}^Z_S(X)) \simeq \mathrm{h}_Z(X_Z)$ (since i^*_{Spc} commutes with colimits).

Construction 4.1.3 Let $X \in \mathrm{Sm}_{/S}$ and $t\colon Z \hookrightarrow X$ an S–morphism, ie a partially defined section of $X \to S$. Then t corresponds by adjunction to a canonical morphism $\tau\colon \mathrm{pt}_S = \mathrm{h}_S(S) \to i^{\mathrm{Spc}}_*(\mathrm{h}_S(X_Z))$, and we define the fibred space $\mathrm{h}_S(X, t) \in \mathrm{Spc}(S)$ as the fibre of the unit map

$$\mathrm{h}^Z_S(X) \to i^{\mathrm{Spc}}_* i^*_{\mathrm{Spc}}(\mathrm{h}^Z_S(X)) \simeq i^{\mathrm{Spc}}_*(\mathrm{h}_Z(X_Z)) \tag{4-1}$$

at the point τ. Thus for any $Y \in \mathrm{Sm}_{/S}$, the space $\Gamma(Y, \mathrm{h}_S(X, t))$ is contractible when Y_Z is empty, and otherwise is given by the fibre of the restriction map

$$\mathrm{Maps}_S(Y, X) \to \mathrm{Maps}_Z(Y_Z, X_Z)$$

at the point defined by the composite $Y_Z \to Z \xhookrightarrow{t} X_Z$.

Remark 4.1.4 Informally speaking, points of the space $\Gamma(Y, \mathrm{h}_S(X, t))$ (when Y_Z is nonempty) are pairs (f, α), with $f\colon Y \to X$ an S–morphism and α a commutative triangle

$$\begin{tikzcd} Y_Z \arrow[r, "f_Z"] \arrow[d] & X_Z \\ Z \arrow[ur, hook, "t"'] & \end{tikzcd}$$

We refer to α informally as a *Z–trivialization* of f.

Remark 4.1.5 For a smooth morphism $p\colon T \to S$, there is a canonical isomorphism

$$p^*_{\mathrm{Spc}}(\mathrm{h}_S(X,t)) \simeq \mathrm{h}_T(X_T, t_T),$$

where $t_T\colon Z_T \hookrightarrow X_T$ is the base change of t along p. This follows from the fact that the functor p^*_{Spc} commutes with limits and colimits.

We will deduce Theorem 3.2.2 from the following result:

Proposition 4.1.6 *Let $Z \hookrightarrow S$ be a closed immersion of spectral algebraic spaces with quasicompact open complement. Let $p\colon X \to S$ be a smooth morphism of spectral algebraic spaces, and $t\colon Z \hookrightarrow X$ an S–morphism. Then Nisnevich-locally on X, the space $\mathrm{h}_S(X,t)$ is motivically contractible. That is, the morphism $\mathrm{h}_S(X,t) \to \mathrm{pt}_S$ is a motivic equivalence.*

The proof will be completed in Section 4.2.

4.2 Motivic contractibility of $\mathbf{h}_S(X,t)$

In this subsection we prove Proposition 4.1.6. We will need the local structure theory of quasismooth closed immersions.

Proposition 4.2.1 *Let $k\colon Y \hookrightarrow X$ be a closed immersion of spectral algebraic spaces. Let $\mathcal{L}_{Y/X}$ denote the relative cotangent complex. Then the following are equivalent:*

(i) *The morphism k is locally of finite presentation, and the quasicoherent $\mathcal{O}_Y$–module $\mathcal{L}_{Y/X}[-1]$ is locally free of finite rank.*

(ii) *The morphism k is almost of finite presentation, and the quasicoherent $\mathcal{O}_Y$–module $\mathcal{L}_{Y/X}[-1]$ is locally free of finite rank.*

(iii) *The morphism of underlying classical schemes $k_{\mathrm{cl}}\colon Y_{\mathrm{cl}} \hookrightarrow X_{\mathrm{cl}}$ is of finite presentation, and the quasicoherent $\mathcal{O}_Y$–module $\mathcal{L}_{Y/X}[-1]$ is locally free of finite rank.*

(iv) *Nisnevich-locally on X, there exists a morphism $f\colon X \to \boldsymbol{A}^n_{\boldsymbol{S}}$ fitting in a cartesian square*

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(\boldsymbol{S}) \arrow[r, "\{0\}"] & \boldsymbol{A}^n_{\boldsymbol{S}} \end{tikzcd}$$

where the lower horizontal arrow is the inclusion of the origin in spectral affine space.

Definition 4.2.2 If $k\colon Y \hookrightarrow X$ satisfies one of the equivalent conditions of Proposition 4.2.1, then we say that it is *quasismooth*, and write $\mathcal{N}_{Y/X} := \mathcal{L}_{Y/X}[-1]$ for its *conormal sheaf*.

Proof of Proposition 4.2.1 The claim being local, we may assume that X is affine and that $\mathcal{L}_{Y/X}[-1]$ is free of rank $n \geqslant 0$. The equivalence between the first three conditions follows from [12, Theorem 7.4.3.18]. The implication (iv) $\Longrightarrow$ (i) is obvious. Suppose that (i) holds and choose a basis $df_1, \dots, df_n$ for $\Gamma(Y, \mathcal{L}_{Y/X}[-1])$. If $\mathcal{I}$ denotes the fibre of the morphism $\mathcal{O}_X \to k_*\mathcal{O}_Y$, then there is a canonical isomorphism $\pi_0(k^*\mathcal{I}) \simeq \pi_1(\mathcal{L}_{Y/X})$ of $\pi_0(\mathcal{O}_Y)$–modules [12, Theorem 7.4.3.1]. The global sections df_i correspond to global sections $\overline{f_i}$ of $k^*(\mathcal{I})$, which we can lift along the surjection $\mathcal{I} \to k_*k^*(\mathcal{I})$ to global sections f_i of $\mathcal{I}$. By Nakayama's lemma we may assume that these f_i generate $\pi_0(\mathcal{I})$ as a $\pi_0(\mathcal{O}_X)$–module. Therefore, they determine a morphism $f\colon X \to \boldsymbol{A}^n_{\boldsymbol{S}}$ and a commutative square

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec}(\boldsymbol{S}) \arrow[r, "\{0\}"] & \boldsymbol{A}^n_{\boldsymbol{S}} \end{tikzcd}$$

which is cartesian on underlying classical schemes. To show that it is cartesian itself, it will suffice by [12, Corollary 7.4.3.4] to show that the relative cotangent complex $\mathcal{L}_{Y/V}$ of the morphism $Y \to V := X \times_{\boldsymbol{A}^n_{\boldsymbol{S}}} \operatorname{Spec}(\boldsymbol{S})$ vanishes. But this follows immediately from an inspection of the exact triangle

$$\mathcal{L}_{V/X}|Y \to \mathcal{L}_{Y/X} \to \mathcal{L}_{Y/V},$$

where both first terms are isomorphic to a shifted free module $\mathcal{O}_Y^{\oplus n}[1]$. $\square$

We will actually use a slight variant of Proposition 4.2.1, which concerns the case of a smooth morphism admitting a globally defined section (such a section is automatically quasismooth).

Lemma 4.2.3 *Let $p\colon X \to S$ be a smooth morphism of spectral algebraic spaces, and suppose it admits a section $s\colon S \hookrightarrow X$. Then, Nisnevich-locally on X, there exists an S–morphism $f\colon X \to \boldsymbol{A}^n_S$ fitting in a cartesian square*

$$\begin{tikzcd} S \arrow[r, hook, "s"] \arrow[d, equal] & X \arrow[d, "f"] \\ S \arrow[r, hook, "\{0\}"] & \boldsymbol{A}^n_S \end{tikzcd}$$

Moreover, the morphism f is étale on some Zariski neighbourhood of s.

Proof There is a canonical isomorphism $\mathcal{L}_{S/X}[-1] \simeq \mathcal{L}_{X/S}|S$. By the assumption, $\mathcal{L}_{X/S}$ is free of rank n, so the same holds for the $\mathcal{O}_S$–module $\mathcal{N}_{S/X} = \mathcal{L}_{S/X}[-1]$. Then the first claim is proven exactly as the implication (i) $\Longrightarrow$ (iv) of Proposition 4.2.1. For the second, note that the canonical isomorphism $s^*\mathcal{L}_{X/\boldsymbol{A}^n_S} \simeq \mathcal{L}_{S/S} \simeq 0$ shows that f is étale on the image of s (since f is of finite presentation). In other words, s factors through the étale locus of f. □

Next we apply the structure theory to lift partially defined sections (in a weak sense).

Lemma 4.2.4 *Let $i\colon Z \hookrightarrow S$ be a closed immersion of spectral algebraic spaces, $p\colon X \to S$ a smooth morphism of spectral algebraic spaces, and $t\colon Z \hookrightarrow X$ an S–morphism. Then, Nisnevich-locally on X, there exists a spectral algebraic space Y over X such that the composite $Y \to S$ is étale, and induces an isomorphism $Y_Z \to Z$:*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "t"] & Y \arrow[d] \\ X_Z \arrow[r, "i'"] \arrow[d] & X \arrow[d, "p"] \\ Z \arrow[r, "i"] & S \end{tikzcd}$$

Proof Applying Lemma 4.2.3 to the smooth morphism $X_Z \to Z$ with section $Z \hookrightarrow X_Z$ induced by t, we obtain a cartesian square

$$\begin{tikzcd} Z \arrow[r, "t"] \arrow[d, equal] & X_Z \arrow[d, "f"] \\ Z \arrow[r, "\{0\}"] & \boldsymbol{A}^n_Z \end{tikzcd}$$

The morphism f is determined by a set of global sections $f_1, \dots, f_n$ of $\mathcal{O}_{X_Z}$; lifting them along the surjection $\mathcal{O}_X \to i'_*(\mathcal{O}_{X_Z})$, we obtain global sections g_i of $\mathcal{O}_X$ which determine a morphism $g\colon X \to \boldsymbol{A}^n_S$. We define $Y_0 := X \times_{\boldsymbol{A}^n_S} S$ so that we have a commutative square

$$\begin{tikzcd} Z \arrow[r, "i''"] \arrow[d] & Y_0 \arrow[d] \\ X_Z \arrow[r, "i'"] & X \end{tikzcd}$$

where the morphism $Z \to (Y_0)_Z$ is an isomorphism on underlying classical schemes. The exact triangle

$$\mathcal{L}_{(Y_0)_Z/Y_0}|Z \to \mathcal{L}_{Z/Y_0} \to \mathcal{L}_{Z/(Y_0)_Z}$$

shows that $\mathcal{L}_{Z/(Y_0)_Z}$ vanishes, so it follows from [12, Corollary 7.4.3.4] that the square is cartesian. Then we have a canonical isomorphism $(i'')^*\mathcal{L}_{Y_0/S} \simeq \mathcal{L}_{Z/Z} \simeq 0$, which shows that $Y_0 \to S$ is étale on the image of Z. Thus we may take $Y \hookrightarrow Y_0$ to be the étale locus of $Y_0 \to S$ to conclude. □

We are now ready to return to our study of the space of Z–trivialized maps. We first deal with the case of vector bundles:

Lemma 4.2.5 *Let E be a vector bundle over S with zero section $s\colon S \hookrightarrow E$. Then the space $\mathrm{h}_S(E, s_Z)$ is motivically contractible, where $s_Z\colon Z \hookrightarrow E_Z$ denotes the base change of s along $i\colon Z \hookrightarrow S$.*

Proof The map $\varphi\colon \mathrm{h}_S(E, s_Z) \to \mathrm{pt}_S$ admits a section $\sigma\colon \mathrm{pt}_S \to \mathrm{h}_S(E, s_Z)$, induced by the composite $\mathrm{h}_S(S) \xrightarrow{s} \mathrm{h}_S(E) \to \mathrm{h}_S^Z(E)$. It will suffice to exhibit an $\boldsymbol{A}^1$–homotopy

$$\gamma\colon \mathrm{h}_S(\boldsymbol{A}^1_S) \times \mathrm{h}_S(E, s_Z) \to \mathrm{h}_S(E, s_Z)$$

between $\sigma \circ \varphi$ and the identity. The canonical action of $\boldsymbol{A}^1_S$ on E gives rise to the vertical maps in the commutative square

$$\begin{tikzcd}
\mathrm{h}_S(\boldsymbol{A}^1_S) \times \mathrm{h}_S^Z(E) \arrow[r] \arrow[d] & \mathrm{h}_S(\boldsymbol{A}^1_S) \times i_*^{\mathrm{Spc}}(\mathrm{h}_Z(E_Z)) \arrow[d] \\
\mathrm{h}_S^Z(E) \arrow[r] & i_*^{\mathrm{Spc}}(\mathrm{h}_Z(E_Z))
\end{tikzcd}$$

The homotopy γ is the map induced on fibres (given informally by the assignment $(\lambda\colon Y \to \boldsymbol{A}^1_S,\ f\colon Y \to E) \mapsto (\lambda \cdot f\colon Y \to E)$ on Y–sections). □

Our final ingredient is a certain invariance property for the construction $\mathrm{h}_S(X, t)$:

Lemma 4.2.6 *Let X and X' be smooth spectral algebraic spaces over S, and let $t\colon Z \hookrightarrow X$ and $t'\colon Z \hookrightarrow X'$ be S–morphisms. Suppose $f\colon X' \to X$ is an étale S–morphism such that the square*

$$\begin{tikzcd}
Z \arrow[r, hook, "t'"] \arrow[d, equal] & X'_Z \arrow[d, "f_Z"] \\
Z \arrow[r, hook, "t"] & X_Z
\end{tikzcd}$$

is cartesian. Then the induced morphism of fibred spaces

$$\varphi\colon \mathrm{h}_S(X', t') \to \mathrm{h}_S(X, t)$$

is a Nisnevich-local equivalence.

Proof The claim is that the induced morphism of Nisnevich sheaves $\mathrm{L}_{\mathrm{Nis}}(\varphi)$ is invertible, so it will suffice to show that it is 0–truncated (ie its diagonal is a monomorphism) and 0–connected (ie it is an effective epimorphism and so is its diagonal).

Step 1 To show that $\mathrm{L}_{\mathrm{Nis}}(\varphi)$ is 0–truncated, it suffices to show that φ is already 0–truncated (since $\mathrm{L}_{\mathrm{Nis}}$ is exact). For this, it suffices to show that for every $Y \in \mathrm{Sm}_{/S}$, the induced morphism of spaces of Y–sections

$$\Gamma(Y, \varphi)\colon \Gamma(Y, \mathrm{h}_S^Z(X', t')) \to \Gamma(Y, \mathrm{h}_S^Z(X, t))$$

is 0–truncated. We may assume Y_Z is not empty; then this is the morphism induced on fibres in the diagram

$$\begin{tikzcd}
\Gamma(Y, \mathrm{h}_S^Z(X', t')) \arrow[r] \arrow[d, dashed] & \mathrm{Maps}_S(Y, X') \arrow[r] \arrow[d] & \mathrm{Maps}_Z(Y_Z, X'_Z) \arrow[d] \\
\Gamma(Y, \mathrm{h}_S^Z(X, t)) \arrow[r] & \mathrm{Maps}_S(Y, X) \arrow[r] & \mathrm{Maps}_Z(Y_Z, X_Z)
\end{tikzcd}$$

Note that the two right-hand vertical morphisms are 0–truncated: p is itself 0–truncated since it is étale, and since the Yoneda embedding commutes with limits, the induced morphism $\mathrm{h}_S(X') \to \mathrm{h}_S(X)$ is also 0–truncated. It follows that the left-hand vertical morphism is also 0–truncated for each Y.

Step 2 To show that $\mathrm{L}_{\mathrm{Nis}}(\varphi)$ is an effective epimorphism, it suffices to show that for every $Y \in \mathrm{Sm}_{/S}$ (with Y_Z not empty), any Y–section of $\mathrm{h}_S^Z(X, t)$ can be lifted Nisnevich-locally along φ. Let f be a Y–section of $\mathrm{h}_S^Z(X, t)$, ie a Z–trivialized morphism $f\colon Y \to X$. Let $q\colon Y' \to Y$ denote the base change of $p\colon X' \to X$ along f:

$$\begin{tikzcd}
Y' \arrow[r, "q"] \arrow[d, "g"] & Y \arrow[d, "f"] \\
X' \arrow[r, "p"] & X
\end{tikzcd}$$

Then note that

$$\begin{tikzcd}
q^{-1}(Y_U) \arrow[r, hook] \arrow[d] & Y' \arrow[d, "q"] \\
Y_U \arrow[r, hook] & Y
\end{tikzcd}$$

is a Nisnevich square. Indeed, the closed immersion $Y_Z \hookrightarrow Y$ is complementary to $Y_U \hookrightarrow Y$, and it is clear that $q^{-1}(Y_Z) \to Y_Z$ is invertible because in the diagram

$$\begin{tikzcd} q^{-1}(Y_Z) \arrow[r] \arrow[d] & Y_Z \arrow[d] \\ Z \arrow[r, "\mathrm{id}_Z"] \arrow[d, hook, "t"] & Z \arrow[d, hook, "t'"] \\ X'_Z \arrow[r, "p_Z"] & X_Z \end{tikzcd}$$

the lower square and the composite square are cartesian, and hence so is the upper square. It now suffices to show that the restriction of f to either component of this Nisnevich cover lifts to $\mathrm{h}_S^Z(X', t')$. The restriction $f|Y'$ lifts to a section of $\mathrm{h}_S^Z(X', t')$ given by $g\colon Y' \to X'$. The restriction $f|Y_U$ admits a lift tautologically: since $(Y_U) \times_S Z = \varnothing$, the spaces $\mathrm{h}_S^Z(X, t)(Y_U)$ and $\mathrm{h}_S^Z(X', t')(Y_U)$ are both contractible.

Step 3 It remains to show that the diagonal $\Delta_{\mathrm{L}_{\mathrm{Nis}}(\varphi)}$ of $\mathrm{L}_{\mathrm{Nis}}(\varphi)$ is an effective epimorphism, or equivalently that $\mathrm{L}_{\mathrm{Nis}}(\Delta_\varphi)$ is. For every $Y \in \mathrm{Sm}_{/S}$, the diagonal induces a morphism of spaces

$$\Gamma(Y, \mathrm{h}_S^Z(X', t')) \to \Gamma(Y, \mathrm{h}_S^Z(X', t')) \times_{\Gamma(Y, \mathrm{h}_S^Z(X, t))} \Gamma(Y, \mathrm{h}_S^Z(X', t')).$$

It suffices to show that for each Y (with Y_Z not empty), any point of the target lifts Nisnevich-locally to a point of the source. Choose a point of the target, given by two Z–trivialized morphisms $f\colon Y \to X'$ and $g\colon Y \to X'$, and an identification $\alpha\colon p \circ f \simeq p \circ g$. Let $Y_0 \hookrightarrow Y$ denote the open immersion defined as the equalizer of the pair (f, g); note that the closed immersion $Y_Z \hookrightarrow Y$ factors through Y_0. Thus the open immersions $Y_0 \hookrightarrow Y$ and $Y_U \hookrightarrow Y$ form a Zariski cover of Y. It is clear that the point (f, g, α) lifts after restriction to Y_0 by definition, and after restriction to Y_U since $Y_U \times_S Z = \varnothing$, so the claim follows. □

We are now ready to conclude the proof of Proposition 4.1.6.

Proof of Proposition 4.1.6 The question being local on X, we may assume by Lemma 4.2.4 that there exists a Nisnevich square

(4-2)
$$\begin{tikzcd} Y_U \arrow[r, hook] \arrow[d] & Y \arrow[d, "q"] \\ U \arrow[r, hook, "j"] & S \end{tikzcd}$$

where $j\colon U \hookrightarrow S$ is the complement of i, and q factors through $p\colon X \to S$. By the Nisnevich separation property (Proposition 2.5.7), it will suffice to show that $j^*\,\mathrm{h}_S(X,t)$ and $q^*\,\mathrm{h}_S(X,t)$ are motivically contractible. By Remark 4.1.5, we have $j^*\,\mathrm{h}_S(X,t) \simeq \mathrm{h}_U(X_U, t_U)$. But, since U is complementary to Z, t_U is the inclusion of the empty scheme, so $\mathrm{h}_U(X_U, t_U)$ is contractible by construction. Next consider $q^*\,\mathrm{h}_S(X,t) \simeq \mathrm{h}_Y(Y \times_S X, t')$, where $t'\colon Y_Z \hookrightarrow (Y \times_S X)_Z$ is the base change of t. By construction there exists a section $t''\colon Y \hookrightarrow Y \times_S X$ which lifts t' (since q factors through X). Hence, by Lemmas 4.2.3 and 4.2.6, we have a motivic equivalence

$$\mathrm{h}_Y(Y \times_S X, t') \simeq \mathrm{h}_Y(\boldsymbol{A}^n_Y, z),$$

where $z\colon Y_Z \hookrightarrow \boldsymbol{A}^n_{Y_Z}$ is the zero section. Now the claim follows from Lemma 4.2.5. □

4.3 The proof

We conclude this section by proving the localization theorem (Theorem 3.2.2). Let $i\colon Z \hookrightarrow S$ be a closed immersion of spectral algebraic spaces with quasicompact open complement $j\colon U \hookrightarrow S$. We wish to show that for every motivic space $\mathcal{F} \in \boldsymbol{H}(S)$, the canonical morphism

$$\mathcal{F} \sqcup_{j^{\boldsymbol{H}}_{\sharp}\, j^*_{\boldsymbol{H}}(\mathcal{F})} \mathrm{M}_S(U) \to i^{\boldsymbol{H}}_*\, i^*_{\boldsymbol{H}}(\mathcal{F}) \tag{4-3}$$

is invertible. In what follows below, we will omit the decorations $\boldsymbol{H}$.

Using Proposition 2.4.4 we may reduce to the case where $\mathcal{F}$ is the motivic localization $\mathrm{M}_S(X)$ of some affine $X \in \mathrm{Sm}_{/S}$ that admits an étale morphism $X \to \boldsymbol{A}^n_S$ for some $n \geqslant 0$, since all functors involved commute with sifted colimits (Theorem 3.1.1). In this case the morphism (4-3) is canonically identified with the morphism

$$\mathrm{M}_S(X) \sqcup_{\mathrm{M}_S(X_U)} \mathrm{M}_S(U) \to i^{\boldsymbol{H}}_*\, \mathrm{M}_Z(X_Z),$$

where we write $X_U = X \times_S U$ and $X_Z = X \times_S Z$. Note that the source of this morphism is the motivic localization of the space $\mathrm{h}^Z_S(X)$ (Construction 4.1.1). Hence it suffices to show that the morphism of fibred spaces

$$\mathrm{h}^Z_S(X) \to i^{\mathrm{Spc}}_*\, \mathrm{h}_Z(X_Z)$$

is a motivic equivalence. By universality of colimits, it suffices to show that for every $Y \in \mathrm{Sm}_{/S}$ and every morphism $\mathrm{h}_S(Y) \to i^{\mathrm{Spc}}_*\, \mathrm{h}_Z(X_Z)$, corresponding to an S–morphism $t\colon Z \to X$, the base change

$$\mathrm{h}^Z_S(X) \times_{i^{\mathrm{Spc}}_*\, \mathrm{h}_Z(X_Z)} \mathrm{h}_S(Y) \to \mathrm{h}_S(Y)$$

is invertible. If $p\colon Y \to S$ denotes the structural morphism, we have $\mathrm{h}_S(Y) \simeq p_\sharp^{\mathrm{Spc}} \mathrm{h}_Y(Y)$, so that this morphism is identified, by the smooth projection formula (Proposition 2.5.13), with a morphism

$$p_\sharp^{\mathrm{Spc}}(p_{\mathrm{Spc}}^* \mathrm{h}_S^Z(X) \times_{p_{\mathrm{Spc}}^* i_*^{\mathrm{Spc}} \mathrm{h}_Z(X_Z)} \mathrm{h}_Y(Y)) \to p_\sharp^{\mathrm{Spc}} \mathrm{h}_Y(Y).$$

If k (resp. q) denotes the base change of i (resp. p) along p (resp. i), then by Remark 4.1.5 and the smooth base change formula $p^* i_* \simeq k_* q^*$ (Proposition 2.5.9), we see that this morphism is the image by $p_\sharp$ of the morphism

$$\mathrm{h}_Y^{Y_Z}(X \times_S Y) \times_{k_*^{\mathrm{Spc}} \mathrm{h}_{Y_Z}((X \times_S Y)_Z)} \mathrm{h}_Y(Y) \to \mathrm{h}_Y(Y). \tag{4-4}$$

Now the source is nothing but the space $\mathrm{h}_Y(X \times_S Y, t_Y)$, where $t_Y\colon Z \times_S Y \to X \times_S Y$ is the base change of t along p. Hence we conclude by Proposition 4.1.6.

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