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Hausdorff dimension of boundaries of relatively hyperbolic groups

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We study the Hausdorff dimension of the Floyd and Bowditch boundaries of a relatively hyperbolic group, and show that, for the Floyd metric and shortcut metrics, they are both equal to a constant times the growth rate of the group.

In the proof, we study a special class of conical points called uniformly conical points and establish that, in both boundaries, there exists a sequence of Alhfors regular sets with dimension tending to the Hausdorff dimension and these sets consist of uniformly conical points.

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1 Introduction

1.1 Main results

The main goal of the paper is to calculate the Hausdorff dimension of the limit set of a geometrically finite action of a finitely generated group G on a compactum X. Every action $G \curvearrowright X$ we consider is a *convergence* action, ie the induced action on the space of the distinct triples is discontinuous. We say that $G \curvearrowright X$ is *minimal* if X coincides with the limit set $\Lambda_X G$ (or ΛG if X is fixed) of the action, which is the set of the accumulation points of every orbit Gx for $x \in X$.

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A point $\xi \in X$ is called *conical* if there exists a sequence of elements $g_n \in G$ for $n \geq 1$ such that the closure of $\{(g_n\xi, g_n\eta) : n \geq 1\}$ in X^2 is disjoint from the diagonal $\Delta(X^2) = \{(x,x) : x \in X\}$ for any $\eta \in X \setminus \xi$. If, in addition, the set of elements $\{g_n g_{n+1}^{-1} : n \geq 1\}$ is in a uniformly bounded distance from the identity, then ξ is called *uniformly conical*. A quantitative version of an L–uniformly conical point for $L \geq 0$ is given in Definition 2.2.

The action of a subgroup $H < G$ on X is *parabolic* if H fixes a point $p \in X$, called a *parabolic fixed point*. The parabolic action is *bounded parabolic* if H acts properly and cocompactly on $X \setminus \{p\}$. We will always assume that the action of the whole group G is nonparabolic, so there is no global fixed point.

A minimal nonparabolic action $G \curvearrowright X$ is called *geometrically finite* (or *relatively hyperbolic*) if every point $x \in X$ is either conical or bounded parabolic (see Definition 2.2). The stabilizer of a parabolic point is a *maximal parabolic* subgroup of G. We denote by $\mathcal{P}$ the set of maximal parabolic subgroups and call it the *peripheral system* for the action. A group is called *relatively hyperbolic* with respect to $\mathcal{P}$ if G admits a geometrically finite action on X with the peripheral system $\mathcal{P}$. If the compactum X on which G acts is metrizable then the action is geometrically finite if and only if the induced action on the space of distinct pairs is cocompact (we say in this case that the action on X is 2–*cocompact*); see V Gerasimov [12]. If the opposite is not stated, we will always assume that a relatively hyperbolic group is finitely generated and so X is metrizable.

Let G be a group with a finite generating set S. Assume that $1 \notin S$ and $S = S^{-1}$. Consider the word metric d_S on G. Denote $B(n) = \{g \in G : d_S(1,g) \leq n\}$ for $n \geq 0$. The *growth rate* $\delta_{G,S}$ of G relative to S is the limit

$$\delta_{G,S} = \lim_{n \to \infty} \frac{\log \# B(n)}{n}.$$

Recall that Floyd completion of a group G generated by S is the Cauchy completion of the Cayley graph $\mathscr{G}(G,S)$ equipped with the distance ρ_λ^o obtained by rescaling the length of an edge $e \in \mathscr{G}(G,S)$ by a scalar function $\lambda^{d(e,o)}$ for a fixed $\lambda \in (0,1)$ and a basepoint $o \in G$. The distance ρ_λ^o is called the Floyd distance at o, and we use the notation ρ if o and λ are clear from the context (see Section 2.2 for more details). We denote by $\overline{G}_\lambda$ and $\partial_\lambda G$ the corresponding Floyd completion and its boundary, respectively. By Gerasimov's theorem [13, Proposition 3.4.6] for every finitely generated relatively hyperbolic group the space $\partial_\lambda G$ is the universal pullback space for every geometrically finite action of $G \curvearrowright X$ in the sense that there exists an equivariant continuous mapping $F\colon \partial_\lambda G \to X$ (called the Floyd map).

A Karlsson proved that the action of G on the compact space $\overline{G}_\lambda$ is a convergence action [22]. Let $\partial^{c}_\lambda G$ (resp. $\partial^{uc}_\lambda G$) denote the set of all (resp. all uniformly) conical points for the action. We denote by Hdim_ρ the Hausdorff dimension with respect to $\rho = \rho_{\lambda,o}$. The first main result of the paper is the following:

Theorem 1.1 *Let G be a relatively hyperbolic group with a finite generating set S. There exists a constant $0 < \lambda_0 < 1$ such that*

$$\mathrm{Hdim}_\rho(\partial_\lambda G) = \mathrm{Hdim}_\rho(\partial^{c}_\lambda G) = \mathrm{Hdim}_\rho(\partial^{uc}_\lambda G) = -\frac{\delta_{G,S}}{\log \lambda}$$

for any $\lambda \in [\lambda_0, 1)$.

Remark For a hyperbolic group the Floyd metric is bilipschitz equivalent to the visual metric on the Gromov boundary (with appropriate choices of parameters). Even though this result seems to be a folklore, we have not found the corresponding reference in the literature. We provide a proof of it in the appendix. As a consequence the result of M Coornaert [5] for the hyperbolic groups is a partial case of Theorem 1.1.

Note that the action of G on the Floyd boundary $\partial_\lambda G$ is not necessarily geometrically finite, as is shown in Yang [38] for Dunwoody's inaccessible groups. In particular, the Floyd boundary is not in general homeomorphic to the limit set ΛG. So it is natural to ask if an analogous result to Theorem 1.1 is true for ΛG.

Consider a minimal geometrically finite action of G on a compact $X = \Lambda G$. It is shown in Gerasimov and Potyagailo [14] that the Floyd metric ρ transferred by the Floyd map $F\colon \partial_\lambda G \to \Lambda G$ is a metric on ΛG, called the shortcut metric, and is denoted by $\overline{\rho}$ (see Section 2.2).

Our next goal is to calculate the Hausdorff dimension $\mathrm{Hdim}_{\overline{\rho}}$ of ΛG with respect to $\overline{\rho}$. Denote by $\Lambda^{uc} G$ the set of uniformly conical points of ΛG. The following theorem provides the same conclusion for the shortcut metric as in the case of the Floyd metric:

Theorem 1.2 *Let G be a group with a finite generating set S acting geometrically finitely on a compactum $X = \Lambda G$. Then there exists a constant $0 < \lambda_0 < 1$ such that*

$$\mathrm{Hdim}_{\overline{\rho}}(\Lambda G) = \mathrm{Hdim}_{\overline{\rho}}(\Lambda^{uc} G) = -\frac{\delta_{G,S}}{\log \lambda}$$

for any $\lambda \in [\lambda_0, 1)$.

The above theorems imply the following:

Corollary 1.3 *For any shortcut metric $\bar{\rho}$, the Hausdorff dimension of the limit set of every relatively hyperbolic action of a group G is constant and is equal to*

$$\mathrm{Hdim}_{\rho}(\partial_\lambda G) = \mathrm{Hdim}_{\bar{\rho}}(\Lambda G) = -\frac{\delta_{G,S}}{\log \lambda}$$

for any $\lambda \in [\lambda_0, 1)$, where $\lambda_0 \in (0,1)$ is a fixed number.

We say that a metric space X is *Ahlfors Q–regular* for a constant $Q > 0$ if there exists a Borel measure μ on X such that

$$\mu(B(x,r)) \asymp r^Q$$

for any open ball $B(x,r)$ centered at $x \in X$ of radius $r > 0$, where the symbol $\asymp$ denotes the bilipschitz equivalence between two quantities, $C^{-1} r^Q \le \mu(B(x,r)) \le C r^Q$ for a uniform constant C.

Our next main result shows that the Hausdorff dimension of the Floyd boundary and of the limit set of a relatively hyperbolic action can be well approximated by a sequence of Ahlfors regular subsets.

Theorem 1.4 *Let G be a finitely generated relatively hyperbolic group with a finite generating set S. Then there exists a sequence of Ahlfors Q_i–regular subsets X_i in $\partial_\lambda G$ or ΛG such that X_i consists of uniformly conical points, $0 < Q_i < -\delta_{G,S}/\log \lambda$ and $Q_i \to -\delta_{G,S}/\log \lambda$ as $i \to \infty$.*

The proof of Theorem 1.4 is based on the existence of an L–transitional geodesic tree $\mathcal{T} = \mathcal{T}(L) \subset G$ (Lemma 3.7) depending on a parameter $L \gg 0$. Every vertex of $\mathcal{T}$ is a central point of a geodesic interval, whose size depends on L, and which belongs to a neighborhood of a left coset (*horosphere*) gP where $P \in \mathcal{P}$ (see Section 2.4). We show that the endpoints of such a tree are L–uniformly conical (Lemma 2.11). However, it is not true in general that every uniformly conical point appears as an endpoint of an L–transitional tree for a bounded L (see the discussion after Lemma 2.11). The proof of Theorem 1.4 shows that the Hausdorff dimension of the endpoints of L–transitional trees well approximate the Hausdorff dimension of the Floyd boundary (or the limit set) if $L \to \infty$. We recapitulate all these facts in the following:

Corollary 1.5 *There exists a sequence T_i of L_i–transitional trees such that $X_i = \partial T_i$ from the statement of Theorem 1.4 are Ahlfors Q_i–regular spaces.*

In the fractal geometry, there is another useful notion of dimension called *box-counting dimension* of metric spaces X (also known as Minkowski–Bouligand dimension). Let $N(\epsilon)$ denote the maximal number of pairwise disjoint balls with radius ϵ contained in X. Then the *box-counting* dimension is defined, when it exists, as

$$\mathrm{Bdim}(X) = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log 1/\epsilon}.$$

It is a general fact that the Hausdorff dimension is less than the box-counting dimension (see Falconer [9]). Thus, Theorems 1.1 and 1.2 give a lower bound on the box-counting dimension. In the following result, we prove that they are in fact equal:

Theorem 1.6 *Under the assumption of Theorem 1.2,*

$$\mathrm{Bdim}_{\rho}(\partial_{\lambda} G) = \mathrm{Bdim}_{\overline{\rho}}(\Lambda G) = -\frac{\delta_{G,S}}{\log \lambda}$$

for any $\lambda \in [\lambda_0, 1)$, where $\lambda_0 \in (0,1)$ is a fixed number.

Remark A similar conclusion was proved for the limit sets of geometrically finite Kleinian groups by B Stratmann and M Urbański in [30]. Their proof makes use of global measure formulas due to D Sullivan [33] (see Stratmann and S Velani [31] for another proof). Although an Ahlfors Q–regular space has the box-counting dimension equal to Q for any $Q > 0$, Corollary 1.5 does not imply the upper bound for the whole limit set (or the Floyd boundary).

Let us mention some problems related to our study.

For hyperbolic groups, Coornaert [5] proved that the Gromov boundary is Ahlfors regular with respect to the visual metric; in fact, he showed that the class of Patterson–Sullivan measures (PS measures in short) on the Gromov boundary coincides with the Hausdorff measures of the right dimension, up to a bounded constant. In view of Theorem 1.4 and Corollary 1.5, it is natural to ask whether the Floyd boundary (or the limit set) is itself Ahlfors regular for the Floyd (or shortcut) metric.

Recall that a metric space is *doubling* if every ball of radius R can be covered by a uniform number of balls of radius $\frac{1}{2}R$. J Mackay and A Sisto [23] proved that the Bowditch boundary endowed with the visual metric is doubling if and only if the peripheral subgroups are virtually nilpotent.

Starting from the action on the Cayley graph, the second author developed in [36] a theory of Patterson–Sullivan measures on the Floyd and Bowditch boundaries (see also a discussion in Section 1.2).

Recall also that a measure μ is called *doubling* if

$$\mu(B(x,R)) \prec \mu\big(B\big(x,\tfrac{1}{2}R\big)\big)$$

for any $x \in X$ and $R > 0$. It is clear that a metric space carrying a doubling measure must be doubling. Although we suspect that PS measures are *not* Ahlfors-regular on $\partial_\lambda G$ and ΛG, we do expect that the following is true:

Question 1.7 *The PS measures on the Floyd boundary and Bowditch boundary equipped with the Floyd/shortcut metric are doubling.*

A positive answer would provide a striking contrast with the visual metric, where the nilpotency of the peripheral subgroups must be imposed.

We believe that the first step to attack the question is to provide an explicit description of the metric balls with respect to the Floyd and shortcut metrics. For that purpose, we conduct a detailed study of the geodesics with respect to the shortcut metric in the completion $\mathscr{G}(G,S) \cup \Lambda G$. This generalizes the result of Gerasimov and Potyagailo [15] that the geodesics of the Floyd metrics are approximated by so-called *tight paths*.

Considering *generalized tight paths* (see Definition 5.10) we show that they approximate the geodesics with respect to the shortcut metric. We refer to Proposition 5.12 for a precise statement. The following result is an application of this proposition providing useful approximation formulas for the Floyd and shortcut metrics on the essential parts of the corresponding spaces.

Let $\partial^{\mathrm{uc}}_{L,o} G$ and $\Lambda^{\mathrm{uc}}_{L,o} G$ denote subsets of uniformly conical points in $\partial^{\mathrm{uc}}_{\lambda} G$ and $\Lambda^{\mathrm{uc}} G$ depending on the above parameter L (see Section 2.4 for the precise definitions).

Proposition 5.14 *Under the assumptions of Theorem 1.1 there exists $0 < \lambda_0 < 1$ such that for any $L > 0$ and $\lambda \in [\lambda_0, 1)$ we have*

$$\rho_{\lambda,o}(\xi,\eta) \asymp_L \lambda^n \quad \textit{for all } \xi \neq \eta \in \partial^{\mathrm{uc}}_{L,o} G$$

and

$$\bar{\rho}_{\lambda,o}(\xi,\eta) \asymp_L \lambda^n \quad \textit{for all } \xi \neq \eta \in \Lambda^{\mathrm{uc}}_{L,o} G,$$

where $n = d(o,[\xi,\eta])$.

1.2 Historical remarks and motivations

We provide here a short history of the study of the Hausdorff dimension of the limit set of various convergence actions: Kleinian, hyperbolic and relatively hyperbolic.

The identification of the Hausdorff dimension with the critical exponent of Poincaré series was first established by S Patterson [25]. He introduced a probability measure on the limit set of the convex–cocompact Fuchsian groups, and proved that up to a constant it is equal to the Hausdorff measure. Sullivan generalized this result and constructed such measures (called since then *Patterson–Sullivan measures*) on the limit sets of geometrically finite Kleinian groups acting on the hyperbolic space $\mathbb{H}^n$ of dimension n [32]. To finish the discussion of the case of Kleinian groups, we note the result of C Bishop and P Jones, who proved in [1] that for a nonelementary Kleinian group acting on the hyperbolic 3–space the Hausdorff dimension of the conical limit set is equal to the critical exponent of the Poincaré series (compare with our Theorems 1.1 and 1.2). The latter results were generalized by F Paulin [26] to discrete groups of isometries of Riemannian manifolds of strictly negative curvature.

Coornaert has generalized the results of Patterson and Sullivan to the class of word-hyperbolic groups [5]. In particular he proved that the Hausdorff dimension of the (Gromov) boundary of such a group with respect to the visual metric is equal to the critical exponent of the Poincaré series.

A natural question arises whether Coornaert's theorem holds for the class of relatively hyperbolic groups. However, it was shown by M Burger and S Mozes that if G is a closed subgroup of the isometry group of a CAT(-1) space X and the parabolic subgroups of G are not amenable then the critical exponent is infinite [3, Proposition 1.6]. Such an example of a relatively hyperbolic group whose parabolic subgroups contain noncyclic free subgroups was constructed by D Gaboriau and Paulin [11, Example 1, page 189]. By [26] it then follows that the Hausdorff dimension of the limit set for the action of such a group with respect to the visual metric is infinite too. So in order to generalize Coornaert's theorem to the class of relatively hyperbolic groups one must replace the visual metric by a different one.

The Floyd metric obtained by a rescaling procedure of the word metric is a natural candidate as it extends to the Floyd compactification of a group. Furthermore, by a theorem of Gerasimov there exists an equivariant and continuous map from the Floyd boundary $\partial_\lambda G$ to the limit set of any relatively hyperbolic action of G [13]. In particular, if G is hyperbolic, the Floyd and Gromov boundaries are bilipschitz equivalent for some exponential Floyd function.

M Bourdon has observed (private communication) that the Hausdorff dimension of the Floyd boundary of a relatively hyperbolic group, calculated with respect to the Floyd metric obtained with the exponential rescaling function λ^n with $\lambda \in (0,1)$, is always bounded above by $-\delta_{G,S}/\log\lambda$ (see Lemma 4.1). However, the question of whether it admits a lower strictly positive bound which is equal to the same constant remained open. This was our first motivation, giving rise to Theorem 1.1. Theorem 1.2 is then obtained by transferring the Floyd metric from the Floyd boundary $\partial_\lambda G$ to the limit set ΛG of the geometrically finite action using the above Gerasimov map.

The lower bound estimate for the Hausdorff dimension in Theorems 1.1 and 1.2 follows from Theorem 1.4, providing the approximation of the boundary points by Ahlfors regular subsets X_i. These subsets entirely consist of uniformly conical points which form the space of ends of subtrees of the Cayley graph of G. Note that the idea of such an approximation by trees is quite standard in both settings: hyperbolic (see eg Gromov [19, Section 6.1]) or Kleinian (see Bishop and Jones [1]). However, these constructions of trees essentially use the hyperbolicity of the ambient space. The latter property is not true for a relatively hyperbolic group: the Cayley graph is not in general hyperbolic and the relative Cayley graph is hyperbolic but the action on the set of vertices is not proper. The approximating trees constructed in the paper admit certain periodicity, allowing us to obtain a Patterson–Sullivan measure μ on X_i also having periodic properties. Theorem 1.4 then shows that these measures converge to the Hausdorff measure on a subset of uniformly conical points whose dimension coincides with the full Hausdorff dimension of the ambient space.

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2 Preliminaries

2.1 Notation and conventions

Let (Y, d) be a geodesic metric space. Given a subset X and a number $r \geq 0$, let $N_r(X) = \{y \in Y : d(y, X) \leq r\}$. For $x \in Y$ let $B(x, r) = N_r(\{x\})$. Sometimes, we will write $B_d(x, r)$ to emphasize the metric d.

Given a point $y \in Y$ and a closed subset $X \subset Y$, let $\mathbf{Proj}_X(y)$ be the set of points x in X such that $d(y, x) = d(y, X)$. The *projection* of a subset $A \subset Y$ to X is then $\mathbf{Proj}_X(A) = \bigcup_{a \in A} \mathbf{Proj}_X(a)$.

We always consider a rectifiable path α in Y with arc-length parametrization. Denote by $\ell(\alpha)$ the length of α, and by α_- and α_+ the initial and terminal points of α, respectively. Let $x, y \in \alpha$ be two points which are given by parametrization. Then denote by $[x, y]_\alpha$ the parametrized subpath of α going from x to y. We also denote by $[x, y]$ a choice of a geodesic in Y between $x, y \in Y$.

A path α is called a c*–quasigeodesic* for $c \geq 1$ if

$$\ell(\beta) \leq c \cdot d(\beta_-, \beta_+) + c$$

for any rectifiable subpath β of α.

Let α and β be two paths in Y. Denote by $\alpha \cdot \beta$ (or simply $\alpha\beta$) the concatenated path provided that $\alpha_+ = \beta_-$.

A path α going from α_- to α_+ induces a first-to-last order as follows. Given a property (P), a point z on α is called the *first point* satisfying (P) if z is among the points w on α with the property (P) such that $\ell([\alpha_-, w]_\alpha)$ is minimal. The *last point* satisfying (P) is defined in a similar way (replacing $[\alpha_-, w]_\alpha$ by $[w, \alpha_+]_\alpha$).

Let f and g be real-valued functions with domain understood in the context. Then $f \prec_{c_i} g$ means that there is a constant $C > 0$ depending on parameters c_i such that $f < Cg$, and $\succ_{c_i}$ is defined similarly. We use the symbol $\asymp_{c_i}$ if both inequalities are true. For simplicity, we omit c_i if they are some universal constants.

Denote by $\|\cdot\|$ the diameter of a set in a metric space. Recall the notion of Hausdorff measures in a metric space.

Definition 2.1 Let X be a subset in a metric space (Y, d). Given numbers $\epsilon, s \geq 0$, define

$$\mathcal{H}^s_\epsilon(X) = \inf\left\{\sum \|U_i\|^s : X \subset \bigcup_{i=1}^{\infty} U_i, U_i \subset Y, \ \|U_i\| \leq \epsilon\right\}.$$

Define $\mathcal{H}^s(X) = \lim_{\epsilon \to 0} \mathcal{H}^s_\epsilon(X)$, the *s–dimensional Hausdorff measure* of X. The *Hausdorff dimension* of X is defined as

$$\mathrm{Hdim}_d(X) = \inf\{s \geq 0 : \mathcal{H}^s(X) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(X) = \infty\}.$$

By convention, set $\inf \varnothing = \sup\{s \in \mathbb{R}_{\geq 0}\} = \infty$. Thus, $\mathrm{Hdim}_d X \in [0, \infty]$. Note that $\mathcal{H}^s(X)$ may be zero for $s = \mathrm{Hdim}_d X$.

2.2 Floyd boundary and relative hyperbolicity

Let G be a group with a finite generating set S. Assume that $1 \notin S$ and $S = S^{-1}$. Let $\mathscr{G}(G, S)$ be the *Cayley graph* of G with respect to S. Denote by d_S (or simply by d if there is no ambiguity) the word metric on $\mathscr{G}(G, S)$.

Fix $0 < \lambda < 1$ and a basepoint $o \in G$. We define a Floyd metric $\rho_{\lambda,o}$ as follows. The *Floyd length* $\mathfrak{l}_{\lambda,o}(e)$ of an edge e in $\mathscr{G}(G, S)$ is λ^n, where $n = d(o, e)$. The Floyd length $\mathfrak{l}_{\lambda,o}(\gamma)$ of a path γ is the sum of Floyd lengths of its edges. This induces a length metric $\rho_{\lambda,o}$ on $\mathscr{G}(G, S)$, which is the infimum of Floyd lengths of all possible paths between two points.

Let $\overline{G}_\lambda$ be the Cauchy completion of G with respect to $\rho_{\lambda,o}$. The complement $\partial_\lambda G$ of $\mathscr{G}(G, S)$ in $\overline{G}_\lambda$ is called the *Floyd boundary* of G. The $\partial_\lambda G$ is called *nontrivial* if $\#\partial_\lambda G > 2$. We refer the reader to [10; 13; 14; 22] for more details.

By construction, the following equivariant property holds:

$$\rho_{\lambda,o}(x, y) = \rho_{\lambda,go}(gx, gy) \tag{1}$$

for any $g \in G$. The Floyd metrics with different basepoints are related by a bilipschitz inequality,

$$\lambda^{d(o,o')} \leq \frac{\rho_{\lambda,o}(x, y)}{\rho_{\lambda,o'}(x, y)} \leq \lambda^{-d(o,o')} \tag{2}$$

for any two points $o, o' \in G$.

We now recapitulate several standard definitions concerning geometrically finite convergence actions which will be often used later.

Definition 2.2 Let X be a compact metrizable space on which G admits a minimal and nontrivial convergence action by homeomorphisms.

(1) A point $\xi \in X$ is called *conical* if there exists a sequence of elements $g_n \in G$ for $n \geq 1$ such that the closure of $\{g_n(\xi, \eta) : n \geq 1\}$ in X^2 is disjoint from the diagonal $\Delta(X^2) = \{(x, x) : x \in X\}$ for any $\eta \in X \setminus \xi$. If, in addition, there exists $L > 0$ such that $d(1, g_n g_{n+1}^{-1}) \leq L$, then ξ is called *L–uniformly conical* (or *uniformly conical* if the constant L is not important).

(2) A point $\xi \in X$ is called *bounded parabolic* if the stabilizer G_ξ of ξ in G is infinite, and acts properly and cocompactly on $X \setminus \xi$. The subgroup G_ξ is called *maximal parabolic*.

(3) A convergence group action of G on X is called *geometrically finite* if every limit point $\xi \in X$ is either a conical point or a bounded parabolic point.

As was mentioned in the introduction, a pair $(G, \mathcal{P})$ is *relatively hyperbolic* if G admits a geometrically finite group action on a compact metrizable space X such that $\mathcal{P}$ coincides with the collection of maximal parabolic subgroups (peripheral system). Using the relative Cayley graph one can construct the limit set ΛG of the action with the boundary of this graph [2, Section 8]. We will often call the *Bowditch boundary* the limit set ΛG of a geometrically finite action. Bowditch proved that if G is finitely generated then ΛG up to an equivariant homeomorphism depends only on the pair $(G, \mathcal{P})$ [2, Theorem 9.4]. We also note the same result still holds in general case when G is not finitely generated [17, Corollary 6.1.e].

The following result establishes a universal *pullback* property of the Floyd boundary:

Proposition 2.3 [13, Corollary 1.5] *Suppose $(G, \mathcal{P})$ is a relatively hyperbolic pair. Then there exists $0 < \lambda_0 < 1$ such that for any $\lambda \in [\lambda_0, 1)$ there exists a continuous G–equivariant surjective map,* (called the *Floyd map*),

$$F_\lambda \colon \partial_\lambda G \to \Lambda G.$$

Let G_p be the stabilizer of a parabolic point $p \in X$ for the action $G \curvearrowright X = \Lambda G$. Denote by $\Lambda_{\partial_\lambda G}(G_p)$ and $\partial_\lambda G_p$ the limit set of G_p for its action on the Floyd boundary $\partial_\lambda G$ of G and the Floyd boundary of G_p, respectively. The following result precisely describes the kernel of the Floyd map:

Proposition 2.4 [14, Theorem A] *Under the assumption of Proposition 2.3,*

$$F_\lambda^{-1}(p) = \Lambda_{\partial_\lambda G}(G_p)$$

for any parabolic point p in ΛG. Moreover, $F_\lambda^{-1}(p)$ consists of one point if p is a conical point.

Remark The above statement admits a stronger form; namely, Corollary 7.7 of [15] proves that if in addition the Floyd rescaling function satisfies

$$\frac{f(n)}{f(2n)} \le \text{const}$$

(eg if $f = 1/P$ where P is a polynomial of degree at least 2) then

$$F_\lambda^{-1}(p) = \partial_\lambda G_p.$$

We equip ΛG with a *shortcut metric* as follows: Let

$$\omega = \{(\eta, \xi) \in \partial_\lambda G \times \partial_\lambda G : F_\lambda(\xi) = F_\lambda(\eta)\}$$

be the relation on $\partial_\lambda G$ given by the Floyd map $F_\lambda \colon \partial_\lambda G \to \Lambda G$. For any $\xi, \eta \in \overline{G}_\lambda$, define a pseudodistance $\tilde\rho_{\lambda,o}(\xi,\eta)$ on $\overline{G}_\lambda$ to be

$$(3)\quad \tilde\rho_{\lambda,o}(\xi,\eta) = \inf_{n\ge 1}\left\{\sum_{i=1}^{n} \rho_{\lambda,o}(\xi_i,\eta_i) : (\eta_i, \xi_{i+1}) \in \omega,\ 1 \le i < n,\ \xi_1 = \xi,\ \eta_n = \eta\right\}.$$

We have

$$(4)\qquad \tilde\rho_{\lambda,o}(\xi,\eta) \le \rho_{\lambda,o}(\xi,\eta) \quad \text{for all } \xi, \eta \in \overline{G}_\lambda,$$

and it is a maximal pseudometric on $\overline{G}_\lambda \times \overline{G}_\lambda$ satisfying this inequality and vanishing on ω. It is shown in [13, Proposition 8.3.1] that the space $\widetilde{\Lambda G} := \Lambda G \sqcup \mathscr{G}(G,S)$ (called the *attractor sum*) is compact. The action $G \curvearrowright \widetilde{\Lambda G}$ is convergence and such that its restriction on $\mathscr{G}(G,S)$ is the identity and on ΛG it coincides with the initial action. Furthermore, the Floyd map F_λ extends to an equivariant continuous map (denoted by the same symbol),

$$F_\lambda \colon \overline{G}_\lambda \to \widetilde{\Lambda G},$$

such that $F_\lambda|_G \equiv \mathrm{id}$. Pushing forward $\tilde\rho_{\lambda,o}$ with F_λ, we obtain a *shortcut pseudometric* on $\widetilde{\Lambda G}$,

$$(5)\qquad \bar\rho_{\lambda,o}(x,y) = \tilde\rho_{\lambda,o}(F_\lambda^{-1}(x), F_\lambda^{-1}(y)) \quad \text{for all } x, y \in \widetilde{\Lambda G},$$

which turns out to be a real metric on $\widetilde{\Lambda G}$ (see the remark after Lemma 3.2 in [14] for details). By the above construction, one can easily see that the shortcut metrics $\overline{\rho}_{\lambda,o}$ satisfy the properties (1) and (2) too.

Convention 2.5 Since from now on we will always suppose that $\lambda \in [\lambda_0, 1)$, where λ_0 is given by Proposition 2.3. We omit the index λ in $\mathfrak{l}_{\lambda,o}, \rho_{\lambda,o}$ and $\overline{\rho}_{\lambda,o}$ if λ is given in the context.

Finally, we recall the following visibility lemma:

Lemma 2.6 (visibility lemma [22, Lemma 1]) *There is a function $\phi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for any $v \in G$ and any geodesic γ in $\mathscr{G}(G, S)$, we have if $\mathfrak{l}_v(\gamma) \geq \kappa$, then $d(v, \gamma) \leq \phi(\kappa)$.*

Remark The same result is valid for quasigeodesics or more general Θ–geodesics where $\Theta\colon \mathbb{N} \to G$ is a polynomial distortion function [14, Lemma 5.1].

2.3 Floyd geodesics

In this subsection, we provide a few basic tools to study Floyd geodesics. We will either present a complete proof or give an exact reference to a statement which is claimed. The results obtained below will be used further on in the paper.

We say that a path $\alpha\colon \mathbb{Z} \to \mathscr{G}(G, S)$ *ends at* $\xi \in \partial_\lambda G$ if $\xi = \lim_{n\to\infty} \alpha(n)$. Write in this case $\alpha_+ = \xi$, and $\alpha_- = \lim_{n\to-\infty} \alpha(n)$. It follows from Lemma 2.6 that every geodesic ray ends at a point of the Floyd boundary. Moreover, $\overline{G}_\lambda$ is a geodesic metric space and is a visual boundary: any two distinct points $\xi, \eta \in \overline{G}_\lambda$ are connected by a bi-infinite word geodesic belonging to the Cayley graph [14, Proposition 2.4].

The following lemma states that word geodesic rays are also Floyd and shortcut geodesics:

Lemma 2.7 *Let $o \in G$ be a basepoint and γ be a geodesic ray with $\gamma_- = o$. Then, for any $v \in \gamma$, we have*

$$\mathfrak{l}_{\lambda,o}([v, x]_\gamma) = \overline{\rho}_{\lambda,o}(v, y)$$

and

$$\mathfrak{l}_{\lambda,o}([v, x]_\gamma) = \rho_{\lambda,o}(v, x),$$

where $x = \gamma_+ \in \partial_\lambda G$ and $y = F(x) \in \Lambda G$, where F is the Floyd map given in Proposition 2.3.

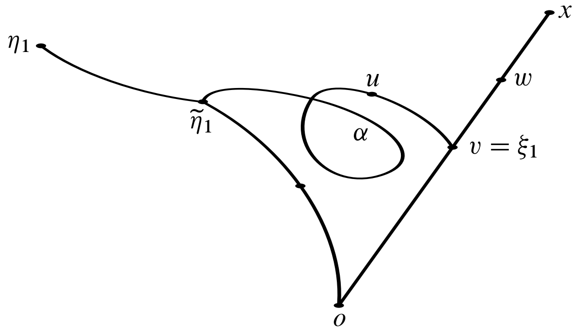

Figure 1

Proof We only prove the result for the shortcut metric. In the case of the Floyd metric a straightforward calculation shows that the geodesic ray γ as well as every subray is also a $\rho_{\lambda,o}$–Floyd geodesic.

By definition (3) of a shortcut metric, for any $n \in \mathbb{N}$, there exist pairs $(\eta_i, \xi_{i+1}) \in \omega$ for $1 \le i < m$ such that

$$\bar{\rho}_{\lambda,o}(v, y) \ge \sum_{1 \le i \le m} \rho_{\lambda,o}(\xi_i, \eta_i) - \frac{1}{2n},$$

where $\xi_1 = v$ and $\eta_m = x$. Every geodesic ray $[o, \eta_1]$ is also a Floyd geodesic so we can choose $\tilde{\eta}_1 \in [o, \eta_1]$ such that $\rho_{\lambda,o}(\tilde{\eta}_1, \eta_1) \le \frac{1}{2n}$. It follows that

$$\bar{\rho}_{\lambda,o}(v, y) \ge \rho_{\lambda,o}(v, \tilde{\eta}_1) - \frac{1}{n}.$$

Choose $w \in [v, y]_\gamma$ such that $d(v, w) = d(v, \tilde{\eta}_1) = n$. Then

$$\rho_{\lambda,o}(v, \tilde{\eta}_1) \ge \rho_{\lambda,o}(v, w). \tag{6}$$

Indeed, connect v and $\tilde{\eta}_1$ by a curve α. There exists a point $u = \alpha(t_0)$ such that the subcurve $\alpha' = [v, u]_\alpha$ contains exactly n edges. Since γ is a word geodesic, for the k^{th} edge $e \in \alpha'$ and the k^{th} edge $e_1 \in [v, w]_\gamma$ we have $\mathfrak{l}_{\lambda,o}(e) \ge \mathfrak{l}_{\lambda,o}(e_1)$ for $k \in \{0, \dots, n\}$. Then $\mathfrak{l}_{\lambda,o}(\alpha) \ge \mathfrak{l}_{\lambda,o}(\alpha') \ge \rho_{\lambda,o}(v, w)$. So (6) follows.

We have

$$\bar{\rho}_{\lambda,o}(v, y) \ge \rho_{\lambda,o}(v, \tilde{\eta}_1) - \frac{1}{n} \ge \rho_{\lambda,o}(v, w) - \frac{1}{n} \ge \mathfrak{l}_{\lambda,o}([v, x]_\gamma) - \frac{\lambda^{n+d(o,v)}}{1-\lambda} - \frac{1}{n}.$$

Passing to the limit, we obtain

$$\bar{\rho}_{\lambda,o}(v, y) \ge \mathfrak{l}_{\lambda,o}([v, x]_\gamma) = \rho_{\lambda,o}(v, x).$$

Since $\rho_{\lambda,o}(v, x) \ge \bar{\rho}_{\lambda,o}(v, y)$, we conclude that $\mathfrak{l}_{\lambda,o}([v, \gamma_+]_\gamma) = \bar{\rho}_{\lambda,o}(v, \gamma_+)$. □

2.4 Transitional paths and uniformly conical points

In this subsection we shall give a description of uniformly conical points in ΛG using the geometry of a Cayley graph.

Let $(G, \mathcal{P})$ be a relatively hyperbolic pair. Let $\mathbb{P} = \{gP : g \in G,\ P \in \widetilde{\mathcal{P}}\}$, where $\widetilde{\mathcal{P}}$ is a maximal set of nonconjugate subgroups in $\mathcal{P}$. Following [14] we call the elements of $\mathbb{P}$ *horospheres*.

Definition 2.8 [21, Definition 8.9] Fix $\epsilon, R > 0$. Let γ be a path in $\mathscr{G}(G, S)$ and $v \in \gamma$ a vertex. Given $X \in \mathbb{P}$, we say that v is (ϵ, R)*–deep* in X if

$$\gamma \cap B(v, R) \subset N_\epsilon(X).$$

If v is not (ϵ, R)–deep in any $X \in \mathbb{P}$, then v is called an (ϵ, R)*–transition point* of γ.

The following lemma together with Lemma 2.6 will be invoked several times:

Lemma 2.9 (1) *For any $c \geq 1$ and $R > 0$, there exist $\epsilon = \epsilon(c)$, $\kappa = \kappa(\epsilon, R) > 0$ such that for any c–quasigeodesic γ and an (ϵ, R)–transitional point v in γ, we have*

$$\rho_v(\gamma_-, \gamma_+) \geq \overline{\rho}_v(\gamma_-, \gamma_+) > \kappa.$$

(2) *For any $c \geq 1$ and $\kappa, \epsilon > 0$ there exists $R = R(c, \kappa, \epsilon) > 0$ such that for any c–quasigeodesic γ and a point $v \in \gamma$ with $\overline{\rho}_v(\gamma_-, \gamma_+) > \kappa$, we have that v is an (ϵ, R)–transitional point of γ.*

Proof (2) Suppose not; then there are $c \geq 1$ and $\kappa, \epsilon > 0$ such that for all n, there exist c–quasigeodesics γ_n and $v_n \in \gamma_n$ such that v_n is (ϵ, n)–deep and $\overline{\rho}_{v_n}((\gamma_n)_-, (\gamma_n)_+) > \kappa$. Up to a normalization we may assume that $v_n = v = \gamma_n(0)$. Then $\gamma_n((-n, n)) \subset N_\epsilon(X_n)$ for $X_n \in \mathbb{P}$. By compactness of uniform quasigeodesics in the Tychonoff topology, we obtain a limit horocycle α such that $\alpha_\pm = q$ and every part of α belongs to γ_n for sufficiently large n (see [16, Proposition 5.9] for more details). Then the diameter of $\partial(\gamma_n \cap \alpha)$ with respect to the distance $\overline{\rho}_v$ tends to 0. As γ_n are geodesics whose interior points are all in the graph we must have $\overline{\rho}_v((\gamma_n)_-, (\gamma_n)_+) \to 0$, which is a contradiction.

(1) By [14, Corollary 3.9] there exists a constant $\epsilon = \epsilon(c)$ such that for every $X \in \mathbb{P}$ any c–quasigeodesic with endpoints in X lies in $N_\epsilon(X)$ (all horospheres are uniformly quasiconvex). For the constants c and $\epsilon = \epsilon(c)$ the statement now follows from [16, Corollary 5.10], following a similar argument as above. □

We introduce a special class of paths, which plays an important role in the present study.

Definition 2.10 Given $\epsilon, R, L > 0$, a path γ in $\mathscr{G}(G,S)$ is called (ϵ,R,L)*–transitional* (or simply *transitional* if the choice of the constants is not important) if for any point $v \in \gamma$, there exists an (ϵ,R)–transitional point $w \in \gamma$ such that $\ell([v,w]_\gamma) \le L$.

We say that an infinite path γ in $\mathscr{G}(G,S)$ is *eventually* (ϵ,R,L)–transitional if there exists $v \in \gamma$ such that $[v,\gamma_+)_\gamma$ is (ϵ,R,L)–transitional.

We fix the constant $\epsilon = \epsilon(1) > 0$ given by Lemma 2.9(1). The following lemma characterizes uniformly conical points as the endpoints of transitional geodesic rays:

Lemma 2.11 *Let $(G,\mathcal{P})$ be a relatively hyperbolic pair. There exists $R > 0$ for which the following property is true:*

A point $\xi \in \Lambda G$ is L–uniformly conical for some $L > 0$ if and only if some (or any) geodesic ray ending at ξ is eventually an (ϵ,R,L)–transitional geodesic ray.

Proof $(\Longrightarrow)$ Since G acts geometrically finitely on ΛG, it follows from Theorem 1C of [34] that there exists $\delta > 0$ such that for any conical point $\xi \in \Lambda G$, there exists a sequence $(g_n) \subset G$ such that for all points $\eta \in (\Lambda G \cup G) \setminus \xi$ one has $\bar\rho_1(g_n\xi, g_n\eta) > \delta$. Let $r_0 := \phi\big(\frac{1}{2}\delta\big)$, where ϕ is given by Lemma 2.6.

Assume that ξ is an L–uniformly conical point for some $L > 0$. Let $\gamma = [\gamma_-,\xi)$ be a geodesic ray ending at ξ and $(g_n) \subset G$ be the above sequence taken for the pair (γ_-,ξ). Then $\bar\rho_1(g_n\xi, g_n\gamma_-) = \bar\rho_{g_n^{-1}}(\xi,\gamma_-) > \frac{1}{2}\delta$ and $d(1, g_n g_{n+1}^{-1}) \le L$ for all $n \ge 1$. By Lemma 2.6, $\gamma \cap B(g_n^{-1}, r_0) \ne \varnothing$ for $n \ge 1$. Let $v_n \in \gamma$ be such that $d(v_n, g_n^{-1}) < r_0$. By the inequality (2), $\bar\rho_{v_n}(\gamma_-,\xi) > \kappa$, where $\kappa = \lambda^{r_0} \cdot \frac{1}{2}\delta$ is a uniform constant. Moreover, $d(v_n, v_{n+1}) \le L + 2r_0$.

Hence, Lemma 2.9(2) gives rise to a uniform constant R for which v_n are all (ϵ,R)–transitional for $n \ge 1$.

$(\Longleftarrow)$ Let γ be an (ϵ,R,L)–transitional geodesic ray at $\xi = \gamma_+$, and v_n for $n \ge 0$ a sequence of (ϵ,R)–transitional points in γ such that $d(v_n,v_{n+1}) \le L$ and $v_n \to \xi$. Then $\bar\rho_{v_n}(\gamma_-,\xi) \ge \kappa$, where $\kappa > 0$ is given by Lemma 2.9(1). Write $g_n := v_n^{-1}$. Then $\bar\rho_1(g_n\gamma_-, g_n\xi) \ge \kappa$. In other words, $\{(g_n\gamma_-, g_n\xi)\}$ lies outside a uniform neighborhood of the diagonal $\Delta(\Lambda G^2)$.

Since the action is convergence, the point ξ is conical. As $d(1, g_n g_{n+1}^{-1}) \le L$, it is uniformly conical. □

Remarks (1) The proof of the "$\Longleftarrow$" direction equally applies to a conical point in the Floyd boundary $\partial_\lambda G$ without assuming the geometrical finiteness of the action.

(2) The existence of the uniform constant $\delta > 0$ which measures the size of a compact fundamental set for the cocompact action of G on the set of distinct pairs was only used to prove the implication "$\Longrightarrow$" (in order to get a uniform constant R). The existence of such a constant implies that the action of G on a metrizable space ΛG is 2–cocompact; the converse statement that a 2–cocompact and nonelementary convergence action is geometrically finite is shown in [12], and its proof does not request the metrizability of the space $X = \Lambda G$.

(3) As a corollary we see that for each $L > 0$ the set of L–uniformly conical points is G–invariant, although this is not clear at all from the dynamical definition.

Corollary 2.12 *Let $\epsilon = \epsilon(1) > 0$ given by Lemma 2.9(1). For any $R, L > 0$, an (ϵ, R, L)–transitional geodesic ray ends at a uniformly conical point $\xi \in \partial_\lambda G$.*

As another consequence of the proof, we have the following result:

Corollary 2.13 *Let $G \curvearrowright X$ be a geometrically finite action. Then, for any uniformly conical point $\xi \in X$, there exists a constant $L > 0$ with the following property: there is a sequence of elements $g_n \in G$ such that for any geodesic γ ending at ξ, we have*

$$[v, \xi[_\gamma \subset \bigcup_{n \geq 1} B(g_n, L)$$

for some $v \in \gamma$.

Remark In the setting of Kleinian groups, this property is used to define uniformly conical points; see [29]. The corollary also holds for "quasigeodesics" instead of "geodesics".

We set up some notation for future discussions about uniformly conical points.

Let ϵ and R be given by Lemma 2.11. Denote by $\Lambda^{\mathrm{uc}}_L G$ the set of uniformly conical points $\xi \in \Lambda G$ such that there exists an (ϵ, R, L)–transitional geodesic ray γ ending at ξ. It is obvious that $\Lambda^{\mathrm{uc}}_L G$ is a G–invariant set.

Fixing a basepoint $o \in G$, denote by $\Lambda^{\mathrm{uc}}_{L,o} G$ the set of all uniformly conical points $\xi \in \Lambda^{\mathrm{uc}}_L G$ where a geodesic γ between o and ξ is (ϵ, R, L)–transitional.

Clearly, $G \cdot \Lambda^{\mathrm{uc}}_{L,o} G = \Lambda^{\mathrm{uc}}_{L} G$. Thus, the set $\Lambda^{\mathrm{uc}}_{L,o} G$ can be thought as a fundamental domain for the action of G on the set $\Lambda^{\mathrm{uc}}_{L} G$.

Similarly, we define the set of uniformly conical points $\partial^{\mathrm{uc}}_{L,o} G$ and $\partial^{\mathrm{uc}}_{L} G$ on the Floyd boundary $\partial_\lambda G$. By Proposition 2.4, there exists a one-to-one correspondence between $\Lambda^{\mathrm{uc}}_{L} G$ and $\partial^{\mathrm{uc}}_{L} G$.

2.5 Contracting property

Recall that $\| \cdot \|$ denotes the diameter of a set in a metric space.

Definition 2.14 For $c \geq 1$, a subset X is called *c–contracting* in a metric space Y if there exist $\mu_c, D_c > 0$ such that

$$\|\mathbf{Proj}_X(\gamma)\| < D_c \tag{7}$$

for any c–quasigeodesic γ in Y with $N_{\mu_c}(X) \cap \gamma = \varnothing$.

A collection of c–contracting subsets is referred to as a *c–contracting system* if μ_c and D_c depend only on c.

A system $\mathbb{X}$ has *a bounded intersection property* if for any $\epsilon > 0$ there exists $\mathcal{R} = \mathcal{R}(\epsilon) > 0$ such that

$$\|N_\epsilon(X) \cap N_\epsilon(X')\| < \mathcal{R}$$

for any two distinct $X, X' \in \mathbb{X}$.

In what follows, our discussion applies to the Cayley graph of a relatively hyperbolic group $(G, \mathcal{P})$ with a finite generating set S. In particular, we are interested in the contracting system with bounded intersection given by the following lemma:

Lemma 2.15 [16] *Let $\mathbb{P} = \{gP : g \in G, P \in \widetilde{\mathcal{P}}\}$, where $\widetilde{\mathcal{P}}$ is a complete set of conjugacy representatives in $\mathcal{P}$. There exists $\mathcal{R}\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that the collection $\mathbb{P}$ is a c–contracting system with the $\mathcal{R}$–bounded intersection for each $c \geq 1$.*

Proof The contracting property of the system $\mathbb{P}$ is proven in [16, Proposition 8.5], and the bounded intersection is in [16, Proposition 5.6]; the last property can be also deduced from [8, Theorem 4.1]. □

The following lemma will be often used further on:

Lemma 2.16 *Let $\mathbb{P}$ be the collection of horospheres in Lemma 2.15. For any $c \geq 1$, there exists $\epsilon_c = \epsilon(c) > 0$ such that for every c–quasigeodesic γ in $\mathscr{G}(G, S)$ and $\epsilon \geq \epsilon_c$ we have:*

For each $R \geq 0$, there exists $L = L(\epsilon, R) > 0$ such that the condition

$$\max\{d(\gamma_-, X), d(\gamma_+, X)\} < \epsilon$$

for some $X \in \mathbb{P}$ implies that every point $z \in \gamma$ satisfying $d(z, \gamma_-), d(z, \gamma_+) > L$ is (ϵ_c, R)–deep in X.

Proof The result is proved in [36, Lemma 2.8] for geodesics. We provide below a proof to make precise the choice of the constants.

By Lemma 2.15, let μ_c and D_c be the constants such that for any $X \in \mathbb{P}$, for any c–quasigeodesic outside $N_{\mu_c}(X)$, the diameter of its projection to X is bounded above by D_c.

Set $\epsilon_c := c(2\mu_c + D_c) + c$. If a c–quasigeodesic has two endpoints in $N_{\mu_c}(X)$ for $X \in \mathbb{P}$, then it lies in $N_{\epsilon_c}(X)$. Indeed, if $x, y \in \gamma$ satisfy

$$\max\{d(x, X), d(y, X)\} \leq \mu_c$$

and $(x, y)_\gamma \cap N_{\mu_c}(X) = \varnothing$, then by Lemma 2.15, $d(x, y) \leq 2\mu_c + D_c$. Since γ is c–quasigeodesic, we have $\ell([x, y]_\gamma) \leq \epsilon_c$ and $[x, y]_\gamma \subset N_{\epsilon_c}(X)$.

Set $L = c(2\epsilon + D_c) + c + R$ for $\epsilon \geq \epsilon_c$. We first claim $\gamma \cap N_{\mu_c}(X) \neq \varnothing$. Otherwise, we obtain, using projection,

$$2L \leq \ell(\gamma) \leq c\,d(\gamma_-, \gamma_+) + c \leq c(2\epsilon + D_c) + c.$$

This gives a contradiction by the choice of L. Thus, there exist the entry point x and the exit point y of γ in $N_{\mu_c}(X)$.

By the same argument one obtains

$$\max\{\ell([\gamma_-, x]_\gamma), \ell([y, \gamma_+]_\gamma)\} \leq c(\epsilon + \mu_c + D_c) + c < L.$$

Since $\min\{d(z, \gamma_-), d(z, \gamma_+)\} > L$, we have $z \in [x, y]_\gamma$. Then we obtain

$$\min\{d(x, z), d(z, y)\} \geq L - c(\epsilon + \mu_c + D_c) - c > R.$$

By definition of ϵ_c, we have $[x, z]_\gamma \subset N_{\epsilon_c}(X)$ and $[z, y]_\gamma \subset N_{\epsilon_c}(X)$. So z is (ϵ_c, R)–deep in X. $\square$

Remark By the proof, we actually have $\epsilon_c > \mu_c$, where μ_c is uniform for every $X \in \mathbb{P}$ by Lemma 2.15.

In what follows, we take constants ϵ and R as in Convention 2.17.

Convention 2.17 (about ϵ_c, R_c) When talking about (ϵ_c, R_c, L)–transitional c–quasigeodesics, or (ϵ_c, R_c)–transitional and (ϵ_c, R_c)–deep points in a c–quasigeodesic, we assume without explicitly specifying the quantifiers

(1) $\epsilon_c = \epsilon(c) > \mu_c$ to satisfy Lemmas 2.9 and 2.16, where μ_c is given by Definition 2.14, and

(2) $R_c > \mathcal{R}(\epsilon)$, where $\mathcal{R}$ is given by Lemma 2.15.

Besides the peripheral cosets (horospheres), transitional quasigeodesics provide another source of contracting subsets.

Lemma 2.18 (transitional geodesic is contracting) *For any numbers $c, L \geq 0$, every (ϵ_c, R_c, L)–transitional c–quasigeodesic γ is c–contracting.*

Proof The argument we present below works for a general (ϵ_c, R_c, L) c–quasigeodesic. For notational simplicity we consider only the case when $c = 1$ and drop the index 1 in ϵ_1 and R_1 correspondingly.

Let $\kappa = \kappa(\epsilon, R)$ given by Lemma 2.9 and ϕ given by Lemma 2.6. By Lemma 2.6, there exists $D_0 = \phi\left(\frac{1}{4}\kappa\right) > 0$ such that for any $v \in G$, a geodesic segment outside the ball $B(v, D_0)$ has $\mathfrak{f}_v$–Floyd length less than $\frac{1}{4}\kappa$.

Let $D = 2(L+2D_0+1)$ and $\mu = \phi\left(\frac{1}{2}\kappa\right)$. Let β be a geodesic such that $\beta \cap N_\mu(\gamma) = \varnothing$. Let $x, y \in \mathbf{Proj}_\gamma(\beta)$ be such that $d(x, y) = \|\mathbf{Proj}_\gamma(\beta)\|$. We are going to prove that $d(x, y) \leq D$. Suppose by contradiction that $d(x, y) > D$.

Assume that x and y are projection points of $\tilde{x}, \tilde{y} \in \beta$, respectively. Observe that

$$2d(z, [x, \tilde{x}]) \geq d(z, x), \quad 2d(z, [y, \tilde{y}]) \geq d(z, y) \tag{8}$$

for any $z \in [x, y]_\gamma$. We only prove the first inequality; the second one is completely analogous. Let $m \in [x, \tilde{x}]$ such that $d(z, m) = d(z, [x, \tilde{x}])$. Note that $d(m, z) + d(m, \tilde{x}) \geq d(x, \tilde{x})$ by the shortest point property. Since $d(x, \tilde{x}) = d(x, m) + d(m, \tilde{x})$, we obtain $d(m, z) \geq d(x, m)$. Then $d(z, x) \leq d(z, m) + d(m, x) \leq 2d(z, m)$, implying (8).

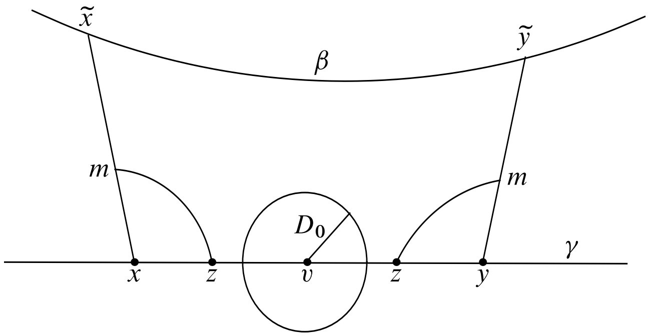

Figure 2

Since $d(x,y) > D$, there exists $z \in [x,y]_\gamma$ such that

$$\min\{d(z,x), d(z,y)\} > \tfrac{1}{2}D = L + 2D_0 + 1.$$

Since γ is (ϵ, R, L)–transitional, one of the intervals $[x,z]_\gamma$ or $[z,y]_\gamma$ contains an (ϵ, R)–transitional point v such that $\min\{d(x,v), d(y,v)\} > 2D_0$. Hence, by (8), $\min\{d(v,[x,\tilde{x}]), d(v,[y,\tilde{y}])\} > D_0$. By the choice of $D_0 = \phi\big(\tfrac{1}{4}\kappa\big)$, we have

$$\max\{\rho_v(x,\tilde{x}), \rho_v(y,\tilde{y})\} < \tfrac{1}{4}\kappa.$$

On the other hand, v is (ϵ, R)–transitional, so $\rho_v(x,y) \ge \kappa$ by Lemma 2.9. Hence, $\rho_v(\tilde{x},\tilde{y}) > \tfrac{1}{2}\kappa$ and thus $d(v,\beta) \le \mu$, which is impossible. □

For a c–quasigeodesic, we denote by $\epsilon_c = \epsilon(c)$ and $R_c = \mathcal{R}(\epsilon_c)$ any numbers satisfying Convention 2.17 (in particular, ϵ_1 and R_1 correspond to geodesics). In the following proposition we will establish the "thinness" of a triangle whose two sides are transitional geodesics.

Proposition 2.19 (transitional triangle is thin) *For any $L, c > 0$ there exist constants $D = D(c)$, $M = M(L,c)$, $L' = L'(L,c) > 0$ with the following properties:*

Let α_1 and α_2 be (ϵ_1, R_1, L)–transitional geodesic rays issuing at o and ending at $\xi \ne \eta \in \Lambda G$, respectively. Then, for any c–quasigeodesic γ with $\gamma_- \in \alpha_1$ and $\gamma_+ \in \alpha_2$, the following holds:

1. *γ is (ϵ_c, R_c, L')–transitional.*
2. *If the length of γ is sufficiently large then there exists an (ϵ_c, R_c)–transitional point $z \in \gamma$ such that $d(z, \alpha_1 \cup \alpha_2) \le D$ and $d(z, \alpha_i) \le M$ for $i = 1, 2$.*
3. *Let $d(o, [\xi,\eta])$ denote the distance from o to a geodesic between ξ and η. If $\min\{d(\gamma_-, o), d(\gamma_+, o)\} \gg 0$. Then $|d(o,[\xi,\eta]) - d(o,\gamma)| \le M$.*

Remark In (2) D is a uniform constant not depending on L; this will play a crucial role in establishing Proposition 5.14 below.

The statement (2) could be deduced from a more general statement, Proposition 4.6(3) of [28]. We provide a proof in our concrete setting for completeness and to illustrate our methods.

Proof Let $\kappa = \kappa(\epsilon_c, R_c)$ be given by Lemma 2.9 and $D = \phi\left(\frac{1}{2}\kappa\right)$, where ϕ is given by Lemma 2.6. The constant L' will be computed below.

(1) Given a point x in γ, assume that x is (ϵ_c, R_c)–deep in some $X \in \mathbb{P}$. Let x_- and x_+ be the entry and exit points of γ in $N_{\epsilon_c}(X)$, respectively.

Observe first that x_- and x_+ are (ϵ_c, R_c)–transitional in γ. Indeed, if not, there exists $Y \in \mathbb{P}$ such that x_- is (ϵ_c, R_c)–deep in Y. Then $Y \neq X$ by the choice of x_- as the entry point of γ in $N_{\epsilon_c}(X)$. Since $d(x, x_-) \geq R_c$, we have $\|N_{\epsilon_c}(X) \cap N_{\epsilon_c}(Y)\| \geq R_c > \mathcal{R}(\epsilon_c)$ by Convention 2.17. This contradicts Lemma 2.15.

To find a constant L' we will find the upper bound on L' for which the opposite inequality

$$\min\{\ell([x, x_-]_\gamma), \ell([x, x_+]_\gamma)\} > L' \tag{9}$$

is not valid. So suppose (9) is true; then $\ell([x_-, x_+]_\gamma) \geq 2L'$. Since x_- and x_+ are (ϵ_c, R_c)–transitional, by Lemma 2.9 we have

$$\min\{\rho_{x_-}(\gamma_-, \gamma_+), \rho_{x_+}(\gamma_-, \gamma_+)\} > \kappa.$$

By the triangle inequality,

$$\max\{\rho_{x_-}(\gamma_-, o), \rho_{x_-}(o, \gamma_+)\} \geq \tfrac{1}{2}\kappa,$$

and the same for ρ_{x_+}. Then $\max\{d(x_-, \alpha_1 \cup \alpha_2), d(x_+, \alpha_1 \cup \alpha_2)\} \leq D = \phi\left(\frac{1}{2}\kappa\right)$. For concreteness consider the case that

$$d(x_-, \alpha_1), d(x_+, \alpha_2) \leq D; \tag{10}$$

the other cases are similar and even easier.

Project x_- and x_+ to $x'_-, x'_+ \in X$ such that $d(x_-, x'_-), d(x_+, x'_+) \leq \epsilon_c$. So

$$d(x'_-, \alpha_1), d(x'_+, \alpha_2) \leq \epsilon_c + D$$

and $N_{D+\epsilon_c}(X) \cap \alpha_i \neq \varnothing$ for $i = 1, 2$.

Let $w \in X$ be a projection point of o to X. We claim that

$$d(w, \alpha_i) \leq D_2 := \max\{D + \epsilon_c + D_1, \mu_1 + D_1\} \quad \text{for } i = 1, 2, \tag{11}$$

where $\mu_1, D_1 > 0$ are given for a 1–contracting $X \in \mathbb{P}$ such that (7) holds.

Indeed, if, first, $o \in N_{\mu_1}(X)$ then there is nothing to prove. If not, there are two more cases; if $\alpha_i \cap N_{\mu_1}(X) = \varnothing$, then by the contracting property we have $d(w, \alpha_i) \leq D + \epsilon_c + D_1$. If $\alpha_i \cap N_{\mu_1}(X) \neq \varnothing$ then the projection on X of the maximal connected subcurve of α_i, situated outside of $N_{\mu_1}(X)$ and containing o, gives $d(w, \alpha_i) \leq \mu_1 + D_1$. So (11) follows.

Since $\epsilon_c + D + D_2 > \epsilon_1$, we consider the function $L_0 = L(\epsilon_c + D + D_2, R_1 + L)$ coming from Lemma 2.16. Set

$$L' = 2c(D_2 + D + L_0 + 2\epsilon_c) + c^2. \tag{12}$$

Since γ is a c–quasigeodesic, we have

$$d(x'_-, x'_+) \geq \frac{\ell([x_-, x_+]_\gamma)}{c} - c - 2\epsilon_c \geq \frac{2L'}{c} - c - 2\epsilon_c \geq 4(\epsilon_c + D + D_2 + L_0).$$

Since $\max\{d(x'_-, \alpha_1), d(x'_+, \alpha_2)\} \leq \epsilon_c + D$, we obtain, from (11),

$$\|\alpha_1 \cap N_{\epsilon_c + D + D_2}(X)\| \geq d(x'_-, w), \quad \|\alpha_2 \cap N_{\epsilon_c + D + D_2}(X)\| \geq d(x'_+, w).$$

We have $d(x'_-, w) + d(x'_+, w) \geq d(x'_-, x'_+) \geq 4(\epsilon_c + D + D_2 + L_0)$. Thus,

$$\max_{i=1,2} \|\alpha_i \cap N_{\epsilon_c + D + D_2}(X)\| \geq 2(\epsilon_c + D + D_2 + L_0).$$

Hence, α_i contains a subcurve of length at least $2L_0$ such that its endpoints lie in $N_{\epsilon_c + D + D_2}(X)$. By the choice of L_0 and Lemma 2.16, α_i contains an $(\epsilon_1, R_1 + L)$–deep point in X. This gives a contradiction, as α_i is (ϵ_1, R_1, L)–transitional. So for the value of L' chosen in (12) the inequality (9) is not valid. The statement (1) is proved.

(2) By the statement (1), γ is (ϵ_c, R_c, L')–transitional. Hence, Lemma 2.18 implies that γ is contracting. By the projection argument (used to prove (11)) we have a constant $D_3 = D_3(\epsilon_c, R_c, L') > 0$ such that for any projection point v of o to γ we have $d(v, \alpha_i) \leq D_3$ for $i = 1, 2$.

Remark We need a new constant D_3 (and not D_2 used above) since we project now on γ and not on a horosphere.

Recall that $D = \phi\left(\frac{1}{2}\kappa\right)$. By Lemma 2.6, for any $z \in G$, a geodesic segment outside $B(z, D)$ has $\mathfrak{l}_z$–Floyd length less than $\frac{1}{2}\kappa$.

The curve γ is quasigeodesic and its length is sufficiently large. So, by continuity of the distance function $d(v, x)$ for $x \in \gamma$ we find a point $z' \in \gamma$ such that $D + D_3 + L' \le d(v, z') \le D + D_3 + L' + 1$. Since γ is (ϵ_c, R_c, L')–transitional, by Definition 2.10 there exists an (ϵ_c, R_c)–transitional point $z \in \gamma$ for which $d(z', z) \le l([z, z']_\gamma) \le L'$. We obtain

$$D + D_3 \le d(v, z) \le D + D_3 + 2L' + 1. \tag{13}$$

Then

$$d(z, \alpha_i) \le d(z, v) + d(v, \alpha_i) \le M$$

with $M = 2L' + 2D_3 + D + 1$ for $i = 1, 2$.

To prove the first claim of (2) assume for definiteness that $z \in [v, \gamma_+]$. Lemma 2.9 yields $\rho_z(v, \gamma_+) \ge \kappa$.

Let $z_2 \in \alpha_2$ be such that $d(v, z_2) \le D_3$. We have $d(z, [v, z_2]) \ge d(z, v) - d(z_2, v) \ge D + D_3 - D_3 = D$. By Lemma 2.6, $\rho_z(v, z_2) < \frac{1}{2}\kappa$ and so

$$\rho_z(z_2, \gamma_+) \ge \rho_z(v, \gamma_+) - \rho_z(v, z_2) \ge \tfrac{1}{2}\kappa.$$

Lemma 2.6 gives

$$d(z, \alpha_2) \le D.$$

The statement (2) is proved.

(3) Since ξ and η are distinct, by Lemma 2.6 there exist $n_0 = n_0(\xi, \eta)$, $r = r(\xi, \eta) > 0$ such that if

$$\min\{d(\gamma_-, o), d(\gamma_+, o)\} > n_0,$$

then $d(o, \gamma) \le r$. In the proof of (2), we projected o to a point v in γ, and found an (ϵ_c, R_c)–transitional point $z \in \gamma$ such that $d(v, z) \le M$.

Since $d(o, z) \le M + r$ (and these constants do not depend on γ) up to increasing n_0, by Lemma 2.6 we have

$$\max\{\overline{\rho}_z(\xi, \gamma_-), \overline{\rho}_z(\eta, \gamma_+)\} \le \tfrac{1}{4}\kappa.$$

The point $z \in \gamma$ is (ϵ_c, R_c)–transitional; thus, $\overline{\rho}_z(\gamma_-, \gamma_+) \ge \kappa$, and so $\overline{\rho}_z(\xi, \eta) \ge \frac{1}{2}\kappa$. Consequently, $d(z, [\xi, \eta]) \le D$, which yields

$$d(o, [\xi, \eta]) \le d(o, z) + d(z, [\xi, \eta]) \le d(o, \gamma) + M + d(z, [\xi, \eta]) \le d(o, \gamma) + M + D.$$

The statement (2) holds for the geodesic $[\xi,\eta]$ as well, so the above argument works for $[\xi,\eta]$ and an (ϵ_c, R_c)–transitional point $z \in [\xi,\eta]$ correspondingly. As a consequence, $d(z,\gamma) \le D$. Thus, $d(o,\gamma) \le d(o,z) + d(z,\gamma) \le d(o,[\xi,\eta]) + M + D$.

Since D is a uniform constant not depending on L, we put $M := M + D$. Then the statements of (2) and (3) are both valid for the same constant M. The proposition is proved. $\square$

Claim (3) of the proposition and Lemma 2.11 imply:

Corollary 2.20 *Suppose $(G,\mathcal{P})$ is a relatively hyperbolic pair. Then, for any $c \ge 1$ and $L > 0$, there exists $M = M(c,L)$ such that for any $\xi,\eta \in \Lambda^{\mathrm{uc}}_{L,o}G$ or $\xi,\eta \in \partial^{\mathrm{uc}}_{L,o}G$, we have $|d(o,[\xi,\eta]) - d(o,\gamma)| \le M$, where γ is a c–quasigeodesic with the endpoints on the corresponding geodesic rays converging to ξ and η.*

3 Patterson–Sullivan measures on ends of a geodesic tree

In this section, we shall construct an iterated transitional tree having several nice properties which will allow us to carry out the Patterson construction on this tree. The space of ends of the tree equipped with the Patterson–Sullivan measure will give rise to an Ahlfors regular subset of the boundary.

3.1 Iterated transitional trees

Let $(G,\mathcal{P})$ be a relatively hyperbolic pair and $\mathscr{G}(G,S)$ the Cayley graph of G with respect to S. The existence of large transitional trees is established in [36, Theorem 5.9]. The main difference of the construction below is that these trees will be equipped with certain periodicity. For this reason we call them *iterated transitional trees*. We start by recalling several results from [36].

Definition 3.1 (partial cone) For $\epsilon, R \ge 0$, the *partial cone* $\Omega_{\epsilon,R}(g)$ at $g \in G$ is the set of elements $h \in G$ such that there exists a geodesic $\gamma = [1,h]$ containing g and one of the following holds:

(1) $d(1,h) \le d(1,g) + 2R$.

(2) γ contains an (ϵ, R)–transitional point v such that $d(v,g) \le 2R$.

For $\Delta \geq 0, n \geq 0$, define

$$A(g,n,\Delta) = \{h \in G : n - \Delta \leq d(1,h) - d(1,g) < n + \Delta\}$$

for any $g \in G$. For simplicity we write $A(n,\Delta) := A(1,n,\Delta)$. For $r, \epsilon, R, \Delta > 0$, define

$$\Omega_{\epsilon,R}(g,n,\Delta) = \Omega_{\epsilon,R}(g) \cap A(g,n,\Delta)$$

for any $g \in G$ and $n \geq 0$.

For fixed $\epsilon, R > 0$, two partial cones $\Omega_{\epsilon,R}(g)$ and $\Omega_{\epsilon,R}(g')$ are of the *same type* if

$$g' g^{-1} \cdot \Omega_{\epsilon,R}(g) = \Omega_{\epsilon,R}(g').$$

By abuse of language, we say that g and g' have the same partial cone types.

The following result generalizes the result of Cannon [4] for hyperbolic groups:

Lemma 3.2 (finiteness of partial cone types [36, Lemma B.1]) *There exist $\epsilon, R_0 > 0$ such that for any $R > R_0$, there are at most $M = M(\epsilon, R)$ types among all (ϵ, R)–partial cones $\{\Omega_{\epsilon,R}(g) : g \in G\}$.*

The following is a key technical result of [36, Lemma 5.8]:

Lemma 3.3 *There exist $\epsilon, R, \Delta, \theta, L_0 > 0$ with the following property:*

For any $L > L_0$ there exists a subset $\widehat{G}$ of G such that

$$\#(\Omega_{\epsilon,R}(g,L,\Delta) \cap \widehat{G}) > \theta \cdot \exp(L \cdot \delta_{G,S}), \quad 1 \in \widehat{G} \tag{14}$$

for any $g \in \widehat{G}$.

Convention 3.4 (ϵ, R, Δ) Until the end of Section 3, the constants $\epsilon, R, \Delta > 0$ are given by Lemmas 3.2 and 3.3, and satisfy Convention 2.17.

The following terminology comes from [1], which was certainly very motivating for us:

Definition 3.5 (iterated tree set) For given $L > 0$, an *L–iterated tree set T* in G is a union of a sequence of sets T_i for $i \geq 0$ in G defined inductively as follows:

Let $T_0 = \{1\}$. Assume that T_i is defined for $i \geq 0$. The set of *children* $T(x)$ of $x \in T_i$ is a subset in $\Omega_{\epsilon,R}(x,L,\Delta)$. Then T_{i+1} is the union of children of all $x \in T_i$.

Recall that a subset Z of a metric space (X,d) is called C*–separated* if $d(z_1,z_2)\geq C$ for every pair of distinct points $\{z_1,z_2\}\subset Z$. The following fact is elementary:

Lemma 3.6 *Let (X,d) be a proper metric space on which a group $G\subset \mathrm{Isom}(X)$ acts properly. For any orbit Go with $o\in X$ and $C>0$, there exists a constant $\theta=\theta(Go,C)>0$ with the following property:*

For any finite set Y in Go, there exists a C–separated subset $Z\subset Y$ such that $\#Z\geq\theta\cdot\#Y$.

Proof Let Z be a maximal C–separated set in Y. We have $Y\subset N_C(Z)$. Since the action of G on (X,d) is proper, any ball of radius C contains at most N points of Go. The result follows for $\theta:=1/N$. □

An (ϵ,R,L)*–transitional geodesic tree* $\mathcal{T}$ rooted at o in $\mathscr{G}(G,S)$ is a tree subgraph with a distinguished vertex o such that every branch in $\mathcal{T}$ originating at o is a (ϵ,R,L)–transitional geodesic in $\mathscr{G}(G,S)$.

In order to obtain a useful theory of Patterson–Sullivan measures, certain symmetry on the iterated tree set is required. This is the content of the following key result:

Lemma 3.7 (existence of iterated transitional trees) *There exist constants L_0, C_0, t_0, $n_0>0$ such that for $L>L_0$ and $C>C_0$ there are $\theta=\theta(C)>0$ and $L'=L'(L)>0$ and an iterated tree set T parametrized by (ϵ,R,L) with the following properties:*

(1) $x^{-1}T(x)=y^{-1}T(y)$ *for any* $x\in T_t$, $y\in T_{t+n_0}$ *and* $t\geq t_0$.
(2) $x^{-1}T(x)=y^{-1}T(y)$ *for any* $x,y\in T_t$ *and* $t\geq t_0$.
(3) $\#T(x)\geq\theta\cdot\exp(\delta_{G,S}L)$ *for any* $x\in T$.
(4) $T(x)$ *is C–separated for any* $x\in T$.
(5) *There exists an (ϵ,R,L')–transitional geodesic tree $\mathcal{T}$ rooted at 1 in $\mathscr{G}(G,S)$ such that the vertex set $\mathcal{T}^0$ contains T and lies in $N_{L'}(T)$.*
(6) *Any two distinct infinite branches originating at 1 terminate at distinct endpoints in ΛG.*

Proof Set $L_0=\Delta$, and all other constants will be defined in the proof. We divide the proof into three steps for the reader's convenience.

Step 1 At this step we construct the iterated tree set T with properties (1)–(3). The construction proceeds by an induction argument. Set $T_0=\{1\}$ to start.

Let M be the number of (ϵ, R)–partial cone types in G given by Lemma 3.2, and $\widehat{G}$ the set given by Lemma 3.3. Then there exists $T_1 \subset \Omega_{\epsilon,R}(1, L, \Delta) \cap \widehat{G}$ such that every element in T_1 has the same partial cone type and the inequality (14) holds for $g = 1$, where the constant θ is divided by M. By Lemma 3.6, we can also arrange that T_1 is C–separated, where θ is further decreased and depends on C (given in Step 3 below).

Fix some $x_1 \in T_1$. Up to dividing θ by M again, we choose Y to be a subset of $\Omega_{\epsilon,R}(x_1, L, \Delta) \cap \widehat{G}$ such that the inequality (14) holds for Y and every element in Y has the same partial cone type. By the same reason, we can choose Y to be C–separated. Since all $x \in T_1$ are of same type as x_1, we could define

$$T(x) := x x_1^{-1} Y \subset \Omega_{\epsilon,R}(x, L, \Delta).$$

Then all elements in the union $T_2 := \bigcup_{x \in T_1} T(x)$ have the same partial cone types. We note that Y is chosen to be contained in $\widehat{G}$, but T_2 may not be in $\widehat{G}$.

We repeat the same argument to construct T_i for $i \geq 3$, with a sequence of divisions of θ. By construction, all elements in the constructed T_i are of the same partial cone type. Since there are at most M partial cone types, we obtain that there are $1 \leq t_0, n_0 \leq M$ such that all elements in T_{t_0} and $T_{t_0+n_0}$ have the same cone type. It follows that the set of levels $\{T_i\}_i$ becomes periodic for $i \geq t_0$ with the period of length n_0: the levels between $T_{t_0+kn_0}$ and $T_{t_0+(k+1)n_0}$ repeat those between T_{t_0} and $T_{t_0+n_0}$ for all $k \geq 1$. So we obtain $x^{-1}T(x) = y^{-1}T(y)$ for any $x \in T_t$, $y \in T_{t+n_0}$ and $t \geq t_0$. This also implies that we need to divide θ by M at most n_0 times, so θ in the inequality (14) can be chosen uniformly for all $T(x)$ for $x \in T_i$ and $i \geq 1$. Thus, the set T satisfies the properties (1)–(3).

Step 2 Using the iterated tree set T, we will now construct a geodesic graph $\mathcal{T}$.

Without loss of generality assume that $\theta < 1$. The root of $\mathcal{T}$ is $\mathcal{T}_0 = \{1\}$. Assume that $\mathcal{T}_i$ is defined for $i \geq 0$ and for each terminal vertex $x \in \mathcal{T}_i$, denote by γ_x the geodesic $[1, x]$ in $\mathcal{T}_i$. Since $T_i(x)$ is a subset of $\Omega_{\epsilon,R}(x, L, \Delta) \cap \widehat{G}$, by definition of a partial cone for $L_0 > R$ and for every $y \in T(x)$ we can choose a geodesic $[x, y]$ so that $[1, y] = \gamma_x \cdot [x, y]$ is a geodesic in $\mathscr{G}(G, S)$ containing an (ϵ, R)–transitional point v which is $2R$–close to x.

We set

$$\mathcal{T}_{i+1} = \bigcup_{x \in \mathcal{T}_i} \bigcup_{y \in T(x)} \gamma_x \cdot [x, y]. \tag{15}$$

Let $\mathcal{T} = \lim_{i\to\infty} \mathcal{T}_i$. By construction, each geodesic ray originating at 1 is (ϵ, R, L')–transitional, and $T \subset \mathcal{T}^0 \subset N_{L'}(T)$.

Remark A priori, the geodesic $[1, y]$ is not unique but any choice gives rise to the same constants. Indeed, suppose we have two such geodesics γ_y and γ'_y both passing through x and such that γ_y contains an (ϵ, R)–transitional point v $2R$–close to x. Then, by Lemmas 2.9 and 2.6, the geodesic γ'_y also contains an (ϵ, R)–transitional point which has a uniform distance from v only depending on ϵ and R. Up to increasing R by this uniform amount, the geodesic γ'_y is (ϵ, R, L')–transitional for $L' := L + 2R + \Delta$.

Step 3 We now prove that $\mathcal{T}$ is a geodesic tree rooted at 1 in $\mathscr{G}(G, S)$.

Indeed, if it is not a tree, there exist two distinct geodesics α_1 and α_2 in $\mathcal{T}$ with the same endpoints $x, w \in T$ such that the length of α_1 and α_2 is minimal among all such choices. Assume that x is closer to 1 than w. Consider two points $y_i \in \alpha_i \cap T(x)$ for $i = 1, 2$. By the choice of α_1 and α_2, we have $y_1 \neq y_2$. Then, by construction, $d(y_i, x) \geq L - \Delta$ for $i = 1, 2$. Moreover, there exists an (ϵ, R)–transitional point $z_1 \in \alpha_1$ such that $d(y_1, z_1) \leq 2R$.

Let $D_0 = \phi(\kappa)$, where ϕ is given by Lemma 2.6 and $\kappa = \kappa(\epsilon, R)$ by Lemma 2.9. There exists $z_2 \in \alpha_2$ such that $d(z_1, z_2) \leq D_0$ and then $d(y_1, z_2) \leq 2R + D_0$. We can choose $\tilde{y}_2 \in \alpha_2$ such that $d(x, \tilde{y}_2) = d(x, y_1)$. Hence, $d(z_2, \tilde{y}_2) = |d(x, z_2) - d(x, \tilde{y}_2)| = |d(x, z_2) - d(x, y_1)| \leq 2R + D_0$. It follows that $d(y_1, \tilde{y}_2) \leq 2(2R + D_0)$.

Since y_1 and y_2 lie in the annulus $A(x, L, \Delta)$, we get $|d(x, y_1) - d(x, y_2)| \leq 2\Delta$, then $d(y_2, \tilde{y}_2) = |d(x, \tilde{y}_2) - d(x, y_2)| \leq 2\Delta$. It follows that $d(y_1, y_2) \leq 2(2R + D_0 + \Delta)$. Choosing now the constant C to be greater than

$$C_0 := 2\left(2R + \phi\left(\tfrac{1}{2}\kappa\right) + \Delta\right), \tag{16}$$

we obtain that $T(x)$ is C–separated in $\Omega_{\epsilon,R}(x, L, \Delta)$, and

$$d(y_1, y_2) \geq C_0 > 2(2R + D_0 + \Delta),$$

which is a contradiction. Thus, $\mathcal{T}$ is a rooted geodesic tree, satisfying assertion (5).

To prove (6), we argue by contradiction. Assume that two distinct geodesic rays α_1 and α_2 in $\mathcal{T}$ terminate at the same boundary point $\xi \in \Lambda G$. Since α_1 is (ϵ, R, L')–transitional, we proceed by the same argument as above in proving that $\mathcal{T}$ contains no loop, and arrive at a contradiction with the assumption that $T(x)$ is C–separated. The proof of the lemma is thus complete. $\square$

Remarks (1) By Lemma 2.11 the boundary of the tree $\mathcal{T}$ (in ΛG or in $\partial_\lambda G$) constructed above consists of uniformly conical points.

(2) The constant C_0 in (16) is bigger than we really need in the above proof (it is enough to replace $\phi(\frac{1}{2}\kappa)$ by the smaller term $D_0 = \phi(\kappa)$) but we do need such a constant in the next lemma.

In the next two lemmas, we shall derive more properties of the sets T and $\mathcal{T}$ constructed in Lemma 3.7. To this end, we recall the notion of Poincaré series.

For a subset $X \subset G$ and a point $o \in G$, set

$$\Theta_X(s, o) = \sum_{g \in X} \exp(-s d(o, g)), \quad s \geq 0.$$

Define the *critical exponent* of $\Theta_X(s, o)$ to be

$$\delta_{X,S} = \limsup_{n \to \infty} \frac{\log \#(B(o, n) \cap X)}{n}, \tag{17}$$

where S is a fixed finite symmetric generating set of G, and $B(o, n)$ is the ball in the word metric of radius n centered at o.

It is an elementary fact that $\Theta_X(s, o)$ converges for $s > \delta_{X,S}$ and diverges for $s < \delta_{X,S}$.

Recall that the bilipschitz equivalence $\asymp_{\mathrm{const}}$ between two functions means that they are comparable up to a constant (see Section 2). We have the following:

Lemma 3.8 *Under the same assumptions as in Lemma 3.7, we have*

$$\Theta_T(s, x) \asymp_L \Theta_T(s, y)$$

for any $x, y \in T$ and $s \geq 0$ whenever one of the series converges.

Proof Let $\Omega(x)$ be a *cone* at $x \in T$, which is the union of all points $y \in T$ such that the unique geodesic $[1, y]$ in the geodesic tree $\mathcal{T}$ contains x.

Claim *The Poincaré series of T is bilipschitz equivalent to that of any cone at a vertex in T:*

$$\Theta_T(s, 1) \asymp_L \Theta_{\Omega(x)}(s, x) \tag{18}$$

for any $x \in T$.

Proof of the claim Let $S_n(o)$ denote the sphere of radius n centered at o in G. We have

$$\Theta_X(s,o) = \sum_{n\geq 0} \#(S_n(o)\cap X)\cdot \exp(-ns), \quad s \geq 0.$$

Hence, to compare $\Theta_T(s,1)$ with $\Theta_{\Omega(x)}(s,x)$, it is enough to show that the numbers $\#(S_n(1)\cap T)$ and $\#(S_n(x)\cap\Omega(x))$ are uniformly bilipschitz equivalent for all $x\in T$. Recall that by Lemma 3.7(1), after a finite time t_0, the set T is periodic with a fixed period n_0. So it is enough to show (18) for $x\in T$ such that $t_0\leq d(1,x)\leq n_0+t_0$. Obviously we have

$$\#(S_n(x)\cap\Omega(x)) \leq \#(S_{n+d(1,x)}(1)\cap T).$$

Furthermore, by Lemma 3.7(2), the cones based at all points of T_t have the same type for every fixed t. So if C denotes the cardinality of $T_{n_0+t_0}$, then

$$\#(S_{n+d(1,x)}(1)\cap T) \leq C\cdot\#(S_n(x)\cap\Omega(x)).$$

Hence,

$$\#(S_n(1)\cap T) \asymp \#(S_n(x)\cap\Omega(x))$$

for all x in T. The claim follows. □

To complete the proof of the lemma, by (18), it suffices to establish

$$\Theta_T(s,1) \asymp_L \Theta_{T\setminus\Omega(x)}(s,x) \quad (x\neq 1), \tag{19}$$

as (18) and (19) would imply $\Theta_T(s,1)\asymp_L \Theta_T(s,x)$ for all $x\in T$.

For $y\in T\setminus\Omega(x)$, let o be the farthest point from 1 such that $o\in T$ and $[1,o]\subset[1,x]\cap[1,y]$, where the geodesics $[1,x]$ and $[1,y]$ are in the geodesic tree $\mathcal{T}$. The point o will be referred to as the *branch point* of $[1,x]$ and $[1,y]$.

By construction of the tree $\mathcal{T}$ the geodesic rays $[1,x]$, $[1,y]$ are parts of distinct infinite (ϵ,R,L')–transitional geodesics (where $L'=L'(L)$), which by Lemma 3.7(6) end at distinct boundary points. By Proposition 2.19, $[x,y]$ is transitional and so is contracting by Lemma 2.18.

Claim *There exists a uniform constant $D=D(L)>0$ such that $d(o,[x,y])\leq D$.*

Proof of the claim Let $z\in[x,y]$ be the projection of o to a geodesic $[x,y]$ in the Cayley graph $\mathscr{G}(G,S)$. By the contracting property of $[x,y]$ it follows from the inequality (11) that there exists $D_1 = D_1(\epsilon, R, L')$ such that

$$\max\{d(z,[o,x]), d(z,[o,y])\} \le D_1.$$

So, let $x_1 \in [o,x]$ and $y_1 \in [o,y]$ be such that $d(z,x_1) \le D_1$ and $d(z,y_1) \le D_1$.

Set $d(o,z) = d$; then

$$\min\{d(o,x_1), d(o,y_1)\} \ge d - D_1. \tag{20}$$

Let $w \in [o,x_1] \cap T(o)$, where $T(o) \subset \Omega_{\epsilon,R}(o,L,\Delta)$. Then $d(o,w) < L+\Delta$. Furthermore, since $x \in T$ there exists an (ϵ,R)–transitional point $x_2 \in [o,x]$ such that $d(w,x_2) \le 2R$, and so $d(o,x_2) \le L+\Delta+2R$. Using (20) we deduce

$$d(x_2,[x_1,y_1]) \ge d(x_1,o) - d(x_2,o) - 2D_1 \ge K, \tag{21}$$

where $K = d - 3D_1 - L - \Delta - 2R$.

We assert that

$$K \le \phi\left(\tfrac{1}{2}\kappa\right), \tag{22}$$

where κ and $\phi\left(\frac{1}{2}\kappa\right)$ are universal constants given by Lemmas 2.9 and 2.6, respectively. Indeed, suppose (22) is not true; then $d(x_2,[x_1,y_1]) \ge \phi\left(\frac{1}{2}\kappa\right)$. By Lemma 2.6 we have $\rho_{x_2}(x_1,y_1) \le \frac{1}{2}\kappa$. Since x_2 is transitional, Lemma 2.9 yields $\rho_{x_2}(o,x_1) \ge \kappa$. It follows that $\rho_{x_2}(o,y_1) \ge \frac{1}{2}\kappa$, and thus $d(x_2,[o,y]) = d(x_2,\tilde{x}_2) \le \phi\left(\frac{1}{2}\kappa\right)$ for some $\tilde{x}_2 \in [o,y]$.

Following the argument of Step 3 of Lemma 3.7 we choose a vertex $w' \in [o,y]$ such that $d(o,w') = d(o,w)$. Then $d(\tilde{x}_2,w') = |d(o,\tilde{x}_2) - d(o,w)| \le d(\tilde{x}_2,w) \le 2R + \phi\left(\frac{1}{2}\kappa\right)$. Then $d(w',w) \le d(w',\tilde{x}_2) + d(\tilde{x}_2,w) \le 2\left(2R + \phi\left(\frac{1}{2}\kappa\right)\right)$. Let $w'' \in T(o) \cap [o,y]$. Since $w \in \Omega_{\epsilon,R}(o,L,\Delta)$, we have $d(w'',w') \le |d(o,w') - d(o,w'')| \le 2\Delta$. Indeed $|(d(o,w') = d(o,w)) - L| \le \Delta$ and also $|d(o,w'') - L| \le \Delta$. Therefore, for the vertices $w'', w \in T(o) \subset T$ we have $d(w,w'') \le 2\left(\phi\left(\frac{1}{2}\kappa\right) + 2R + \Delta\right)$. This is impossible by (16). The obtained contradiction implies that $K \le \phi\left(\frac{1}{2}\kappa\right)$ and by definition of K (see (21)), we have

$$d(o,z) = d \le D = 3D_1 + \phi\left(\tfrac{1}{2}\kappa\right) + L + \Delta + 2R.$$

The claim is proved. □

The second claim implies

$$d(o,x) + d(o,y) \ge d(x,y) \ge d(o,x) + d(o,y) - 2D. \tag{23}$$

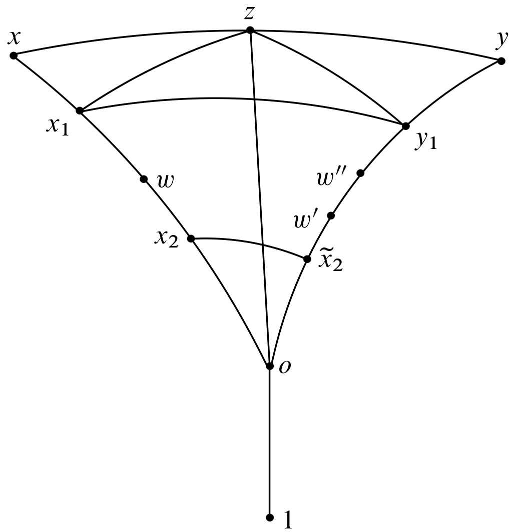

Figure 3

Given $o \in [1, x) \cap T$, we denote by Y_o the set of elements $y \in T \setminus \Omega(x)$ such that $o \in [1, x]$ is the branch point of $[1, y]$ and $[1, x]$ in $\mathcal{T}$. The argument of the first claim also yields

$$\Theta_{\Omega(o)}(s, o) \asymp_L \Theta_{Y_o}(s, o).$$

Then (18) and (23) imply

$$\sum_{y \in Y_o} \exp(-sd(x, y)) \asymp_L \exp(-sd(o, x)) \cdot \Theta_T(s, 1)$$

for every $o \in [1, x) \cap T$. By construction of T in Lemma 3.7 the sequence of points $[1, x) \cap T$ has the property that any two consecutive points have distance between $L - \Delta$ and $L + \Delta$. Summing up over all $o \in [1, x) \cap T$, we get

$$\sum_{y \in T \setminus \Omega(x)} \exp(-sd(x, y)) \asymp_L \sum_{0 \le k < d(1,x)} \exp(-sk) \cdot \Theta_T(s, 1) \asymp_L \Theta_T(s, 1),$$

which proves (19). The lemma is proved. □

Lemma 3.9 *Under the same assumptions as in Lemma 3.7, the Poincaré series $\Theta_T(s, 1)$ is divergent at $s = \delta_{T,S}$. Furthermore, $\lim_{L \to \infty} \delta_{T,S} = \delta_{G,S}$.*

Proof This is inspired by the proof of Proposition 4.1 in [7]. Consider the annulus set in T,

$$A_T(g, n, 3\Delta_0) := A(g, n, 3\Delta_0) \cap T,$$

where $\Delta_0 := \Delta + L + 2R$ and $n \geq 0$. Observe that there exists $c > 1$ such that

$$c^{-1} \cdot \#A_T(g', n, 3\Delta_0) \leq \#A_T(g, n, 3\Delta_0) \leq c \cdot \#A_T(g', n, 3\Delta_0) \tag{24}$$

for any $g, g' \in T$ and $n \geq 0$. Indeed, it is a direct consequence of Lemma 3.7 that T has certain periodicity. Moreover, we claim that:

Claim *The following inequality holds:*

$$\#A_T(1, n+m, 3\Delta_0) \leq c \cdot \#A_T(1, n, 3\Delta_0) \cdot \#A_T(1, m, 3\Delta_0)$$

for $n, m \geq 0$.

Proof of the claim For $h \in A_T(n+m, 3\Delta_0)$, we connect 1 and h by a geodesic $[1, h]$ in $\mathcal{T}$. Assume that $d(1, h) = m + n + 3\Delta_1$ for some $|\Delta_1| \leq \Delta_0$. Let $z \in [1, h]$ be such that $d(1, z) = n + \frac{3}{2} \cdot \Delta_1$. Note that z might not be in T. However, by Lemma 3.7(5), there exists $w \in T$ such that $d(z, w) \leq L'$ where L' was fixed to be $\Delta_0 = \Delta + L + 2R$ (see Step 2 in the proof of Lemma 3.7). Then $d(w, h) \leq m + 3\Delta_0$. This implies that $w \in A(1, n, 3\Delta_0)$ and $h \in A_T(w, m, 3\Delta_0)$. The conclusion thus follows from (24). □

Define $a_n = c \cdot \#A_T(1, n, 3\Delta_0)$. The above claim implies that $a_{n+m} \leq a_n a_m$. So the sequence $(\log a_n)_n$ is subadditive. Then, by Fekete's lemma, $\lim_{n\to\infty} (\log a_n)/n = \inf\{(\log a_n)/n : n \geq 1\}$. Since $(a_n)_n$ is nondecreasing we have $a_n \leq \sum_{0\leq i\leq n} a_i \leq n a_n$. So

$$\delta_{T,S} = \limsup_{n\to\infty} \frac{\log \sum_{0\leq i\leq n} a_i}{n} = \lim_{n\to\infty} \frac{\log a_n}{n} = \inf\left\{ \frac{\log a_n}{n} : n \geq 1 \right\}.$$

It follows that $\#A_T(1, n, \Delta_0) \geq c^{-1} \exp(n\delta_{T,S})$ for $n \geq 1$. Observe that

$$\Theta_T(s, 1) \asymp_{L,\Delta} \sum_{n\geq 0} \#A_T(1, n, \Delta_0) \cdot \exp(-sn), \quad s \geq 0,$$

whenever both parts are finite. Thus, $\Theta_T(s, 1)$ is divergent at $s = \delta_{T,S}$.

To prove the second statement we estimate the lower bound of $\delta_{T,S}$. By Lemma 3.7, we notice that

$$\#(B(1, i(L+\Delta)) \cap T) \geq \theta^i \cdot \exp(i \cdot \delta_{G,S} \cdot L)$$

for $i \geq 0$. This implies that

$$\delta_{T,S} \geq \frac{\log \# B_T(1, i(L+\Delta)) \cap T}{i(L+\Delta)} \geq \frac{L \cdot \delta_{G,S} + \log\theta}{L+\Delta}.$$

We obtain $\lim_{L\to\infty} \delta_{T,S} \geq \delta_{G,S}$. Since $\delta_{T,S} \leq \delta_{G,S}$ for all L, the lemma follows. □

3.2 Patterson–Sullivan measures on the space of ends of an iterated transitional tree

In this and the next subsections, for any $L \gg 0$, let T and $\mathcal{T}$ be the iterated tree set and transitional tree, respectively, given by Lemma 3.7. At the same time, assume that they satisfy Lemmas 3.8 and 3.9.

We denote by the common notation ∂T the limit set of T in either the Bowditch boundary ΛG or in the Floyd boundary $\partial_\lambda G$. In this subsection, we shall construct a Patterson–Sullivan measure on ∂T.

Consider the set $\mathcal{M}(\widetilde{T})$ of finite Borel measures on the compact space $\widetilde{T} := T \cup \partial T$, which is endowed with the weak-convergence topology. Then $\mu_n \to \mu$ for $\mu_n \in \mathcal{M}(\widetilde{T})$ if and only if $\liminf_{n\to\infty} \mu_n(U) \geq \mu(U)$ for any open set $U \subset \widetilde{T}$. Note that a set of uniformly bounded measures in $\mathcal{M}(\widetilde{T})$ is relatively compact.

We first construct a family of measures $\{\mu_v^s\}_{v\in T} \subset \mathcal{M}(\widetilde{T})$ supported on T. Set

$$\mu_v^s = \frac{1}{\Theta_T(s,1)} \sum_{g\in T} \exp(-s d(v,g)) \cdot \mathrm{Dirac}(g), \tag{25}$$

where $s > \delta_{T,S}$ and $v \in T$. By Lemma 3.8, the measures $\{\mu_v^s\}_{v\in T}$ are bounded by a uniform constant depending on L.

By Lemma 3.9, $\Theta_T(s,v)$ is divergent at $s = \delta_{T,S}$ for any $v \in T$. Choose $s_i \to \delta_{T,S}$ such that $\mu_v^{s_i}$ converge in $\mathcal{M}(\widetilde{T})$. The limit measures $\mu_v = \lim \mu_v^{s_i}$ are called *Patterson–Sullivan measures* at v. Clearly, $\{\mu_v\}_{v\in G}$ are absolutely continuous with respect to each other.

In the sequel, we will write "PS measures" as shorthand for Patterson–Sullivan measures.

A horofunction cocycle $B_\xi : G \times G \to \mathbb{R}$ at conical points $\xi \in \Lambda G$ or $\xi \in \partial_\lambda G$ was studied in [36]. The precise definition is not relevant here, but we have the following estimation:

Lemma 3.10 [36, Lemma 2.20] *For any $L > 0$ there exists $C = C(L) > 0$ such that the following holds:*

Fix $\xi \in \partial T$. For any $x, y \in G$, there is a neighborhood V of ξ in $\overline{G}_\lambda$ or $G \cup \Lambda G$ such that

$$|B_\xi(x,y) - B_z(x,y)| < C \quad \textit{for all } z \in V \cap G,$$

where $B_z(x,y) := d(z,x) - d(z,y)$.

Remarks (on the proof) The above statement is proved in [36, Lemma 2.20] for a conical point of the Bowditch boundary, where the constant C is universal (not depending on L). In our setting, by Lemma 3.7 there exists an (ϵ, R, L')–transitional ray in the tree $\mathcal{T}$ ending at ξ in ∂T. Then by Lemma 2.11 the constant R is uniform for every $\xi \in \partial T$. So the same proof as [36, Lemma 2.20] works to produce a constant $C = C(L)$.

We have to warn the reader that the constant $C > 0$ cannot be made uniform for all conical points for the action $G \curvearrowright \partial_\lambda G$ on the Floyd boundary as the action is not necessarily geometrically finite (see the discussion after Lemma 2.11).

With the help of Lemma 3.10, the following can be proven exactly as [5, Théorème 5.4]:

Lemma 3.11 *PS measures $\{\mu_g\}_{g \in T}$ on ∂T satisfy the property*

$$\frac{d\mu_g}{d\mu_h}(\xi) \asymp_L \exp(-\delta_{T,S} B_\xi(g,h)) \tag{26}$$

for μ_h–ae points $\xi \in \partial T$ and any $g, h \in T$.

3.3 Shadow lemma

We shall establish a shadow lemma for $\{\mu_g\}_{g \in T}$ on ∂T. We first introduce two notions of shadows in the Floyd boundary and Bowditch boundary. In this subsection, we are actually interested in (the part of) shadows in ∂T, ie the intersection with ∂T.

By abuse of notation, the following notions of (strong) shadows belonging to which boundary should be clear in the context.

Definition 3.12 (shadow) The *shadow* $\Pi_r(g)$ is the set of points $\xi \in \Lambda G$ (or $\xi \in \partial_\lambda G$) such that for some geodesic $[1,\xi]$ we have $[1,\xi] \cap B(g,r) \neq \varnothing$. The *strong shadow* $\mathit{\Pi}_r(g)$ is the set of points $\xi \in \Lambda G$ (or $\xi \in \partial_\lambda G$) such that for any geodesic $[1,\xi]$ we have $[1,\xi] \cap B(g,r) \neq \varnothing$.

Lemma 3.13 (shadow lemma) *There exists $r_0 > 0$ such that*

$$\exp(-\delta_{T,S} d(1,g)) \prec \mu_1(\Pi_r(g) \cap \partial T) \prec_r \exp(-\delta_{T,S} d(1,g))$$

for any $r > r_0$ and $g \in T$.

Remark In [36] the shadow lemma was proved for the whole group G. The current lemma describes the shadows of the points $g \in T$ in terms of $\delta_{T,S}$.

Proof By Lemmas 3.10 and 3.11, there exist $C_1 = C_1(L)$, $C_2 = C_2(L) > 0$ such that

(27) $$C_1 \exp(-\delta_{T,S} d(1,g)) \le \frac{d\mu_1}{d\mu_g}(\xi) \le C_2 \exp(-\delta_{T,S} d(1,g))$$

for μ_1–ae points $\xi \in \partial T$. So in order to estimate $\mu_1(\Pi_r(g) \cap \partial T)$ we can do it for $\mu_g(\Pi_r(g) \cap \partial T)$.

Claim *Given any $\epsilon > 0$, there is a constant $r_0 > 0$ such that*

$$\mu_g(\partial T \setminus \Pi_r(g)) < \epsilon$$

for all $g \in T$ and $r > r_0$.

Proof of the claim Note that $\Pi_r(g)$ is a closed set. We consider the *convex cone* $\mathcal{C}(\partial T \setminus \Pi_r(g))$ of $\partial T \setminus \Pi_r(g)$, which consists of all geodesic rays in $\mathcal{T}$ originating at 1 and terminating at a point in $\partial T \setminus \Pi_r(g)$. Let V be the set of vertices of T in $\mathcal{C}(\partial T \setminus \Pi_r(g))$.

For any $x \in V$, consider the branch point o of $[1,x]$ and $[1,g]$ in $\mathcal{T}$ (defined in the proof of Lemma 3.8). Since $x \notin \mathcal{C}(\Pi_r(g))$, we have $d(g,o) > r$. By a similar argument to that of Lemma 3.8, we get

$$\Theta_V(s,g) = \sum_{x\in V} \exp(-sd(x,g)) \asymp \sum_{r\le k\le d(1,g)} \exp(-sk)\cdot\Theta_T(s,1).$$

So,

$$\mu_g^s(V) = \frac{\Theta_V(s,g)}{\Theta_T(s,1)} \asymp \sum_{r\le k\le d(1,g)} \exp(-sk),$$

which tends to 0 when $r \to \infty$ and $s > \delta_{T,S}$.

Thus, the μ_g^s–measure of the open set $V \cup (\partial T \setminus \Pi_r(g))$ can be arbitrarily small for r large enough, and so is $\mu_g(\partial T \setminus \Pi_r(g))$. This proves the claim. □

By Lemma 3.8, $\{\mu_g(\partial T)\}_{g\in T}$ are bounded above and below by uniform constants depending on L. Let $\eta_1 = \frac{1}{2}\inf\{\mu_g(\partial T) : g \in T\} > 0$ and $\eta_2 = \sup\{\mu_g(\partial T) : g \in T\} < \infty$. By the above claim, there is a constant $r_0 > 0$ such that

(28) $$\eta_1 < \mu_g(\Pi_r(g) \cap \partial T) < \eta_2 \quad \text{for all } r > r_0$$

for all $g \in T$. So (27) implies that

$$\eta_1 C_1 \exp(-\delta_{T,S} d(1,g)) \le \mu_1(\Pi_r(g) \cap \partial T) \le \eta_2 C_2 \exp(-\delta_{T,S} d(1,g))$$

for all $g \in G$. The lemma is proved. □

The above shadow lemma allows one to estimate the PS measure of shadows. Our goal is now to express the latter one in terms of the PS measure of balls on the boundary ∂T. Below, we use the symbol $\lfloor s \rfloor$ to denote the integer part of $s \in \mathbb{R}$. Denote by $B_{\rho_{\lambda,1}}(\xi, t)$ (resp. $B_{\bar\rho_{\lambda,1}}(\xi, t)$) the ball in ∂T around $\xi \in \partial T$ of radius t with respect to the metric $\rho_{\lambda,1}$ (resp. $\bar\rho_{\lambda,1}$).

Until the end of this subsection let us fix $\lambda_0 \in (0,1)$ given by Proposition 2.3. For any $\lambda \in [\lambda_0, 1)$, we consider the family of Floyd metrics $\{\rho_{\lambda,v}\}_{v\in G}$ and the corresponding shortcut metrics $\{\bar\rho_{\lambda,v}\}_{v\in G}$. The following two lemmas are general facts without involving the ∂T:

Lemma 3.14 (shadows $\subseteq$ balls) *For any $r > 0$, $\lambda \in [\lambda_0, 1)$ and $\xi \in \Lambda G$, there exists $C = C(r, \lambda)$ such that*

$$\Pi_r(g) \subset B_{\bar\rho_{\lambda,1}}(\xi, Ct)$$

for any $0 < t < \lambda$, where $g \in [1, \xi]$ is chosen so that $d(1, g) = \lfloor \log_\lambda t \rfloor$. The same conclusion holds for $\rho_{\lambda,1}$ with $\xi \in \partial_\lambda G$.

Proof Let $\eta \in \Pi_r(g)$ so that $d(g, [1,\eta]) \le r$ for some geodesic $[1,\eta]$. Consequently, there exists $w \in [1,\eta]$ such that $d(1, w) = d(1, g)$ and $d(g, w) \le 2r$. By Lemma 5.1, any segment of $[1, \xi]$ is a Floyd geodesic with respect to $\rho_{\lambda,1}$, so $\rho_{\lambda,1}(\xi, g) = \lambda^{d(1,g)}/(1-\lambda)$. Let α be a word geodesic between w and g. Every edge of α is in the word distance at least $d(1,g) - 2r$ from 1. So the Floyd length of α is at least $2r \cdot \lambda^{d(1,g)-2r}$. We obtain

$$\begin{aligned}\bar\rho_{\lambda,1}(\xi,\eta) \le \rho_{\lambda,1}(\xi,\eta) &\le \rho_{\lambda,1}(g,\xi) + \rho_{\lambda,1}(w,\eta) + \rho_{\lambda,1}(g,w)\\ &\le 2\left(\frac{1}{1-\lambda} + \frac{r}{\lambda^{2r}}\right)\cdot \lambda^{d(g,1)}.\end{aligned}$$

Setting $C = 2\lambda^{-1}(1/(1-\lambda) + r/\lambda^{2r})$, we see that $\bar\rho_{\lambda,1}(\xi,\eta) \le \rho_{\lambda,1}(\xi,\eta) \le Ct$. This completes the proof. □

The following lemma deals with the strong shadow $\mathit{\Pi}_r(g)$, which makes the statement stronger:

Lemma 3.15 (transitional balls $\subseteq$ shadows) *For any $\lambda \in [\lambda_0, 1)$ and $\epsilon, R, L > 0$, there exist constants $C = C(\lambda, \epsilon, R, L) > 0$ and $r = r(\lambda, \epsilon, R, L) > 0$ with the following property:*

Given $\xi \in \partial_\lambda G$, let $g \in [1,\xi]$ be a point within L–distance of an (ϵ, R)–transition point on $[1,\xi]$. Then

$$B_{\rho_{\lambda,1}}(\xi, Ct) \subset \Pi_r(g)$$

for $t := \lambda^{d(1,g)}$. The same conclusion holds for $\bar{\rho}_{\lambda,1}$ with $\xi \in \Lambda G$.

Proof Let $\kappa = \kappa(\lambda,\epsilon,R)$ be the constant given by Lemma 2.9(1). Let z be the (ϵ,R)–transition point of $[1,\xi]$ which is L–close to g. Then $\bar{\rho}_{\lambda,z}(1,\xi) \geq \kappa$, and by property (2), $\rho_{\lambda,g}(1,\xi) \geq \bar{\rho}_{\lambda,g}(1,\xi) \geq \kappa \cdot \lambda^L$.

Define $2C = \kappa \cdot \lambda^L$ and $r = \phi_\lambda(C)$. Let $\eta \in B_{\rho_{\lambda,1}}(\xi, Ct)$. Using property (2) again, it follows that $\rho_{\lambda,g}(\eta,\xi) \leq \lambda^{-d(g,1)}\rho_{\lambda,1}(\eta,\xi) \leq C$. Thus, $\rho_{\lambda,g}(1,\eta) \geq C$, and $d(g,[1,\eta]) \leq r$ by Lemma 2.6. Hence, $B_{\rho_{\lambda,1}}(\xi, Ct) \subset \Pi_r(g)$, proving the lemma. □

From the previous two lemmas, one easily derives the following:

Lemma 3.16 (shadows $\asymp$ balls) *Let r_0 given by Lemma 3.13. For any $\lambda \in [\lambda_0, 1)$ and $L > 0$, there exist $r = r(L,\lambda) > r_0$ and $C = C(L,\lambda) > 1$ with the following property:*

For any $\xi \in \partial T$ and $0 < t < \lambda$,

$$B_{\rho_{\lambda,1}}(\xi, C^{-1}t) \subset \Pi_r(g) \subset B_{\rho_{\lambda,1}}(\xi, Ct) \tag{29}$$

and

$$B_{\bar{\rho}_{\lambda,1}}(\xi, C^{-1}t) \subset \Pi_r(g) \subset B_{\bar{\rho}_{\lambda,1}}(\xi, Ct), \tag{30}$$

where $g \in [1,\xi]$ is chosen so that $d(1,g) = \lfloor \log_\lambda t \rfloor$.

Proof For any $0 < t < \lambda$, let $g \in [1,\xi]$ be such that $d(1,g) = \lfloor \log_\lambda t \rfloor$. Thus,

$$\lambda^{d(1,g)+1} < t \leq \lambda^{d(1,g)}.$$

By construction of T (see Lemma 3.7(5)), we know that $[1,\xi]$ is (ϵ,R,L)–transitional.

Let $C_1 = C(\lambda,\epsilon,R,L)$ and $r = r(\lambda,\epsilon,R,L) > r_0$ be given by Lemma 3.15. Then $B_{\bar{\rho}_{\lambda,1}}(\xi, C_1 t) \subset \Pi_r(g)$. This proves the first inclusions of (29) and (30) for $C = C_1$.

Let $C_2 = C(r,\lambda)$ be given by Lemma 3.14. Then $\Pi_r(g) \subset B_{\bar{\rho}_{\lambda,1}}(\xi, C_2 t)$, and so the second inclusions of (29) and (30) follow.

Setting $C = \max\{C_1^{-1}, C_2\}$, we complete the proof of the lemma. □

3.4 Proof of Theorem 1.4

We recapitulate the main results of the previous subsections in the following:

Proposition 3.17 *There exists $\lambda_0 > 0$ such that for every $\lambda \in [\lambda_0, 1)$ and $L \gg 0$, there exist an L–iterated tree set T and a PS measure μ_1 on ∂T satisfying*

$$\mu_1(B_{\rho_{\lambda,o}}(\xi, t) \cap \partial T) \asymp_{\lambda,L} t^{-\delta_{T,S}/\log\lambda}, \tag{31}$$

for any $\xi \in \partial T$ and $0 < t < \lambda$.

Proof The existence of the tree T is proved in Lemma 3.7. Lemmas 3.13 and 3.16 and direct calculations imply that ∂T is Alhfors Q–regular for $Q = -\delta_{T,S}/\log\lambda$ (see the definition in the introduction). Hence, (31) follows. □

By Lemma 2.11, ∂T consists of uniformly conical points, so Proposition 3.17 implies the first claim of the theorem. The statement $\lim_{i\to\infty} Q_i = -\delta_{G,S}/\log\lambda$ is proved in Lemma 3.9. Theorem 1.4 is proved.

4 Proofs of Theorems 1.1 and 1.2

We consider the Floyd metric on $\partial_\lambda G$ and shortcut metric on ΛG, where the corresponding Theorems 1.1 and 1.2 are proved with the same argument.

The following lemma giving the upper bound for the Hausdorff dimension is due to Marc Bourdon. We notice that it is a general fact which is true for a finitely generated group G without assuming that it is relatively hyperbolic.

Lemma 4.1 (Bourdon, private communication) *For every $\lambda \in (0, 1)$, the Hausdorff dimension $\mathrm{Hdim}_{\rho_{\lambda,1}}$ of $\partial_\lambda G$ (resp. $\mathrm{Hdim}_{\bar\rho_{\lambda,1}}$ of ΛG) with respect to the Floyd metric $\rho_{\lambda,1}$ (resp. to the shortcut metric $\bar\rho_{\lambda,1}$) is bounded above by $-\delta_{G,S}/\log\lambda$.*

Proof To give an upper bound, it suffices to prove that $\mathcal{H}^s(\partial_\lambda G) = 0$ for any fixed $s > -\delta_{G,S}/\log\lambda$.

Define $S_n = \{g \in G : d(1, g) = n\}$. For any $g \in S_n$, define the cone $\Omega_g := \{\xi \in \partial_\lambda G : g \in [1, \xi]\}$, where $[1, \xi]$ is a geodesic between 1 and ξ.

For any $\xi \in \partial_\lambda G$, consider the point $x \in [1, \xi] \cap S_n$. By Lemma 2.7, the subray $[x, \xi)$ is a $\rho_{\lambda,1}$–Floyd geodesic. So $\rho_{\lambda,1}(x, \xi) = \lambda^n/(1-\lambda)$ for any $\xi \in \partial_\lambda G$. Thus, $\{\Omega_g : g \in S_n\}$ is an ε–covering of $\partial_\lambda G$, where $\varepsilon := 2\lambda^n/(1-\lambda)$.

For any $t \in (-\delta_{G,S}/\log\lambda, s)$, we have $-t\log\lambda > \delta_{G,S}$ and so $\#S_n \prec_t \lambda^{-tn}$ for $n \geq 1$. We obtain, for all $n \geq 1$,

$$\mathcal{H}^s(\partial_\lambda G) \leq \sum_{g \in S_n} \varepsilon^s \prec \lambda^{(s-t)n},$$

which then tends 0 as $n \to \infty$. Thus, $\mathcal{H}^s(\partial_\lambda G) = 0$ for any $s > -\delta_{G,S}/\log\lambda$. The lemma is proved. □

So the upper bound on the Hausdorff dimension of $\partial_\lambda G$ and ΛG in Theorems 1.1 and 1.2 is proved. In the remainder of the proofs, we aim to establish the lower bound for the Hausdorff dimension.

Taking into account Proposition 3.17, there exists a universal $\lambda_0 > 0$ such that for each $L \gg 0$, there exist an L–iterated tree T and a PS measure μ_1 on ∂T such that (31) holds and $\delta_{T,S} \to \delta_{G,S}$ as $L \to \infty$.

The following lemma shows that the PS measures constructed in Section 3 are actually the Hausdorff measures on ∂T with respect to the Floyd metric $\rho_{\lambda,1}$ restricted on ∂T. Modulo Proposition 3.17, this result is standard (see [20, Exercise 8.11]). We provide a proof for the reader's convenience.

Lemma 4.2 *Let μ_1 be a PS measure on ∂T in ΛG or $\partial_\lambda G$. Let $\sigma = -\delta_{T,S}/\log\lambda$. Then*

$$\mathcal{H}_\sigma(A) \asymp_L \mu_1(A).$$

for any subset $A \subset \partial T$.

Proof In the proof, we assume that ∂T is a subset of the Bowditch boundary. The proof for $\partial T \subset \partial_\lambda G$ is similar.

Let $\mathcal{B}$ be an ε–covering of A for $\varepsilon > 0$. Then $\mu_1(A) \leq \sum_{B \in \mathcal{B}} \mu_1(B)$. Let $\varepsilon \to 0$. By Proposition 3.17, we obtain that $\mu_1(A) \prec_L \mathcal{H}_\sigma(A)$.

For the other inequality, we need to make use of the following well-known covering result. Let B be a metric ball of radius $\mathrm{rad}(B)$ in a proper metric space X. Denote by $5B$ the union of all balls of radius $2\cdot\mathrm{rad}(B)$ intersecting B, so that $\|5B\| \leq 10\cdot\mathrm{rad}(B)$. Then, by [24, Theorem 2.1], for a family of balls $\mathcal{B}$ in X with uniformly bounded radii there exists a subfamily $\mathcal{B}' \subset \mathcal{B}$ of pairwise disjoint balls such that

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{B \in \mathcal{B}'} 5B. \tag{32}$$

Note that μ_1 and $\mathcal{H}_\sigma$ are Radon measures. Then for any $\tau > 0$ there exists a compact set K and an open set U such that $K \subset A \subset U$ with $\mathcal{H}_\sigma(U \setminus K) < \tau$ and $\mu_1(U \setminus K) < \tau$.

Set $\epsilon_0 := \rho_1(K, \Lambda G \setminus U) > 0$. For any $0 < \epsilon < \epsilon_0$, let $\mathcal{B}$ be an ε–covering of K. By (32) and Proposition 3.17, there exists a pairwise disjoint subfamily $\mathcal{B}'$ of $\mathcal{B}$ such that

$$H_\sigma(K) \le \sum_{B \in \mathcal{B}'} (\|5B\|)^\sigma \le \sum_{B \in \mathcal{B}'} (10 \cdot \mathrm{rad}(B))^\sigma \prec_L \mu_1(U).$$

The condition $\tau \to 0$ yields $H_\sigma(A) \prec \mu_1(A)$. $\square$

Remark The measure μ_1 is unique in the following sense: if μ_1 and μ_1' are two PS measures, then $d\mu_1/d\mu_1'$ is bounded from above and below.

Lemma 4.2 proves that the Hausdorff dimension of ∂T is equal to σ. Since ∂T is a subset of the set of uniformly conical points in $\partial_\lambda G$ and ΛG, the dimension $\sigma = -\delta_{T,S}/\log\lambda$ of ∂T gives a lower bound of $\mathrm{Hdim}_{\rho_{\lambda,1}}(\partial^{\mathrm{uc}}_\lambda G)$ and $\mathrm{Hdim}_{\bar\rho_{\lambda,1}}(\Lambda^{\mathrm{uc}} G)$.

Letting $L \to \infty$, we have $\delta_{T,S} \to \delta_{G,S}$ by Proposition 3.17. So,

$$\mathrm{Hdim}_{\rho_{\lambda,1}}(\partial^{\mathrm{uc}}_\lambda G) \ge -\frac{\delta_{G,S}}{\log\lambda}$$

and

$$\mathrm{Hdim}_{\rho_{\lambda,1}}(\Lambda^{\mathrm{uc}} G) \ge -\frac{\delta_{G,S}}{\log\lambda}.$$

The proofs of Theorems 1.1 and 1.2 are complete.

4.1 Box-counting dimension

As an application of Theorems 1.1 and 1.2, we are able to identify the box-counting dimension with Hausdorff dimension. In the proof, we need the help of a class of Patterson–Sullivan measures defined on the whole Bowditch boundary ΛG. This contrasts with the ones constructed on the ends of iterated trees.

For notational simplicity, we still denote by $\{\mu_v : v \in G\}$ the PS measures on ΛG which were constructed in [36] using the action of G on the Cayley graph $\mathscr{G}(G,S)$. There it was proved that the PS measures are $\delta_{G,S}$–dimensional quasiconformal measures without atoms. As a consequence, a shadow lemma is derived in this setting (see Lemma 4.3 below).

In the same way, we can apply the construction of PS measures for the Floyd compactification $\partial_\lambda G \cup \mathscr{G}(G,S)$. By Proposition 2.4, there exists a surjective map $F\colon \partial_\lambda G \to \Lambda G$, where the preimage of a parabolic point coincides with the limit set of the corresponding parabolic subgroup in $\partial_\lambda G$. The proof that PS measures on ΛG have no atom at a parabolic point $p \in \Lambda G$ [36, Lemma 4.10] actually establishes that $\mu_1(F^{-1}(p)) = 0$. This implies that PS measures on $\partial_\lambda G$ have no atoms either. Following the proof of [36, Lemmas 4.1 and 4.2] the shadow lemma also holds for PS measures on $\partial_\lambda G$.

Summarizing the above discussion, the following shadow lemma holds for PS measures $\{\mu_v : v \in G\}$ on the Bowditch boundary ΛG and the Floyd boundary $\partial_\lambda G$:

Lemma 4.3 *There exists $r_0 > 0$ such that*

$$\mu_1(\Pi_r(g)) \asymp_r \exp(-\delta_{G,S} d(1,g)) \asymp_r \mu_1(\mathit{\Pi}_r(g))$$

for any $g \in G$ and $r > r_0$.

We are ready to identify the box-counting dimension.

Proof of Theorem 1.6 We only consider the Floyd boundary below. The case of a shortcut metric on ΛG is similar. It is a well-known fact that the Hausdorff dimension is a lower bound of the box-counting dimension (see [9, formulas 3.17]). Thus, $\mathrm{Bdim}_{\rho_{\lambda,1}}(\partial_\lambda G) \geq Q := \delta_{G,S}/(-\log\lambda)$ by Theorem 1.1.

For the upper bound, consider a maximal collection $P(\epsilon)$ of pairwise disjoint balls with radius ϵ contained in $\partial_\lambda G$. Let $r > 0$ be a constant satisfying Lemma 4.3, and $C > 0$ given by Lemma 3.14. Thus, each ball B of radius ϵ centered at $\xi \in \partial_\lambda G$ contains a shadow $\Pi_r(g)$, where g is an element on $[1,\xi]$ such that $d(1,g) = \lfloor \log_\lambda(\epsilon/C) \rfloor$. Noting the relation

$$\exp(-\delta_{G,S} d(1,g)) \asymp \left(\frac{\epsilon}{C}\right)^{\delta_{G,S}/(-\log\lambda)},$$

we obtain from Lemma 4.3 that $\mu_1(B) \succ \epsilon^Q$ and, thus,

$$1 = \mu_1(\partial_\lambda G) \geq \sum_{B \in P(\epsilon)} \mu_1(B) \succ \#P(\epsilon) \cdot \epsilon^Q.$$

This implies that the box-counting dimension

$$\mathrm{Bdim}_{\rho_{\lambda,1}}(\partial_\lambda G) = \lim_{\epsilon \to 0} \frac{\log \#P(\epsilon)}{\log 1/\epsilon}$$

is bounded above by Q. The result is proved. □

5 Tight paths and Floyd metrics

In this section, we shall develop a detailed understanding of shortcut geodesics via a class of well-controlled paths called (generalized) tight paths. We start by recalling the approximation of the Floyd geodesic through the tight paths introduced by Gerasimov and Potyagailo in [15].

5.1 Tight paths and Floyd geodesic

We note that a Floyd geodesic between ξ and η does not necessarily belong to the graph (eg an example of such situation is given by the Floyd geodesic $[n,+\infty]\cup[-\infty,-n]$ between $-n$ and n for the group $\mathbb{Z}+\mathbb{Z}$). A method to overcome this problem was proposed in [15]. It consists in introducing a special type of paths called *tight paths* (see Definition 5.2 below) situated in the Cayley graph which will well approximate the Floyd geodesics.

This notion is actually motivated by the following result:

Lemma 5.1 [15, Lemma 7.2] *For any $l>0$, there exists $0<\lambda_0<1$ such that the following property holds for any $\lambda\in[\lambda_0,1)$:*

Let $x,y\in\mathscr{G}(G,S)$ such that $d(x,y)\le l$, and p be a path with $\alpha_-=x$ such that $\ell(\alpha)\ge d(x,y)+1$. Then $\mathfrak{l}_{\lambda,o}(\alpha)>\rho_{\lambda,o}(x,y)$. In particular, the $\rho_{\lambda,o}$–geodesic between x,y is a geodesic in $\mathscr{G}(G,S)$

The class of tight paths is defined as follows:

Definition 5.2 For $c\ge 1, l>0$, a path γ is called (c,l)*–tight* path if, for any two points $x,y\in\gamma$ with $d(x,y)\le l$, the subpath $[x,y]_\gamma$ is a c–quasigeodesic.

Remark This definition is a partial case of [15, Definition 6.1], where a local quasigeodesicity is requested outside of the horospheres only and an additional condition is assumed for the horospheres. So, if a path is tight in the sense of Definition 5.2, it is also tight in the sense of [15, Definition 6.1], but not necessarily vice versa. In particular, we can use all results proven in [15]. In addition, the above definition implies that every subpath of a tight path is a tight path itself, which is not always true in the general case. This stability of the tightness for subpaths will be used often below.

We also stress that the above definition does not coincide with the standard notion of local (quasi)geodesicity, where the assumption that the length of a subpath (and not its diameter) is small implies its (quasi)geodesicity.

We consider the following *shortening procedure*, implicitly introduced in Lemma 7.4 of [15]: Consider two points $x, y \in \overline{G}_\lambda$; we take a sequence of paths γ_n in $\mathscr{G}(G,S)$ such that $(\gamma_n)_- \to x$, $(\gamma_n)_+ \to y$ and

$$\mathfrak{l}_{\lambda,o}(\gamma_n) \to \rho_{\lambda,o}(x,y).$$

For every $l > 0$ we can choose $\lambda_0 \in (0,1)$ such that γ_n is an l–local geodesic. Indeed, if a segment between two points of γ_n at distance at most l is not a geodesic, then it can be replaced by a geodesic. Applying this procedure several times, we obtain an l–local geodesic, still denoted by γ_n, whose Floyd length is not increased by Lemma 5.1 (see Lemma 5.3 for more details).

Recall that a Floyd geodesic in the Floyd completion does not in general belong to the Cayley graph and the shortening procedure as above allows one to approximate them by local geodesics in the graph. Furthermore, the following lemma shows that this approximation can be done using the tight paths:

Lemma 5.3 [15, Corollary 7.5] *For any $l > 0$ there exists $\lambda_0 \in (0,1)$ such that for every $\lambda \in (\lambda_0, 1)$, if the Floyd geodesic $\gamma \subset \overline{G}_\lambda$* (with respect to the metric $\rho_{\lambda,o}$) *joining two distinct points x and y in $\overline{G}_\lambda$ does not belong to the Cayley graph $\mathscr{G}(G,S)$, then for $\varepsilon > 0$ there exists a tight path $\widetilde{\gamma} \subset \mathscr{G}(G,S)$ such that $|\mathfrak{l}_{\lambda,o}(\widetilde{\gamma}) - \mathfrak{l}_{\lambda,o}(\gamma)| < \varepsilon$.*

In what follows, to reduce cumbersome quantifiers, we continue to use Convention 2.17 without explicit mention of the constants ϵ and R, which depend on the parameter $c > 0$ in tight paths.

5.2 Truncation of tight paths

We shall introduce a truncation of a tight path so that it becomes as a quasigeodesic. Before doing so, we recall another result about tight paths from [15].

It is well known that in hyperbolic spaces, a sufficiently "long" local geodesic becomes globally a quasigeodesic. This property in general fails for the Cayley graph of a relatively hyperbolic group. The next result shows that a tight path is a generalization of local geodesics in the relative setting.

Lemma 5.4 *For any $c \ge 1$, there exist $\kappa = \kappa(c)$, $l_0 = l_0(c) > 0$ with the following property:*

Let γ be a (c,l)–tight path for $l \ge l_0$. Then $\rho_v(\gamma_-, \gamma_+) \ge \kappa$ for any (ϵ, R)–transitional point $v \in \gamma$.

Comments on the proof The statement that $\rho_v(\gamma_-, \gamma_+) \ge \kappa$ is first established in [15, Proposition 6.7] for a special subsequence of transitional vertices $v := v_n \in \gamma$. Then it is shown in the proof of [15, Theorem B] that the tightness of a path implies that the statement of the lemma is true for every transitional vertex of γ (up to decreasing the constant κ). □

We call below a sequence of points $z_i = \gamma(t_i)$ of a length-parametrized path γ *well ordered* if $t_i > t_{i-1}$ for $t_i \in \mathbb{Z}$.

The following lemma is an intermediate step in the proof of Lemma 5.8 below, which is the main result of this subsection.

Lemma 5.5 (transitional tight path is quasigeodesic) *For any $c, L \ge 0$, there exist $l_0 = l_0(L)$, $c' = c'(c) \ge 1$ with the following property:*

Let γ be a (c, l)–tight path for $l \ge l_0$. Assume that γ is an (ϵ, R, L)–transitional path, where ϵ and R satisfy Convention 2.17. Then γ is a c'–quasigeodesic.

Proof By Lemma 5.4, there exists $\kappa = \kappa(c) \ge 0$ such that $\rho_x(\gamma_-, \gamma_+) \ge \kappa$ for every (ϵ, R)–transitional point $x \in \gamma$. Set $D_0 = \phi\big(\frac{1}{2}\kappa\big)$. Choose $l_0 \ge 2(L + D_0)$.

Since any subpath of γ is (c, l)–tight, it is enough to prove that there exists a linear bound for $\ell(\gamma)$ with respect to $d(\gamma_-, \gamma_+)$. Let α be a geodesic with the same endpoints as γ. The idea of the proof is to find two sequences of well-ordered points in γ and α, respectively, which are uniformly close.

Since γ is (ϵ, R, L)–transitional, there exists a maximal set of (ϵ, R)–transitional well-ordered points $\{z_i : 1 \le i \le n\}$ in γ such that

$$\ell([z_i, z_j]_\gamma) \ge 2cD_0 + c$$

for $i \ne j$ and

$$\ell([z_i, z_{i+1}]_\gamma) \le 2(L + cD_0) + c$$

for $1 \le i < n$. Indeed, let z_1 be the first (ϵ, R)–transitional point in γ. Suppose z_i is chosen for $i \ge 1$ with $\ell([z_i, \gamma_+]_\gamma) \ge 2cD_0 + c$. If $\ell([z_i, \gamma_+]) \le 2(L + cD_0) + c$ then $z_{i+1} = \gamma_+$. Consider the point z in $[z_i, \gamma_+]_\gamma$ such that $\ell([z_i, z]_\gamma) = L + 2cD_0 + c$. If z is (ϵ, R)–transitional in γ, then set $z_{i+1} = z$. Otherwise, there exists an (ϵ, R)–transitional point z_{i+1} such that $\ell([z, z_{i+1}]_\gamma) \le L$ and $\ell([z_i, z_{i+1}]_\gamma) \le 2(L + cD_0) + c$.

By Lemma 5.4, there exists $\kappa > 0$ such that

$$\rho_{z_{i+1}}(z_i, \gamma_+) \geq \kappa$$

for any $1 \leq i < n$. By Lemma 2.6, there exists $w_1 \in \alpha$ such that $d(z_1, w_1) \leq D_0$. We now choose other w_i inductively for $i \geq 1$.

Suppose $w_i \in \alpha$ is chosen so that $d(z_i, w_i) \leq D_0$. We borrow the following argument that the points w_i are well ordered on α from [15, Lemma 7.2].

Recall that $[z_i, z_j]_\gamma$ is c–quasigeodesic, so $d(z_{i+1}, z_i) \geq 2D_0$. Thus, we obtain $[z_i, w_i] \cap B(z_{i+1}, D_0) = \varnothing$. By the choice of $D_0 = \phi\big(\frac{1}{2}\kappa\big)$, we know that for any $v \in G$, any geodesic outside $B(v, D_0)$ has $\mathfrak{l}_v$–length at most $\frac{1}{2}\kappa$. So $\rho_{z_{i+1}}(w_i, z_i) \leq \frac{1}{2}\kappa$ and then $\rho_{z_{i+1}}(w_i, \gamma_+) \geq \frac{1}{2}\kappa$. Thus, there exists $w_{i+1} \in [w_i, \alpha_+]_\alpha$ such that $d(z_{i+1}, w_{i+1}) \leq D_0$. Up to increasing D_0 by 1, we can assume that $d(w_i, w_{i+1}) \geq 1$. Hence, the obtained points w_i are well ordered on α.

As $l_0 > 2(L + D_0)$, $[z_i, z_{i+1}]_\gamma$ is a c–quasigeodesic by the tightness property. Since w_i are well ordered on α, we see that γ is a c'–quasigeodesic for $c' := 2c + 2D_0$. Indeed,

$$\begin{aligned}
\ell(\gamma) &\leq \ell([\gamma_-, z_1]_\gamma) + \sum_{i=1}^{n-1} \ell([z_i, z_{i+1}]_\gamma) + \ell([z_n, \gamma_+]_\gamma) \\
&\leq c \cdot (\ell([\alpha_-, w_1]_\alpha) + D_0) + c + \sum_{i=1}^{n-1} \big(c \cdot (\ell([w_i, w_{i+1}]_\alpha) + 2D_0) + c\big) \\
&\qquad\qquad\qquad\qquad + c \cdot (\ell([w_n, \alpha_+]_\alpha) + D_0) + c \\
&\leq (2c + 2D_0)\ell(\alpha) \leq c' \cdot d(\gamma_-, \gamma_+),
\end{aligned}$$

where $n + 1 \leq \ell(\alpha)$ is used. The proof is complete. □

The following lemma will be used often later:

Lemma 5.6 (bounded overlap) *For $c \geq 1$ and (ϵ, R) given by Convention 2.17, there exist $K_0, l_0 > 0$ with the following property:*

Let γ be a (c, l)–tight path for $l \geq l_0$. Assume that β_1 and β_2 are two maximal connected segments of γ such that $(\beta_i)_-, (\beta_i)_+ \in N_\epsilon(X_i)$ for some $X_i \in \mathbb{P}$ for $i = 1, 2$. Then $\ell(\beta_1 \cap \beta_2) \leq K_0$. In particular, the endpoints of β_i are (ϵ, R)–transitional for $i = 1, 2$.

Proof By Definition 5.2, a subpath of a tight path is itself tight. Then, by [15, Proposition 7.6], it follows that there exists $l_0 > 0$ such that for all $l \geq l_0$ the elements of $\mathbb{P}$ are uniformly quasiconvex with respect to the system of (c,l)–tight paths. This implies that there exists a uniform constant $\varepsilon = \varepsilon(\epsilon, c) > 0$ such that $\beta_i \subset N_\varepsilon(X_i)$ for $i = 1, 2$. By Lemma 2.15 we find a constant $R_0 = \mathcal{R}(\varepsilon) > 0$ such that $\|N_\varepsilon(X) \cap N_\varepsilon(X')\| \leq R_0$ for every distinct $X, X' \in \mathbb{P}$. Assume that $l_0 > c R_0 + c$. Since β_i are l–local c–quasigeodesics for $l > l_0$, it follows that $\ell(\beta_1 \cap \beta_2) \leq K_0 := c R_0 + c$. By Convention 2.17 we have $R > R_0$, so the endpoints of β_i are transitional for both curves β_1 and β_2. Indeed, if eg $x \in \partial\beta_1 \cap \beta_2$ then x is (ϵ, R)–transitional for β_2 as $d(x, \partial\beta_2) \leq R_0 < R$; furthermore, it is automatically (ϵ, R)–transitional with respect to β_1 being one of its endpoints. $\square$

Remark By the bounded intersection of $\mathbb{P}$, this lemma holds trivially if γ is a quasigeodesic. However, the tight path γ above is a local quasigeodesic only.

Let γ be a (c,l)–tight path. Let $\epsilon = \epsilon(c)$ given by Convention 2.17 and K_0 given by Lemma 5.6. For $K > K_0$, we consider all maximal connected segments β_i in γ for $1 \leq i \leq m$ such that $\ell(\beta_i) \geq K$ and $(\beta_i)_-, (\beta_i)_+ \in N_\epsilon(X_i)$ for some $X_i \in \mathbb{P}$. Consequently, $X_i \neq X_j$ for $i \neq j$. These (β_i, X_i) shall be referred to as (ϵ, K)–*components* of γ.

We stress that by the argument of Lemma 5.6 the segment β_i belongs to $N_\epsilon(X_i)$ for a uniform $\epsilon > 0$ and unique X_i.

We now introduce a modification of a tight path to make the obtained path a quasigeodesic.

Definition 5.7 (truncation of a tight path) Let γ be a (c,l)–tight path for $c \geq 1$ and $l > 0$. Consider all (ϵ, K)–components (β_i, X_i) for $1 \leq i \leq m$ for a fixed $K > 2K_0$, where $K_0 > 0$ is given by Lemma 5.6.

Set $y_1 = (\beta_1)_-$ and $x_2 = (\beta_1)_+$. If $\beta_i \cap \beta_{i-1} = \varnothing$ for $i \geq 2$, let $y_i = (\beta_i)_-$ and $x_{i+1} = (\beta_i)_+$; otherwise, set $y_i = x_{i-1}$ and $x_{i+1} = (\beta_i)_+$. Replace $[y_i, x_{i+1}]_\gamma$ by a geodesic segment $[y_i, x_{i+1}]$ for each $i \geq 1$.

The path $\overline{\gamma}$ obtained in this way is called a K–*truncation* of γ.

Remark The following observation is elementary and useful: Every β_i produces an $\left(\epsilon, \frac{1}{2}K\right)$–deep point in X_i in the truncation path $\overline{\gamma}$. Consequently, if $\overline{\gamma}$ does not contain an (ϵ, R)–deep point, then $d((\beta_i)_-, (\beta_i)_+) \leq 2R$ for all β_i.

The following lemma is the main result of this subsection. It provides a further generalization of Lemma 5.5 to the truncated tight paths.

Lemma 5.8 (truncation is quasigeodesic) *For any $c \geq 1$, there exist $l_0 = l_0(c)$, $K = K(c)$, $c' = c'(c) > 0$ with the following property: for any $l \geq l_0$, the K–truncation of a (c, l)–tight path is a c'–quasigeodesic.*

Proof Let $K > 2K_0$ be a fixed integer, where K_0 is given by Lemma 5.6. Let $\overline{\gamma}$ be the K–truncation of a (c, l)–tight path γ. Keeping the notation as in Definition 5.7, we have by Lemma 5.6 that y_i and x_{i+1} for $1 \leq i \leq m$ are (ϵ, R)–transitional points in γ. Furthermore, since $[x_i, y_i]_\gamma$ contains no (ϵ, K)–components for $1 \leq i < m$, we see that $[x_i, y_i]_\gamma$ is an (ϵ, R, L)–transitional path for $L := \frac{1}{2}K$. By Lemma 5.5, there exist $l_0 = l_0(L)$, $c_0 = c_0(c) \geq 1$ such that $[x_i, y_i]_\gamma$ is a c_0–quasigeodesic.

Before proving the full generality, we first prove that a subpath of $\overline{\gamma}$ is quasigeodesic. Let β be a geodesic between the point x_i with an arbitrary point $z \in [y_i, x_{i+1}]$. We show below that $[x_i, z]_{\overline{\gamma}}$ is a quasigeodesic.

Since y_i is (ϵ, R)–transitional, $\rho_{y_i}(x_i, x_{i+1}) \geq \kappa$, where κ is given by Lemma 5.4. Hence, $d(y_i, [x_i, x_{i+1}]) \leq D_0 := \phi(\kappa)$, where ϕ is the function given by Lemma 2.6. By the triangle inequality we have $d(x_i, y_i) + d(y_i, x_{i+1}) \leq d(x_i, x_{i+1}) + 2D_0$. Since $d(y_i, z) + d(z, x_{i+1}) = d(y_i, x_{i+1})$, we obtain

$$d(x_i, y_i) + d(y_i, z) \leq d(x_i, x_{i+1}) - d(z, x_{i+1}) + 2D_0 \leq d(x_i, z) + 2D_0.$$

Finally,

$$\ell[x_i, z]_{\overline{\gamma}} = \ell[x_i, y_i]_\gamma + \ell[y_i, z] \leq c_0 d(x_i, y_i) + c_0 + d(y_i, z) \leq c_0 d(x_i, z) + c_0 + 2c_0 D_0.$$

So $[x_i, z]_{\overline{\gamma}}$ is a c_1–quasigeodesic for $c_1 := c_0(2D_0 + 1)$. The same argument also shows the c_1–quasigeodesicity of $[z, y_{i+1}]_{\overline{\gamma}}$.

We have that any subpath $\widetilde{\gamma}$ of the truncated path $\overline{\gamma}$ is the union of three types of c_1–quasigeodesic subpaths: (a) $\gamma_i = [x_i, y_i]_\gamma$, (b) $\beta_i = [y_i, x_{i+1}]$, and (c) $\delta = [x_i, z]_{\overline{\gamma}}$ or $\delta = [z, y_{i+1}]_{\overline{\gamma}}$. Both vertices of the intervals of types (a) and (b) are transitional on the corresponding tight path γ, and every γ_i is (ϵ, R, L)–transitional whereas β_i is a geodesic segment replacing the corresponding (ϵ, K)–component. The path $\widetilde{\gamma}$ can contain at most two intervals δ of type (c) such that one of the endpoints of δ coincides with an endpoint of $\widetilde{\gamma}$ and is an interior point of a geodesic truncation of $\overline{\gamma}$.

Repeating the argument of Lemma 5.5, consider a maximal well-ordered subset V of the transitional vertices $\{v_j \in \tilde{\gamma}\}$ in the set $W := \tilde{\gamma} \cap \{y_i, x_{i+1} : 1 \le i \le m\}$ such that $d(v_j, v_{j+1}) \ge 2D_0$. We connect the endpoints of $\tilde{\gamma}$ by a geodesic α. Then, for each $v_j \in V$, there exists $v'_j \in \alpha$ such that $d(v_j, v'_j) \le D_0$ and by the argument of Lemma 5.5 we have $v'_j \in [v'_{j-1}, v'_{j+1}]_\alpha$. Since V is maximal in W, for any $w \in W$ there exists $v \in V$ such that $d(v, w) \le 2D_0$. So, for the well-ordered collection $W = \{w_1, w_2, \dots, w_n\} \subset \tilde{\gamma}$, there exists another well-ordered set $W' = \{w'_1, w'_2, \dots, w'_n\} \subset \alpha$ such that $d(w_i, w'_i) \le 3D_0$. Then $[\tilde{\gamma}_-, w_1]_{\tilde{\gamma}}$, $[w_i, w_{i+1}]_\gamma$ and $[w_n, \tilde{\gamma}_+]_{\tilde{\gamma}}$ are all c_1–quasigeodesics by the above argument. Due to the well-orderedness of W', we have $d(w'_i, w'_{i+1}) \ge 1$, so $n+1 \le \ell(\alpha)$.

We have

$$\begin{aligned}
\ell(\tilde{\gamma}) &\le c_1\left(d(\tilde{\gamma}_-, w_1) + \sum_{i=1}^{n-1} d(w_i, w_{i+1}) + d(w_n, \tilde{\gamma}_+)\right) + c_1 \\
&\le c_1\left(3D_0 + d(\tilde{\gamma}_-, w'_1) + \sum_{i=1}^{n-1}(d(w'_i, w'_{i+1}) + 3D_0) + d(w'_n, \tilde{\gamma}_+) + 3D_0\right) + c_1 \\
&\le 3D_0(n+1)c_1 + l(\alpha) + c_1 \le c_1(3D_0+1)l(\alpha) + c_1 \le c' d(\tilde{\gamma}_-, \tilde{\gamma}_+) + c',
\end{aligned}$$

where $c' := (1+3D_0)c_1$. The lemma is proved. □

Convention 5.9 For any $c \ge 1$, we will assume further on that $l_0, K > 0$ satisfy both Lemmas 5.4 and 5.6.

5.3 Shortcut metrics and generalized tight paths

The goal of this subsection is to extend Lemma 5.3 for Floyd geodesics to the setting of shortcut geodesics with respect to the shortcut metrics $\{\bar{\rho}_{\lambda,o}\}_{o \in G}$ on ΛG (see Section 2.2). For this purpose we generalize the notion of a tight path as follows:

Definition 5.10 (generalized tight paths and truncations) Let γ be a finite sequence of (c,l)–tight paths γ_i in $\mathscr{G}(G,S)$ for $1 \le i \le n$ such that $(\gamma_i)_+, (\gamma_{i+1})_- \in N_\epsilon(X_i)$ for some $X_i \in \mathbb{P}$, where $X_i \ne X_j$ for $1 \le i \ne j < n$.

We say that γ is a (c,l)*–generalized tight path* if for each pair of entry and exit points y_i and x_{i+1} of γ_i and γ_{i+1}, respectively, in $N_\epsilon(X_i)$ we have $d(y_i, x_{i+1}) \ge l$ for $1 \le i < n$.

Fix $K>0$. For $n>1$, consider the K–truncation $\overline{\gamma}_i$ of $[x_i, y_i]_{\gamma_i}$, where $1 \le i \le n$ (see Definition 5.7). The path

$$\widetilde{\gamma} = \overline{\gamma}_1 \cdot [y_1, x_2] \cdot \overline{\gamma}_2 \cdots [y_{n-1}, x_n] \cdot \overline{\gamma}_n$$

is called the K–*truncation* of a generalized (c,l)–tight path γ.

Remark A generalized tight path is possibly not connected. If it is connected, then it is a tight path in Definition 5.2.

Lemma 5.11 (generalized truncation is quasigeodesic) *For any $c \ge 1$, there exist $l_0, K, M, c' \ge 1$ such that for any $l > l_0$, the K–truncation $\widetilde{\gamma}$ of a (c,l)–generalized tight path γ is a c'–quasigeodesic. Moreover, any $(\epsilon_{c'}, R_{c'})$–transition point of $\widetilde{\gamma}$ lies in the M–neighborhood of γ.*

Proof Let $K = K(c)$ be given by Lemma 5.8. Let $\widetilde{\gamma}$ be the K–truncation of a generalized (c,l)–tight path γ. We keep the notation of Definition 5.10. If $n=1$, the proof is finished by Lemma 5.8. Assume that $n \ge 2$.

By Lemma 5.8, there exists $c_1 > 0$ such that each $\overline{\gamma}_i$ is a c_1–quasigeodesic for each $1 \le i < n$. We prove below that $\overline{\gamma}_i$ and $\overline{\gamma}_{i+1}$ have bounded projection to $N_\epsilon(X_i)$, where $X_i \in \mathbb{P}$.

By Lemma 2.15, $X \in \mathbb{P}$ is c_1–contracting and there exist $\mu_{c_1}, D_{c_1} > 0$ such that (7) holds. Let z be the entry point of $\overline{\gamma}_i$ in $N_{\mu_{c_1}}(X_i)$.

Claim *There exists a constant $C > 0$ such that $d(z, y_i) \le C$.*

Proof of the claim We choose a point w depending on the position of z on $\overline{\gamma}_i$: if $z \in \gamma_i$ then we set $w = z$; otherwise, z belongs to a geodesic segment β coming from an (ϵ, K)–component of γ_i and then we set $w = \beta_+$. The goal of the proof is to bound $d(z,w)$ and $d(w, y_i)$.

Consider the tight subpath $[w, y_i]_{\gamma_i}$ and its K–truncation β_1. By the argument of Lemma 5.8, the path $[z,w] \cdot \beta_1$ is a c_2–quasigeodesic for some $c_2 > 0$.

Since X_i is quasiconvex, there exists $\varepsilon = \varepsilon(\epsilon, c_2) > 0$ such that any c_2–quasigeodesic with two endpoints in $N_\epsilon(X_i)$ lies in $N_\varepsilon(X_i)$. This implies that $[z,w] \cdot \beta_1 \subset N_\varepsilon(X_i)$. We first show that $[w, y_i]_{\gamma_i}$ contains no (ϵ, K)–components. Indeed, if not, there exists an (ϵ, K)–component β' in $[\beta_+, y_i]_{\gamma_i}$ and $Y \in \mathbb{P}$ such that $\beta'_\pm \in N_\epsilon(Y)$ and $d(\beta'_-, \beta'_+) > K > l_0$. Since y_i is the entry point of γ_i in $N_\epsilon(X_i)$, we have $Y \ne X_i$.

Since $\beta'_\pm \in N_\varepsilon(X_i)$, we get $d(\beta'_-, \beta'_+) \le R_0 := \mathcal{R}(\max\{\epsilon, \varepsilon\})$ by Lemma 2.15. This is a contradiction as $l_0 > \mathcal{R}(\max\{\epsilon, \varepsilon\})$.

Since $[z, w] \subset N_\varepsilon(X_i)$, we can have $d(z, w) \neq 0$ only if $[z, w]_{\gamma_i}$ contains an (ϵ, K)–component, which, by the above reasoning, would imply that $d(z, w) \le R_0$.

It remains to bound $d(w, y_i)$. Since $[w, y_i]_{\gamma_i}$ contains no (ϵ, K)–components, we have $\beta_1 = [w, y_i]_{\gamma_i}$. Let $L = L(\varepsilon, R_0)$ be given by Lemma 2.16, then we affirm that $[w, y_i]_{\gamma_i}$ is of length at most l_0 for any $l_0 > c(2L + R_0) - c$. Indeed, suppose it is not true. We have $\beta_1 = [w, y_i]_{\gamma_i} \subset N_\varepsilon(X_i)$ and γ_i is l_0–local c–quasigeodesic. Hence, β_1 contains a subpath α which is a c–quasigeodesic of length l_0 contained in $N_\varepsilon(X_i)$. Thus, $d(\alpha_-, \alpha_+) > 2L + R_0$, and by Lemma 2.16, there exists an interior point in $[w, y_i]_{\gamma_i}$ which is (ϵ_c, R_0)–deep in X_i. This is impossible as y_i is the entry point of γ_i in $N_\epsilon(X_i)$ and $\epsilon \ge \epsilon_c$ by Convention 2.17. Hence, $d(w, y_i) \le C := 2L + R_0$. The claim is proved. □

By the contracting property, Lemma 2.15, we see that

$$\begin{aligned}\|\mathbf{Proj}_{X_i} \overline{\gamma}_i\| &\le \|\mathbf{Proj}_{X_i} [(\overline{\gamma}_i)_-, z]_{\overline{\gamma}_i}\| + d(z, X_i) + d(z, y_i) + d(y_i, X_i) \\ &\le \tau := D_{c_1} + \mu_{c_1} + C + \epsilon.\end{aligned}$$

The same is true for $\mathbf{Proj}_{X_i}(\overline{\gamma}_{i+1})$. Then $\widetilde{\gamma}$ satisfies the following properties:

(1) Each $\overline{\gamma}_i$ is a c_1–quasigeodesic.

(2) $\max\{\mathbf{Proj}_{X_i}(\overline{\gamma}_i), \mathbf{Proj}_{X_i}(\overline{\gamma}_{i+1})\} \le \tau$.

(3) $d(x_i, y_{i+1}) > l$.

These properties imply that $\widetilde{\gamma}$ is (l, c_1, c_1, τ)–admissible in the sense of [37, Section 3]. Therefore, by Corollary 3.2 in [37], there exist $l_0, c' > 0$ depending on c_1 and τ such that for any $l > l_0$ the truncation $\widetilde{\gamma}$ is a c'–quasigeodesic.

It remains only to prove the "Moreover" statement of the lemma. Let $M = L(\epsilon_c, R_{c'})$ be given by Lemma 2.16, where $\epsilon_{c'} \ge \epsilon_c \ge \epsilon_1$ and they all satisfy Convention 2.17. Assume further that $l_0 \ge 2M$. Let z be an $(\epsilon_{c'}, R_{c'})$–transition point of $\widetilde{\gamma}$. We shall prove that $d(z, \gamma) \le M$ according to the following two cases:

Case 1 The point z lies in a geodesic segment coming from some (ϵ_c, K)–component β of a (c, l)–tight path γ_i. Then $\min\{d(\beta_-, z), d(\beta_+, z)\} \le M$. Indeed, if not, then by applying Lemma 2.16 for the geodesic β, we obtain a point $z \in \widetilde{\gamma}$ which is $(\epsilon_1, R_{c'})$–deep in X_j. This is a contradiction, as z is an $(\epsilon_{c'}, R_{c'})$–transitional point in $\widetilde{\gamma}$.

Case 2 The point z lies in $[y_j, x_{j+1}]$ for some j, where $[y_j, x_{j+1}]$ is given in Definition 5.10 of a generalized tight path. By the same reasoning as above, we apply Lemma 2.16 for the geodesic $[y_j, x_{j+1}]$. Then $\min\{d(z, y_j), d(z, x_{j+1})\} \le L_1$.

In both cases, we have proved that z is M–close to a point of γ, so the lemma follows. □

Remark An alternative way to prove the above lemma is to use the arguments of Proposition 6.1 in [16] to prove that $\tilde{\gamma}$ is a curve whose distortion is a quadratic polynomial; then it follows from Proposition 7.8 in [16] that $\tilde{\gamma}$ is linearly distorted.

Proposition 5.12 (approximation by generalized tight paths) *For any $l \ge 0$, there exists $0 < \lambda_0 < 1$ such that the following property holds for any $\lambda \in [\lambda_0, 1)$:*

For any $\xi \ne \eta \in \Lambda G$, there exists a sequence of generalized $(1, l)$–tight paths γ_n with $(\gamma_n)_- \in [o, \xi]$ and $(\gamma_n)_+ \in [o, \eta]$ such that

$$\lim_{n\to\infty} d(o, (\gamma_n)_-) = \lim_{n\to\infty} d(o, (\gamma_n)_+) = \infty$$

and

$$\lim_{n\to\infty} \mathfrak{l}_{\lambda,o}(\gamma_n) = \bar{\rho}_{\lambda,o}(\xi, \eta).$$

Proof By definition of the shortcut metric (3), for any $\varepsilon > 0$, there are finitely many pairs $(\eta_i, \xi_{i+1}) \in \omega$ where $1 \le i < m$ such that

$$\bar{\rho}_{\lambda,o}(\xi, \eta) \ge \sum_{1\le i\le m} \rho_{\lambda,o}(\xi_i, \eta_i) - \tfrac{1}{3}\varepsilon, \tag{33}$$

where $\xi_1 := \xi$, $\eta_m := \eta$. If $m = 1$, the proof is completed by Lemma 5.3. Assume that $m \ge 2$.

Let $\kappa = \min\{\bar{\rho}_{\lambda,o}(\eta_i, \xi_{i+1}) : 1 \le i < m\} > 0$. For each $1 \le i < m$, there exists $X_i \in \mathbb{P}$ such that $\eta_i, \xi_{i+1} \in \partial_\lambda(X_i)$, where $\partial_\lambda(X_i)$ is the topological boundary of X_i in $\partial_\lambda G$.

First we claim that one can choose $\tilde{\xi}_1$, $\tilde{\eta}_m$ and $\tilde{\eta}_i, \tilde{\xi}_{i+1} \in X_i$ for each $1 \le i < m$ such that the following two conditions hold:

(1) $\max\{\bar{\rho}_{\lambda,o}(\tilde{\xi}_i, \xi_i), \bar{\rho}_{\lambda,o}(\tilde{\eta}_i, \eta_i)\} \le \min\{\frac{1}{4}\kappa, \varepsilon/6m\}$ for $1 \le i \le m$.

(2) If there exists a path α between $\tilde{\eta}_i$ and $\tilde{\xi}_{i+1}$ for $1 \le i < m$ such that $\ell(\alpha) \le 3l$, then it has $\mathfrak{l}_o$–length at most $\frac{1}{4}\kappa$.

Indeed, (1) is true for $\tilde{\xi}_i$ and $\tilde{\eta}_i$ sufficiently close to ξ_i and η_i, respectively. To prove (2), let $R = \min\{d(1, \tilde{\xi}_i), d(1, \tilde{\eta}_i) : 1 \le i \le m\}$. We have $d(1, \alpha) \ge R - 3l$. So for sufficiently large R the statement (2) follows from the visibility lemma, Lemma 2.6.

By Lemma 5.3, we can connect $\tilde{\xi}_i$ and $\tilde{\eta}_i$ by a $(1,l)$–tight path γ_i for $1 \le i \le m$ such that $(\gamma_i)_- = \tilde{\xi}_i$ and $(\gamma_i)_+ = \tilde{\eta}_i$ and

$$|\rho_{\lambda,o}(\tilde{\xi}_i, \tilde{\eta}_i) - \mathfrak{l}_{\lambda,o}(\gamma_i)| \le \frac{\varepsilon}{6m}. \tag{34}$$

By the condition (1) above, (33) and (34),

$$\bar{\rho}_{\lambda,o}(\xi,\eta) \ge \sum_{1 \le i \le m} \mathfrak{l}_{\lambda,o}(\gamma_i) - \tfrac{5}{6}\varepsilon. \tag{35}$$

Let y_i and x_{i+1} be the entry and exit points of γ_i and γ_{i+1} in $N_\epsilon(X_i)$, respectively. If $d(y_i, x_{i+1}) \ge l$ for all $1 \le i < m$, then we are done: $\{\gamma_i\}$ gives the generalized tight path. Otherwise, assume that $d(x_{j+1}, y_j) \le l$ for some $1 \le j < m$.

Observe that $\max\{d(\tilde{\eta}_j, y_j), d(\tilde{\xi}_{j+1}, x_{j+1})\} \ge l+1$. Indeed, if not, it follows that $\tilde{\eta}_j$ and $\tilde{\xi}_{j+1}$ are connected by a path of length at most $3l$. By the above condition (2), we have $\rho_o(\tilde{\eta}_j, \tilde{\xi}_{j+1}) \le \frac{1}{4}\kappa$. By condition (1), we have $\rho_o(\eta_j, \xi_{j+1}) \le \frac{3}{4}\kappa$. We arrive at a contradiction with the definition of κ. Thus, we have proved that

$$\max\{d(\tilde{\eta}_j, y_j), d(\tilde{\xi}_{j+1}, x_{j+1})\} \ge l+1.$$

By Lemma 5.1, we obtain

$$\mathfrak{l}_{\lambda,o}([y_j, \tilde{\eta}_j]_{\gamma_j}) + \mathfrak{l}_{\lambda,o}([\tilde{\xi}_{j+1}, x_{j+1}]_{\gamma_{j+1}}) \ge \mathfrak{l}_{\lambda,o}([y_j, x_{j+1}]),$$

which yields

$$\begin{aligned} \mathfrak{l}_{\lambda,o}(\gamma_j) + \mathfrak{l}_{\lambda,o}(\gamma_{j+1}) &\ge \mathfrak{l}_{\lambda,o}([\tilde{\xi}_j, y_j]_{\gamma_j}) + \mathfrak{l}_{\lambda,o}([y_j, x_{j+1}]) + \mathfrak{l}_{\lambda,o}([x_{j+1}, \tilde{\eta}_{j+1}]_{\gamma_{j+1}}) \\ &\ge \rho_{\lambda,o}(\tilde{\xi}_j, \tilde{\eta}_{j+1}). \end{aligned}$$

This implies that we can drop the pair (η_j, ξ_{j+1}) in (33) such that the corresponding inequality in (35) still holds. Precisely, choose a $(1,l)$–tight path α_j between $\tilde{\xi}_j, \tilde{\eta}_{j+1}$ such that

$$|\mathfrak{l}_{\lambda,o}(\alpha_j) - \rho_{\lambda,o}(\tilde{\xi}_j, \tilde{\eta}_{j+1})| \le \frac{\varepsilon}{6m}.$$

So $\mathfrak{l}_{\lambda,o}(\gamma_j) + \mathfrak{l}_{\lambda,o}(\gamma_{j+1}) \ge \mathfrak{l}_{\lambda,o}(\alpha_j) - \varepsilon/(6m)$. It follows, by (35), that

$$\bar{\rho}_{\lambda,o}(\xi,\eta) \ge \sum_{\substack{1 \le i \le m \\ i \ne j, j+1}} \mathfrak{l}_{\lambda,o}(\gamma_i) + \mathfrak{l}_{\lambda,o}(\alpha_j) - \frac{5\varepsilon}{6} - \frac{\varepsilon}{6m}.$$

Consider the new set of $(1,c)$–tight paths γ_i for $i \ne j, j+1$ and α_j. Repeat the above argument for those j for which $d(x_{j+1}, y_j) \le l$. Since m is finite, for every $\varepsilon > 0$ we obtain a generalized tight path γ such that $\bar{\rho}_{\lambda,o}(\xi,\eta) \ge \mathfrak{l}_{\lambda,o}(\gamma) - \epsilon$. □

5.4 Floyd and shortcut metrics on uniformly conical points

A priori, the shortcut metrics as quotient of the Floyd metrics might be distorted in an unexpected way. The main result of this subsection is to show that this distortion is not severe for uniformly conical points.

Fix a basepoint $o \in G$. Recall that, in Section 2.4, $\Lambda^{\mathrm{uc}}_{L,o}G$ denotes the set of uniformly conical points $\xi \in \Lambda G$ for which there exists an (ϵ, R, L)–transitional geodesic ray between o and ξ. Similarly, denote by $\partial^{\mathrm{uc}}_{L,o}G$ the set of uniformly conical points in $\partial_\lambda G$ *based* at o. By Proposition 2.4, there exists a one-to-one correspondence between $\Lambda^{\mathrm{uc}}_{L,o}G$ and $\partial^{\mathrm{uc}}_{L,o}G$.

The following is a version of Proposition 2.19 for generalized tight paths:

Proposition 5.13 *There exist $l_0, D > 0$ such that for any $L > 0$, there exists $M = M(L) > 0$ with the following property:*

Let $\alpha_1 = [o, \xi]$ and $\alpha_2 = [o, \eta]$, where ξ and η are two distinct points of $\Lambda^{\mathrm{uc}}_{L,o}G$ (or $\partial^{\mathrm{uc}}_{L,o}G$, respectively). Let γ be a generalized $(1, l)$–tight path for some $l \geq l_0$ with $\gamma_- \in \alpha_1$ and $\gamma_+ \in \alpha_2$. If $d(o, \gamma_-), d(o, \gamma_+) \gg 0$, then there exists $z \in \gamma$ such that $d(z, \alpha_1 \cup \alpha_2) \leq D$ and $d(z, \alpha_i) \leq M$ for $i = 1, 2$. Moreover, $|d(o, z) - d(o, [\xi, \eta])| \leq M$.

Proof Let $l_0, c' \geq c$ and $M, K > 0$ be given by Lemma 5.11 such that the K–truncation $\tilde{\gamma}$ of a generalized (c, l)–tight γ for $l \geq l_0$ is a c'–quasigeodesic. By Proposition 2.19, there exists an $(\epsilon_{c'}, R_{c'})$–transitional point z in $\tilde{\gamma}$ such that the conclusion of this proposition holds. If the point z lies on γ, then we are done. Otherwise, by the "Moreover" statement of Lemma 5.11, we have $d(z, \gamma) \leq M$. The proposition now follows from Proposition 2.19. □

The main result of this subsection is the following:

Proposition 5.14 (visual Floyd/shortcut metric) *There exists $0 < \lambda_0 < 1$ such that the following holds for any $L > 0$ and $\lambda \in [\lambda_0, 1)$:*

We have

$$\rho_{\lambda,o}(\xi, \eta) \asymp_L \lambda^n \quad \text{for all } \xi \neq \eta \in \partial^{\mathrm{uc}}_{L,o}G$$

and

$$\bar{\rho}_{\lambda,o}(\xi, \eta) \asymp_L \lambda^n \quad \text{for all } \xi \neq \eta \in \Lambda^{\mathrm{uc}}_{L,o}G,$$

where $n = d(o, [\xi, \eta])$.

Proof Let us consider the shortcut metric case only. The Floyd metric case is similar and even easier.

Let α_1 and α_2 be two (ϵ_1, R_1, L)–transitional geodesic rays originating at o and terminating at ξ and η, respectively.

Let $l_0, D > 0$ be given by Proposition 5.13, and we choose $\lambda_0 \in (0,1)$ verifying Proposition 5.12 for $l = l_0$. Then, by Propositions 5.13 and 5.12, there exists $M = M(L) > 0$ such that the following holds:

(1) For each $k > 0$, there exists a sequence of generalized $(1, l_0)$–tight paths γ_k with $(\gamma_k)_- \in \alpha_1, (\gamma_k)_+ \in \alpha_2$ and such that $(\gamma_k)_- \to \xi$, $(\gamma_k)_+ \to \eta$ and

$$|\mathfrak{l}_{\lambda,o}(\gamma_k) - \bar{\rho}_{\lambda,o}(\xi,\eta)| \le 1/k. \tag{36}$$

(2) There exists $z_k \in \gamma_k$ such that $d(z_k, \alpha_1 \cup \alpha_2) \le D$ and $d(z_k, \alpha_i) \le M$ for $i = 1, 2$. Moreover, $|d(o, z_k) - d(o, [\xi,\eta])| \le M$.

Let $u_k := (\gamma_k)_-$ and $v_k := (\gamma_k)_+$.

Upper bound Choose $x_k \in \alpha_1$ and $y_k \in \alpha_2$ such that $\max\{d(z_k, x_k), d(z_k, y_k)\} \le M$. Then for every point $t \in [x_k, z_k] \cup [z_k, y_k]$ we have

$$d(o,t) \ge d(o, z_k) - M \ge n - 2M.$$

Hence,

$$\max\{\mathfrak{l}_{\lambda,o}([x_k, z_k]), \mathfrak{l}_{\lambda,o}([y_k, z_k])\} \le M \cdot \lambda^{n-2M}.$$

We also have

$$\max\{\mathfrak{l}_{\lambda,o}([x_k,\xi]_{\alpha_1}), \mathfrak{l}_{\lambda,o}([y_k,\eta]_{\alpha_2})\} \le \frac{\lambda^{\min\{d(o,x_k), d(o,y_k)\}}}{1-\lambda} \le \frac{\lambda^{n-2M}}{1-\lambda}.$$

It follows that

$$\begin{aligned} \bar{\rho}_{\lambda,o}(\xi,\eta) &\le \mathfrak{l}_{\lambda,o}([x_k,\xi]_{\alpha_1}) + \mathfrak{l}_{\lambda,o}([y_k,\eta]_{\alpha_2}) + \mathfrak{l}_{\lambda,o}([x_k,z_k]) + \mathfrak{l}_{\lambda,o}([y_k,z_k]) \\ &\le 2\lambda^{n-2M}\Big(\frac{1}{1-\lambda} + M\Big). \end{aligned} \tag{37}$$

Let $C_1 := 2\lambda^{-2M}(1/(1-\lambda) + M)$. Then $\bar{\rho}_{\lambda,o}(\xi,\eta) \le C_1 \lambda^n$.

Lower bound Since $d(z_k, \alpha_1 \cup \alpha_2) \le D$, there exists $w_k \in \alpha_1 \cup \alpha_2$ such that $d(z_k, w_k) \le D$. Assume that $w_k \in \alpha_2$ for concreteness.

By Lemma 2.7, any segment of α_2 is a Floyd geodesic. Since $v_k \to \eta$ and $d(o, w_k) \le n + D$, we can assume that $w_k \in [o, v_k]_{\alpha_2}$ for all $k \gg 0$. So

$$\rho_{\lambda,o}(v_k, w_k) = \mathfrak{l}_{\lambda,o}([w_k, v_k]_{\alpha_2}) = \frac{\lambda^{d(o,w_k)}}{1-\lambda} - \frac{\lambda^{d(o,v_k)}}{1-\lambda} \ge \frac{\lambda^{d(o,z_k)+D}}{1-\lambda} - \frac{\lambda^{d(o,v_k)}}{1-\lambda}.$$

We have

$$\rho_{\lambda,o}(v_k, w_k) \le \mathfrak{l}_{\lambda,o}(\gamma_k) + \mathfrak{l}_{\lambda,o}([w_k, z_k]).$$

Since $d(z_k, w_k) \le D$, we have

$$\mathfrak{l}_{\lambda,o}([w_k, z_k]) \le D \cdot \lambda^{d(o,z_k)-D}.$$

Thus,

$$\begin{aligned} \mathfrak{l}_{\lambda,o}(\gamma_k) &\ge \rho_{\lambda,o}(v_k, w_k) - \mathfrak{l}_{\lambda,o}([w_k, z_k]) \\ &\ge \mathfrak{l}_{\lambda,o}([v_k, w_k]_{\alpha_2}) - \mathfrak{l}_{\lambda,o}([w_k, z_k]) \\ &\ge \left(\frac{\lambda^D}{1-\lambda} - \frac{D}{\lambda^D}\right)\lambda^{d(o,z_k)} - \frac{\lambda^{d(o,v_k)}}{1-\lambda} \\ &\ge \left(\frac{\lambda^D}{1-\lambda} - \frac{D}{\lambda^D}\right)\cdot \lambda^M \cdot \lambda^n - \frac{\lambda^{d(o,v_k)}}{1-\lambda}. \end{aligned} \tag{38}$$

Since D does not depend on L by Proposition 5.13, there exists $1 > \lambda_0 > 0$ such that

$$\frac{\lambda^D}{1-\lambda} - \frac{D}{\lambda^D} \ge \frac{\lambda_0^D}{1-\lambda_0} - \frac{D}{\lambda_0^D} > 0 \tag{39}$$

for any $\lambda \in [\lambda_0, 1)$. Let $C_2 := (\lambda_0^D/(1-\lambda_0) - (D)/\lambda_0^D)\cdot\lambda^M > 0$. Note that $d(o, v_k) \to \infty$ as $k \to \infty$. By (36) and (38), passing to the limit when $k \to \infty$, we obtain

$$\bar\rho_{\lambda,o}(\xi, \eta) \ge C_2 \lambda^n$$

for any $\xi \ne \eta \in \Lambda^{\mathrm{uc}}_{L,o} G$ and any $L > 0$. The proof is then complete. □

Remarks (1) The fact that the constant D does not depend on L is crucial for the choice of λ_0 in (39).

(2) This lemma gives an asymptotic formula for two uniformly conical points with respect to the Floyd metric and shortcut metric. This could be used to give an alternative proof of Lemma 3.16, but cannot be derived from (the proof of) Lemma 3.16.

Appendix Visual metrics and Floyd metrics are bilipschitz equivalent

The aim of the appendix is to give a short proof that the visual Gromov metric $\nu_{a,o}$ and the Floyd metric $\rho_{\lambda,o}$ on the boundary ∂X of a δ–hyperbolic graph (X, d) are bilipschitz equivalent for some choice of parameters a and λ. This fact, mentioned in the introduction, is often considered as folklore; however, we have not found a complete proof of it in the literature — see eg [19, Lemma 7.2.M] and the key inequality after it, or [5, formula (1.3)]; in both cases the fact is stated without proof.

Recall the definition of the Gromov visual metric ν on ∂X. For a real parameter $a > 0$ set $\delta_{a,o}(\xi,\eta) = e^{-a(\xi|\eta)}$, where $(\xi\,|\,\eta)$ denotes the Gromov product for the basepoint o. Let

$$\nu_{a,o}(\xi,\eta) = \inf\left\{\sum_{i=0}^{n} \delta_{a,o}(c_{i-1},c_i) : \{c_i\} \in \mathcal{C}_{\xi,\eta}\right\}, \tag{40}$$

where $\mathcal{C}_{\xi,\eta}$ is the set of chains of points in ∂X such that $c_0 = \xi$ and $c_n = \eta$. If $a < (\ln 2)/(6\delta)$ then $\nu_{a,o}$ is a metric on ∂X satisfying the inequality [18, Proposition 7.10]

$$(3 - 2e^{a\delta}) \cdot \delta_{a,o}(\xi,\eta) \le \nu_{a,o}(\xi,\eta) \le \delta_{a,o}(\xi,\eta). \tag{41}$$

Remark An inequality similar to (41) where the metric ν is replaced by the Floyd metric (which is our goal now) is formally stated in [27, page 5] but unclear justification is given.

For the Gromov product the following inequality is true [6, Lemma 2.7]:

$$e^{-4a\delta} \cdot \delta_{a,o}(\xi,\eta) \le e^{-ad(o,[\xi,\eta])} \le \delta_{a,o}(\xi,\eta), \tag{42}$$

where $d(o,[\xi,\eta])$ is the distance in X from o to the union of all geodesics between ξ and η. The inequalities (41) and (42) imply

$$\nu_{a,o}(\xi,\eta)(\xi,\eta) \asymp_{C_1} e^{-a\cdot d(o,[\xi,\eta])} \tag{43}$$

for the constant $C_1 = \max\{e^{4a\delta}, (3-2e^{a\delta})^{-1}\}$.

The following proposition provides the bilipschitz equivalence between the visual metric and the Floyd metric on the boundary of X.

Proposition A.1 *Let (X,d) be a δ–hyperbolic graph. There exists a constant a_0 such that for any $a \in (0,a_0]$ there exists a constant C for which*

$$\nu_{a,o}(\xi,\eta) \asymp_C \rho_{\lambda,o}(\xi,\eta) \quad \text{for all } \xi,\eta \in \partial X, \tag{44}$$

where $\lambda = e^{-a}$.

Proof Similarly to (40), we introduce the metric $\widetilde{\nu}_{a,o}(x,y)$ defined with the chains formed by the graph vertices. Then the boundary ∂X equipped with the metric (40) can be seen as the remainder $\widehat{X} \setminus X$, where $\widehat{X}$ is the Cauchy completion of the graph X equipped with the distance $\widetilde{\nu}_{a,o}(x,y)$ (see eg [35, Section IV.22]). So $\widetilde{\nu}_{a,o}(\xi,\eta)=\lim_{x\to\xi,\, y\to\eta}\widetilde{\nu}_{a,o}(x,y)$, where

$$\widetilde{\nu}_{a,o}(x,y)=\inf\left\{\sum_{i=0}^{n}\delta_{a,o}(x_{i-1},x_i): x_0=x,\ x_n=y,\ x_i\in X\right\}. \tag{40$'$}$$

Similarly, the formulas (42)–(44) are still true if we replace the boundary points ξ and η by vertices x and y in the graph X and use the distance $d(o,[x,y])$ between o and a geodesic $[x,y]\subset X$. Thus, we obtain

$$\widetilde{\nu}_{a,o}(x,y)\le\inf\left\{\sum_{i=0}^{n}C_1\cdot\lambda^{d(o,[x_{i-1},x_i])}: x_0=x,\ x_n=y,\ x_i\in X\right\}. \tag{45}$$

The chains used in (40$'$) are more general than those in the definition of the Floyd distance, for which the distance between neighboring vertices must be 1. Therefore, we obtain the upper bound $\widetilde{\nu}_{a,o}(x,y)\le C_1\cdot\rho_{\lambda,o}(x,y)$. Passing to the limit we have $\widetilde{\nu}_{a,o}(\xi,\eta)\le C_1\cdot\rho_{\lambda,o}(\xi,\eta)$ for $\xi,\eta\in\partial X$.

Since the metric $\widetilde{\nu}_{a,o}$ is also bilipschitz equivalent to the right-hand side of (43) (see eg [35, Proposition 22.8]) the metrics $\widetilde{\nu}_{a,o}$ and $\nu_{a,o}$ are bilipschitz equivalent on ∂X, so, up to changing the constant, we have

$$\nu_{a,o}(\xi,\eta)\le C\cdot\rho_{\lambda,o}(\xi,\eta),\quad \xi,\eta\in\partial X.$$

To prove the opposite inequality, we need to use the δ–thin triangle property. Consider a geodesic triangle with vertices o, x, y in X. There exists a δ–center c on $[x,y]$ such that $d(c,[o,x])\le\delta$ and $d(c,[o,y])\le\delta$. For notational simplicity we ignore a small uniform difference between different hyperbolicity constants (see eg [18, Proposition 2.21]), and denote all of them by $\delta>0$. Since $|d(o,c)-d(o,[x,y])|$ is uniformly bounded above, to simplify the notation again we assume that $d(o,c)=d(o,[x,y])$.

Choose $x'\in[o,x]$ and $y'\in[o,y]$ so that $\max\{d(c,x'),d(c,y')\}\le\delta$. We have $\min\{d(o,[x',c]),d(o,[y',c])\ge d(o,[x,y])-\delta$. Hence, the Floyd length $\mathfrak{l}_{\lambda,o}([x',c])$ of $[x',c]$ is at most

$$\delta\cdot\lambda^{d(o,[x',c])}\le\delta\cdot\lambda^{d(o,[x,y])-\delta},$$

and similarly for $\mathfrak{l}_{\lambda,o}([y',c])$. Since $[x',x]$ and $[y',y]$ are the Floyd $\rho_{\lambda,o}$–geodesics (Lemma 2.7), we have

$$\mathfrak{l}_{\lambda,o}([x',x])=\frac{\lambda^{d(o,x')}}{1-\lambda}\le\frac{\lambda^{d(o,[x,y])-\delta}}{1-\lambda},$$

and the same for $\mathfrak{l}_{\lambda,o}([y', y])$. Summing all up we obtain the following estimation for the Floyd length of $[x, y]$:

$$\begin{aligned}\rho_{\lambda,o}(x,y) &\le \mathfrak{l}_{\lambda,o}([x',x]) + \mathfrak{l}_{\lambda,o}([x',c]) + \mathfrak{l}_{\lambda,o}([y',c]) + \mathfrak{l}_{\lambda,o}([y',y]) \\ &\le 2\frac{\lambda^{d(o,[x,y])-\delta}}{1-\lambda} + 2\delta\cdot\lambda^{d(o,[x,y])-\delta} \\ &\le C_2\cdot\lambda^{d(o,[x,y])}\end{aligned}$$

for the constant $C_2 = \max\{2/(\lambda^\delta(1-\lambda)), 2\delta/\lambda^\delta\}$. Passing to the limits when $x \to \xi \in \partial X$ and $y \to \eta \in \partial X$ and using (43), we obtain (44) for the constant $C = \max\{C_1, C_2\}$. □

References

[1] **C J Bishop**, **P W Jones**, *Hausdorff dimension and Kleinian groups*, Acta Math. 179 (1997) 1–39 MR

[2] **B H Bowditch**, *Relatively hyperbolic groups*, Internat. J. Algebra Comput. 22 (2012) art. id. 1250016 MR

[3] **M Burger**, **S Mozes**, *CAT(−1)–spaces, divergence groups and their commensurators*, J. Amer. Math. Soc. 9 (1996) 57–93 MR

[4] **J W Cannon**, *The combinatorial structure of cocompact discrete hyperbolic groups*, Geom. Dedicata 16 (1984) 123–148 MR

[5] **M Coornaert**, *Mesures de Patterson–Sullivan sur le bord d'un espace hyperbolique au sens de Gromov*, Pacific J. Math. 159 (1993) 241–270 MR

[6] **M Coornaert**, **T Delzant**, **A Papadopoulos**, *Géométrie et théorie des groupes: les groupes hyperboliques de Gromov*, Lecture Notes in Mathematics 1441, Springer (1990) MR

[7] **F Dal'Bo**, **M Peigné**, **J-C Picaud**, **A Sambusetti**, *On the growth of quotients of Kleinian groups*, Ergodic Theory Dynam. Systems 31 (2011) 835–851 MR

[8] **C Druțu**, **M Sapir**, *Tree-graded spaces and asymptotic cones of groups*, Topology 44 (2005) 959–1058 MR

[9] **K Falconer**, *Fractal geometry: mathematical foundations and applications*, Wiley, Chichester (1990) MR

[10] **W J Floyd**, *Group completions and limit sets of Kleinian groups*, Invent. Math. 57 (1980) 205–218 MR

[11] **D Gaboriau**, **F Paulin**, *Sur les immeubles hyperboliques*, Geom. Dedicata 88 (2001) 153–197 MR

[12] **V Gerasimov**, *Expansive convergence groups are relatively hyperbolic*, Geom. Funct. Anal. 19 (2009) 137–169 MR

[13] **V Gerasimov**, *Floyd maps for relatively hyperbolic groups*, Geom. Funct. Anal. 22 (2012) 1361–1399 MR

[14] **V Gerasimov**, **L Potyagailo**, *Quasi-isometric maps and Floyd boundaries of relatively hyperbolic groups*, J. Eur. Math. Soc. 15 (2013) 2115–2137 MR

[15] **V Gerasimov**, **L Potyagailo**, *Non-finitely generated relatively hyperbolic groups and Floyd quasiconvexity*, Groups Geom. Dyn. 9 (2015) 369–434 MR

[16] **V Gerasimov**, **L Potyagailo**, *Quasiconvexity in relatively hyperbolic groups*, J. Reine Angew. Math. 710 (2016) 95–135 MR

[17] **V Gerasimov**, **L Potyagailo**, *Similar relatively hyperbolic actions of a group*, Int. Math. Res. Not. 2016 (2016) 2068–2103 MR

[18] **E Ghys**, **P de la Harpe**, *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progr. Math. 83, Birkhäuser, Boston, MA (1990) MR

[19] **M Gromov**, *Hyperbolic groups*, from "Essays in group theory" (S Gersten, editor), Math. Sci. Res. Inst. Publ. 8, Springer (1987) 75–263 MR

[20] **J Heinonen**, *Lectures on analysis on metric spaces*, Springer (2001) MR

[21] **G C Hruska**, *Relative hyperbolicity and relative quasiconvexity for countable groups*, Algebr. Geom. Topol. 10 (2010) 1807–1856 MR

[22] **A Karlsson**, *Free subgroups of groups with nontrivial Floyd boundary*, Comm. Algebra 31 (2003) 5361–5376 MR

[23] **J M Mackay**, **A Sisto**, *Quasi-hyperbolic planes in relatively hyperbolic groups*, preprint (2012) arXiv

[24] **P Mattila**, *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*, Cambridge Studies in Advanced Mathematics 44, Cambridge Univ. Press (1995) MR

[25] **S J Patterson**, *The limit set of a Fuchsian group*, Acta Math. 136 (1976) 241–273 MR

[26] **F Paulin**, *On the critical exponent of a discrete group of hyperbolic isometries*, Differential Geom. Appl. 7 (1997) 231–236 MR

[27] **M J Puls**, *The first L^p–cohomology of some groups with one end*, Arch. Math. (Basel) 88 (2007) 500–506 MR

[28] **A Sisto**, *On metric relative hyperbolicity*, preprint (2012) arXiv

[29] **B O Stratmann**, *The exponent of convergence of Kleinian groups: on a theorem of Bishop and Jones*, from "Fractal geometry and stochastics, III" (C Bandt, U Mosco, M Zähle, editors), Progr. Probab. 57, Birkhäuser, Basel (2004) 93–107 MR

[30] **B Stratmann**, **M Urbański**, *The box-counting dimension for geometrically finite Kleinian groups*, Fund. Math. 149 (1996) 83–93 MR

[31] **B Stratmann**, **S L Velani**, *The Patterson measure for geometrically finite groups with parabolic elements, new and old*, Proc. London Math. Soc. 71 (1995) 197–220 MR

[32] **D Sullivan**, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Études Sci. Publ. Math. 50 (1979) 171–202 MR

[33] **D Sullivan**, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta Math. 153 (1984) 259–277 MR

[34] **P Tukia**, *Conical limit points and uniform convergence groups*, J. Reine Angew. Math. 501 (1998) 71–98 MR

[35] **W Woess**, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics 138, Cambridge Univ. Press (2000) MR

[36] **W Y Yang**, *Patterson–Sullivan measures and growth of relatively hyperbolic groups*, preprint (2013) arXiv

[37] **W Y Yang**, *Growth tightness for groups with contracting elements*, Math. Proc. Cambridge Philos. Soc. 157 (2014) 297–319 MR

[38] **W Y Yang**, *Peripheral structures of relatively hyperbolic groups*, J. Reine Angew. Math. 689 (2014) 101–135 MR

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