Holomorphic curves in exploded manifolds
Regularity

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The category of exploded manifolds is an extension of the category of smooth manifolds; for exploded manifolds, some adiabatic limits appear as smooth families. This paper studies the $\bar\partial$ equation on variations of a given family of curves in an exploded manifold. Roughly, we prove that the $\bar\partial$ equation on variations of an exploded family of curves behaves as nicely as the $\bar\partial$ equation on variations of a smooth family of smooth curves, even though exploded families of curves allow the development of normal-crossing or log-smooth singularities. The resulting regularity results are foundational to the author's construction of Gromov–Witten invariants for exploded manifolds.

1 Introduction

Exploded manifolds, introduced in Parker [18], are a generalization of smooth manifolds akin to the generalization of complex manifolds given by log schemes, or the generalization of smooth manifolds given by Melrose's b-structure on manifolds with boundary and corners.

A key feature of exploded manifolds is that normal-crossing or log-smooth degenerations — singular from the perspective of smooth manifolds — become smooth when viewed in the category of exploded manifolds. Bubbling and node-formation on the domain of holomorphic curves seems singular from the perspective of smooth manifolds, but happens within smooth families of the exploded version of holomorphic curves. In this paper, we study the regularity of the $\overline{\partial}$ equation on such families of curves. The $\overline{\partial}$ equation is elliptic and well behaved when we restrict to the case of a single domain. For a smooth family of domains, the regularity of such an elliptic differential equation is also well understood. Our goal is to understand the regularity of the $\overline{\partial}$ equation on apparently singular families of domains allowing bubbling and node-formation.

Exploded manifolds provide a natural regularity to desire in families where bubbling and node-formation occurs; in this paper, we prove that the $\overline{\partial}$ equation has our desired regularity. This natural regularity allows us to define in [20; 30] Gromov–Witten invariants of suitable exploded manifolds, working on a moduli stack with a natural smooth structure. Gromov–Witten invariants of suitable exploded manifolds do not change in families, so using a degeneration that is smooth from the perspective of exploded manifolds, Gromov–Witten invariants of a compact smooth symplectic manifold can be computed using an exploded manifold corresponding to a singular smooth manifold. The resulting computation is a gluing formula involving Gromov–Witten invariants relative normal crossing divisors. Such relative Gromov–Witten invariants are naturally defined within the category of exploded manifolds, and roughly equivalent to Ionel's GW invariants relative to normal crossing divisors from [12], and log Gromov–Witten invariants defined by Gross and Siebert in [5] and Abramovich and Chen in [3; 1]. For comparisons of these different methods of defining Gromov–Witten invariants relative to normal crossing divisors, see Parker [19; 22].

One novel feature of exploded manifolds is that each exploded manifold $\boldsymbol{B}$ is a set with several[1] relevant topologies: a small scale, on which the exploded manifold generally

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[1]Even though there are several topologies to keep in mind, we use the topology on $\boldsymbol{B}$ induced from $\lceil \boldsymbol{B} \rceil$ when referring to topological notions such as continuous functions or open subsets.

looks like an infinite disjoint union of smooth manifolds, a large scale, on which the exploded manifold looks like a union of integral affine polytopes, $\underline{\boldsymbol{B}}$, and a third topology, in which the exploded manifold looks like a stratified smooth space, $\lceil \boldsymbol{B} \rceil$. On the large scale, holomorphic curves look like tropical curves — piecewise linear maps of metric graphs — and the gluing formula referred to above is a sum of contributions corresponding to tropical curves. This tropical gluing formula is proved by the author in [28]; for a description of the formula in simple cases without using the language of exploded manifolds, see [24], and for examples, see [23; 27].

Despite the strange features described above, differential geometry and topology works for exploded manifolds roughly like it does for smooth manifolds, so the reader may read the remainder of the introduction pretending that exploded manifolds are just manifolds. In the rest of the paper, some familiarity with the definitions in [18] or [26] will be required.

This paper studies the $\bar{\partial}$ equation on a space of variations of a given family of curves in the category of exploded manifolds. In various guises, this is a key step in most constructions of Gromov–Witten invariants of general compact symplectic manifolds; see Chen, Li and Wang [2], Fukaya and Ono [4], Li and Tian [13], Liu and Tian [14], McDuff [16] and Ruan [31]. In Section 2.4, we define some Banach norms on this space of variations — Banach norms involving many local choices. Then in Section 3, the $\bar{\partial}$ equation is studied on the corresponding Banach spaces of variations of our family of curves. Our final results are stated using the regularity $C^{\infty,\underline{1}}$, defined in [18, Section 7].[2] With extra, less-analytic, work, the results stated in the remainder of this section are used by the author in [20; 21; 30] to describe the $\bar{\partial}$ equation on the moduli stack of $C^{\infty,\underline{1}}$ curves, and to construct the virtual fundamental class of the moduli space of holomorphic curves and Gromov–Witten invariants of exploded manifolds.

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[2]This regularity $C^{\infty,\underline{1}}$ should be thought of as a generalization of smooth functions with an exponential decay condition of all weights $\delta < 1$ at cylindrical ends.

1.1 Statement of results

The technical heart of this paper analyzes a family of (nonlinear) elliptic differential operators, $\overline{\partial}$, over a $C^{\infty,1}$ family of curves[3] $(\hat{C}, j) \to F$. Let us first describe the form of the linearization, $D\overline{\partial}$, of $\overline{\partial}$. Both $\overline{\partial}$ and $D\overline{\partial}$ send sections of a complex vectorbundle X to sections of $Y := (T^*_{\mathrm{vert}}\hat{C} \otimes X)^{(0,1)}$.[4] The form of $D\overline{\partial}$ is

$$(\star') \qquad\qquad D\overline{\partial}(\psi) = E'(\psi) + \tfrac{1}{2}(\nabla_{\mathrm{vert}}\psi + J \circ \nabla_{\mathrm{vert}}\psi \circ j),$$

where

- ∇_{vert} is the restriction of a $C^{\infty,1}$ connection[5] ∇ on X to $T_{\mathrm{vert}}\hat{C} \subset T\hat{C}$, the vertical tangent space of $\hat{C}$;

- J indicates the complex structure on X; and

- the map

$$E' \colon X \to Y$$

  is a $C^{\infty,1}$ map of vectorbundles, vanishing on the edges[6] of curves in $\hat{C}$.

Our linearized operator determines a linear map

$$D\overline{\partial} \colon X^{\infty,1} \to Y^{\infty,1},$$

where $X^{\infty,1}$ is the sheaf of $C^{\infty,1}$ sections of X, and $Y^{\infty,1}$ is the sheaf of $C^{\infty,1}$ sections of Y vanishing on edges of curves in $\hat{C} \to F$. The map $D\overline{\partial}$ above is then a map of sheaves of $C^{\infty,1}(F, \mathbb{R})$–modules over F. We can restrict $D\overline{\partial}$, $X^{\infty,1}$

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[3]Curves are used in the sense of Definition 8.3 of [18]; in particular, a curve is a complete 2–dimensional exploded manifold C with a complex structure j, and a curve in an exploded manifold B is a map $f \colon C \to B$. A family $\hat{C} \to F$ is a kind of proper submersion, discussed in Section 10 of [18]. See Section 1.2 for a brief discussion of the local structure of such a family of curves $\hat{C} \to F$, including (co)tangent bundles and $C^{\infty,1}$ regularity.

[4]For any family $\hat{C} \to F$, let $T_{\mathrm{vert}}\hat{C}$ and $T^*_{\mathrm{vert}}\hat{C}$ respectively denote the vertical tangent and cotangent space. Our vectorbundle Y is the subbundle of $T^*_{\mathrm{vert}}C \otimes_{\mathbb{R}} X$ fixed by $j \otimes J$, where J is the complex structure on X.

[5]A $C^{\infty,1}$ connection on a vectorbundle V over an exploded manifold B is a linear map ∇ from $C^{\infty,1}$ sections of V to $C^{\infty,1}$ sections of $V \otimes T^*B$ that satisfies the usual derivation condition, and such that $\nabla_v w = 0$ for any $\mathbb{R}$–nil vector v and $C^{\infty,1}$ section w of V. An $\mathbb{R}$–nil vector v is one for which $vf = 0$ for all smooth or $C^{\infty,1}$ functions f. In particular, on a curve, the nonzero $\mathbb{R}$–nil vectors are the nonzero vectors on edges. See Section 6 of [18] for a discussion of TB.

[6]The tropical part of a curve C is a graph $\underline{C}$. As in Definition 8.3 of [18], an edge of a curve refers to a stratum of C over one of the edges of this graph $\underline{C}$. The smooth part of a C is a nodal curve $\lceil C \rceil$ with marked points. Edges of C correspond to the nodes and marked points of this nodal curve $\lceil C \rceil$.

and $Y^{\infty,\underline{1}}$ to any open subset of X, but we can also restrict them to any individual curve $f \in \boldsymbol{F}$ to determine a map

$$D\bar{\partial}(f): X^{\infty,\underline{1}}(f) \to Y^{\infty,\underline{1}}(f),$$

which is an elliptic differential operator, satisfying the usual Fredholm and regularity properties.

Although $X^{\infty,\underline{1}}$ and $Y^{\infty,\underline{1}}$ are not obviously vectorbundles over $\boldsymbol{F}$, because they are not obviously locally trivial,[7] they have the following natural notion of finite-rank subvectorbundle:

Definition 1.1 A rank-k subvectorbundle K of $X^{\infty,\underline{1}}$ (or $Y^{\infty,\underline{1}}$) is a locally free, rank-k subsheaf whose restriction to $X^{\infty,\underline{1}}(f)$ (or $Y^{\infty,\underline{1}}(f)$) is a k–dimensional linear subspace $K(f)$ for all $f \in \boldsymbol{F}$.

Theorem 1.2 *Given any curve $f \in \boldsymbol{F}$, the differential operator $D\bar{\partial}(f)$ has closed image, finite-dimensional kernel and cokernel, and index*

$$\dim \ker D\bar{\partial}(f) - \dim \operatorname{coker} D\bar{\partial}(f) = 2c_1 - 2n(g-1),$$

where c_1 is the integral of the first Chern class of X over the curve f, n is the complex rank of X and g is the genus of the curve f.

Given any finite-rank subvectorbundle $V \subset Y^{\infty,\underline{1}}$ such that $D\bar{\partial}(f)$ is transverse to $V(f) \subset Y^{\infty,\underline{1}}(f)$, there exists a neighborhood $\boldsymbol{F}'$ of $f \in \boldsymbol{F}$, on which $D\bar{\partial}$ surjects onto $Y^{\infty,\underline{1}}(\boldsymbol{F}')/V$ with kernel a finite-rank subvectorbundle $K = D\bar{\partial}^{-1}(V)$ of $X^{\infty,\underline{1}}(\boldsymbol{F}')$.

The first part of Theorem 1.2 simply tells us that, restricted to any individual curve, $D\bar{\partial}$ is Fredholm and has the expected index. The second part is a "gluing and regularity" result. It tells us that we can study $D\bar{\partial}$ with a finite-dimensional, $C^{\infty,\underline{1}}$ approximation $K \to V$, where K and V are finite-rank $C^{\infty,\underline{1}}$ vectorbundles. A proof of Theorem 1.2 is at the end of Section 3.4. The analogous theorem for the nonlinear operator $\bar{\partial}$ is that $\bar{\partial}^{-1}(V)$ is a $C^{\infty,\underline{1}}$ manifold of the expected dimension.

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[7]This is related to the possibly insurmountable difficulty of putting a Banach orbifold structure on the moduli space of curves of a given regularity close to a nodal curve, and the need for different analytic structures such as polyfolds; see [6; 7; 8; 9; 10; 11]. We use standard analysis on Banach space completions of $X^{\infty,\underline{1}}$ and $Y^{\infty,\underline{1}}$ involving unnatural choices, but obtain results independent of such choices.

The nonlinear operators we consider are in the form

$$(\star) \quad \overline{\partial}(v) = E(v) + \tfrac{1}{2}\big(H'(v) \circ \nabla_{\mathrm{vert}} v + J \circ H'(v) \circ \nabla_{\mathrm{vert}} v \circ j\big) := E(v) + H(v)(\nabla_{\mathrm{vert}} v),$$

where:

- E is a (nonlinear) $C^{\infty,\underline{1}}$ map such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ E\ } & Y \\
 & \searrow & \downarrow \\
 & & \widehat{C}
\end{array}
$$

  commutes and such that, restricted to edges of curves in $\widehat{C}$, the image of E is the zero-section of Y.

- H' is a (nonlinear) $C^{\infty,\underline{1}}$ map to the space of invertible, $\mathbb{R}$–linear endomorphisms of X such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ H'\ } & X \otimes_{\mathbb{R}} X^* \\
 & \searrow & \downarrow \\
 & & \widehat{C}
\end{array}
$$

  commutes, and such that H' restricted to the zero-section is the identity.

Theorem 3.2 implies that $\overline{\partial}$ is continuously differentiable, at least when F is extendable. The derivative of $\overline{\partial}$ at the zero-section is a family of linear elliptic differential operators in the form of $(\star')$. Moreover, the derivative of $\overline{\partial}$ at any other section is in the form of $(\star')$ with a different J, so Theorem 1.2 still applies.

Our nonlinear regularity theorem becomes easier to state if $D\overline{\partial}$ is injective; we can achieve this by restricting the domain of $\overline{\partial}$. In particular, choose some finite set of nonintersecting sections $s_1, \ldots, s_m$ of $\widehat{C} \to F$ that avoid the edges of curves in $\widehat{C}$. Now define $X^{\infty,\underline{1}}$ to be the sheaf of $C^{\infty,\underline{1}}$ sections of X that vanish at the image of s_i. Theorem 1.2 still holds for $D\overline{\partial}$ with such a restricted domain, but the index of $D\overline{\partial}$ is now $2c_1 - 2n(g - 1 + m)$. By choosing enough such sections, we may assume that $D\overline{\partial}(f)$ is injective, and apply the following theorem:

Theorem 1.3 *Given a curve f in F and a finite-rank subvectorbundle $V \subset Y^{\infty,\underline{1}}$ such that $\overline{\partial} f \in V(f)$ and $D\overline{\partial}(f)$ is injective and complementary to $V(f) \subset Y^{\infty,\underline{1}}(f)$,*

there exists a neighborhood F' of $f \in F$ and a solution $v \in X^{\infty,1}(F')$ to the equation

$$\bar{\partial} v = 0 \quad \mod V.$$

Moreover, there exists a neighborhood O of $0 \in X^{\infty,1}(F')$ such that v is the unique solution to $\bar{\partial} v = 0 \mod V$ within O in the following sense. Given any curve $f' \in F'$, let $v(f')$ and $O(f')$ be the restriction of v and O to $X^{\infty,1}(f')$. Then $v(f')$ is the unique solution to the equation $\bar{\partial} v(f') = 0 \mod V(f')$ within $O(f')$.

So, when we parametrize curves by a fixed family of domains, the moduli space of solutions to $\bar{\partial} = 0 \mod V$ is regular when $D\bar{\partial}$ is transverse to V. Note that our fixed family can include bubbling and node-formation behavior, so this regularity theorem includes gluing analysis. When we no longer have a fixed family of domains, this theorem is key for proving the regularity of the moduli stack of solutions to $\bar{\partial} = 0 \mod V$; for example, where $D\bar{\partial}$ is transverse to zero, the moduli stack of holomorphic curves is locally a $C^{\infty,1}$ orbifold. For details and precise statements in the setting of the moduli stack of curves, see [20]. Theorem 1.3 follows immediately from Theorem 3.14 stated on page 1685.

1.2 Local model for families of curves

This section illustrates some key differences between exploded manifolds and smooth manifolds; the reader already familiar with exploded manifolds may skip it. Consider the standard local model for node formation,

$$\mathbb{C}^2 \xrightarrow{z_1 z_2} \mathbb{C}.$$

Away from 0, this map is a submersion with fiber a cylinder, but over 0 the fiber is a pair of planes, joined at 0. The corresponding local model for node formation in the category of exploded manifolds is

$$(T^1_{[0,\infty)})^2 = T^2_{[0,\infty)^2} \xrightarrow{\tilde{z}_1 \tilde{z}_2} T^1_{[0,\infty)}.$$

Every exploded manifold B comes with a natural map to a stratified smooth space called its smooth part $\lceil B \rceil$. For example, an exploded curve C has smooth part $\lceil C \rceil$ a nodal Riemann surface, with strata given by nodes and marked points.[8] The smooth

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[8] The curve C also has a tropical part $\underline{C}$ consisting of a complete metric graph with a vertex for every component of the nodal curve $\lceil C \rceil$, an edge for every node of $\lceil C \rceil$, and a semi-infinite edge for every marked point of $\lceil C \rceil$. So, nodes and marked points of $\lceil C \rceil$ correspond to internal and external edges of $\underline{C}$, and we will often refer to such strata of C as "edges".

part of the above exploded node-formation model is the standard node-formation model (stratified by where coordinate functions are 0). In particular $\lceil T^1_{[0,\infty)} \rceil = \mathbb{C}$, and the smooth part $\lceil \tilde{z} \rceil$ of the standard coordinate on $T^1_{[0,\infty)}$ is the standard coordinate on $\mathbb{C}$. In fact, $\mathbb{R}$–valued smooth or continuous functions on an exploded manifold $\boldsymbol{B}$ factor through its smooth part $\boldsymbol{B} \to \lceil \boldsymbol{B} \rceil$, so the smooth or continuous functions on our exploded node-formation model are the same as those on the standard node-formation model. Moreover, although there are several topologies on $\boldsymbol{B}$, by default, we use the topology induced by the map $\boldsymbol{B} \to \lceil \boldsymbol{B} \rceil$.

The tangent sheaf, however, is different. The tangent space of $T^1_{[0,\infty)}$ has a standard trivialization as $\mathbb{R}^2 \times T^1_{[0,\infty)}$, with basis the real and imaginary parts of $\tilde{z}\, \partial/\partial\tilde{z}$; similarly, a basis for the cotangent space is the real and imaginary parts of $\tilde{z}^{-1} d\tilde{z}$. The derivative of our exploded node-formation model is constant in this standard basis, sending

$$a\tilde{z}_1 \frac{\partial}{\partial \tilde{z}_1} + b\tilde{z}_1 \frac{\partial}{\partial \tilde{z}_2} \quad \text{to} \quad (a+b)\tilde{z} \frac{\partial}{\partial \tilde{z}}.$$

So, unlike the standard node-formation model, our exploded node-formation model is a submersion.

A second difference is what replaces $0 \in \mathbb{C}$ and the singular fiber of the standard node-formation model. On $T^1_{[0,\infty)}$, our coordinate $\tilde{z}$ takes values in $\mathbb{C}^* t^{[0,\infty)} \subset \mathbb{C}^* t^{\mathbb{R}}$. As a group, $\mathbb{C}^* t^{\mathbb{R}}$ is $(\mathbb{C}^*, \times) \times (\mathbb{R}, +)$: this group action preserves our standard basis for the tangent space. The smooth part map sends points ct^0 to c, and everything else to 0. In some sense, $0 \in \mathbb{C}$ is replaced by an infinite set of cylinders, $\mathbb{C}^*$, one for every point in $(0, \infty)$. Accordingly, the singular fiber of the standard node-formation model is replaced by many exploded manifold fibers, one for each $ct^a \in \mathbb{C}^* t^{(0,\infty)}$. One way to think of this ct^a is as an infinitesimal gluing parameter: the fiber over ct^a has two natural coordinates $\tilde{z}_1$ and $\tilde{z}_2$, related by $\tilde{z}_2 = ct^a \tilde{z}_1^{-1}$.

In this simple case, let us consider the natural $C^{\infty,\underline{1}}$ regularity used in this paper. We can think of our exploded manifold $T^1_{[0,\infty)}$ replacing $\mathbb{C} \setminus 0$ with a manifold with a cylindrical end, and replacing $0 \in \mathbb{C}$ with lots of "cylinders at infinity". A standard setup for analysis on manifolds with cylindrical ends uses Sobolev spaces with exponential weights at ends, however, as cylindrical coordinates correspond to $\log z$, an exponential weight δ corresponds to using a weight $|z|^{-\delta}$. The regularity we will achieve in this paper is $C^{\infty,\underline{1}}$. Let us describe $C^{\infty,\underline{1}}$ functions on $T^1_{[0,\infty)}$ using the coordinate $z = \lceil \tilde{z} \rceil$ on the smooth part of $T^1_{[0,\infty)}$. A continuous $\mathbb{R}$–valued function $h(z)$ is $C^{\infty,\underline{1}}$ if $(h(z) - h(0))|z|^{-\delta}$ extends to a continuous function for all $\delta < 1$, and the same

applies to any number of derivatives of h, using the real or imaginary part of $z \, \partial/\partial z$. How about $C^{\infty,1}$ functions on $\boldsymbol{T}^2_{[0,\infty)^2}$? Such functions $h(z_1, z_2)$ have

$$(h(z_1, z_2) - h(0, 0))(|z|_1 + |z|_2)^{-\delta},$$

$$(h(z_1, z_2) - h(z_1, 0))|z_2|^{-\delta},$$

$$(h(z_1, z_2) - h(0, z_2))|z_1|^{-\delta},$$

$$(h(z_1, z_2) - h(z_1, 0) - h(0, z_2) + h(0, 0))|z_1 z_2|^{-\delta}$$

extending to continuous functions for all $\delta < 1$, and the same holds for all derivatives of h using our standard basis consisting of the real and imaginary parts of $z_i \, \partial/\partial z_i$. The different conditions above correspond to different sets of strata of $[0, \infty)^2$. In this case, the first expression above is redundant, but the fourth expression might seem excessive. It ensures that each fiberwise-constant $C^{\infty,1}$ function on $\boldsymbol{T}^2_{[0,\infty)^2}$ is the pullback of a $C^{\infty,1}$ function on $\boldsymbol{T}^1_{[0,\infty)}$. This is true more generally: given a family of exploded manifolds, $C^{\infty,1}$ functions on the base correspond to fiberwise-constant $C^{\infty,1}$ functions on the total space; moreover, any $C^{\infty,1}$ function on a fiber extends to a $C^{\infty,1}$ function on the total space.

Most "singular" behavior in our family of curves $\widehat{C} \to F$ is captured in the above exploded node-formation model. Locally on $\widehat{C}$, the family $\widehat{C} \to F$ is either locally trivial, or locally modeled on a base-change of our node-formation model, given by a fiber product diagram

(1)
$$\begin{array}{ccc}
\mathbb{R}^k \times \boldsymbol{T}^{m+1}_Q & \longrightarrow & \boldsymbol{T}^2_{[0,\infty)^2} \\
\downarrow & & \downarrow \\
\mathbb{R}^k \times \boldsymbol{T}^m_P & \longrightarrow & \boldsymbol{T}^1_{[0,\infty)}
\end{array}$$

When $P = [0, \infty)^n$, $\boldsymbol{T}^m_P$ is just $(\boldsymbol{T}^1_{[0,\infty)})^m$; if P is a more complicated polytope, cut from $[0, \infty)^M$ by integral affine equations $\sum \alpha_i x_i = c$, the exploded manifold $\boldsymbol{T}^m_P$ is the subset of $(\boldsymbol{T}^1_{[0,\infty)})^M$ cut out by the corresponding monomial equations $\prod \tilde{z}_i^{\alpha_i} = \mathfrak{t}^c$, and the $C^{\infty,1}$ functions on $\boldsymbol{T}^m_P$ consist of the restriction of $C^{\infty,1}$ functions from $(\boldsymbol{T}^1_{[0,\infty)})^M$.

1.3 Geometric setup

This section explains how Theorems 1.2 and 1.3 apply to the $\overline{\partial}$ equation on the space of variations of a given family of curves $\widehat{f}$ in a smooth family of exploded

manifolds $\hat{\boldsymbol{B}} \to \boldsymbol{B}_0$. To discuss holomorphic curves in such a family $\hat{\boldsymbol{B}}$, we require an almost complex structure J on each fiber.[9]

A family of curves $\hat{f}$ in a family of exploded manifolds $\hat{\boldsymbol{B}} \to \boldsymbol{B}_0$ is a commutative diagram

(2)
$$
\begin{array}{ccc}
(\hat{\boldsymbol{C}}, j) & \xrightarrow{\hat{f}} & (\hat{\boldsymbol{B}}, J) \\
\downarrow{\scriptstyle \pi_F} & & \downarrow \\
\boldsymbol{F} & \longrightarrow & \boldsymbol{B}_0
\end{array}
$$

where π_F is a family of exploded manifolds, and j is a family of almost complex structures such that each fiber is a curve; see Definition 8.3 of [18]. Such a family has regularity $C^{\infty,\underline{1}}$ if j, and all maps, are $C^{\infty,\underline{1}}$. By variations of $\hat{f}$, we mean variations of the map $\hat{f}$ such that all other maps in the above commutative diagram are fixed. Locally, we can identify an open neighborhood of $\hat{f}$ in the space of such variations with $C^{\infty,\underline{1}}$ sections of the vectorbundle $\hat{f}^* T_{\mathrm{vert}} \hat{\boldsymbol{B}}$. The complex vectorbundle X in the previous section is $\hat{f}^* T_{\mathrm{vert}} \hat{\boldsymbol{B}}$ with its almost complex structure, J.

Given such a family of curves, use

$$
d_{\mathrm{vert}} \hat{f} \colon T_{\mathrm{vert}} \hat{\boldsymbol{C}} \to T_{\mathrm{vert}} \hat{\boldsymbol{B}}
$$

to indicate $d\hat{f}$ restricted to the vertical tangent space, $T_{\mathrm{vert}} \hat{\boldsymbol{C}} \subset T\hat{\boldsymbol{C}}$. This family $\hat{f}$ is a family of holomorphic curves if $d_{\mathrm{vert}} \hat{f} \circ j = J \circ d_{\mathrm{vert}} \hat{f}$. Define a map

$$
\bar{\partial} \hat{f} \colon T_{\mathrm{vert}} \hat{\boldsymbol{C}} \to T_{\mathrm{vert}} \hat{\boldsymbol{B}} \quad \text{by} \quad \bar{\partial} \hat{f} := \tfrac{1}{2}(d_{\mathrm{vert}} f + J \circ d_{\mathrm{vert}} f \circ j).
$$

We shall also consider $\bar{\partial} \hat{f}$ as a section of the vectorbundle

(3)
$$
Y(\hat{f}) := (T_{\mathrm{vert}}^* \hat{\boldsymbol{C}} \otimes \hat{f}^* T_{\mathrm{vert}} \hat{\boldsymbol{B}})^{(0,1)},
$$

which is the subbundle of $T_{\mathrm{vert}}^* \hat{\boldsymbol{C}} \otimes_{\mathbb{R}} \hat{f}^* T_{\mathrm{vert}} \hat{\boldsymbol{B}}$ fixed by $j \otimes J$. In fact, $Y(\hat{f})$ is the pullback of a vectorbundle over the universal curve over the moduli stack of

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[9]Almost complex structures are discussed in Section 8 of [18]. For this paper, we shall not require that J be civilized in the sense of Definition 8.2 of [25], and also not require the stronger condition of being $\bar{\partial}$–log compatible, Definition 3.1 of [25]. This stronger condition is used in [25] to prove compactness for the moduli stack of holomorphic curves, and if it were assumed, the results of the present paper could conceivably be improved, with "smooth" replacing $C^{\infty,\underline{1}}$. Such an improvement would not be particularly important, because the category of $C^{\infty,\underline{1}}$ exploded manifolds behaves similarly to the category of smooth exploded manifolds.

$C^{\infty,1}$ curves in $\hat{B} \to B_0$;[10] given any map of families of curves $\hat{f} \to \hat{g}$, there is a corresponding map of vectorbundles $Y(\hat{f}) \to Y(\hat{g})$.

When we locally identify variations of $\hat{f}$ with sections of $\hat{f}^* T_{\mathrm{vert}} \hat{B}$, we can also choose a trivialization of Y over this space of variations. Then the $\bar{\partial}$ equation takes the form $(\star)$, and we can apply our regularity theorems.

Definition 1.4 Given a $C^{\infty,1}$ family of curves

$$
\begin{array}{ccc}
\hat{C} & \xrightarrow{\hat{f}} & \hat{B} \\
\downarrow & & \downarrow \\
F & \longrightarrow & B_0
\end{array}
$$

a trivialization is

(i) a $C^{\infty,1}$ map $\mathcal{F}$ such that the diagram

$$
\begin{array}{ccc}
\hat{f}^* T_{\mathrm{vert}} \hat{B} & \xrightarrow{\mathcal{F}} & \hat{B} \\
\downarrow & & \downarrow \\
F & \longrightarrow & B_0
\end{array}
$$

commutes and such that

(a) $\mathcal{F}$ restricted to the zero-section equals $\hat{f}$,

(b) $T\mathcal{F}$ restricted to the natural inclusion $\hat{f}^* T_{\mathrm{vert}} \hat{B} \subset T(\hat{f}^* T_{\mathrm{vert}} \hat{B})$ over the zero-section is the identity,

(c) $T\mathcal{F}$ restricted to the vertical tangent space at any point of $\hat{f}^* T_{\mathrm{vert}} \hat{B}$ is injective;

(ii) and a $C^{\infty,1}$ vectorbundle map Φ that fits into the commutative diagram

$$
\begin{array}{ccc}
\mathcal{F}^* T_{\mathrm{vert}} \hat{B} & \xrightarrow{\Phi} & \hat{f}^* T_{\mathrm{vert}} \hat{B} \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
\hat{f}^* T_{\mathrm{vert}} \hat{B} & \xrightarrow{\pi} & \hat{C}
\end{array}
$$

and which is a J–preserving isomorphism on each fiber and is the identity when the vectorbundle $\mathcal{F}^* T_{\mathrm{vert}} \hat{B} \to \hat{f}^* T_{\mathrm{vert}} \hat{B}$ is restricted to the zero-section of $\hat{f}^* T_{\mathrm{vert}} \hat{B}$.

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[10] See Section 11 of [18] for a short discussion of the moduli stack of $C^{\infty,1}$ curves. Roughly speaking, this moduli stack is a category with objects consisting of $C^{\infty,1}$ families of curves. A more detailed analysis of this moduli stack is contained in [20].

For example, we may construct a trivialization by extending $\hat{f}$ to a map $\mathcal{F}$ satisfying the above conditions (for instance by choosing a smooth connection on $T_{\mathrm{vert}}\hat{\boldsymbol{B}}$ and reparametrizing the exponential map on a neighborhood of the zero-section in $f^*T_{\mathrm{vert}}\hat{\boldsymbol{B}}$), and letting Φ be given by parallel transport along a linear path to the zero-section using a smooth J–preserving connection on $T_{\mathrm{vert}}\hat{\boldsymbol{B}}$.

Definition 1.5 A trivialization allows us to define $\overline{\partial}$ of a section

$$v\colon \hat{\boldsymbol{C}} \to \hat{f}^*T_{\mathrm{vert}}\hat{\boldsymbol{B}}$$

as follows: $\mathcal{F} \circ v$ is a family of curves

$$\mathcal{F} \circ v\colon \hat{\boldsymbol{C}} \to \hat{\boldsymbol{B}},$$

so $\overline{\partial}(\mathcal{F} \circ v)$ is a section of $Y(\mathcal{F} \circ v) = (T_{\mathrm{vert}}^*\hat{\boldsymbol{C}} \otimes (\mathcal{F} \circ v)^*T_{\mathrm{vert}}\hat{\boldsymbol{B}})^{(0,1)}$. Applying the map Φ to the second component of $\overline{\partial}(\mathcal{F} \circ v)$ defines a section $\overline{\partial}v$ of $Y(\hat{f})$.

Lemma 1.7 below implies that $\overline{\partial}$ as defined above is in the form $(\star)$.

Given a trivialization for a family $\hat{f}$, we can define the following "simple perturbation" of the $\overline{\partial}$ equation on variations of $\hat{f}$.

Definition 1.6 Given a trivialization for $\hat{f}$, a simple perturbation of $\overline{\partial}$ is a map $\overline{\partial}'$ from sections $\hat{\boldsymbol{C}} \to \hat{f}^*T_{\mathrm{vert}}\hat{\boldsymbol{B}}$ to sections of $Y(\hat{f})$ such that

$$\overline{\partial}'v = \overline{\partial}(v) + \Psi(v),$$

where Ψ is a (possibly nonlinear) $C^{\infty,\underline{1}}$ map such that the diagram

$$\hat{f}^*T_{\mathrm{vert}}\hat{\boldsymbol{B}} \xrightarrow{\ \Psi\ } Y(\hat{f})$$

$$\searrow \qquad \downarrow$$

$$\hat{\boldsymbol{C}}$$

commutes and such that the image of Ψ restricted to edges of curves in $\hat{\boldsymbol{C}}$ is contained in the zero-section of $Y(\hat{f})$.

Lemma 1.7 *Let $\hat{f}$ be a $C^{\infty,\underline{1}}$ family of curves with a choice of trivialization. Then any simple perturbation $\overline{\partial}'$ of $\overline{\partial}$ is in the form $(\star)$. In particular, in any coordinate chart U on $\hat{\boldsymbol{C}}$ where $\hat{f}^*T_{\mathrm{vert}}\hat{\boldsymbol{B}}$ is trivialized as $\mathbb{R}^n \times U$, consider a section as a map $v\colon U \to \mathbb{R}^n$. In such a coordinate chart, there is the following formula for $\overline{\partial}'$:*

$$\overline{\partial}'v(u) = E(v(u), u) + H(v(u), u)(d_{\mathrm{vert}}v),$$

where

$$H(d_{\text{vert}}v) := \tfrac{1}{2}(H' \circ d_{\text{vert}}v + J \circ H' \circ d_{\text{vert}}v \circ j)$$

and E, H and H' are as described in $(\star)$.

Proof The map $E(x, u)$ is $\overline{\partial}'$ of the constant section $u \mapsto (x, u)$. This is a $C^{\infty,\underline{1}}$ map, because $\widehat{f}$, J and j are $C^{\infty,\underline{1}}$, and it vanishes on the edges of fibers of $\widehat{C} \to F$ because every $C^{\infty,\underline{1}}$ curve is automatically holomorphic restricted to any edge, and Definition 1.6 tells us that simple perturbations do not modify $\overline{\partial}$ on edges of curves.

The tensor H' is given by the formula

$$H'(x, u) := \Phi(x, u) \circ d_{\text{vert}}\mathcal{F}(x, u),$$

where Φ and $\mathcal{F}$ are as in Definition 1.4. To interpret $H'(x, u)$ as an endomorphism of $\widehat{f}^* T_{\text{vert}}\widehat{B}$ at x, identify the vertical tangent space of $\widehat{f}^* T_{\text{vert}}\widehat{B} \to \widehat{C}$ at (x, u) with $\widehat{f}^* T_{\text{vert}}\widehat{B}$ at u. Both $\Phi(x, u)$ and the vertical derivative of $\mathcal{F}$ are injective, and both are equal to the identity at the zero-section, so the same holds for H'. Because Φ and $\mathcal{F}$ are $C^{\infty,\underline{1}}$, H' is $C^{\infty,\underline{1}}$ too. Direct computation gives that $\overline{\partial}'v$ obeys the above formula.

Given any $C^{\infty,\underline{1}}$ connection ∇ on $\widehat{f}^* T_{\text{vert}}\widehat{B}$, $\nabla_{\text{vert}} - d_{\text{vert}}$ is a $C^{\infty,\underline{1}}$ section of $T_{\text{vert}}^*\widehat{C} \otimes \widehat{f}^* T_{\text{vert}}\widehat{B} \otimes \widehat{f}^* T_{\text{vert}}^*\widehat{B}$, which vanishes on edges of curves in $\widehat{C} \to F$ because $\nabla_v w$ always vanishes when v is a $\mathbb{R}$–nil vector, and the vectors on edges of a curve are $\mathbb{R}$–nil.

It follows that $E'(v, u) := H(v, u)(\nabla_{\text{vert}}v - d_{\text{vert}}v)$ is a $C^{\infty,\underline{1}}$ map satisfying the conditions on E, so exchanging d_{vert} for ∇_{vert} gives $(\star)$. It also follows that an operator is in the form above on coordinate charts if and only if it is in the form $(\star)$. $\square$

2 Some norms on sections of vectorbundles

In what follows, we define some norms on the space of sections of a real vectorbundle $V \to \widehat{C}$ over a family of curves $\widehat{C} \xrightarrow{\pi_F} F$. Applied to the vectorbundles X and Y, these norms will determine a series of Banach-space completions $X_{k,\delta}$ and $Y_{k,\delta}$ of (bounded subspaces) in $X^{\infty,\underline{1}}$ and $Y^{\infty,\underline{1}}$ so that $\overline{\partial}$ determines a well-behaved, continuously differentiable map $X_{k,\delta} \to Y_{k,\delta}$, and so that convergence in $C^{\infty,\underline{1}}$ is implied by convergence in $\|\cdot\|_{k,\delta}$ for all natural numbers k and $\delta < 1$. We will prove our desired $C^{\infty,\underline{1}}$ regularity by proving regularity in each of these norms.

2.1 e_S and Δ_S

For defining and working with our norms, we need to recall some basic notions from [18]. In particular, we need the notation for a coordinate chart on an exploded manifold from Section 3 of [18], the notion of a stratum of P or T_P^m from Section 4, and the following definitions from Section 7.

Definition 2.1 (the operator e_S) Given any C^0 function h on $\mathbb{R}^n \times T_P^m$ and a stratum $S \subset P$, define

$$e_S(h)(x, \tilde{z}_1, \dots, \tilde{z}_m) := h(x, \tilde{z}_1 \mathfrak{t}^{(a_1 - \underline{\tilde{z}}_1)/2}, \dots, \tilde{z}_m \mathfrak{t}^{(a_m - \underline{\tilde{z}}_m)/2}),$$

where $(a_1, \dots, a_m)$ is any point in S, and $\underline{c\mathfrak{t}^x} := x$.

So $e_S h(x, \tilde{z})$ samples the function h at a point with tropical part halfway between $\underline{\tilde{z}}$ and the point $a \in S$. Note that $e_S h$ does not depend on the choice of the point $a \in S$, because a different choice of $a \in S$ would ask us to sample h at a topologically indistinguishable point.

For example consider $T_2^2 := T_{[0,\infty)^2}^2$. The polytope $[0, \infty)^2$ has a zero-dimensional stratum S_0, two one-dimensional strata

$$S_1 := (0, \infty) \times 0 \quad \text{and} \quad S_2 := 0 \times (0, \infty)$$

and one two-dimensional stratum $S_3 := (0, \infty)^2$.

If we have a function $h \in C^0(T_2^2)$, then $e_{S_0} h = h$, but

$$e_{S_1} h(z_1, z_2) = h(0, z_2), \quad e_{S_2} h(z_1, z_2) = h(z_1, 0), \quad e_{S_3} h(z_1, z_2) = h(0, 0).$$

As a second example, consider $T_{[0,1]}^1$. Smooth or continuous functions on $T_{[0,1]}^1$ are generated by $\zeta_1 = \lceil \tilde{z} \rceil$ and $\zeta_2 = \lceil \mathfrak{t}^1 \tilde{z}^{-1} \rceil$. There are three strata of $[0, 1]$ to consider, 0, 1 and $(0, 1)$:

$$e_0 \zeta_1 = \zeta_1, \qquad e_0 \zeta_2 = 0, \qquad e_0 h(\zeta_1, \zeta_2) = h(\zeta_1, 0),$$

$$e_1 \zeta_1 = 0, \qquad e_1 \zeta_2 = \zeta_2, \qquad e_1 h(\zeta_1, \zeta_2) = h(0, \zeta_2),$$

$$e_{(0,1)} \zeta_1 = 0, \qquad e_{(0,1)} \zeta_2 = 0, \qquad e_{(0,1)} h(\zeta_1, \zeta_2) = h(0, 0).$$

We can consider $T_{[0,1]}^1$ as the subset of T_2^2 where $\tilde{z}_1 \tilde{z}_2 = \mathfrak{t}^1$. From this perspective we can relate the above two examples by $e_0 = e_{S_1}$ and $e_1 = e_{S_2}$.

In general, the smooth or continuous functions on T_P^m are generated by the smooth monomials: functions ζ_i of the form $\lceil \mathfrak{t}^{a_i} \tilde{z}^{\alpha^i} \rceil := \lceil \tilde{\zeta}_i \rceil$. For any stratum $S \subset P$ one of the following two options hold:

- $e_S \zeta_i = 0$, ζ_i vanishes on the stratum of T_P^m corresponding to S and $\underline{\tilde{\zeta}_i} > 0$ on S,[11] or

- $e_S \zeta_i = \zeta_i$ and ζ_i is nowhere 0 on the stratum of T_P^m corresponding to S, and $\underline{\tilde{\zeta}_i} = 0$ on S.

The operation e_S on a continuous function h on T_P^m is then

$$e_S h(\zeta_1, \ldots, \zeta_n) = h(e_S \zeta_1, \ldots, e_S \zeta_n).$$

Of course, this implies that if h is smooth or continuous, $e_S h$ is too.

Note that the operations e_{S_i} commute and $e_{S_i} e_{S_i} = e_{S_i}$. More generally, $e_{S_i} e_{S_j} = e_{S'}$, where S' is the smallest stratum of P whose closure contains both S_i and S_j.

Definition 2.2 If I is a nonempty collection of strata $\{S_1, \ldots, S_n\}$, use the notation

$$e_I h := e_{S_1}(e_{S_2}(\cdots e_{S_n} h)) \quad \text{and} \quad \Delta_I h := \left(\prod_{S_i \in I} (\mathrm{id} - e_{S_i}) \right) h.$$

For an empty collection of strata, define $e_\varnothing h = h$ and $\Delta_\varnothing h = h$.

For example,

$$\Delta_{\{S_1, S_2\}} h(z_1, z_2) := (1 - e_{S_1})(1 - e_{S_2}) h(z_1, z_2)$$
$$:= h(z_1, z_2) - h(0, z_2) - h(z_1, 0) + h(0, 0).$$

Note that if $S \in I$, then $e_S \Delta_I = 0$, so Δ_I is an operator for constructing functions that vanish on all strata in I. In the above example, this corresponds to

$$\Delta_{\{S_1, S_2\}} h(z_1, 0) = 0 = \Delta_{\{S_1, S_2\}} h(0, z_2).$$

2.2 Allowable coordinate charts

We will define our norms using a class of allowable coordinate charts on a family of curves. These coordinate charts are in a rather rigid form so that Δ_S is defined, and to simplify gluing analysis and facilitate defining a global version $\tilde{\Delta}_S$ of Δ_S.

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[11] For this paper, the tropical part of a function is $\mathbb{R}$–valued instead of $\mathfrak{t}^{\mathbb{R}}$–valued, so $\underline{c\mathfrak{t}^a} := a$. One complication is that $\mathfrak{t}$ is infinitesimal, so $a > 0$ is equivalent to $\mathfrak{t}^a < \mathfrak{t}^0$.

Definition 2.3 An allowable coordinate chart on a family of exploded curves $\hat{C} \xrightarrow{\pi_F} F$ is a coordinate chart on $\hat{C} \xrightarrow{\pi_F} F$ satisfying the following requirements:

- The coordinate chart on F is some open subset $U \subset \mathbb{R}^k \times T_P^m$ such that $e_S h$ is defined on U for all strata S if h is defined on U, and such that U is contained in a compact[12] subset of $\mathbb{R}^k \times T_P^m$.

- The coordinate chart on $\hat{C}$ and the map π_F is some restriction of a standard projection

$$\mathbb{R}^k \times T_Q^{m+1} \xrightarrow{\pi} \mathbb{R}^k \times T_P^m$$

  such that:

  - In standard coordinates, π is given by

  $$(x, \tilde{z}_1, \ldots, \tilde{z}_{m+1}) = (x, \tilde{z}_2, \ldots, \tilde{z}_{m+1}).$$

  - The polytope $Q = \underline{\mathbb{R}^k \times T_Q^{m+1}}$ is defined by the equations

  $$(\underline{\tilde{z}_2}, \ldots, \underline{\tilde{z}_{m+1}}) \in P, \quad \underline{\tilde{z}_1} \geq 0,$$

  and possibly

  $$\underline{\mathfrak{t}^a \tilde{z}^{(-1, \beta_2, \ldots, \beta_{m+1})}} \geq 0;$$

  in this case, let $\tilde{z}^\beta := \mathfrak{t}^a \tilde{z}^{(-1, \beta_2, \ldots, \beta_{m+1})}$.

- The coordinate chart $\tilde{U}$ on $\hat{C}$ is in one of the following forms:

  (I) If the polytope Q is not a product of P with $[0, \infty)$, then

  $$\tilde{U} := \pi^{-1} U \cap \{|\tilde{z}_1| < c, \; |\tilde{z}^\beta| < c\}$$

  and $|\tilde{z}_1 \tilde{z}^\beta| < c^2/16$ on U.

  (II) If the polytope Q is $P \times [0, \infty)$, $\tilde{U}$ is in one of the following forms:

  (a) $\tilde{U}$ is the intersection of $\pi^{-1} U$ with the set where the coordinate $\tilde{z}_1$ on $T_{[0,\infty)}^1$ takes values in some open ball compactly contained in $\mathbb{C}^* \mathfrak{t}^0$;

  (b) $\tilde{U}$ is the intersection of $\pi^{-1} U$ with the set where $|\tilde{z}_1| < c$.

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[12] Topological notions refer to the topology induced from the smooth part of an exploded manifold; see [18].

The nodes of curves in our family are covered by charts of type (I). We will be doing gluing analysis in these charts—the rigid conditions on the shape of such a chart are artifacts of our chosen gluing construction; for example, the condition $|\tilde{z}_1 \tilde{z}^\beta| < c^2/16$ is there to ensure that some cutoff functions do not interact. Such an allowable coordinate chart always exists around any point in $C \to F$. The condition on its tropical part follows from the requirement that $Q \to P$ be surjective, and have derivative (restricted to each stratum of Q) surjective on integral vectors; see Definition 10.1 of [18].

2.3 Weight functions w_S and w_0

Our norms generalize the notion (on manifolds with cylindrical ends) of Sobolev spaces with exponential weights. On a manifold with a cylindrical end, we only need a single weight function; this situation corresponds to a coordinate chart modeled on $\mathbb{R}^k \times T^1_{[0,\infty)}$. For more general coordinate charts T^m_P, we need lots of weight functions. In particular, we need a weight function w_S for any set of strata S of P. This weight function should vanish on all these strata, but decay at least as slowly as any smooth function vanishing on the same strata. We need to use these weight functions when analyzing a family of vertical differential operators on an allowable coordinate chart, $\tilde{U} \to U$; for this we must control terms in w_S that are not lifted from U by using the algorithm below.

Definition 2.4 Given any allowable coordinate chart $\tilde{U}$, and any collection of strata $S := \{S_j \subset \tilde{U} \subset Q\}$, define a weight function w_S as follows: choose a set of generators $\{\zeta_i\}$ for the monomial ideal[13] of smooth monomials ζ on T^{m+1}_Q so that $e_{S_j}\zeta = 0$ for all $S_j \in S$. Choose these generators for our monomial ideal using the following algorithm: Choose a finite set of generators for the monoid of smooth monomials on T^{m+1}_Q, a set consisting of $\lceil \tilde{z}_1 \rceil$, $\lceil \tilde{z}^\beta \rceil$ and lifts of smooth monomials on T^m_P. Include in our set $\{\zeta_i\}$ all the generators which are in our monomial ideal, then all products of two unused generators in our monomial ideal. Continue this, adding all products of k generators in our monomial ideal, but with no subproduct in our monomial ideal.

Now define

$$w_S := \left(\sum |\zeta_i| \right).$$

[13]This monomial ideal is an ideal within the monoid of smooth monomials on T^{m+1}_Q; the set $\{\zeta_i\}$ generates this monomial ideal in the sense that any smooth monomial ζ with $e_{S_j}\zeta = 0$ for all $S_j \in S$ is the product of some ζ_i with another smooth monomial.

This weight function $w_{\mathcal{S}}$ vanishes on all strata in $\mathcal{S}$, and has the following property: If h is any smooth function, then for any $\delta < 1$, the function $w^{-\delta} \Delta_{\mathcal{S}} h$ extends to a continuous function that vanishes on the strata in $\mathcal{S}$. So the smooth function $\Delta_{\mathcal{S}} h$ vanishes on the strata in $\mathcal{S}$, and decays faster than $w_{\mathcal{S}}^{\delta}$ for any $\delta < 1$; see [18, Lemma 7.4].

A special case of $w_{\mathcal{S}}$ Let $\mathcal{S}_0$ be the set of strata not projecting homeomorphically onto strata of $\underline{U}$. These are the strata corresponding to the edges of the exploded curves in our family.

For example, in the case of coordinate charts in the form of (I) above, then

$$w_{\mathcal{S}_0} := \left(|\lceil \tilde{z}_1 \rceil| + |\lceil \tilde{z}^{\beta} \rceil| + \sum_i |\zeta_i| \right),$$

where $\{\zeta_i\}$ is some finite set of generators for the monomial ideal of smooth monomials on U that vanish on all strata on which $\lceil \tilde{z}_1 \tilde{z}^{\beta} \rceil$ vanishes. This weighting function $w_{\mathcal{S}_0}$ satisfies the following: if h is a smooth or $C^{\infty,1}$ function on T_Q^{m+1} vanishing where $\lceil \tilde{z}_1 \rceil = \lceil \tilde{z}^{\beta} \rceil = 0$, then for all $\delta < 1$, $h w_{\mathcal{S}_0}^{-\delta}$ extends to a continuous function on T_Q^{m+1}, also vanishing where $\lceil \tilde{z}_1 \rceil = \lceil \tilde{z}^{\beta} \rceil = 0$. Another way to say this is as follows: if $e_{\mathcal{S}} h = 0$ for all S in $\mathcal{S}_0$, then $h w_{\mathcal{S}_0}^{-\delta}$ extends to a continuous function so that $e_S(h w_{\mathcal{S}_0}^{-\delta}) = 0$.

For uniform control over the norm of a cutting map used in gluing analysis, we replace $w_{\mathcal{S}_0}$ with a simpler weighting function w_0 that does not include these extra $|\zeta_i|$ terms.

Definition 2.5 On allowable coordinate charts of type (I), define

$$w_0 := |\lceil \tilde{z}_1 \rceil| + |\lceil \tilde{z}^{\beta} \rceil|,$$

and on allowable coordinate charts of type (II), define

$$w_0 := |\lceil \tilde{z}_1 \rceil|.$$

The following lemma ensures that replacing $w_{\mathcal{S}_0}$ with w_0 is not too dangerous.

Lemma 2.6 *Given any allowable coordinate chart, there exists a positive number q and a constant c such that*

$$c w_0 \geq w_{\mathcal{S}_0}^q.$$

Proof On allowable coordinate charts of type (II), $w_0 = w_{\mathcal{S}_0}$, so we can reduce to the case of charts of type (I). On such a chart, $w_{\mathcal{S}_0} := \left(|\lceil \tilde{z}_1 \rceil| + |\lceil \tilde{z}^{\beta} \rceil| + \sum_i |\zeta_i| \right)$,

where $\{\zeta_i\}$ is some finite generating set for the smooth monomials vanishing where $\lceil \widetilde{z}_1 \widetilde{z}^\beta \rceil = 0$. Each such ζ_i is the smooth part of some exploded monomial $\widetilde{\zeta}_i$ whose tropical part is a nonnegative integral affine function $\underline{\widetilde{\zeta}}_i : P \to [0, \infty)$, strictly positive on every stratum where $\underline{\widetilde{z}_1 \widetilde{z}^\beta}$ is strictly positive. If there are no strata on which $\underline{\widetilde{\zeta}}_i = 0$, then ζ_i is identically 0, and can be safely discarded. Otherwise, there exists some positive integer q such that

(4)
$$q\underline{\widetilde{\zeta}}_i \geq \underline{\widetilde{z}_1 \widetilde{z}^\beta}.$$

Such an estimate holds on any ray within P starting at a point on the largest stratum on which $\underline{\widetilde{\zeta}}_i$ vanishes, and holds with a uniform constant because the set of rays normal to this stratum at a given point is compact. As P is contained within the span of vectors within this stratum and such normal rays, it follows that the above estimate (4) holds for a uniform q on all of P. As $\{\zeta_i\}$ is finite, we can choose q so that (4) holds for all i. It follows that the product of any q such $\widetilde{\zeta}_i$ will have tropical part bounded by $\underline{\widetilde{z}_1 \widetilde{z}^\beta}$, and therefore be the product of $\widetilde{z}_1 \widetilde{z}^\beta$ with a monomial $\widetilde{\zeta}$ whose tropical part is nonnegative on P. The corresponding product of ζ_i will therefore be $\lceil \widetilde{\zeta} \rceil \lceil \widetilde{z}_1 \rceil \lceil \widetilde{z}^\beta \rceil$, and on our coordinate chart, its absolute value will be bounded by a constant times $|\lceil \widetilde{z}_1 \rceil|$. Similarly any term appearing in $w_{\mathcal{S}_0}^q$ will be bounded by a constant times either $|\lceil \widetilde{z}_1 \rceil|$ or $|\lceil \widetilde{z}^\beta \rceil|$. As there are only finitely many such terms, our required estimate follows:

$$cw_0 \geq w_{\mathcal{S}_0}^q. \qquad \qquad \square$$

Lemma 2.6 implies that given any smooth or $C^{\infty,1}$ function h and $\delta' < 1/q$, the function $w_0^{-\delta'} \Delta_{\mathcal{S}_0} h$ extends to a continuous function vanishing on all strata in $\mathcal{S}_0$.

2.4 Norms in allowable coordinate charts

Given a finite collection of allowable coordinate charts, we now define a series of norms on vector-valued functions on these coordinate charts $\widetilde{U} \subset \mathbb{R}^n \times \boldsymbol{T}_Q^{m+1}$. In all that follows, use the standard metric in which $\partial/\partial x_i$ and the real and imaginary parts of $\widetilde{z}_i \, \partial/\partial \widetilde{z}_i$ are orthonormal. In the case of a chart on a single curve, these norms $\|\cdot\|_{k,\delta}$ and $\|\cdot\|_{k-1,\delta}^1$ are analogous to L_k^p with exponential weights on cylindrical ends. In the more general case of a chart on a family of curves, these norms are also analogous to weighted L_k^p norms, however we treat vertical directions specially, because we are analyzing a family of vertical differential operators. In particular, the norm $\|v\|_{k,\delta}^1$ controls k derivatives of v in addition to one vertical derivative.

(i) Choose some exponent $p > 2$ and weight $0 < \delta < 1$. Also, choose some number K so that, for each coordinate chart, Lemma 2.6 holds for some $q < K$. Then let

$$\delta' := \frac{1}{K}(1 - \delta)$$

and define the norm

$$\|\nu\|_{\delta'} := \sup_{x \in \pi(\widetilde{U})} \left(\int_{\pi^{-1}(x) \cap \widetilde{U}} |w_0^{-\delta'} \nu|^p \right)^{\frac{1}{p}}.$$

So $\|\nu\|_{\delta'}$ is the supremum, over all fibers, of a weighted L^p norm of ν restricted to a fiber. This is only defined when $\Delta_{\mathcal{S}_0} \nu = \nu$, so $e_S \nu = 0$ for all $S \in \mathcal{S}_0$, or equivalently ν vanishes on edges of the curves in our family.

(ii) Define

$$\|\nu\|_{0,\delta} := \|\nu\|_{\delta'} + \max_{\mathcal{S}} (\|w_{\mathcal{S}}^{-\delta} \Delta_{\mathcal{S}} \nu\|_{\delta'}).$$

The maximum is taken over all collections $\mathcal{S}$ of substrata of $\underline{\widetilde{U}}$. (Where $w_{\mathcal{S}}$ and $\Delta_{\mathcal{S}} \nu$ both vanish, we use the convention that $w_{\mathcal{S}}^{-\delta} \Delta_{\mathcal{S}} \nu = 0$ for the purposes of calculating these norms.)

(iii) Define

$$\|\nu\|_{k,\delta} := \|\nu\|_{k-1,\delta} + \|d\nu\|_{k-1,\delta}.$$

These norms should be thought of as suitable generalizations of L_k^p with exponential weights.

(iv) For this last collection of norms, we shall use the notation $d_{\mathrm{vert}} \nu$ to refer to $d\nu$ restricted to the vertical tangent space — spanned by the real and imaginary parts of $\widetilde{z}_1 \, \partial/\partial\widetilde{z}_1$. Define

$$\|\nu\|_{\delta'}^1 := \sup|\nu| + \|d_{\mathrm{vert}} \nu\|_{\delta'},$$

$$\|\nu\|_{0,\delta}^1 := \|\nu\|_{\delta'}^1 + \max_{\mathcal{S}}(\sup|w_{\mathcal{S}}^{-\delta} \Delta_{\mathcal{S}} \nu| + \|w_{\mathcal{S}}^{-\delta} \Delta_{\mathcal{S}} d_{\mathrm{vert}} \nu\|_{\delta'}),$$

$$\|\nu\|_{k,\delta}^1 := \|\nu\|_{k-1,\delta}^1 + \|d\nu\|_{k-1,\delta}^1.$$

Recall the following definition of $C^{k,\delta}$ for any $0 < \delta < 1$.

Definition 2.7 ($C^{k,\delta}$ and $C^{\infty,\underline{\delta}}$ regularity) Define $C^{0,\delta}$ to be the same as C^0. A sequence of smooth functions $f_i \in C^\infty(\mathbb{R}^n \times T_P^m)$ converges to a continuous function f in $C^{k,\delta}(\mathbb{R}^n \times T_P^m)$ if the following conditions hold:

(i) Given any collection I of at most k nonzero strata, the sequence of functions

$$|w_I^{-\delta} \Delta_I (f_i - f)|$$

converges to 0 uniformly on compact subsets of $\lceil \mathbb{R}^n \times T_P^m \rceil$ as $i \to \infty$. (This includes the case where our collection of strata is empty and $f_i \to f$ uniformly on compact subsets.)

(ii) For any smooth vectorfield v, $v(f_i)$ converges to some function vf in $C^{k-1,\delta}$.

Define $C^{k,\delta}(\mathbb{R}^n \times T_P^m)$ to be the closure of C^∞ in C^0 with this topology.[14]

Define $C^{\infty,\delta}$ to be the intersection of $C^{k,\delta}$ for all k.

Define $C^{\infty,\underline{\delta}}$ to be the intersection of $C^{\infty,\delta'}$ for all $\delta' < \delta$.

Note that any $C^{\infty,\underline{1}}$ function v restricted to any compact subset will have $\|v\|_{k,\delta}^1$ finite, and if $\Delta_{\mathcal{S}_0} v = v$, then $\|v\|_{k,\delta}$ will also be finite. Standard Sobolev estimates imply that if $\|v\|_{k,\delta}$ or $\|v\|_{k,\delta}^1$ is finite for all k, and all $\delta < 1$, then v is $C^{\infty,\underline{1}}$.

We shall often need the following observations about the behavior of products in our norms.

Lemma 2.8 (A) *For I_1 and I_2 any collection of strata of an allowable coordinate chart, $w_{I_1} w_{I_2} w_{I_1 \cup I_2}^{-1}$ is bounded.*

(B) *The following product formula for $\Delta_{\mathcal{S}}$ holds:[15]*

$$\Delta_{\mathcal{S}}(\phi\psi) = \sum_{I \subset \mathcal{S}} (e_{\mathcal{S}-I} \Delta_I \phi) \Delta_{\mathcal{S}-I} \psi.$$

(C) *On any allowable coordinate chart and for any $k \in \mathbb{N}$, there exists some constant c, depending only on k and our choices defining the above norms, such that, if at least one of ϕ or ψ is a real-valued function,*

$$\|\phi\psi\|_{k,\delta} \le c \|\phi\|_{k,\delta} \|\psi\|_{k,\delta}^1.$$

(D) *On any allowable coordinate chart and for any $k \in \mathbb{N}$, there exists some constant c, depending only on k and our choices defining the above norms, such that, if at least one of ϕ or ψ is a real-valued function,*

$$\|\phi\psi\|_{k,\delta}^1 \le c \|\phi\|_{k,\delta}^1 \|\psi\|_{k,\delta}^1.$$

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[14] This notion of convergence induces a well-behaved topology because it is equivalent to convergence in some countable sequence of norms. In particular, a subset is closed in the induced topology if and only if it is sequentially closed.

[15] Throughout this paper, the symbol $\subset$ means $\subseteq$. Proper subsets will be indicated by the symbol $\subsetneq$.

Proof To see item (A), note that w_{I_i} is a finite sum of absolute values of smooth monomials vanishing on strata in I_i, so $w_{I_1} w_{I_2}$ is a finite sum of absolute values of smooth monomials vanishing on strata in $I_1 \cup I_2$. The weight function $w_{I_1 \cup I_2}$ is a sum of absolute values of generators for the monomial ideal of such smooth monomials, so on $\mathbb{R}^n \times \boldsymbol{T}_Q^m$, every item in the former sum is a continuous function times an element of the latter sum. As allowable coordinate charts are always compactly contained inside $\mathbb{R}^n \times \boldsymbol{T}_Q^m$, it follows that $w_{I_1} w_{I_2} w_{I_1 \cup I_2}^{-1}$ is bounded.

To prove item (B), note first that if S denotes a single stratum,

$$\Delta_S \phi \psi = \phi \psi - (e_S \phi)(e_S \psi) = (\Delta_S \phi)\psi + (e_S \phi)\Delta_S \psi,$$

so the required identity holds if S consists of a single stratum. Suppose now that $\mathcal{S} = \mathcal{S}' \cup \{S\}$, and the required identity holds for $\mathcal{S}'$. Then

$$\begin{aligned}
\Delta_{\mathcal{S}} \phi \psi &= \Delta_S \left(\sum_{I \subset \mathcal{S}'} (e_{\mathcal{S}'-I} \Delta_I \phi)\Delta_{\mathcal{S}'-I}\psi \right) \\
&= \sum_{I \subset \mathcal{S}'} (e_{\mathcal{S}'-I} \Delta_S \Delta_I \phi)\Delta_{\mathcal{S}'-I}\psi + (e_S e_{\mathcal{S}'-I} \Delta_I \phi)\Delta_S \Delta_{\mathcal{S}'-I}\psi \\
&= \sum_{I \subset \mathcal{S}} (e_{\mathcal{S}-I} \Delta_I \phi)\Delta_{\mathcal{S}-I}\psi.
\end{aligned}$$

So by induction, the required identity holds for any set of strata $\mathcal{S}$.

To prove item (C), note that

$$\|\phi \psi\|_{\delta'} \leq \|\phi\|_\delta \sup|\psi| \leq \|\phi\|_{\delta'} \|\psi\|_{\delta'}^1.$$

Next, we use item (B), then item (A), then the above observation to show the following:

$$\begin{aligned}
\|w_{\mathcal{S}}^{-\delta} \Delta_{\mathcal{S}}(\phi\psi)\|_{\delta'} &\leq \sum_{I \subset \mathcal{S}} \|w_{\mathcal{S}}^{-\delta}(e_{\mathcal{S}-I} \Delta_I \phi)\Delta_{\mathcal{S}-I}\psi\|_{\delta'} \\
&\leq c \sum_{I \subset \mathcal{S}} \|w_I^{-\delta}(e_{\mathcal{S}-I} \Delta_I \phi)w_{\mathcal{S}-I}^{-\delta}\Delta_{\mathcal{S}-I}\psi\|_{\delta'} \\
&\leq c \sum_{I \subset \mathcal{S}} \|w_I^{-\delta} e_{\mathcal{S}-I} \Delta_I \phi\|_{\delta'} \sup|w_{\mathcal{S}-I}^{-\delta}\Delta_{\mathcal{S}-I}\psi|.
\end{aligned}$$

The constant c above depends on the collection of strata $\mathcal{S}$, but there are only a finite number of strata, and because $\Delta_{\mathcal{S}} \Delta_{\mathcal{S}} = \Delta_{\mathcal{S}}$, we need only consider sets of distinct

strata. Therefore, for a different constant c,[16]

$$\|\phi\psi\|_{0,\delta} \le c\|\phi\|_{0,\delta}\|\psi\|_{0,\delta}^1.$$

Now we can use induction on the number of derivatives. Suppose that the required inequality holds for $k-1$ derivatives. Then

$$\begin{aligned}
\|\phi\psi\|_{k,\delta} &= \|\phi\psi\|_{k-1,\delta} + \|d(\phi\psi)\|_{k-1,\delta} \\
&\le c'\big(\|\phi\|_{k-1,\delta}\|\psi\|_{k-1,\delta}^1 + \|d\phi\|_{k-1,\delta}\|\psi\|_{k-1,\delta}^1 + \|\phi\|_{k-1,\delta}\|d\psi\|_{k-1,\delta}^1\big) \\
&\le 3c'\|\phi\|_{k,\delta}\|\psi\|_{k,\delta}^1.
\end{aligned}$$

Therefore, by induction the required inequality holds for all k.

We prove item (D) similarly as follows:

$$\sup|\phi\psi| \le \sup|\phi|\sup|\psi|,$$

so as above, we may estimate using item (B) and item (A):

$$\begin{aligned}
\sup|w_{\mathcal{S}}^{-\delta}\Delta_{\mathcal{S}}(\phi\psi)| &\le \sum_{I \subset \mathcal{S}} \sup|w_{\mathcal{S}}^{-\delta}(e_{\mathcal{S}-I}\Delta_I\phi)\Delta_{\mathcal{S}-I}\psi| \\
&\le c\sum_{I \subset \mathcal{S}} \sup|w_I^{-\delta}(e_{\mathcal{S}-I}\Delta_I\phi)w_{\mathcal{S}-I}^{-\delta}\Delta_{\mathcal{S}-I}\psi| \\
&\le c\sum_{I \subset \mathcal{S}} \sup|w_I^{-\delta}e_{\mathcal{S}-I}\Delta_I\phi|\sup|w_{\mathcal{S}-I}^{-\delta}\Delta_{\mathcal{S}-I}\psi|.
\end{aligned}$$

We also have from item (C),

$$\begin{aligned}
\|d_{\text{vert}}(\phi\psi)\|_{k,\delta} &\le \|(d_{\text{vert}}\phi)\psi\|_{k,\delta} + \|\phi d_{\text{vert}}\psi\|_{k,\delta} \\
&\le c\|d_{\text{vert}}\phi\|_{k,\delta}\|\psi\|_{k,\delta}^1 + c\|\phi\|_{k,\delta}^1\|d_{\text{vert}}\psi\|_{k,\delta} \\
&\le 2c\|\phi\|_{k,\delta}^1\|\psi\|_{k,\delta}^1.
\end{aligned}$$

Therefore, for some new constant c we get the required inequality:

$$\|\phi\psi\|_{0,\delta}^1 \le c\|\phi\|_{0,\delta}^1\|\psi\|_{0,\delta}^1.$$

---

[16]Throughout this paper, constants c in such inequalities will depend on the choice of coordinate chart, norm and maybe a collection of strata, but they will never depend on ϕ or ψ. There are many inequalities, and we will reuse c, even though it refers to a different constant in different inequalities.

The general case now follows by induction because if it holds for $k - 1$,

$$
\begin{aligned}
\|\phi\psi\|_{k,\delta}^1 &= \|\phi\psi\|_{k-1,\delta}^1 + \|d(\phi\psi)\|_{k-1,\delta}^1 \\
&\leq c'\big(\|\phi\|_{k-1,\delta}^1\|\psi\|_{k-1,\delta}^1 + \|d\phi\|_{k-1,\delta}^1\|\psi\|_{k-1,\delta}^1 + \|\phi\|_{k-1,\delta}^1\|d\psi\|_{k-1,\delta}^1\big) \\
&\leq 3c'\|\phi\|_{k,\delta}^1\|\psi\|_{k,\delta}^1. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square
\end{aligned}
$$

We use the adjective extendable to grant noncompact sets some compact-set superpowers. In particular:

Definition 2.9   (i)   An extendable open subset of a topological space is an open subset contained inside a compact subset of that space.

  (ii) An extendable allowable coordinate chart is an allowable coordinate chart that is also an extendable subset of some larger allowable coordinate chart.

 (iii) An extendable function is a function which is defined on an extendable subset, and which is the restriction of a function defined on a larger, compact domain.

 (iv) An extendable vectorbundle is a vectorbundle which is the restriction of some vectorbundle to an extendable subset.

  (v)   An extendable function on an extendable vectorbundle is the restriction of some function to an extendable vectorbundle.

Lemma 2.10   *If, on some extendable allowable coordinate chart $\widetilde{U}$, v is a $\mathbb{R}^n$–valued function with $\|v\|_{k,\delta}^1$ finite and E is a $C^{\infty,1}$ extendable function on the extendable vectorbundle $\mathbb{R}^n \times \widetilde{U}$, then $\|E(v)\|_{k,\delta}^1$ is bounded. If $\Delta_{S_0} E = E$, then $\|E(v)\|_{k,\delta}$ is bounded. These bounds can be chosen to depend continuously on v in the $\|\cdot\|_{k,\delta}^1$ topology.*

Proof   For a single stratum, $\Delta_S(E(v)) = (\Delta_S E)(v) + \Delta_S((e_S E)(v))$, so if $\mathcal{S}$ is a collection of strata,

$$
\tag{5} \Delta_{\mathcal{S}}(E(v)) = \sum_{I \subset \mathcal{S}} \Delta_I((e_I \Delta_{\mathcal{S}-I} E)(v)).
$$

Because $\|v\|_{k,\delta}^1$ is bounded, $\sup|v|$ is bounded, so, for $E(v)$, we may restrict attention to a fiberwise-bounded subset of $\mathbb{R}^n \times \widetilde{U} \to \widetilde{U}$. Here, we have bounds on E and all its derivatives, because E and $\widetilde{U}$ are extendable. In the following, let D_I indicate the derivative with respect to $\partial/\partial t_i$ for all $i \in I$:

$$
\tag{6} \Delta_I((e_I E)(v)) = \int_0^1 \cdots \int_0^1 D_I\Big((e_I E)\Big(\prod_{i \in I}(e_{S_i} + t_i \Delta S_i)v\Big)\Big) \prod_{i \in I} dt_i.
$$

To estimate $\Delta_I(e_I E)(v)$, estimate this integrand. Writing $\phi_I := \prod_{i \in I}(e_{S_i} + t_i \Delta S_i)v$, we have

$$(7) \qquad D_I(e_I E)(\phi_I) = \sum_{\bigsqcup_{j=1}^n I_j = I} (D^n e_I E)(D_{I_1}\phi_I) \cdots (D_{I_n}\phi_I)$$

$$= \sum_{\bigsqcup_{j=1}^n I_j = I} (D^n e_I E)(\Delta_{I_1}\phi_{I-I_1}) \cdots (\Delta_{I_n}\phi_{I-I_n}).$$

The sum above is over all partitions of I. The above equations (6) and (7) also hold with $\Delta_{S-I} E$ replacing E; note also that $D^n e_I \Delta_{S-I} E = \Delta_{S-I} e_I D^n E$. Using the equations (5), (6) and (7) and Lemma 2.8(A), we get that $\sup|w_S^{-\delta} \Delta_S(E(v))|$ is bounded by a constant times the expression

$$(8) \qquad \sum_{I \subset S} \sum_{\bigsqcup_{j=1}^n I_j = I} \sup|w_{S-I}^{-\delta} \Delta_{S-I} e_I (D^n E)(\phi_I)| \prod_{j=1}^n \sup|w_{I_j}^{-\delta} \Delta_{I_j}\phi_{I-I_j}|.$$

The first term in each of the above summands is finite because E is extendable and in $C^{\infty,\underline{1}}$, and the other terms are bounded by $\|v\|_{0,\delta}^1$. Our estimate for each of these terms can be chosen to depend continuously on v in the $\|\cdot\|_{0,\delta}^1$ topology. (The estimate of the first term can be chosen continuous in the supremum topology, which is weaker than the $\|\cdot\|_{0,\delta}^1$ topology.)

Similar to the above, if $\Delta_{S_0} E = E$, we may bound $\|w_S^{-\delta} \Delta_S(E(v))\|_{\delta'}$ by a constant times

$$(9) \qquad \sum_{I \in S} \sum_{\bigsqcup_{j=1}^n I_j = I} \sup|w_0^{-\delta'-\epsilon} w_{S-I}^{-\delta} \Delta_{S-I} e_I (D^n E)(\phi_I)| \prod_{j=1}^n \sup|w_{I_j}^{-\delta} \Delta_{I_j}\phi_{I-I_j}|$$

for some $\epsilon > 0$. The first term is bounded for ϵ small enough using Lemma 2.6:

$$\sup|w_0^{-\delta'-\epsilon} w_{S-I}^{-\delta} \Delta_{S-I}(D^n E)| \leq c \sup|w_{S_0}^{-q(\delta'+\epsilon)} w_{S-I}^{-\delta} \Delta_{S-I}(D^n E)|$$

$$= c \sup|w_{S_0}^{-q(\delta'+\epsilon)} w_{S-I}^{-\delta} \Delta_{S_0 \cup (S-I)}(D^n E)|$$

$$\leq c' \sup|w_{S_0 \cup (S-I)}^{-q(\delta'+\epsilon)-\delta} \Delta_{S_0 \cup (S-I)}(D^n E)|.$$

We may choose ϵ and q so that the above inequality holds and $q(\delta' + \epsilon) + \delta < 1$, so the final term in the inequality above is bounded. The other terms on the right-hand side of (9) are bounded by $\|v\|_{0,\delta}^1$. Again, the bounds can be chosen continuous in the $\|\cdot\|_{0,\delta}^1$ topology. Therefore, in this case $\|E(v)\|_{0,\delta}$ is bounded, and the bound can be chosen to depend continuously on v in the $\|\cdot\|_{0,\delta}^1$ topology.

In the case that $\Delta_{S_0} E$ is not necessarily equal to E, we know that $\sup |w_S^{-\delta} \Delta_S(E(v))|$ is bounded, and we must show that $\|d_{\text{vert}}(E(v))\|_{0,\delta}$ is bounded. To this end, note that $d_{\text{vert}}(E(v)) = DE(v)(d_{\text{vert}} v) + (d_{\text{vert}} E)(v)$. The second term can be dealt with by observing that $d_{\text{vert}} E$ is in $C^{\infty,\underline{1}}$ and $\Delta_{S_0} d_{\text{vert}} E = d_{\text{vert}} E$, therefore, as argued above, $\|(d_{\text{vert}} E)(v)\|_{0,\delta}$ is bounded. The first term can be dealt with in the same way as the product was dealt with in Lemma 2.8(C). Note that in that argument, the only part of the norm $\|\cdot\|_{0,\delta}^1$ used was the part involving the supremum. Following this argument, we can bound $\|DE(v)(d_{\text{vert}} v)\|_{0,\delta}$ by the product of $\|d_{\text{vert}} v\|_{0,\delta}$ with $\sum_S \sup |w_S^{-\delta} \Delta_S(DE(v))|$, which as argued above is bounded. Again, our bounds may be chosen to depend on v continuously in the $\|\cdot\|_{0,\delta}^1$ topology.

We have now shown that if $\|v\|_{0,\delta}^1$ is bounded, $\|E(v)\|_{0,\delta}^1$ is bounded, and if $\Delta_{S_0} E = E$, then $\|E(v)\|_{0,\delta}$ is bounded, and these bounds can be chosen to depend continuously on v in the $\|\cdot\|_{0,\delta}^1$ topology. For induction, suppose that the equivalent statement holds for $\|\cdot\|_{k,\delta}^1$ and $\|\cdot\|_{k,\delta}$. Note that $d(E(v))(\cdot) = DE(v)(0, dv(\cdot)) + (DE)(v)(\cdot, 0)$. If $\|v\|_{k+1,\delta}^1$ is bounded, the first term is a composition of the $C^{\infty,\underline{1}}$ function $DE(\cdot)(0, \cdot)$ and (v, dv), which has $\|(v, dv)\|_{k,\delta}^1$ bounded, and the second term is a composition of the $C^{\infty,\underline{1}}$ function $DE(\cdot)(\cdot, 0)$ with v. By our inductive assumption, the $\|\cdot\|_{k,\delta}^1$ norm of both these terms is bounded, and $\|d(E(v))\|_{k+1,\delta}^1$ is bounded. Similarly, $\|d(E(v))\|_{k+1,\delta}$ is bounded if $\|v\|_{k+1,\delta}^1$ is bounded and $\Delta_{S_0} E = E$, as in that case $\Delta_{S_0} DE = DE$. All these bounds can be chosen to depend continuously on v in the $\|\cdot\|_{k+1,\delta}^1$ topology. By induction, we have proved the lemma for all k. $\square$

Corollary 2.11 *If E is an extendable $C^{\infty,\underline{1}}$ function on $\mathbb{R}^n \times \widetilde{U}$, then $E(v)$ in the $\|\cdot\|_{k,\delta}^1$ topology depends continuously on v in the $\|\cdot\|_{k,\delta}^1$ topology. If $\Delta_{S_0} E = E$, then $E(v)$ in the $\|\cdot\|_{k,\delta}$ topology depends continuously on v in the $\|\cdot\|_{k,\delta}^1$ topology.*

Proof We have

$$E(v_1) - E(v_2) = \int_0^1 DE(v_1 + t(v_2 - v_1))(v_2 - v_1)\, dt,$$

so using Lemma 2.8(D),

$$\|E(v_1) - E(v_2)\|_{k,\delta}^1 \le c \|v_1 - v_2\|_{k,\delta}^1 \int_0^1 \|DE(v_1 + t(v_2 - v_1))\|_{k,\delta}^1 \, dt.$$

As DE is $C^{\infty,\underline{1}}$, Lemma 2.10 tells us that $\|DE(v')\|_{k,\delta}^1$ can be bounded uniformly for v' in a $\|\cdot\|_{k,\delta}^1$–neighborhood of v_1. Therefore, if v_2 is in this neighborhood,

$$\|E(v_1) - E(v_2)\|_{k,\delta}^1 \le c \|v_1 - v_2\|_{k,\delta}^1.$$

So, $E(\nu)$ in the $\|\cdot\|^1_{k,\delta}$ topology depends continuously on ν in the $\|\cdot\|^1_{k,\delta}$ topology.

Similarly, using [Lemma 2.8(C)](#),

$$\|E(\nu_1) - E(\nu_2)\|_{k,\delta} \le c\|\nu_1 - \nu_2\|^1_{k,\delta} \int_0^1 \|DE(\nu_1 + t(\nu_2 - \nu_1))\|_{k,\delta}\, dt.$$

As DE is $C^{\infty,1}$ and $\Delta_{\mathcal{S}_0} DE = DE$ if $\Delta_{\mathcal{S}_0} E = E$, the norm $\|DE(\nu')\|_{k,\delta}$ can be bounded uniformly for ν' in a $\|\cdot\|^1_{k,\delta}$–neighborhood of ν_1 if $\Delta_{\mathcal{S}_0} E = E$. Therefore, if ν_2 is in this neighborhood,

$$\|E(\nu_1) - E(\nu_2)\|_{k,\delta} \le c\|\nu_1 - \nu_2\|^1_{k,\delta}.$$

So, if $\Delta_{\mathcal{S}_0} E = E$, then $E(\nu)$ in the $\|\cdot\|_{k,\delta}$ topology depends continuously on ν in the $\|\cdot\|^1_{k,\delta}$ topology. $\qquad\square$

2.5 Equivalent norms using lifted sets of strata

We now describe equivalent norms for $\|\cdot\|_{0,\delta}$ and $\|\cdot\|^1_{0,\delta}$. These new norms only involve weighting functions $w_{\mathcal{S}}$ with $d_{\text{vert}} w_{\mathcal{S}} = 0$, making estimates involving vertical derivatives easier. For this, we need the following concepts:

Definition 2.12 Given an allowable coordinate chart $\widetilde{U} \xrightarrow{\pi} U$ and a set of strata $\mathcal{S}$ of U, the lift $\widetilde{\mathcal{S}}$ of $\mathcal{S}$ is a set of strata of $\widetilde{U}$ defined as

$$\widetilde{\mathcal{S}} := \{S \mid \underline{\pi}(S) \in \mathcal{S}\}.$$

For our purposes, two sets of strata $\mathcal{S}$ and $\mathcal{S}'$ will act identically if $\Delta_{\mathcal{S}} = \Delta_{\mathcal{S}'}$, so define a lifted set of strata $\mathcal{S}$ to be a set of strata of $\widetilde{U}$ with the following property: if, for some strata T, the projection $\underline{\pi}(T)$ is in $\underline{\pi}(\mathcal{S})$, then $\Delta_T \Delta_{\mathcal{S}} = \Delta_{\mathcal{S}}$.

For any set of strata $\mathcal{S}$ in $\widetilde{U}$, define the complement $\mathcal{S}^c$ to be the set of strata S' such that $\Delta_{S'} \Delta_{\mathcal{S}} \ne \Delta_{\mathcal{S}}$, and $\underline{\pi}(S') = \underline{\pi}(S)$ for some $S \in \mathcal{S}$.

Use $\mathcal{S}^c = \varnothing$ when $\mathcal{S}$ is a lifted set of strata.

If $\mathcal{S}^c = \varnothing$, the vertical derivative $d_{\text{vert}} w_{\mathcal{S}} = 0$ because $w_{\mathcal{S}}$ is the lift of some function from U. Note also that $(\mathcal{S} \cup \mathcal{S}^c)^c = \varnothing$.

Lemma 2.13 *On any allowable coordinate chart, given a stratum S_j and $\overline{S}_j \in S_j^c$, there exists a constant $c > 0$ such that*

$$|e_{\overline{S}_j} \Delta_{S_j} \phi| \le c\|e_{\overline{S}_j} d_{\text{vert}}\phi\|_\delta e_{\overline{S}_j} w_0^\delta.$$

Proof The left-hand side of the above inequality is $|e_{\overline{S}_j}\phi - e_{\overline{S}_j}e_{S_j}\phi|$. This equals the difference between the ϕ on some fiber of the coordinate chart and ϕ on the edge contained in the same fiber. We shall bound this difference using a standard Sobolev estimate on this fiber. We will then get a uniform bound using the bounded geometry of allowable coordinate charts.

Without losing generality, we may assume that the part of this fiber of interest has coordinate $\tilde{z}_1$, and the smooth component of the fiber of interest is $\{0 < |\lceil\tilde{z}_1\rceil| < c\} \subset \mathbb{C}$, where c is a constant depending only on the coordinate chart, and not the particular fiber. Use cylindrical coordinates $\lceil\tilde{z}_1\rceil = e^{t+i\theta}$. Denote ϕ restricted to this smooth component of this fiber simply as ϕ.

We are interested in bounding $|\phi(t,\theta) - \phi(-\infty,\theta)|$ in terms of $\|e_{\overline{S}_j}d_{\mathrm{vert}}\phi\|_\delta e_{\overline{S}_j}w_0^\delta$. In the case we have restricted ourselves to, $e_{\overline{S}_j}w_0 = |\lceil\tilde{z}_1\rceil| = e^t$. Using this observation and the definition of $\|\cdot\|_\delta$, we have

$$\|e_{\overline{S}_j}d_{\mathrm{vert}}\phi\|_\delta \geq \left(\int_{\{t<\log c\}}(e^{-\delta t}|d\phi|)^p\,dt\,d\theta\right)^{\frac{1}{p}}.$$

(The inequality sign above is there because the right-hand side is considering only one fiber. The ϕ on the right-hand side is ϕ restricted to this fiber, so $e_{\overline{S}_j}d_{\mathrm{vert}}\phi = d\phi$.) So long as $p > 2$, a Sobolev estimate implies that there exists some constant $c_1 > 0$ such that

$$\left(\int_{\{x-1<t<x\}}|d\phi|^p\,dt\,d\theta\right)^{\frac{1}{p}} \geq c_1 \sup_{\{x-1<t_i<x\}}|\phi(t_1,\theta_1) - \phi(t_2,\theta_2)|.$$

Therefore,

$$\sup_{\{x-1<t_i<x\}}|\phi(t_1,\theta_1) - \phi(t_2,\theta_2)| \leq c_1 e^{\delta x}\|e_{\overline{S}_j}d_{\mathrm{vert}}\phi\|_\delta,$$

so

$$\sup_{\{t_i<x\}}|\phi(t_1,\theta_1) - \phi(t_2,\theta_2)| \leq c_1\frac{e^{\delta x}}{1-e^{-\delta}}\|e_{\overline{S}_j}d_{\mathrm{vert}}\phi\|_\delta.$$

The above implies the required estimate:

$$|\phi(t,\theta) - \phi(-\infty,\theta)| \leq c_1\frac{e^{\delta t}}{1-e^{-\delta}}\|e_{\overline{S}_j}d_{\mathrm{vert}}\phi\|_\delta$$
$$\leq c'\|e_{\overline{S}_j}d_{\mathrm{vert}}\phi\|_\delta e_{\overline{S}_j}w_0^\delta. \qquad \square$$

Definition 2.14 Define the norm $\|v\|_{\delta\&\delta'}$ by using the $\|\cdot\|_{\delta'}$ norm on smooth manifold fibers of $\pi\colon \widetilde{U} \to U$ and the norm $\|\cdot\|_{\delta+\delta'}$ on fibers with tropical part not equal to a point:

$$\|v\|_{\delta\&\delta'} :=$$

$$\max\left\{ \sup_{\underline{\pi^{-1}(x)=\text{point}}} \left(\int_{\pi^{-1}(x)\cap\widetilde{U}} |w_0^{-\delta'}v|^p \right)^{\frac{1}{p}}, \sup_{\underline{\pi^{-1}(x)\neq\text{point}}} \left(\int_{\pi^{-1}(x)\cap\widetilde{U}} |w_0^{-\delta-\delta'}v|^p \right)^{\frac{1}{p}} \right\}.$$

Define the norm $\|v\|_{\delta\&\delta'}^1 = \sup|v| + \|d_{\text{vert}}v\|_{\delta\&\delta'}$.

We shall see that $\|v\|_{\delta\&\delta'}$ is equivalent to the norm in equation (10) below.

Lemma 2.15 (i) *On any allowable coordinate chart, the norm $\|v\|_{0,\delta}$ is equivalent to the following norm using the lifts $\widetilde{S}$ of sets of strata S in $\underline{U}$:*

$$\|v\|_{\delta'} + \max_{S} \|w_{\widetilde{S}}^{-\delta} \Delta_{\widetilde{S}} v\|_{\delta\&\delta'}.$$

(ii) *On any allowable coordinate chart, the norm $\|v\|_{0,\delta}^1$ is equivalent to the norm*

$$\|v\|_{\delta'}^1 + \max_{S} \|w_{\widetilde{S}}^{-\delta} \Delta_{\widetilde{S}} v\|_{\delta\&\delta'}^1.$$

Proof The strata on which the higher weight $\delta + \delta'$ is used for the $\|\cdot\|_{\delta\&\delta'}$ norm are the strata T such that $T^c \neq \varnothing$. Therefore, the norm $\|v\|_{\delta\&\delta'}$ is equivalent to the norm

(10) $$\|v\|_{\delta'} + \max_{T^c\neq 0} \|e_T v\|_{\delta+\delta'}$$

because $\|e_T v\|_{\delta+\delta'}$ is equivalent to taking the supremum involved in the $\|\cdot\|_{\delta+\delta'}$ norm just over the stratum T.

Therefore, to show that $\|v\|_{0,\delta}$ and $\|v\|_{0,\delta}^1$ dominate the two new norms above, it is enough to show that $\|v\|_{0,\delta}$ and $\|v\|_{0,\delta}^1$ respectively dominate $\|e_T w_{\widetilde{S}}^{-\delta} \Delta_{\widetilde{S}} v\|_{\delta'+\delta}$ and $\|e_T w_{\widetilde{S}}^{-\delta} \Delta_{\widetilde{S}} v\|_{\delta'+\delta}^1$.

Claim *If $T^c \neq \varnothing$, there exists a constant c such that*

(11) $$e_T w_{\widetilde{S}\cup S_0} \leq c e_T w_{\widetilde{S}} w_0.$$

(Here $\widetilde{S}$ is a lifted set of strata, and S_0 indicates the set of strata on which w_0 disappears.)

The above claim holds trivially if $T \in \mathcal{S}_0$ or $e_T \Delta_{\widetilde{S}} = 0$, as then both $e_T w_{\widetilde{S} \cup S_0}$ and $e_T w_{\widetilde{S}} w_0$ are 0. As $T^c \neq \varnothing$, assuming that $T \notin \mathcal{S}_0$ gives that $e_T w_0 = |\zeta_0|$, where ζ_0 is a smooth monomial either equal to $\lceil \widetilde{z}_1 \rceil$ or $\lceil \widetilde{z}^\beta \rceil$. Suppose now that ζ is one of the smooth monomials used in the definition of $w_{\widetilde{S} \cup S_0}$. For any smooth monomial, either $e_S \zeta = 0$ or $e_S \zeta = \zeta$. Because $\zeta = 0$ on S_0, if $e_T \zeta \neq 0$, then ζ must be $\zeta_0^n \zeta'$ where ζ' is some lifted smooth monomial and $n \geq 1$. Moreover, ζ' must disappear on $\widetilde{S}$; this is because both $\widetilde{S}$ and ζ' are lifted, so if ζ' is nonzero on some stratum $S \in \widetilde{S}$, it is nonzero on all $S' \in S^c \subset \widetilde{S}$, but $\zeta_0 \neq 0$ on one such S' and therefore $\zeta \neq 0$ on S' — a contradiction because $\zeta_0^n \zeta' = 0$ on all $S' \in \widetilde{S}$. As ζ' vanishes on all strata in $\widetilde{S}$, it is bounded by some constant times $w_{\widetilde{S}}$, so $\zeta = e_T \zeta$ is bounded by some constant times $e_T w_{\widetilde{S}} w_0$. Applied to all monomials ζ appearing in $w_{\widetilde{S} \cup S_0}$, this proves the above claim that $e_T w_{\widetilde{S} \cup S_0}$ is bounded by a constant times $e_T w_{\widetilde{S}} w_0$.

Our inequality (11) implies that $\|e_T w_{\widetilde{S}}^{-\delta} \Delta_{\widetilde{S}} v\|_{\delta' + \delta}$ is dominated by $\|w_{\widetilde{S} \cup S_0}^{-\delta} \Delta_{\widetilde{S} \cup S_0} v\|_{\delta'}$ and $\|e_T w_{\widetilde{S}}^{-\delta} \Delta_{\widetilde{S}} v\|_{\delta' + \delta}^1$ is dominated by

$$\|w_{\widetilde{S} \cup S_0}^{-\delta} \Delta_{\widetilde{S} \cup S_0} d_{\text{vert}} v\|_{\delta'} + \sup |w_{\widetilde{S}}^{-\delta} \Delta_{\widetilde{S}} v|,$$

so our new norms are dominated by $\|v\|_{0,\delta}$ and $\|v\|_{0,\delta}^1$, respectively.

We must show that $\|v\|_{0,\delta}$ and $\|v\|_{0,\delta}^1$ are dominated by the new norms above. For I an arbitrary collection of strata, we need to bound $\|w_I^{-\delta} \Delta_I v\|_{\delta'}$ with these new norms. Let $\widetilde{S}$ be the largest lifted collection of strata such that $\Delta_{\widetilde{S}} \Delta_I = \Delta_I$. Then $\Delta_I = \Delta_{\widetilde{S}} \prod \Delta_S$, where the product is over strata $S \in I$ not contained in $\widetilde{S}$ (which implies that $S^c \neq \varnothing$). Simple computation gives the identity

$$\Delta_S = \Delta_{S \cup S^c} + \sum_{\overline{S} \in S^c} e_{\overline{S}} \Delta_S.$$

Therefore,

$$(12) \qquad \Delta_I = \sum_{I' \subset (I - \widetilde{S})} \left(\prod_{S \in I - \widetilde{S} - I'} \sum_{\overline{S} \in S^c} e_{\overline{S}} \Delta_S \right) \Delta_{\widetilde{S} \cup I' \cup (I')^c}.$$

Note that if $\overline{S} \in S^c$, then $e_{\overline{S}} \Delta_S \Delta_{S_0} = e_{\overline{S}} \Delta_{S_0}$. If $\|v\|_{\delta'}$ is finite, $\Delta_{S_0} v = v$, so if $\|v\|_{\delta'}$ is finite we can replace $e_{\overline{S}} \Delta_S$ with $e_{\overline{S}}$ in the above equation (12), getting

$$(13) \qquad \Delta_I v = \sum_{I' \subset (I - \widetilde{S})} \left(\prod_{S \in I - \widetilde{S} - I'} \sum_{\overline{S} \in S^c} e_{\overline{S}} \right) \Delta_{\widetilde{S} \cup I' \cup (I')^c} v \quad \text{if } \Delta_{\widetilde{S}_0} v = v.$$

To bound $\|w_I^{-\delta}\Delta_I v\|_{\delta'}$ and $\|w_I^{-\delta}\Delta_I v\|_{\delta'}^1$, using the above decomposition of Δ_I, we also need the estimate

$$(14) \qquad cw_I \geq \left(\prod_{S \in I - \widetilde{S} - I'} e_{\overline{S}} \right) w_{\widetilde{S} \cup I' \cup (I')^c} w_0.$$

To prove the above inequality, suppose that ζ is one of the smooth monomials involved in the right-hand side, so ζ vanishes on all strata in $\mathcal{S}_0 \cup \widetilde{S} \cup I' \cup (I')^c$, but does not vanish on $\overline{S}$. In particular, $\zeta = \lceil \widetilde{\zeta} \rceil$ where $\widetilde{\zeta} = 0$ on some $\overline{S} \in S^c$, and $\widetilde{\zeta} > 0$ on $S^0 \in \mathcal{S}_0$, where $\underline{\pi}(S) = \underline{\pi}(S^0)$. Therefore, $\widetilde{\zeta} > 0$ on S. This is valid for all $S \in I - \widetilde{S} - I'$. As all other strata in I appear in $\widetilde{S} \cup I' \cup (I')^c$, it follows that ζ vanishes on I, so is bounded by a constant times w_I. The inequality (14) follows.

Using the decomposition (13), the above inequality (14) and the fact that $w_I \geq cw_{I \cup I^c}$, we get that the following inequality holds, so long as $\Delta_{\mathcal{S}_0} v = v$:

$$\|w_I^{-\delta}\Delta_I v\|_{\delta'} \leq c\|w_{I \cup I^c}^{-\delta}\Delta_{I \cup I^c} v\|_{\delta'}$$
$$+ c \sum_{I' \subsetneq (I - \widetilde{S})} \left\| \left(\prod_{S \in I - \widetilde{S} - I'} \sum_{\overline{S} \in S^c} e_{\overline{S}} \right) w_{\widetilde{S} \cup I' \cup (I')^c}^{-\delta} w_0^{-\delta} \Delta_{\widetilde{S} \cup I' \cup (I')^c} v \right\|_{\delta'}$$
$$= c\|w_{I \cup I^c}^{-\delta}\Delta_{I \cup I^c} v\|_{\delta'}$$
$$+ c \sum_{I' \subsetneq (I - \widetilde{S})} \left\| \left(\prod_{S \in I - \widetilde{S} - I'} \sum_{\overline{S} \in S^c} e_{\overline{S}} \right) w_{\widetilde{S} \cup I' \cup (I')^c}^{-\delta} \Delta_{\widetilde{S} \cup I' \cup (I')^c} v \right\|_{\delta' + \delta}.$$

(In the above inequality, we also used the fact that $e_S w_I \leq w_I$.) All the terms above are dominated by the first new norm, and we have proved that $\|v\|_{0,\delta}$ is dominated by the first new norm.

To dominate $\|w_I^{-\delta}\Delta d_{\mathrm{vert}} v\|_{\delta'}$ by the second new norm, we can use the above inequality. Now dominate $\sup|w_I^{-\delta}\Delta_I v|$ using (12) and (14) as follows:

$$(15) \quad \sup|w_I^{-\delta}\Delta_I v|$$
$$\leq c \sup|w_{I \cup I^c}^{-\delta}\Delta_{I \cup I^c} v|$$
$$+ c \sum_{I' \subsetneq (I - \widetilde{S})} \sup \left| \sum \left(\prod_{S \in I - \widetilde{S} - I'} e_{\overline{S}} \right) w_{\widetilde{S} \cup I' \cup (I')^c}^{-\delta} w_0^{-\delta} \Delta_{I - \widetilde{S} - I'} \Delta_{\widetilde{S} \cup I' \cup (I')^c} v \right|.$$

The unspecified sum (above and in the equation below) is over all choices of $\overline{S} \in S^c$ for all $S \in I - \widetilde{S} - I'$. We must bound the terms appearing in the sum above in (15).

Lemma 2.13 gives that $|e_{\overline{S}}\Delta_S\phi| \le c\|e_{\overline{S}}d_{\text{vert}}\phi\|_{\delta+\delta'}e_{\overline{S}}w_0^{\delta+\delta'}$. Use this inequality for a single $S \in I - \widetilde{S} - I'$, and remove the other terms in $\Delta_{I-\widetilde{S}-I'}$ by observing that $\|\Delta_{S'}\phi\| \le 2\|\phi\|$. We have

$$\sup\left|\sum\left(\prod_{S\in I-\widetilde{S}-I'} e_{\overline{S}}\right)w_{\widetilde{S}\cup I'\cup(I')^c}^{-\delta}w_0^{-\delta}\Delta_{I-\widetilde{S}-I'}\Delta_{\widetilde{S}\cup I'\cup(I')^c}\nu\right|$$

$$\le c\left\|\left(\prod_{S\in I-\widetilde{S}-I'}\sum_{\overline{S}\in S^c} e_{\overline{S}}\right)w_{\widetilde{S}\cup I'\cup(I')^c}^{-\delta}\Delta_{\widetilde{S}\cup I'\cup(I')^c}d_{\text{vert}}\nu\right\|_{\delta+\delta'}.$$

The last term is bounded by our second new norm, so this completes the proof that $\|\nu\|_{0,\delta}^1$ is dominated by the second new norm. $\square$

2.6 Norms on an allowable collection of coordinate charts

We are now ready to define our norms on a vectorbundle V over a family of curves $\pi_F\colon \widehat{C} \to F$. To avoid the problems that arise if F is not compact, we define the notion of an "allowable" family, which is extendable and can be covered by an "allowable" collection of extendable allowable coordinate charts. We shall also need a version of this definition when there is a collection of marked points on our family.

Definition 2.16 An allowable collection of coordinate charts on a vectorbundle V over a family $\pi_F\colon \widehat{C} \to F$ is a local trivialization of V over a finite collection of extendable allowable coordinate charts $\pi\colon U_{\alpha,i} \to U_\alpha$, satisfying the following additional conditions:

(1) $\pi_F^{-1}(U_\alpha) = \bigcup_i U_{\alpha,i}$.

(2) The restriction of the vectorbundle V to $U_{\alpha,i}$ is $\mathbb{R}^n \times U_{\alpha,i}$ with the obvious projection.

(3) Coordinate change maps and intersections between $U_{\alpha,i}$ and $U_{\alpha,j}$ satisfy the following conditions:

   (a) If $U_{\alpha,i}$ and $U_{\alpha,j}$ are coordinate charts of type (I) from Definition 2.3 (these are the charts that cover an internal edge of an exploded curve), and $i \ne j$, then they do not intersect.

   (b) If $U_{\alpha,i}$ and $U_{\alpha,j}$ are coordinate charts of type (II), they are the product of an open subset of T_1^1 with U_α. In this case, their intersection in either of these coordinate charts is also the product of an open subset of T_1^1 with U_α, and the coordinate change map is a product of some map between these

subsets of T_1^1 with the identity map on U_α. The coordinate change map between the vectorbundle trivializations over these charts is independent of position in U_α and depends only on T_1^1.

(c) If $U_{\alpha,i}$ is a coordinate chart of type (I) and $U_{\alpha,j}$ is a coordinate chart of type (II), their intersection is as follows: In $U_{\alpha,j}$, it is the product of a subset $O \subset T_1^1$ with U_α. If $U_{\alpha,i}$ is given by $\{|\tilde{z}_1| < c, \, |\tilde{z}^\beta| < c\}$, then the intersection with U_α is a subset of the form $\{\tilde{z}_1 \in O'\}$ or $\{\tilde{z}^\beta \in O'\}$, where $O' \subset \{c/2 < |z| < c\} \in \mathbb{C}$. In either case, we identify this subset with the product $O' \times U_\alpha$. The transition map in this case is the product of a diffeomorphism between O and O' and the identity on U_α. Again, the coordinate change map between the vectorbundle trivializations over these charts is independent of position in U_α and depends only on O.

If our family also has a collection of nonintersecting marked-points sections $F \to \hat{C}$ not intersecting edges of curves in our family, then an allowable collection of coordinate charts is an allowable collection as above with the extra conditions that no marked points are inside coordinate charts of type (I), and in the coordinate charts of type (II) containing marked points, the sections corresponding to these marked points are constant sections $U_\alpha \to T_1^1 \times U_\alpha$.

Outside a neighborhood of internal edges, every family of exploded curves is locally trivial (ignoring complex structures), so it is easy to verify that any single curve $C \neq T$ in a family of curves (with marked points as described above) is covered by an allowable collection of coordinate charts: first cover internal edges with separated charts $U_i \to U$ of type (I) that avoid marked points. On the subset of U_i where $c/2 < \tilde{z}_1 < c$ or $c/2 < \tilde{z}^\beta < c$, there is a canonical map to $U_i \cap C$ given by keeping all coordinates apart from $\tilde{z}^\beta$ or $\tilde{z}_1$ constant; after possibly shrinking c, there is a contractible choice of extension of this map to $\pi_F^{-1} U \setminus \bigcup_i U_i \to C$ trivializing the family $\pi_F^{-1}(U) \to U$ (along with its marked points). We can then choose the remaining charts and transition maps compatible with this trivialization. Moreover, by shrinking and subdividing these coordinate charts if necessary, we may extend these allowable coordinate charts to any finite-rank vectorbundle. Note that the almost complex structure on fibers can not be assumed to be the standard one from the allowable coordinate charts.

Definition 2.17 An allowable family $\hat{C} \xrightarrow{\pi_F} F$ is a subset of a family of curves $\hat{C} \xrightarrow{\pi_F} F'$ covered, along with the vectorbundle X, by an allowable collection of coordinate charts.

In particular, an allowable family of curves

$$
\begin{array}{ccc}
\widehat{C} & \xrightarrow{\ \widehat{f}\ } & \widehat{B} \\
\downarrow & & \downarrow \\
F & \longrightarrow & B_0
\end{array}
$$

is a family such that $\widehat{C} \to F$ with the vectorbundle $\widehat{f}^* T_{\mathrm{vert}} \widehat{B}$ is covered by an allowable collection of coordinate charts.

If $\widehat{C} \to F$ is an allowable family of curves, $T_{\mathrm{vert}}^* \widehat{C}$ is canonically an allowable vectorbundle, trivialized using the standard basis for the cotangent space in coordinates consisting of dx_i and the real and imaginary parts of $\widetilde{z}_i^{-1} \, d\widetilde{z}_i$. So, $T_{\mathrm{vert}}^* \widehat{C} \otimes X$ or $T_{\mathrm{vert}}^* \widehat{C} \otimes \widehat{f}^* T_{\mathrm{vert}} \widehat{B}$ is also canonically an allowable vectorbundle. For the purposes of measuring the norm of sections of $Y := (T_{\mathrm{vert}}^* \widehat{C} \otimes X)^{(0,1)}$, we consider Y as a subbundle of this allowable vectorbundle. Note that Y may not be a constant subbundle using this trivialization as j may not be constant in our local trivialization of $T_{\mathrm{vert}}^* \widehat{C}$.

Definition 2.18 On an allowable family $\widehat{f}$ with vectorbundle V covered by the allowable collection of coordinate charts $U_{\alpha,i} \to U_\alpha$, define the norms

$$
\|\nu\|_{**} := \sum_{U_{\alpha,i}} \|\nu|_{U_{\alpha,i}}\|_{**},
$$

where $**$ stands for the different possible labels for norms defined so far. For example, $\|\nu\|_{k,\delta}^1 := \sum_{U_{\alpha,i}} \|\nu|_{U_{\alpha,i}}\|_{k,\delta}^1$.

Given a choice of weight $\delta_{\alpha,i}$ for each coordinate chart $U_{\alpha,i}$, define the norms

$$
\|\nu\|_{\mathrm{mixed}\,\delta} := \sum_{U_{\alpha,i}} \|\nu|_{U_{\alpha,i}}\|_{\delta_{\alpha,i}} \quad \text{and} \quad \|\nu\|_{\mathrm{mixed}\,\delta}^1 := \sum_{U_{\alpha,i}} \|\nu|_{U_{\alpha,i}}\|_{\delta_{\alpha,i}}^1.
$$

One problem with the norms $\|\cdot\|_{k,\delta}$ is that $w_{\mathcal{S}} \Delta_{\mathcal{S}} \nu$ is only defined on coordinate patches and not globally defined on a fiber. The following definition provides a means of remedying this.

Definition 2.19 Given an allowable collection of coordinate charts $U_{\alpha,i} \to U_\alpha$, and a collection $\mathcal{S}$ of strata in $\underline{U_\alpha}$, define $\widetilde{\Delta}_{\mathcal{S}} \nu$ on $\bigcup_i U_{\alpha,i}$ as follows:

(i)   Choose a smooth cutoff function $\rho \colon T_1^1 \to [0,1]$ so that

$$
\rho(\widetilde{z}) = \begin{cases} 1 & \text{if } |\widetilde{z}| \le \frac{1}{2}, \\ 0 & \text{if } |\widetilde{z}| \ge 1. \end{cases}
$$

(ii) On coordinate charts $U_{\alpha,i}$ of type (I), we have coordinates $\tilde{z}_1$ and $\tilde{z}_\beta$ such that $|\tilde{z}_1| < c$, $|\tilde{z}_\beta| < c$ and $|\tilde{z}_1 \tilde{z}^\beta| < c^2/16$. Use the notation $\tilde{\mathcal{S}}$ for the lift of $\mathcal{S}$ to this chart, $\mathcal{S}^+$ for the collection of strata in $\tilde{\mathcal{S}}$ such that $\underline{\tilde{z}_1} = 0$, and $\mathcal{S}^-$ for the collection of strata in $\tilde{\mathcal{S}}$ such that $\underline{\tilde{z}^\beta} = 0$.

Define $\tilde{\Delta}_{\mathcal{S}} v$ on this chart by

$$\tilde{\Delta}_{\mathcal{S}} v := \rho\left(\frac{2\tilde{z}_1}{c}\right) \rho\left(\frac{2\tilde{z}^\beta}{c}\right) \Delta_{\tilde{\mathcal{S}}} v + \left(1 - \rho\left(\frac{2\tilde{z}_1}{c}\right)\right) \Delta_{\mathcal{S}^+} v + \left(1 - \rho\left(\frac{2\tilde{z}^\beta}{c}\right)\right) \Delta_{\mathcal{S}^-} v.$$

(iii) On all other coordinate charts U_{α_i} of type (II), let $\tilde{\mathcal{S}}$ denote the lift of $\mathcal{S}$ to this coordinate chart, and define

$$\tilde{\Delta}_{\mathcal{S}} v := \Delta_{\tilde{\mathcal{S}}} v.$$

It follows from the types of transition functions allowed for allowable collections of coordinate charts that $\tilde{\Delta}_{\mathcal{S}} v$ is well defined on $\bigcup_i U_{\alpha,i}$ and smooth or $C^{\infty,1}$ if v is. We can now state a fiberwise-global version of Lemma 2.15.

Lemma 2.20 *Restricted to a collection of allowable coordinate charts $U_{\alpha,i}$ over a single chart U_α, the norm $\|v\|_{0,\delta}$ is equivalent to the norm*

$$\|v\|_{\delta'} + \max_{\mathcal{S}} \|w_{\mathcal{S}}^{-\delta} \tilde{\Delta}_{\mathcal{S}} v\|_{\delta \& \delta'},$$

and the norm $\|v\|_{0,\delta}^1$ is equivalent to the norm

$$\|v\|_{\delta'}^1 + \max_{\mathcal{S}} \|w_{\mathcal{S}}^{-\delta} \tilde{\Delta}_{\mathcal{S}} v\|_{\delta \& \delta'}^1.$$

In the above, the maximum is taken over all collections of strata $\mathcal{S}$ in $\underline{U_\alpha}$, and $w_{\mathcal{S}}$ indicates the lift of the weighting function $w_{\mathcal{S}}$ on U_α (which equals $w_{\tilde{\mathcal{S}}}$ in each of the coordinate charts $U_{\alpha,i}$). The norm $\|\cdot\|_{\delta \& \delta'}$ is defined in Definition 2.14.

Proof On all coordinate charts of type (II), this lemma follows immediately from Lemma 2.15, as in this case $\tilde{\Delta}_{\mathcal{S}} = \Delta_{\tilde{\mathcal{S}}}$.

On a coordinate chart of type (I), where $|\tilde{z}_1| < c$ and $|\tilde{z}^\beta| < c$, using the notation from Definition 2.19 we have that

$$\Delta_{\tilde{\mathcal{S}}} v - \tilde{\Delta}_{\mathcal{S}} v$$
$$= \left(1 - \rho\left(\frac{2\tilde{z}_1}{c}\right) \rho\left(\frac{2\tilde{z}^\beta}{c}\right)\right) \Delta_{\tilde{\mathcal{S}}} v - \left(1 - \rho\left(\frac{2\tilde{z}_1}{c}\right)\right) \Delta_{\mathcal{S}^+} v - \left(1 - \rho\left(\frac{2\tilde{z}^\beta}{c}\right)\right) \Delta_{\mathcal{S}^-} v.$$

Use the notation $\check{\rho}^+$ for the function $(1 - \rho(2\tilde{z}_1/c))$ and $\check{\rho}^-$ for $(1 - \rho(2\tilde{z}^\beta/c))$. As $\rho(\tilde{z}) = 1$ when $|\tilde{z}| \le \frac{1}{2}$, and our coordinate chart has $|\tilde{z}_1 \tilde{z}^\beta| < c^2/16$, the region where $\rho(2\tilde{z}_1/c) \ne 1$ is disjoint from the region where $\rho(2\tilde{z}^\beta/c) \ne 1$, so we can rewrite the above equation as

$$(16) \qquad \Delta_{\tilde{S}} v - \tilde{\Delta}_S v = \check{\rho}^+ (\Delta_{\tilde{S}} v - \Delta_{S^+} v) + \check{\rho}^- (\Delta_{\tilde{S}} v - \Delta_{S^-} v).$$

Given any stratum T for which $T^c \ne \varnothing$, we have $e_T \lceil \tilde{z}_1 \tilde{z}^\beta \rceil = 0$. We then obtain the following dichotomy:

$$\text{if } e_T \lceil \tilde{z}_1 \rceil = 0 \quad \text{then } e_T \check{\rho}^+ = 0 \text{ and } e_T \Delta_{\tilde{S}} = e_T \Delta_{S^-},$$
$$\text{if } e_T \lceil \tilde{z}^\beta \rceil = 0 \quad \text{then } e_T \check{\rho}^- = 0 \text{ and } e_T \Delta_{\tilde{S}} = e_T \Delta_{S^+}.$$

Therefore, for any stratum T such that $T^c \ne \varnothing$, in our coordinate chart

$$(17) \qquad e_T \tilde{\Delta}_S = e_T \Delta_{\tilde{S}}.$$

Equation (17) holds on any fiber of our coordinate chart $U_{\alpha,i} \to U_\alpha$ with nontrivial tropical part, so implies

$$(18) \qquad \|w_S^{-\delta} \tilde{\Delta}_S v - w_S^{-\delta} \Delta_{\tilde{S}} v\|_{\delta \& \delta'} = \|w_S^{-\delta} \tilde{\Delta}_S v - w_S^{-\delta} \Delta_{\tilde{S}} v\|_{\delta'},$$
$$(19) \qquad \|w_S^{-\delta} \tilde{\Delta}_S v - w_S^{-\delta} \Delta_{\tilde{S}} v\|_{\delta \& \delta'}^1 = \|w_S^{-\delta} \tilde{\Delta}_S v - w_S^{-\delta} \Delta_{\tilde{S}} v\|_{\delta'}^1.$$

Note that $\Delta_{\tilde{S}} - \Delta_{S^+}$ is a sum of terms $\pm e_T \Delta_{S^+}$ for $T \in \tilde{S} - S^+$, so Equation (16) implies that there exists some constant c' such that on this coordinate chart,

$$\|w_S^{-\delta} \tilde{\Delta}_S v - w_S^{-\delta} \Delta_{\tilde{S}} v\|_{\delta'}$$
$$\le c' \sum_{T \in \tilde{S} - S^+} \|w_S^{-\delta} \check{\rho}^+ e_T \Delta_{S^+} v\|_{\delta'} + c' \sum_{T \in \tilde{S} - S^-} \|w_S^{-\delta} \check{\rho}^- e_T \Delta_{S^-} v\|_{\delta'}.$$

Use the notation $\hat{S}$ to indicate $S^+ \cap S^-$. This is the set of strata within $\tilde{S}$ where $\underline{\tilde{z}}_1 = 0 = \underline{\tilde{z}}^\beta$, or equivalently the set of strata $S \in \tilde{S}$ with $S^c = \varnothing$. If $T \in \tilde{S} - S^+$, then

$$e_T \Delta_{S^+} = e_T \Delta_{\hat{S}} \Delta_{S_0},$$

where S_0 is the set of strata where $\lceil \tilde{z}_1 \rceil = 0 = \lceil \tilde{z}^\beta \rceil$. To see this, note that if $T' \in S^+$, then either $T' \in \hat{S}$ or $e_T e_{T'} \Delta_{S_0} = 0$ because $e_T \lceil \tilde{z}_1 \rceil = 0$ and $e_{T'} \lceil \tilde{z}^\beta \rceil = 0$. Therefore, $e_T \Delta_{S^+} e_T \Delta_{\hat{S}} \Delta_{S_0} = e_T \Delta_{\hat{S}} \Delta_{S_0}$. Similarly, if $S \in S_0$, then $e_S e_T \Delta_{S^+} = 0$, because if T^+ indicates the strata in S^+ with the same projection as T, then $e_S e_{T^+} = e_S e_T$. Therefore, $e_T \Delta_{S^+} e_T \Delta_{\hat{S}} \Delta_{S_0} = e_T \Delta_{S^+}$, so $e_T \Delta_{S^+} = e_T \Delta_{\hat{S}} \Delta_{S_0}$ as required.

Similarly, if $T \in \widetilde{\mathcal{S}} - \mathcal{S}^-$, then $e_T \Delta_{\mathcal{S}^-} = e_T \Delta_{\widehat{\mathcal{S}}} \Delta_{\mathcal{S}_0}$. Therefore, we get

$$
\|w_{\mathcal{S}}^{-\delta} \widetilde{\Delta}_{\mathcal{S}} v - w_{\mathcal{S}}^{-\delta} \Delta_{\widetilde{\mathcal{S}}} v\|_{\delta'}
$$
$$
\leq c' \sum_{T \in \widetilde{\mathcal{S}} - \mathcal{S}^+} \|w_{\mathcal{S}}^{-\delta} \breve{\rho}^+ e_T \Delta_{\widehat{\mathcal{S}} \cup \mathcal{S}_0} v\|_{\delta'} + c' \sum_{T \in \widetilde{\mathcal{S}} - \mathcal{S}^-} \|w_{\mathcal{S}}^{-\delta} \breve{\rho}^- e_T \Delta_{\widehat{\mathcal{S}} \cup \mathcal{S}_0} v\|_{\delta'}.
$$

Claim *If $T \in \widetilde{\mathcal{S}} - \mathcal{S}^+$, then $w_{\mathcal{S}}^{-\delta} \breve{\rho}^+$ is bounded by a constant times $e_T w_{\widehat{\mathcal{S}}}^{-\delta} w_0^{-\delta}$.*

To prove this claim, it suffices to show that $\breve{\rho}^+ e_T w_{\widehat{\mathcal{S}}} w_0$ vanishes on all the strata in $\widetilde{\mathcal{S}}$. If S is a stratum in $\widetilde{\mathcal{S}}$, then either $S \in \widehat{\mathcal{S}}$, $S \in \mathcal{S}_0$, $e_S \lceil \widetilde{z}_1 \rceil = 0$, or $e_S \lceil \widetilde{z}^\beta \rceil = 0$. In the first two cases, $e_S w_{\widehat{\mathcal{S}}} w_0 = 0$. If $e_S \lceil \widetilde{z}_1 \rceil = 0$, then $e_S \breve{\rho}^+ = 0$. If $e_S \lceil \widetilde{z}^\beta \rceil = 0$, then as $e_T \lceil \widetilde{z}_1 \rceil = 0$, we get $e_S e_T w_0 = 0$. Therefore $e_S \breve{\rho}^+ e_T w_{\widehat{\mathcal{S}}} w_0 = 0$, and the above claim follows.

Similarly, if $T \in \widetilde{\mathcal{S}} - \mathcal{S}^-$, then $w_{\mathcal{S}}^{-\delta} \breve{\rho}^-$ is bounded by a constant times $e_T w_{\widehat{\mathcal{S}}}^{-\delta} w_0^{-\delta}$. Therefore,

$$
\|w_{\mathcal{S}}^{-\delta} \widetilde{\Delta}_{\mathcal{S}} v - w_{\mathcal{S}}^{-\delta} \Delta_{\widetilde{\mathcal{S}}} v\|_{\delta'} \leq c'' \sum_{T \in \widetilde{\mathcal{S}} - \widehat{\mathcal{S}}} \|e_T w_{\widehat{\mathcal{S}}}^{-\delta} w_0^{-\delta} \Delta_{\widehat{\mathcal{S}} \cup \mathcal{S}_0} v\|_{\delta'}.
$$

As all the strata T in the above inequality satisfy $T^c \neq \varnothing$, on the strata T the norm $\|\cdot\|_{\delta \& \delta'}$ always uses the higher weight $w_0^{-\delta - \delta'}$. Therefore, we get that

$$
(20) \qquad \|w_{\mathcal{S}}^{-\delta} \widetilde{\Delta}_{\mathcal{S}} v - w_{\mathcal{S}}^{-\delta} \Delta_{\widetilde{\mathcal{S}}} v\|_{\delta'} \leq c''' \|w_{\widehat{\mathcal{S}}}^{-\delta} \Delta_{\widehat{\mathcal{S}}} v\|_{\delta \& \delta'}.
$$

The set of strata $\widehat{\mathcal{S}}$ is the lift of some set of strata which we shall again call $\widehat{\mathcal{S}}$, because each stratum lifts uniquely. With this slight abuse of notation $\widehat{\mathcal{S}}^+ = \widehat{\mathcal{S}} = \widehat{\mathcal{S}}^-$, so $\widetilde{\Delta}_{\widehat{\mathcal{S}}} = \Delta_{\widehat{\mathcal{S}}}$, and we get

$$
(21) \qquad \|w_{\mathcal{S}}^{-\delta} \widetilde{\Delta}_{\mathcal{S}} v - w_{\mathcal{S}}^{-\delta} \Delta_{\widetilde{\mathcal{S}}} v\|_{\delta'} \leq c''' \|w_{\widehat{\mathcal{S}}}^{-\delta} \widetilde{\Delta}_{\widehat{\mathcal{S}}} v\|_{\delta \& \delta'}.
$$

The inequalities (20) and (21) together with (18) and Lemma 2.15 prove that the norm $\|v\|_{0, \delta}$ is equivalent to the norm

$$
\|v\|_{\delta'} + \max_{\mathcal{S}} \|w_{\mathcal{S}}^{-\delta} \widetilde{\Delta}_{\mathcal{S}} v\|_{\delta \& \delta'}.
$$

As we already have the required estimates for the part of the $\|v\|_\delta^1$ norm involving $\|d_{\text{vert}} v\|_\delta$, it remains to estimate

$$
\sup |w_{\mathcal{S}}^{-\delta} \widetilde{\Delta}_{\mathcal{S}} v - w_{\mathcal{S}}^{-\delta} \Delta_{\widetilde{\mathcal{S}}} v|
$$

on our coordinate chart. As argued for the $\|\cdot\|_{\delta'}$ norm above, we get

$$
\sup |w_{\mathcal{S}}^{-\delta} \widetilde{\Delta}_{\mathcal{S}} v - w_{\mathcal{S}}^{-\delta} \Delta_{\widetilde{\mathcal{S}}} v| \leq c' \sum_{T \in \widetilde{\mathcal{S}} - \widehat{\mathcal{S}}} \sup |e_T w_{\widehat{\mathcal{S}}}^{-\delta} w_0^{-\delta} \Delta_{\widehat{\mathcal{S}} \cup \mathcal{S}_0} v|.
$$

We can estimate the right-hand side of the above inequality with Lemma 2.13, as S_0 will contain some stratum whose complement contains T. Therefore,

$$\sup|w_{\mathcal{S}}^{-\delta}\widetilde{\Delta}_{\mathcal{S}}\nu - w_{\mathcal{S}}^{-\delta}\Delta_{\widetilde{\mathcal{S}}}\nu| \leq c'' \sum_{T \in \widetilde{\mathcal{S}} - \widehat{\mathcal{S}}} \|e_T w_{\widehat{\mathcal{S}}}^{-\delta}\Delta_{\widehat{\mathcal{S}}}d_{\text{vert}}\nu\|_{\delta}.$$

So, as argued above, we get the two inequalities

$$(22) \qquad \sup|w_{\mathcal{S}}^{-\delta}\widetilde{\Delta}_{\mathcal{S}}\nu - w_{\mathcal{S}}^{-\delta}\Delta_{\widetilde{\mathcal{S}}}\nu| \leq c\|w_{\widehat{\mathcal{S}}}^{-\delta}\widetilde{\Delta}_{\widehat{\mathcal{S}}}d_{\text{vert}}\nu\|_{\delta\&\delta'},$$

$$(23) \qquad \sup|w_{\mathcal{S}}^{-\delta}\widetilde{\Delta}_{\mathcal{S}}\nu - w_{\mathcal{S}}^{-\delta}\Delta_{\widetilde{\mathcal{S}}}\nu| \leq c\|w_{\widehat{\mathcal{S}}}^{-\delta}\Delta_{\widehat{\mathcal{S}}}d_{\text{vert}}\nu\|_{\delta\&\delta'}.$$

This together with Lemma 2.15 completes the proof of our lemma. $\qquad\square$

The above proof contains the inequalities (20)–(23), which together with equations (18) and (19) imply the following estimates, which will be useful later:

Lemma 2.21 Let $\widehat{\mathcal{S}}$ denote the subset of $\widetilde{\mathcal{S}}$ consisting of all strata $T \in \widetilde{\mathcal{S}}$ such that $T^c = \varnothing$, and also let $\widehat{\mathcal{S}}$ denote the corresponding subset of $\mathcal{S}$. Then there exists some constant c such that the following inequalities hold for all ν for which both sides are defined:

$$\|w_{\mathcal{S}}^{-\delta}\widetilde{\Delta}_{\mathcal{S}}\nu - w_{\widetilde{\mathcal{S}}}^{-\delta}\Delta_{\widetilde{\mathcal{S}}}\nu\|_{\delta\&\delta'} \leq c\|w_{\widehat{\mathcal{S}}}^{-\delta}\Delta_{\widehat{\mathcal{S}}}\nu\|_{\delta\&\delta'} = c\|w_{\widehat{\mathcal{S}}}^{-\delta}\widetilde{\Delta}_{\widehat{\mathcal{S}}}\nu\|_{\delta\&\delta'},$$

$$\|w_{\mathcal{S}}^{-\delta}\widetilde{\Delta}_{\mathcal{S}}\nu - w_{\widetilde{\mathcal{S}}}^{-\delta}\Delta_{\widetilde{\mathcal{S}}}\nu\|_{\delta\&\delta'}^1 \leq c\|w_{\widehat{\mathcal{S}}}^{-\delta}\Delta_{\widehat{\mathcal{S}}}\nu\|_{\delta\&\delta'}^1 = c\|w_{\widehat{\mathcal{S}}}^{-\delta}\widetilde{\Delta}_{\widehat{\mathcal{S}}}\nu\|_{\delta\&\delta'}^1.$$

3 Analysis of the $\overline{\partial}$ equation in families

3.1 $\overline{\partial}$ is continuously differentiable

Throughout this section, we shall be considering an operator $\overline{\partial}$ in the form ($\star$) described on page 1626.

Recall that an allowable family of curves as defined in Definition 2.17 is an extendable family covered by an allowable collection of coordinate charts in which the vectorbundle X or $\widehat{f}^*T_{\text{vert}}\widehat{B}$ is trivialized. So, on the space of sections of X, we may use our norms from Definition 2.18. Similarly, each of our allowable coordinate charts comes with a canonical trivialization of $T_{\text{vert}}^*\widehat{C}$, and therefore a trivialization of $T_{\text{vert}}^*\widehat{C} \otimes X$, so we may define norms of sections of $T_{\text{vert}}^*\widehat{C} \otimes X$ and therefore norms on sections of the subbundle $Y := (T_{\text{vert}}^*\widehat{C} \otimes X)^{(0,1)}$. Note that any family of curves may be shrunk to a compactly contained subfamily to obtain an allowable family of curves.

Definition 3.1 Consider an allowable family of curves with some (possibly empty) collection of nonintersecting marked-point sections $F \to \widehat{C}$ avoiding the edges of the curves in $\widehat{C} \to F$.

Define the Banach space X_{**} to be the $\|\cdot\|_{**}^1$–completion of the subspace of $X^{\infty,\underline{1}}$ with finite $\|\cdot\|_{**}^1$ norm,[17] where $X^{\infty,\underline{1}}$ is the space of $C^{\infty,\underline{1}}$ sections of the vector-bundle X vanishing at marked points, and $**$ stands for the different labels for norms used in Section 2.4.

Define Y_{**} to be the $\|\cdot\|_{**}$–completion of the subspace of $Y^{\infty,\underline{1}}$ with finite norm, where $Y^{\infty,\underline{1}}$ is the space of $C^{\infty,\underline{1}}$ sections of Y vanishing on all edges of curves in $\widehat{C} \to F$.

Given a curve f in $\widehat{C} \to F$, we can restrict our data to the domain C of f. Define $X_\delta(f)$ to be the corresponding Banach space X_δ with this restricted data. If $v \in X_\delta$, define $v(f) \in X_\delta(f)$ to be the restriction of v to the domain C of f. We can similarly define $Y_\delta(f)$.

Note that the norms used for X_{**} control one extra vertical derivative when compared to the norms used for Y_{**}. We consider $\overline{\partial}$ as a map from X_{**} to Y_{**}.

Theorem 3.2 *If $\widehat{f}$ is an allowable $C^{\infty,\underline{1}}$ family, then $\overline{\partial}$ defines a C^1 map from $X_{k,\delta}$ to $Y_{k,\delta}$.*

Proof Our local form for $\overline{\partial}$ is

$$\overline{\partial}v = E(v) + H(v)(d_{\text{vert}}v).$$

By using Lemma 2.8(C), we may estimate in this coordinate chart

$$\|\overline{\partial}v\|_{k,\delta} \leq \|E(v)\|_{k,\delta} + c\|H(v)\|_{k,\delta}^1 \|d_{\text{vert}}v\|_{k,\delta}.$$

Lemmas 2.10 and 1.7 imply that the terms above $\|E(v)\|_{k,\delta}$ and $\|H(v)\|_{k,\delta}^1$ are bounded if $\|v\|_{k,\delta}^1$ is bounded. Similarly, Corollary 2.11 implies that $\overline{\partial}v$ depends continuously on v, therefore $\overline{\partial}$ gives a well-defined map from $X_{k,\delta}$ to $Y_{k,\delta}$.

The derivative is given by the formula

$$(24) \qquad D\overline{\partial}(v)(\psi) = DE(v)(\psi(u)) + DH(v)(\psi)(d_{\text{vert}}v) + H(v)(d_{\text{vert}}\psi).$$

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[17]A section in $X^{\infty,\underline{1}}$ will have finite $\|\cdot\|_{**}^1$–norm if it is extendable, but not in general because we don't have control near the boundary of F.

Using Lemma 2.8(C), estimate this derivative as

$$\|D\overline{\partial}(v)(\psi)\|_{k,\delta} \le c(\|DE(v)\|_{k,\delta} + \|DH(v)\|^1_{k,\delta}\|d_{\text{vert}}v\|_{k,\delta} + \|H(v)\|^1_{k,\delta})\|\psi\|^1_{k,\delta}.$$

The terms $\|H(v)\|^1_{k,\delta}$, $\|DH(v)\|^1_{k,\delta}$ and $\|DE(v)\|_{k,\delta}$ are bounded by Lemma 2.10, therefore $D\overline{\partial}(v)$ defines a bounded linear map from $X_{k,\delta}$ to $Y_{k,\delta}$. We must prove that $D\overline{\partial}(v)$ is continuous in v. We have

$$D\overline{\partial}(v_1)(\psi) - D\overline{\partial}(v_2)(\psi)$$
$$= (DE(v_1) - DE(v_2))(\psi) + (H(v_1) - H(v_2))(d_{\text{vert}}\psi)$$
$$+ DH(v_1)(\psi)(d_{\text{vert}}v_1 - d_{\text{vert}}v_2)$$
$$+ DH(v_1)(\psi)(d_{\text{vert}}v_2) - DH(v_2)(\psi)(d_{\text{vert}}v_2).$$

Using Lemma 2.8(C) on the above, there exists a positive constant c (independent of v_i and ψ) such that

$$c\|D\overline{\partial}(v_1)(\psi) - D\overline{\partial}(v_2)(\psi)\|_{k,\delta}$$
$$\le \|DE(v_1) - DE(v_2)\|_{k,\delta}\|\psi\|^1_{k,\delta} + \|H(v_1) - H(v_2)\|^1_{k,\delta}\|d_{\text{vert}}\psi\|_{k,\delta}$$
$$+ \|DH(v_1)\|^1_{k,\delta}\|\psi\|^1_{k,\delta}\|d_{\text{vert}}v_1 - d_{\text{vert}}v_2\|_{k,\delta}$$
$$+ \|DH(v_1) - DH(v_2)\|^1_{k,\delta}\|d_{\text{vert}}v_2\|_{k,\delta}\|\psi\|^1_{k,\delta}.$$

The term DE is $C^{\infty,1}$ and $\Delta_{S_0}DE = DE$, so Corollary 2.11 implies that for a fixed v_1, the term $\|DE(v_1) - DE(v_2)\|_{k,\delta}$ converges to zero as $\|v_1 - v_2\|^1_{k,\delta} \to 0$. Similarly, by Corollary 2.11, $\|H(v_1) - H(v_2)\|^1_{k,\delta}$ and $\|DH(v_1) - DH(v_2)\|^1_{k,\delta}$ converge to zero as $v_2 \to v_1$ in $\|\cdot\|^1_{k,\delta}$. Lemma 2.10 implies that $\|DH(v_1)\|^1_{k,\delta}$ is bounded. Therefore, $D\overline{\partial}(v_2) \to D\overline{\partial}(v_1)$ as $v_2 \to v_1$ in $\|\cdot\|^1_{k,\delta}$, so $\overline{\partial}\colon X_{k,\delta} \to Y_{k,\delta}$ is C^1, as required. □

The formula (24) for $D\overline{\partial}$ in the proof above shows that it induces a map on $X_\delta(f)$. In other words, the restriction, $D\overline{\partial}(v)(f)$, of $D\overline{\partial}(v)$ to the domain of a curve f only depends on the restriction of v to this curve, $v(f)$. Use the notation $D\overline{\partial}(f)$ to refer to the restriction to f of $D\overline{\partial}$ at the zero-section.

3.2 Results for variations of a single curve

Lemma 3.3 *For any allowable $C^{\infty,1}$ family of curves, and a choice of weight $0 < \delta < 1$, the linearization*

$$D\overline{\partial}(f)\colon X_\delta(f) \to Y_\delta(f)$$

of $\overline{\partial}$ at the zero-section restricted to any curve is Fredholm.

More generally, given a choice of weight $0 < \delta < 1$ for every coordinate chart in some allowable collection of coordinate charts on our family of curves, we may use the norms with mixed weights defined in Definition 2.18, and

$$D\overline{\partial}(f)\colon X_{\text{mixed }\delta}(f) \to Y_{\text{mixed }\delta}(f)$$

is Fredholm.

Proof Note that the norm $\|\cdot\|_\delta$ restricted to a smooth component of an exploded curve is equivalent to the norm L^p with exponential weight on the cylindrical ends given by δ, and $\|\cdot\|^1_\delta$ is equivalent to the L^p norm with the same exponential weight on derivatives plus the sup norm. The mixed version of these norms is similar, but uses different weights at different punctures. So $X_\delta(f)$ or $X_{\text{mixed }\delta}(f)$, restricted to a smooth component, is the corresponding Sobolev space L^1_p with exponential weights δ at punctures times a finite-dimensional space allowing sections which are asymptotic to constants instead of 0 at cylindrical ends (restricted to the subspace which vanishes on our marked points, if appropriate). From the proof of Theorem 3.2, we have the following formula for $D\overline{\partial}(f)$ in local coordinates:

$$D\overline{\partial}(v_0(f))(\psi) = DE(v_0(f))(\psi) + DH(v_0(f))(\psi)(d_{\text{vert}}v_0(f)) + H(v_0(f))(d_{\text{vert}}\psi).$$

Here, v_0 indicates the zero-section, so the middle term is 0. The conditions on H from $(\star)$ give that the last term equals $\frac{1}{2}(d_{\text{vert}}\psi + J \circ d_{\text{vert}}\psi \circ j)$. This expression is only valid in local coordinates, as $d_{\text{vert}}\psi$ needs a local trivialization of $f^*T_{\text{vert}}\widehat{\boldsymbol{B}}$ to make sense. To remedy this, choose a $C^{\infty,1}$ connection ∇ on $T_{\text{vert}}\widehat{\boldsymbol{B}}$. Replacing d_{vert} with ∇_{vert} in the above formula modifies the other term so that

$$(25) \qquad D\overline{\partial}(v_0(f))(\psi) = E'(f)(\psi) + \tfrac{1}{2}(\nabla\psi + J \circ \nabla\psi \circ j),$$

where, restricted to any smooth component of the domain of f, $E'(f)$ is smooth and decays exponentially on cylindrical ends with all weights less than 1. Therefore, $D\overline{\partial}(\psi)$ is $\frac{1}{2}(\nabla\psi + J \circ \nabla\psi \circ j)$ plus a compact operator. It is well known (and proved in [15]) that for $0 < \delta < 1$, the map $\psi \mapsto \frac{1}{2}(\nabla\psi + J \circ \nabla\psi \circ j)$ is a Fredholm operator on the above weighted Sobolev spaces restricted to any smooth component. It follows that $\overline{\partial}$ is Fredholm from $X_\delta(f)$ to $Y_\delta(f)$, and $X_{\text{mixed }\delta}(f)$ to $Y_{\text{mixed }\delta}(f)$. Therefore $D\overline{\partial}(f)$ is the Fredholm operator $\psi \mapsto \frac{1}{2}(\nabla\psi + J \circ \nabla\psi \circ j)$ plus some compact operator, so it is Fredholm. $\square$

Lemma 3.4 *If $\psi \in X_{\delta_0}(f)$ and $D\overline{\partial}(f)(\psi)$ is $C^{\infty,1}$, then ψ is $C^{\infty,1}$.*

Proof Work in a coordinate chart where we can trivialize (X, J). Using the corresponding flat, J–preserving connection, equation (25) implies that

$$D\bar{\partial}(f)(\psi) = E'(\psi) + \bar{\partial}\psi,$$

where $\bar{\partial}\psi$ is the ordinary $\bar{\partial}$ operator applied to ψ considered as a map to $\mathbb{C}^n$, and E' is a $C^{\infty,\underline{1}}$ tensor vanishing on any edge contained in our coordinate chart.

Restricted to the smooth part of our coordinate chart, E' is a smooth tensor, and $D\bar{\partial}(f)(\psi)$ is smooth, so a standard elliptic bootstrapping argument, such as given in [17], implies that ψ is smooth restricted to the smooth part of our coordinate chart.

It remains to analyze the behavior of ψ near edges. Let $\tilde{z}$ be a holomorphic coordinate on our edge and $\lceil\tilde{z}\rceil = e^{t+i\theta}$. (As any $C^{\infty,\underline{1}}$ coordinate change will just change the norm on X_δ in a bounded way, we will assume that $\tilde{z}$ is one of our originally chosen coordinates.) We already know that $\psi(t,\theta)$ is smooth, and as $\psi \in X_{\delta_0}$, Lemma 2.13 implies that there is some constant c such that

$$\psi_{(t,\theta)} - \lim_{t\to-\infty} \psi(t,\theta) \leq ce^{\delta_0 t}.$$

We must show the same for all weights $\delta < 1$ and all derivatives of ψ. As ψ is smooth and bounded as $t \to -\infty$, and E' is $C^{\infty,\underline{1}}$ and converges to 0 as $t \to -\infty$, we have that $E'(\psi)$ is smooth and converges exponentially to 0 as $t \to -\infty$ with every weight less than 1. It follows that $\bar{\partial}\psi$ is smooth and bounded by $e^{\delta t}$ as $t \to -\infty$ for all weights $\delta < 1$. Below, we shall use that $\bar{\partial}\psi \in Y_\delta$ to show that $\psi \in X_\delta$.

We may assume that our coordinates are valid for all $t \leq 0$, and apply Cauchy's integral formula:

$$\psi(t_0, \theta_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\psi(0,\theta)}{1 - e^{t_0}e^{i(\theta-\theta_0)}}\, d\theta + \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^0 \frac{\bar{\partial}\psi(t,\theta)}{1 - e^{t_0-t}e^{i(\theta-\theta_0)}}\, dt\, d\theta.$$

Therefore,

$$\psi(t_0, \theta_0) - \lim_{t\to-\infty}\psi(t,\theta)$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{t_0}e^{i(\theta-\theta_0)}\psi(0,\theta)}{1 - e^{t_0}e^{i(\theta-\theta_0)}}\, d\theta + \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^0 \frac{e^{t_0-t}e^{i(\theta-\theta_0)}\bar{\partial}\psi(t,\theta)}{1 - e^{t_0-t}e^{i(\theta-\theta_0)}}\, dt\, d\theta.$$

The first integral is bounded by a constant times e^{t_0} as $t_0 \to -\infty$. We may bound the second integral by

$$\|\bar{\partial}\psi\|_\delta \frac{1}{2\pi}\left(\int_0^{2\pi}\int_{-\infty}^0 \left(\frac{e^{t_0-t+\delta t}}{|1 - e^{t_0-t}e^{i(\theta-\theta_0)}|}\right)^q dt\, d\theta\right)^{\frac{1}{q}},$$

where $1/q + 1/p = 1$, and $p > 2$ so $q < 2$. It follows that the integral of

$$e^{q(t_0-t)}|1 - e^{t_0-t}e^{i\theta-\theta_0}|^{-q}$$

for t between $t_0 - 1$ and $t_0 + 1$ is finite, so the above integral between $t_0 - 1$ and $t_0 + 1$ is bounded by some constant times $e^{\delta t_0}$. For $t < t_0 - 1$, we may bound the integrand by $(2e^{\delta t})^q$, and for $t > t_0 + 1$, we may bound the integrand by $(2e^{t_0-t+\delta t})^q$. It follows that the above expression is bounded by a constant times $e^{\delta t_0}$. As this holds for all $\delta < 1$, ψ is in X_δ for all $\delta < 1$. (Equivalently, as we are working over a single curve, ψ is in $X_{0,\delta}$ for all $\delta < 1$.)

Suppose for the purpose of induction that $\psi \in X_{k,\delta}$ for all $\delta < 1$, and that if $\overline{\partial}\psi \in Y_{k,\delta}$ for all $\delta < 1$, then $\psi \in X_{k,\delta}$ for all $\delta < 1$. The fact that $\psi \in X_{k,\delta}$ implies that $E'(\psi) \in Y_{k+1,\delta}$, therefore, $\overline{\partial}\psi \in Y_{k+1,\delta}$. Let v indicate either $\partial/\partial t$ or $\partial/\partial\theta$. We have that $\nabla_v\overline{\partial}\psi = \overline{\partial}\nabla_v\psi$, therefore $\overline{\partial}\nabla_v\psi \in Y_{k,\delta}$ for all $\delta < 1$. Our inductive assumption then implies that $\nabla_v\psi \in X_{k,\delta}$ for all $\delta < 1$, so $\psi \in X_{k+1,\delta}$ for all $\delta < 1$.

By induction, $\psi \in X_{k,\delta}$ for all $\delta < 1$ and k, so $\psi \in C^{\infty,\underline{1}}$. $\qquad\square$

Lemma 3.5 *The index of*

$$D\overline{\partial}(f)\colon X_\delta(f) \to Y_\delta(f) \quad \text{or} \quad D\overline{\partial}(f)\colon X_{\text{mixed }\delta}(f) \to Y_{\text{mixed }\delta}(f)$$

is given by

$$2c_1 - 2n(g + s - 1),$$

where c_1 is the first Chern number of the vectorbundle X or $f^*(T_{\text{vert}}\widehat{\boldsymbol{B}}, J)$ over the domain of f, while g is the total genus, s is the number of marked points where sections of X_δ are required to vanish, and n is the complex rank of X. In particular, the index of $D\overline{\partial}(f)$ is invariant in connected families of curves.

Proof This is just the index of the operator $\psi \mapsto \frac{1}{2}(\nabla\psi + J \circ \nabla\psi \circ j)$. This index equals the sum of the indexes of each smooth component minus $2n$ times the number of internal edges, because a section of $X_*(f)$ is equivalent to a choice of section over each smooth component with the extra requirement that these sections match at internal edges, and a section of $Y_*(f)$ is equivalent to a section over each smooth component, with no extra matching condition because sections of $Y_*(f)$ always vanish on edges. By a smooth component, we mean a connected component of the domain of f minus all its edges. Each such component is a smooth punctured Riemann surface. By the Riemann–Roch formula, the index on each smooth component v is $2c_1 - 2n(g_v + s_v - 1)$, where

c_1 is the first Chern number of the pullback of $(T_{\text{vert}}\widehat{\boldsymbol{B}}, J)$ to that component, g_v is the genus of that component, and s_v is the number of marked points on that component where sections of $X_\delta(f)$ are required to vanish. The sum over all components of c_1 and s_v does not change in connected families. The contribution of internal edges to the index is absorbed into the contribution of genus: $(1-g)$ is $\sum_v(1-g_v)$ minus the number of internal edges. To see this, note that the total genus[18] of an exploded curve $\boldsymbol{C}$ is the genus of the underlying nodal curve $\lceil\boldsymbol{C}\rceil$. This genus is defined so that it is invariant in connected families, and in particular does not change when nodes are smoothed. Topologically, smoothing a node corresponds to replacing a pair of disks around each node with an annulus. The Euler characteristic $2-2g$ of the smoothed curve is therefore the Euler characteristic of $\lceil\boldsymbol{C}\rceil$ minus a pair of disks around each nodal point, or $\sum_v(2-2g_v)$ minus the number of disks removed. In conclusion, the total index is $2c_1 - 2n(g+s-1)$, and therefore invariant in connected families. □

Remark 3.6 (Chern classes) The index formula $2c_1 - 2n(g+s-1)$ involves the total first Chern number c_1 of the pullback $(T_{\text{vert}}\widehat{\boldsymbol{B}}, J)$ to a curve. As in the case of smooth manifolds, Chern classes of $(\widehat{\boldsymbol{B}}, J)$ may be defined as differential forms in $\Omega^*(\widehat{\boldsymbol{B}})$ using the Chern–Weil construction of Chern classes. In particular, choose a smooth Hermitian metric and connection on $(T_{\text{vert}}\widehat{\boldsymbol{B}}, J)$. The curvature of this connection is a smooth two-form on $\widehat{\boldsymbol{B}}$ with values in $\mathbb{C}$–linear endomorphisms of $(T_{\text{vert}}\widehat{\boldsymbol{B}}, J)$. In a local frame, this curvature two-form is defined from some smooth functions simply by exterior differentiation, addition and multiplication, so it must vanish on $\mathbb{R}$–nil vectors. The Chern forms are obtained by taking symmetric polynomials in this curvature two-form. In particular, the first Chern form is obtained by multiplying the trace of this curvature two-form by $i/2\pi$. The proof that these Chern forms are closed real differential forms is entirely local, so following the same construction gives closed real differential forms which vanish on $\mathbb{R}$–nil vectors. Therefore, using the notation of [29], these Chern forms are closed differential forms in $\Omega^*(\widehat{\boldsymbol{B}})$ which represent a Chern class in $H^*(\widehat{\boldsymbol{B}})$. The total Chern number c_1 of a curve is then the integral over the curve of the first Chern form of $(T_{\text{vert}}\widehat{\boldsymbol{B}}, J)$.

Proposition 3.7 *In the case that our family $\widehat{\boldsymbol{C}} \to \boldsymbol{F}$ consists of a single curve, if $v \in X_{1/2}$ has $\overline{\partial}v \in C^{\infty,\underline{1}}$, then $v \in C^{\infty,\underline{1}}$.*

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[18]For exploded curves $\boldsymbol{C}$, we could equivalently define this total genus as $\dim H^1(\boldsymbol{C})/2$, which is invariant in families by Proposition 11.4 of [29], and equal to the genus of $\lceil\boldsymbol{C}\rceil$ when $\lceil\boldsymbol{C}\rceil$ has no nodes by Corollary 4.2 of [29].

Proof Standard elliptic bootstrapping, as explained in [17], gives that locally on smooth components of the domain, v is smooth if $\bar\partial v$ is smooth. It remains to analyze these solutions in a neighborhood of edges in order to verify that they are $C^{\infty,1}$. As a $C^{\infty,1}$ coordinate change will not affect the property of v being in $X_{1/2}$, we may use holomorphic coordinates $\lceil \tilde{z} \rceil = e^{t+i\theta}$ on a neighborhood of a puncture in a smooth component of the domain.

Recall that $\bar\partial$ is in the form $(\star)$, so in a local coordinate chart,

$$\bar\partial v = E(v) + \tfrac{1}{2}(H'(v)(dv) + J \circ H'(v)(dv \circ j)),$$

where H' is a $C^{\infty,1}$ map to the bundle of invertible $\mathbb{R}$–linear endomorphisms of X.

Suppose that $v \in X_{k,\delta_0}$. Then $\|E(v)\|_{k+1,\delta}$ is finite for all $\delta < 1$, and the inverse of $H'(v)(\cdot)$ gives a linear map bounded in the $\|\cdot\|_{k+1,\delta}$ norm. Let $J(v)$ indicate the pullback of J over the map $H'(v)$, and more generally, let $J(\cdot)$ indicate the pullback of J over the map $H'(\cdot)$. Of course, $J(\cdot)$ is $C^{\infty,1}$, so Lemma 2.10 implies that $\|J(v)\|^1_{k,\delta}$ is finite if $v \in X_{k,\delta}$. Applying the inverse of $H'(v)(\cdot)$ to $\bar\partial v - E(v)$ gives the following:

If $v \in X_{k,\delta_0}$, then $\left\| \tfrac{1}{2}(dv + J(v) \circ dv \circ j) \right\|_{k+1,\delta}$ is finite for all $\delta < 1$.

Let S be the stratum of our coordinate chart corresponding to the edge where $t \to -\infty$, and let $\bar\partial_0$ indicate the standard $\bar\partial$ operator corresponding to $e_S(J(v))$. We may rewrite the above displayed statement as follows:

If $v \in X_{k,\delta_0}$, then $\left\| \bar\partial_0 v + \tfrac{1}{2}(\Delta_S J(v)) \circ dv \circ j \right\|_{k+1,\delta}$ is finite for all $\delta < 1$.

Let $L^p_k(D(r))$ indicate the L^p norm with k derivatives on a disk of radius r around a point (t, θ).

Suppose that $v \in X_{1/2}$. Then as $t \to \infty$, we have $\|dv\|_{L^p(D(1))} \le e^{t/2}$, and Lemma 2.13 implies that $|\Delta_S J(v)| \le ce^{t/2}$. Therefore, $\|\bar\partial_0 v\|_\delta$ is finite for all $\delta < 1$. The argument of Lemma 3.4 then implies that $v \in X_\delta$ for all $\delta < 1$.

If $v \in X_{k,\delta}$, the fact that $\|J(v)\|_{L^p_{k+2}D(1)}$ is bounded as $t \to -\infty$ allows us to use the following inequality from Proposition B.4.9 in [17]: there exists some positive constant c such that for t large enough,

$$\|\Delta_S v\|_{L^p_{k+2}(D(\frac{1}{2}))} \le c(\|dv + J(v) \circ dv \circ j\|_{L^p_{k+1}(D(1))} + \|\Delta_S v\|_{L^p_{k+1}(D(1))}).$$

It follows that if $v \in X_{k,\delta}$, then as $t \to -\infty$,

$$\|\Delta_S v\|_{L_{k+2}^p D(\frac{1}{2})} \le e^{t\delta}.$$

Therefore, v is in $X_{k+1,\delta'}$ for all $\delta' < \delta$ if $v \in X_{k,\delta}$. As we have already established that $v \in X_{0,\delta}$ for all $\delta < 1$, it follows that $v \in X_{k,\delta}$ for all k and all $\delta < 1$. So v is $C^{\infty,\underline{1}}$, as required. $\qquad\square$

3.3 A linear gluing theorem

Throughout this section, we consider a linear operator in the form $(\star')$ over a family of curves $\widehat{C} \to F$ with a finite collection of nonintersecting marked-point sections $s_i \colon F \to \widehat{C}$ staying away from the edges of curves in $\widehat{C}$; sections of X will vanish at the image of s_i. Throughout, we shall use V to indicate a finite-rank subvectorbundle $V \subset Y^{\infty,\underline{1}}$ as in Definition 1.1. When we have V over an allowable family of curves, we always assume that V is extendable.

Definition 3.8 Any finite-rank subvectorbundle $V \subset Y^{\infty,\underline{1}}$ over an allowable family of curves defines a closed subspace $V_{k,\delta} \subset Y_{k,\delta}$ consisting of all sections contained in $V(f)$ when restricted to $f \in F$. Define the natural projection

$$\pi_V \colon Y_{k,\delta} \to \frac{Y_{k,\delta}}{V_{k,\delta}} := Y/V_{k,\delta}.$$

Note that $V_{k,\delta}$ is a subset of $Y_{k,\delta}$, not Y, so $Y/V_{k,\delta}$ is only to be interpreted as a short-hand for $Y_{k,\delta}/V_{k,\delta}$, not as the quotient of Y by $V_{k,\delta}$, or sections of a vectorbundle Y/V. Similarly define the restriction of π_V to $Y_\delta(f)$. In this case, the projection π_V has finite-dimensional kernel equal to V restricted to f, so

$$\pi_V \circ D\bar{\partial}(f) \colon X(f)_\delta \to Y/V_\delta(f)$$

is still Fredholm.

We now prove a standard gluing theorem. First, we describe a "gluing" and "cutting" map.

Lemma 3.9 *Given any curve f in F, there exist an allowable collection of coordinate charts on neighborhood U of f and for all $f' \in U$, bounded linear maps*

$$G \colon X_{\delta'}(f) \to X_{\delta'}(f'), \quad G \colon Y_{\delta'}(f) \to Y_{\delta'}(f'), \quad C \colon Y_{\delta'}(f') \to Y_{\delta'}(f)$$

such that the following hold:

(i) *Assign a weight $\delta_i \in \{\delta', \delta + \delta'\}$ to each coordinate chart, but use δ' on every chart not containing an edge of both f and f'.[19] Then the maps*

$$G\colon X_{\mathrm{mixed}\,\delta}(f) \to X_{\mathrm{mixed}\,\delta}(f'), \quad G\colon Y_{\mathrm{mixed}\,\delta}(f) \to Y_{\mathrm{mixed}\,\delta}(f'),$$

$$C\colon Y_{\mathrm{mixed}\,\delta}(f') \to Y_{\mathrm{mixed}\,\delta}(f)$$

are bounded, and the norms of such G and C are bounded uniformly independently of $f' \in U$, so long as the charts using the higher weight contain an edge of f and f'.

(ii) *G is a left inverse to C. In other words, $G \circ C\colon Y_{\delta'}(f') \to Y_{\delta'}(f) \to Y_{\delta'}(f')$ is the identity.*

(iii) *For any $\epsilon > 0$, there exists an open neighborhood of f such that for all f' in this neighborhood, so long as the higher weights are only used where f' has an edge, the following hold:*

    (a) *$\|D\overline{\partial}(f') \circ G\psi - G \circ D\overline{\partial}(f)\psi\|_{\mathrm{mixed}\,\delta} \le \epsilon \|\psi\|^1_{\mathrm{mixed}\,\delta}$.*

    (b) *Given a finite-rank subvectorbundle $V \subset Y^{\infty,\underline{1}}(\widehat{f})$, we may additionally ensure that for all $v \in V(f)$,*

$$\|\pi_V G v\|_{\mathrm{mixed}\,\delta} \le \epsilon \|v\|_{\mathrm{mixed}\,\delta}.$$

Proof Choose an allowable collection of coordinate charts lifting a single coordinate chart U on $\boldsymbol{F}$ so that the largest stratum of $\underline{U}$ contains f. Choose these coordinate charts $U_i \to U$ small enough that we can trivialize both X and Y over them as in Definition 2.16. Note that in each coordinate chart, the trivialization of Y may differ from the trivialization induced on $X \otimes T^*_{\mathrm{vert}}\widehat{\boldsymbol{C}}$, because $Y(\widehat{f}) \subset X \otimes T^*_{\mathrm{vert}}\widehat{\boldsymbol{C}}$ may not be a constant subbundle.

Roughly speaking, G is $(1 - \widetilde{\Delta}_S)$, where S is the stratum of $\underline{U}$ containing f, but we shall describe it more accurately below. Let g denote a section of the relevant vectorbundle, written using the above coordinates. Now define the maps G and C in these coordinate charts $\pi_{\boldsymbol{F}}\colon U_i \to U$ as follows.

If U_i is of type (II) from the definition of allowable coordinate charts in Definition 2.3, U_i is the product of an open subset of $\boldsymbol{T}^1_1$ with U. The coordinate $\widetilde{z}_1$ on $\boldsymbol{T}^1_1$ is a coordinate for both $\pi_{\boldsymbol{F}}^{-1}(f) \cap U_i$ and for $\pi_{\boldsymbol{F}}^{-1}(f') \cap U_i$. In this case, we can define

---

[19]The reason we must be cautious about where the higher weight is used is that $C^{\infty,\underline{1}}$ functions are only uniformly bounded using this higher weight on strata containing edges of curves. This is not a problem if the q in Lemma 2.6 is 1, or if $q(\delta + \delta') < 1$.

$G(\psi)(\tilde{z}_1) = \psi(\tilde{z}_1)$ and $C(\psi)(\tilde{z}_1) = \psi(\tilde{z}_1)$. Note that if ψ vanishes at marked points, then $G(\psi)$ does too, as we chose our coordinates so that the position of marked points is independent of U. (This is a condition of being an allowable collection of coordinate charts.) Our assumption that transition maps for our vectorbundle are independent of U ensures that G and C are compatible with coordinate changes between charts of type (II).

If, on the other hand, U_i is a chart of type (I), then U_i is of the form

$$U_i := \pi_{\boldsymbol{F}}^{-1}(U) \cap \{|\tilde{z}_1| < c, \ |\tilde{z}^\beta| < c\}.$$

Choose some smooth cut-off function $\rho\colon \boldsymbol{T}_1^1 \to \mathbb{R}$ satisfying

$$\rho(\tilde{z}) = \begin{cases} 0 & \text{if } |\tilde{z}| > \tfrac{1}{2}c, \\ 1 & \text{if } |\tilde{z}| \le \tfrac{1}{4}c. \end{cases}$$

Recall that $\tilde{z}_1\tilde{z}^\beta$ is a coordinate function on U, and coordinates for $\pi_{\boldsymbol{F}}^{-1}(f)$ are given by $\tilde{z}_1 \in \boldsymbol{T}_1^1$ and $\tilde{z}^\beta \in \boldsymbol{T}_1^1$ so that $\tilde{z}_1\tilde{z}^\beta = \tilde{z}_1\tilde{z}^\beta(f)$, and we have $|\tilde{z}_1| < c$ and $|\tilde{z}^\beta| < c$. We have assumed that $\underline{\tilde{z}_1\tilde{z}^\beta(f)} > 0$ by assuming that f is in the largest stratum of $\underline{U}$. In our coordinates, ψ is some constant x on the edge of $\pi_{\boldsymbol{F}}^{-1}(f)$, where $\lceil \tilde{z}_1 \rceil = 0 = \lceil \tilde{z}^\beta \rceil$. Define $\psi_+(\tilde{z}_1) := \psi(\tilde{z}_1)$ when $|\tilde{z}_1| > |\tilde{z}^\beta|$, and extend ψ_+ to be x elsewhere. Similarly define $\psi_-(\tilde{z}^\beta) = \psi(\tilde{z}^\beta)$ for $|\tilde{z}^\beta| > |\tilde{z}^\alpha|$, and extend ψ_- to be x everywhere else.

Note that $\pi_{\boldsymbol{F}}^{-1}(f') \cap U_i$ has the same coordinates $\tilde{z}_1$ and $\tilde{z}^\beta$, except $\tilde{z}_1\tilde{z}^\beta = \tilde{z}_1\tilde{z}^\beta(f')$. Define G in these coordinate charts as

$$G(\psi)(\tilde{z}_1, \tilde{z}^\beta) := \rho(\tilde{z}^\beta)(\psi_+(\tilde{z}_1) - x) + \rho(\tilde{z}_1)(\psi_-(\tilde{z}^\beta) - x) + x.$$

Our assumptions about the transition maps between allowable collections of coordinate charts from Definition 2.16 ensure that G is compatible with our transition maps, so we have a globally defined map G. Note also that if ψ is a $C^{\infty,1}$ section of $X(f)$ or $Y(f)$, $G(\psi)$ actually defines a $C^{\infty,1}$ extension of ψ to neighborhood of f.

The norm of G as a map from $X_{\text{mixed }\delta}(f)$ to $X_{\text{mixed }\delta}(f')$ is bounded independently of f' in U. Indeed, $\|\psi_+(\tilde{z}_1)\|_{\text{mixed }\delta}^1 \le \|\psi\|_{\text{mixed }\delta}^1$ because $w_0(\tilde{z}_1)$ is smaller at the fiber over f than the fiber at f', and similarly, $\|\psi_-(\tilde{z}^\beta)\|_{\text{mixed }\delta}^1 \le \|\psi\|_{\text{mixed }\delta}^1$. If our coordinate chart contains edges of both f and f', the cutoff function ρ plays no role in the definition of G, because it is always 1 where it multiplies a nonzero function. Otherwise, we always use the lower weight δ_0 in our coordinate chart, and when we multiply $(\psi_+(\tilde{z}_1) - x)$ by $\rho(\tilde{z}^\beta)$, we increase its $\|\cdot\|_{\delta_0}^1$–norm by a factor of

at most $2\|\rho(\tilde{z}^{\beta})\|^{1}_{\delta_0}$, which is finite because $\rho(\tilde{z}^{\beta})$ is smooth. Therefore, the norm of G as a map from $X_{\text{mixed }\delta}(f)$ to $X_{\text{mixed }\delta}(f')$ is bounded by $1+4\|\rho(\tilde{z}^{\beta})\|^{1}_{\delta_0}+4\|\rho(\tilde{z}_0)\|^{1}_{\delta_0}$.

The argument bounding G as a map from $Y_{\text{mixed }\delta}$ is similar; however, to define $G :=$ $Y_{\text{mixed }\delta}(f) \to Y_{\text{mixed }\delta}(f')$ we used a trivialization of the vectorbundle Y over a coordinate chart which does not necessarily coincide with the trivialization of $T^{*}_{\text{vert}}\hat{C} \otimes X$ used to define the $Y_{\text{mixed }\delta}$ norm. This does not matter as Lemma 2.8 implies that any extendable $C^{\infty,1}$ change of coordinates on our vectorbundle gives an equivalent norm to $Y_{\delta'}$, and restricted to curves in our chart containing an edge, it gives an equivalent norm to $Y_{\delta+\delta'}$. Therefore, $G := Y_{\text{mixed }\delta}(f) \to Y_{\text{mixed }\delta}(f')$ is also uniformly bounded for all f' in U so long as the charts using the higher weight contain an edge of f' and f.

Define the cutting map C in these coordinates by

$$C(\psi)(\tilde{z}_1, \tilde{z}^{\beta}) = \begin{cases} \psi(\tilde{z}_1) & \text{if } |\tilde{z}_1| > \sqrt{|\tilde{z}_1 \tilde{z}^{\beta}(f')|} \text{ and } \lceil \tilde{z}_1 \rceil \neq 0, \\ \psi(\tilde{z}^{\beta}) & \text{if } |\tilde{z}^{\beta}| \geq \sqrt{|\tilde{z}_1 \tilde{z}^{\beta}(f')|} \text{ and } \lceil \tilde{z}^{\beta} \rceil \neq 0, \\ 0 & \text{everywhere else.} \end{cases}$$

Recall that, within coordinate charts of type (I), $|\tilde{z}_1 \tilde{z}^{\beta}| < c^2/16$. Therefore $G \circ C$ is the identity, because the cutoff functions involved are 1 in the relevant regions. Our assumptions on transition maps within allowable collections of coordinate charts ensure that C is defined independently of coordinates. Observe also that if our trivialization of $Y(\hat{f})$ in a coordinate chart coincided with the trivialization used to calculate norms, $C : Y_{\text{mixed }\delta}(f') \to Y_{\text{mixed }\delta}(f)$ would have norm bounded by 2 because $w_0 = |\lceil \tilde{z}_1 \rceil| + |\lceil \tilde{z}^{\beta} \rceil|$, so after accounting for the different trivializations, the norm of C is uniformly bounded independently of f' in U so long as charts using the higher weight contain an edge of f' and f. We have now verified the first three items of our lemma.

Recall that $D\bar{\partial}(f)$ is in the form

$$D\bar{\partial}(f)(\psi)(\tilde{z}_1, \tilde{z}^{\beta}) = E'(\tilde{z}_1, \tilde{z}^{\beta}, f)(\psi(\tilde{z}_1, \tilde{z}^{\beta})) + \tfrac{1}{2}(d_{\text{vert}}\psi + J \circ d_{\text{vert}}\psi \circ j)(\tilde{z}_1, \tilde{z}^{\beta}, f).$$

In the above, we write things as depending on $(\tilde{z}_1, \tilde{z}^{\beta})$ even though these coordinates are related over f. Below, we shall sometimes use only one of these coordinates when we wish to emphasize dependence on that coordinate. In the above expression, E' is some linear map which depends in a $C^{\infty,1}$ way on position in U_i and disappears on the edges of fibers of π_F, and J and j also depend in a $C^{\infty,1}$ way on position in U_i. Let us bound the expression $D\bar{\partial}(f') \circ G\psi - G \circ D\bar{\partial}(f)\psi$ in the interesting case that U_i is a chart of type (I).

This calculation is complicated by the difference between the coordinates in which $D\bar{\partial}f$ is expressed and the coordinates in which the gluing map on $Y_{\text{mixed }\delta}$ is defined. In particular let $A(f, \tilde{z}_1, \tilde{z}^\beta, f)(\cdot)$ indicate the coordinate change, so in our current coordinates,

$$G \colon Y_{\text{mixed }\delta}(f) \to Y_{\text{mixed }\delta}(f')$$

is equal to

$$G(v)(\tilde{z}_1, \tilde{z}^\beta) := A^{-1}(f', \tilde{z}_1, \tilde{z}^\beta)\big(\rho(\tilde{z}^\beta)A(f, \tilde{z}_1)v_+(\tilde{z}_1) + \rho(\tilde{z}_1)A(f, \tilde{z}^\beta)v_-(\tilde{z}^\beta)\big).$$

So,

$$
\begin{aligned}
\text{(26)} \quad D\bar{\partial}(f') &\circ G\psi - G \circ D\bar{\partial}(f)\psi \\
&= \big(\text{id} - A^{-1}(f', \tilde{z}_1)A(f, \tilde{z}_1)\big)\rho(\tilde{z}^\beta)(D\bar{\partial}(f)\psi)_+ \\
&\quad + \big(\text{id} - A^{-1}(f', \tilde{z}^\beta)A(f, \tilde{z}^\beta)\big)\rho(\tilde{z}_1)(D\bar{\partial}(f)\psi)_- \\
&\quad + D\bar{\partial}(f') \circ G(\psi) - \rho(\tilde{z}^\beta)(D\bar{\partial}(f)\psi)_+ - \rho(\tilde{z}_1)(D\bar{\partial}(f)\psi)_-.
\end{aligned}
$$

As $\big(\text{id} - A^{-1}(f', \tilde{z}_1)A(f, \tilde{z}_1)\big)\rho(\tilde{z}^\beta)$ is linear, $C^{\infty, 1}$ and vanishes when $f' = f$, the $\|\cdot\|_{\text{mixed }\delta}$ norm of the right-hand side of the top line can be bounded by $\epsilon\|\psi\|^1_{\text{mixed }\delta}$ by choosing f' close enough to f, and the second line may be bounded similarly. It therefore suffices to bound the last line.

The expression we need to bound is linear in ψ. Consider the case that ψ is a constant x. Then, $G(x) = x$, and so our expression is

$$E'(\tilde{z}_1, \tilde{z}^\beta, f')(x) - \rho(\tilde{z}^\beta)E'(\tilde{z}_1, f)_+(x) - \rho(\tilde{z}_1)E'(\tilde{z}^\beta, f)_-(x).$$

This is $C^{\infty, 1}$ and vanishes on the edges of all curves, and vanishes on f, so its $\|\cdot\|_{\text{mixed }\delta}$ norm may be forced to be as small as we like compared to x by choosing f' close to f and requiring that f' always have an edge in any coordinate using the higher weight.

Using the above, we may reduce to the case that ψ restricted to the edge of f in our coordinate chart is 0, in which case

$$G(\psi) = \rho(\tilde{z}^\beta)\psi_+ + \rho(\tilde{z}_1)\psi_-.$$

Now, the last line of (26) breaks into

$$\text{(27)} \qquad\qquad D\bar{\partial}(f')(\rho(\tilde{z}^\beta)\psi_+) - \rho(\tilde{z}^\beta)(D\bar{\partial}(f)\psi)_+$$

and a similar expression involving $\rho(\tilde{z}_1)$ and ψ_-. To bound the last line of (26), it therefore suffices to bound the above expression (27). Expand (27) as

$$
\begin{aligned}
(28) \quad & \rho(\tilde{z}^\beta)(E'(\tilde{z}_1, f') - E'(\tilde{z}_1, f))(\psi_+(\tilde{z}_1)) \\
& + \tfrac{1}{2}\big(d_{\text{vert}}(\rho(\tilde{z}^\beta)\psi_+(\tilde{z}_1)) + \big(J \circ d_{\text{vert}}(\rho(\tilde{z}^\beta)\psi_+(\tilde{z}_1)) \circ j\big)(\tilde{z}_1, f')\big) \\
& \qquad\qquad - \tfrac{1}{2}\rho(\tilde{z}^\beta)(d_{\text{vert}}\psi_+(\tilde{z}_1) + J \circ d_{\text{vert}}\psi_+(\tilde{z}_1) \circ j)(\tilde{z}_1, f).
\end{aligned}
$$

As $\rho(\tilde{z}^\beta)(E'(\tilde{z}_1, f') - E'(\tilde{z}_1, f))$ is $C^{\infty,\underline{1}}$ and vanishes on f and the edges of curves, the top line of the above expression (28) may have its $\|\cdot\|_{\text{mixed }\delta}$ norm bounded by $\epsilon\|\psi\|^1_{\text{mixed }\delta}$ so long as we choose f' close enough to f and require that f' has an edge contained in the charts on which the higher weight is used. It therefore suffices to bound the last two lines of the expression (28).

We may expand these last two lines of (28) multiplied by 2 as follows:

$$
\begin{aligned}
(29) \quad & d\rho(\tilde{z}^\beta)\psi_+(\tilde{z}_1) + (d\rho(\tilde{z}^\beta) \circ j)(J\psi_+(\tilde{z}_1)) \\
& + \rho(\tilde{z}^\beta)\big(d_{\text{vert}}\psi_+(\tilde{z}_1) + J \circ d_{\text{vert}}\psi_+(\tilde{z}_1) \circ j\big)(\tilde{z}_1, f') \\
& \qquad\qquad - \rho(\tilde{z}^\beta)\big(d_{\text{vert}}\psi_+(\tilde{z}_1) + J \circ d_{\text{vert}}\psi_+(\tilde{z}_1) \circ j\big)(\tilde{z}_1, f).
\end{aligned}
$$

The first line of the above expression may be bounded as follows: Note $|d\rho(\tilde{z}^\beta)|$ is bounded, and is supported on the region $\{\tfrac{1}{4}c < |\tilde{z}^\beta| < \tfrac{1}{2}c\}$. By choosing $|\tilde{z}_1\tilde{z}^\beta(f')|$ small enough by choosing f' close to f, we can ensure that $\tilde{z}_1$ is as small as we like in the above region, thus making $\|\psi_+(\tilde{z}_1)\|_{\text{mixed }\delta}$ here as small as we like compared to $\|\psi\|^1_{\text{mixed }\delta}$ because the weight involved in the calculation the norm here (at f') involves a term $|\lceil\tilde{z}^\beta\rceil| \geq \tfrac{1}{4}c$ and Lemma 2.13 gives us a bound on the size of $\psi_+(\tilde{z}_1)$ by a uniform constant times $|\lceil\tilde{z}_1\rceil|^{\delta'}\|\psi\|^1_{\delta'}$. It follows that we can make the $\|\cdot\|_{\text{mixed }\delta}$ norm of the first line as small as we like compared to $\|\psi\|^1_{\text{mixed }\delta}$.

Therefore, it remains to bound the last two lines of the expression (29). These may be expanded as

$$
\begin{aligned}
& \rho(\tilde{z}^\beta)\big((J(\tilde{z}_1, f') - J(\tilde{z}_1, f)) \circ d_{\text{vert}}\psi_+(\tilde{z}_1) \circ j(\tilde{z}_1, f')\big) \\
& \qquad + \rho(\tilde{z}^\beta)\big(J(\tilde{z}_1, f) \circ d_{\text{vert}}\psi_+(\tilde{z}_1) \circ (j(\tilde{z}_1, f') - j(\tilde{z}_1, f))\big).
\end{aligned}
$$

As $(J(\tilde{z}_1, f') - J(\tilde{z}_1, f))$ and $(j(\tilde{z}_1, f') - j(\tilde{z}_1, f))$ are $C^{\infty,\underline{1}}$ and vanish when $f' = f$, the $\|\cdot\|_{\text{mixed }\delta}$ norm of the above expression may be bounded by $\epsilon\|\psi\|^1_{\text{mixed }\delta}$ by choosing f' close enough to f and requiring that the coordinate charts which use the higher weights contain an edge of f'.

Therefore, by choosing our neighborhood of f small enough we can achieve item (iii)(a) from our lemma on charts of type (I). On the other charts of type (II), proving the same thing is similar, but easier as it involves no cutoff functions and just analysis of how A, E', j and J depend on position.

We now just need to prove item (iii)(b) from our lemma. To see this, consider $C^{\infty,\underline{1}}$ sections $v_1, \ldots, v_n$ of V such that $\{v_i(f)\}$ is a basis for $V(f)$. Then $G(v_i(f)) - v_i$ is $C^{\infty,\underline{1}}$ and vanishes on the domain of f and all strata in $\mathcal{S}_0$. Therefore, we can get $\|G(v_i(f))(f') - v_i(f')\|_{\text{mixed } \delta}$ as small as we like by choosing f' close to f. This in turn bounds $\|\pi_V G(v_i(f))\|_{\text{mixed } \delta}$, and item (iii)(b) follows from the linearity of G:

$$\|\pi_V G(v)\|_{\text{mixed } \delta} \leq \epsilon \|v\|_{\text{mixed } \delta}. \qquad \square$$

Theorem 3.10 *Suppose that for some curve f in $\hat{f}$,*

$$D\overline{\partial}(f)\colon X^{\infty,\underline{1}}(f) \to Y^{\infty,\underline{1}}(f)$$

is injective and has image complementary to $V(f) \subset Y^{\infty,\underline{1}}(f)$ for some finite-rank subvectorbundle $V \subset Y^{\infty,\underline{1}}(\hat{f})$.

Then for any $0 < \delta < 1$, there exists an open neighborhood U of $f \in \mathbf{F}$ covered by an allowable collection of coordinate charts, such that, for all $f' \in U$,

$$\pi_V \circ D\overline{\partial}(f')\colon X_{\delta \& \delta'}(f') \to Y/V_{\delta \& \delta'}(f') \quad \text{and} \quad \pi_V \circ D\overline{\partial}(f')\colon X_{\delta'}(f') \to Y/V_{\delta'}(f')$$

are invertible, and the norm of the inverse is uniformly bounded independently of $f' \in U$.

Proof Recall that $X_{\delta \& \delta'}(f')$, defined using the norm of Definition 2.14, uses the weight $w_0^{-\delta'}$ in coordinate charts not containing an edge of f' and the weight $w_0^{-\delta-\delta'}$ in coordinate charts containing an edge of f'.

Choose an allowable collection of coordinate charts containing f so that Lemma 3.9 holds. Let f' be some curve in the image of this allowable collection of coordinate charts. Assign a weight δ' to any coordinate chart that does not contain an edge of f', and a weight $\delta + \delta'$ to any coordinate chart that contains an edge of f'. Consider the corresponding norm with these mixed weights.

Lemma 3.4 implies that the kernel of $\pi_V \circ D\overline{\partial}(f)$ consists of $C^{\infty,\underline{1}}$ sections, so our assumptions and the fact that $\pi_V \circ D\overline{\partial}(f)\colon X_{\text{mixed } \delta}(f) \to Y/V_{\text{mixed } \delta}(f)$ is Fredholm imply that $\pi_V \circ D\overline{\partial}(f)$ has a bounded inverse.

Use the notation
$$Q := (\pi_V \circ D\overline{\partial}(f))^{-1} \circ \pi_V.$$

Assume that U was chosen small enough that the gluing and cutting maps, G and C from Lemma 3.9, satisfy the following conditions:

(i) $\|D\overline{\partial}(f)\|$, $\|Q\|$, $\|G\|$, and $\|C\|$ are bounded by some M.

(ii) $\|D\overline{\partial}(f') \circ G\psi - G \circ D\overline{\partial}(f)\psi\|_{\text{mixed } \delta} \leq \epsilon \|\psi\|^1_{\text{mixed } \delta}$.

(iii)   For all $v \in V(f)$, $\|\pi_V G v\|_{\text{mixed } \delta} \leq \epsilon \|v\|_{\text{mixed } \delta}$.

By choosing U small enough, we may make the ϵ in the conditions above as small as we like while keeping M the same.

Now consider the map $Q(f'): Y_{\text{mixed } \delta}(f') \to X_{\text{mixed } \delta}(f')$ defined by
$$Q(f') := G \circ Q \circ C.$$

By exchanging $D\overline{\partial}(f') \circ G$ with $G \circ D\overline{\partial}(f)$ using the inequality (ii), and using that $G \circ C$ is the identity, and then using that $D\overline{\partial}(f) \circ Q(Cv) - Cv \in V(f)$ and the inequality (iii), we get

$$(30) \quad \|\pi_V(D\overline{\partial}(f') \circ Q(f')v - v)\|_{\text{mixed } \delta}$$
$$= \|\pi_V(D\overline{\partial}(f') \circ G \circ Q \circ Cv - v)\|_{\text{mixed } \delta}$$
$$\leq \epsilon \|QCv\|^1_{\text{mixed } \delta} + \|\pi_V(G(D\overline{\partial}(f) \circ Q \circ Cv - Cv))\|_{\text{mixed } \delta}$$
$$\leq \epsilon M^2 \|v\|_{\text{mixed } \delta} + \epsilon(M^3 + M)\|v\|_{\text{mixed } \delta}.$$

As $V(f')$ is n–dimensional, there exists a linear inclusion
$$i_V: Y/V_{\text{mixed } \delta}(f') \to Y_{\text{mixed } \delta}(f')$$

with norm bounded by 2^n such that $\pi_V \circ i_V$ is the identity. To see this for the case $n = 1$, use the Hahn–Banach theorem to choose a linear functional L on $Y_{\text{mixed } \delta}(f')$ so that $|Lv| = \|v\|_{\text{mixed } \delta}$ for $v \in V(f)$, and $\|L\| = 1$. Then the obvious inclusion with image $\ker L$ is bounded by 2. Repeating this argument n times gives the n–dimensional case.

In particular, if we choose ϵ small enough, the above inequality (30) tells us that
$$\|\pi_V D\overline{\partial}(f') \circ (Q(f') \circ i_V) - \text{Id}\| < \tfrac{1}{2},$$

so a right inverse to $\pi_V D\overline{\partial}(f')$ is given by
$$(Q(f') \circ i_V)(\pi_V D\overline{\partial}(f') \circ (Q(f') \circ i_V))^{-1},$$

which is bounded by $M 2^{n+1}$.

Lemma 3.5 gives that the index of $D\bar{\partial}$ does not change in connected families, so this right inverse must be a genuine inverse to $\pi_V D\bar{\partial}$. We may repeat this argument for the different mixed norms corresponding to whether edges of f' are in our coordinate charts to obtain a uniform bound for the inverse of $\pi_V \circ D\bar{\partial}(f')\colon X_{\delta\&\delta'}(f') \to Y/V_{\delta\&\delta'}(f')$ for all f' in U.

The argument for $\pi_V \circ D\bar{\partial}(f')\colon X_{\delta'}(f) \to Y/V_{\delta'}(f')$ is identical, except the weight δ' is used in all coordinate charts. $\square$

3.4 Linear regularity results in families

Proposition 3.11 *If $\hat{f}$ is covered by an allowable collection of coordinate charts, and, for all f in $\hat{f}$, the maps*

$$\pi_V \circ D\bar{\partial}(f)\colon X_{\delta'}(f) \to Y/V_{\delta'}(f) \quad and \quad \pi_V \circ D\bar{\partial}(f)\colon X_{\delta\&\delta'}(f) \to Y/V_{\delta\&\delta'}(f)$$

are invertible with a uniformly bounded inverse, then

$$\pi_V \circ D\bar{\partial}\colon X_{0,\delta} \to Y/V_{0,\delta}$$

has a bounded inverse.

Proof We need to bound $\|\phi\|_{0,\delta}^1$ in terms of $\|\pi_V \circ D\bar{\partial}\phi\|_{0,\delta}$. Our assumption that the maps $D\bar{\partial}(f)$ have uniformly bounded inverses implies that there exists some constant c independent of ϕ such that

$$(31) \qquad\qquad\qquad \|\phi\|_{\delta'}^1 \le c\|\pi_V \circ D\bar{\partial}\phi\|_{\delta'},$$

$$(32) \qquad\qquad\qquad \|\phi\|_{\delta\&\delta'}^1 \le c\|\pi_V \circ D\bar{\partial}\phi\|_{\delta\&\delta'}.$$

When there are N charts in our allowable collection of coordinate charts, the constant c is $4N$ times our assumed uniform bound, because

$$\|\phi\|_{\delta'}^1 \le 4N \sup_f \|\phi(f)\|_{\delta'}^1 \quad and \quad \|\pi_V \nu\|_{\delta'} \ge \sup_f \|\pi_V \nu(f)\|_{\delta'},$$

and analogous bounds hold using the $\|\cdot\|_{\delta\&\delta'}$ norm.

We will use the equivalent form of $\|\cdot\|_{0,\delta}^1$ from Lemmas 2.15 and 2.20 because this equivalent form involves only weights w_S with $d_{\mathrm{vert}}w_S = 0$. So, in a local coordinate chart, $D\bar{\partial}w_S\phi = w_S D\bar{\partial}\phi$. If $\pi_V D\bar{\partial}\widetilde{\Delta}_S\phi$ was also equal to $\widetilde{\Delta}_S\pi_V D\bar{\partial}\phi$, then the above inequalities (31) and (32) would be adequate to quickly prove our lemma. The main part of the following proof estimates the extent to which this fails to hold.

Restrict attention to the set of coordinate charts U_i over a single chart U in our allowable collection of coordinate charts. Let $\mathcal{S}$ indicate a set of strata in $\underline{U}$, and $\widetilde{\mathcal{S}}$ indicate the lift of $\mathcal{S}$ to a set of strata in $\widetilde{\underline{U}}_i$. Lemma 2.20 tells us that the norm $\|\phi\|_{0,\delta}^1$ is equivalent to the norm

$$\|\phi\|_{\delta'}^1 + \max_{\mathcal{S}} \|w_{\mathcal{S}}^{-\delta} \widetilde{\Delta}_{\mathcal{S}} \phi\|_{\delta \& \delta'}^1.$$

For the purposes of induction, assume that for any collection of strata I in $\underline{U}$ with fewer than $|\mathcal{S}|$ members,

$$(33) \qquad \|w_I^{-\delta} \widetilde{\Delta}_I \phi\|_{\delta \& \delta'}^1 \leq c \|\pi_V D\overline{\partial}\phi\|_{0,\delta}.$$

The case when I has no members is satisfied because of the inequality (32) and Lemma 2.20. We will now bound $\|w_{\mathcal{S}}^{-\delta} \widetilde{\Delta}_{\mathcal{S}} \phi\|_{\delta \& \delta'}^1$ by bounding $\|D\overline{\partial} w_{\mathcal{S}}^{-\delta} \widetilde{\Delta}_{\mathcal{S}} \phi\|_{\delta \& \delta'}$.

Work in a local coordinate chart where we can use the local form $(\star')$ of $D\overline{\partial}$,

$$D\overline{\partial}\phi = E'(\phi) + H(d_{\mathrm{vert}}\phi).$$

The important facts are that E' is a $C^{\infty,\underline{1}}$ tensor such that $\Delta_{\mathcal{S}_0} E' = E'$, and H is a $C^{\infty,\underline{1}}$ tensor.

For any stratum S,

$$\Delta_S(D\overline{\partial}\phi) = D\overline{\partial}\Delta_S\phi + (\Delta_S E')(e_S\phi) + (\Delta_S H)(e_S \, d_{\mathrm{vert}}\phi).$$

Similarly, for the lift $\widetilde{\mathcal{S}}$ of our set of strata $\mathcal{S}$ in $\underline{U}$ to our coordinate chart,

$$(34) \quad D\overline{\partial}\Delta_{\widetilde{\mathcal{S}}}\phi = \Delta_{\widetilde{\mathcal{S}}}(D\overline{\partial}\phi) - \sum_{\varnothing \neq I \subset \widetilde{\mathcal{S}}} \Delta_I E'(e_I \Delta_{(\widetilde{\mathcal{S}}-I)}\phi) + (\Delta_I H)(e_I \Delta_{(\widetilde{\mathcal{S}}-I)} \, d_{\mathrm{vert}}\phi).$$

We now bound the $\|w_{\mathcal{S}}^{-\delta} \cdot\|_{\delta \& \delta'}$ norm of terms in the above sum. Each of the terms in (34) vanishes on all strata in $\widetilde{\mathcal{S}}$, so if ν indicates one of the above terms, $\Delta_{\widetilde{\mathcal{S}}}\nu = \nu$. For such ν, the following inequality holds:

$$(35) \qquad \|w_{\mathcal{S}}^{-\delta} \Delta_{\widetilde{\mathcal{S}}}\nu\|_{\delta \& \delta'} \leq c \|w_{\widetilde{\mathcal{S}} \cup \mathcal{S}_0}^{-\delta} \Delta_{\widetilde{\mathcal{S}}}\nu\|_{\delta'}.$$

In the above, $\mathcal{S}_0$ indicates the set of strata on which w_0 vanishes. To see why the inequality (35) holds, note that it holds on strata T such that $T^c = \varnothing$ because then the norm on the left-hand side is just the $\|\cdot\|_{\delta'}$ norm. The inequality (35) also holds trivially on strata in $\widetilde{\mathcal{S}}$, because both sides are 0 restricted to strata in $\widetilde{\mathcal{S}}$. It remains to show (35) holds on strata T such that $T \notin \widetilde{\mathcal{S}}$ and $T^c \neq \varnothing$. It suffices to show that

in this case $e_T w_{\widetilde{\mathcal{S}} \cup \mathcal{S}_0}$ is bounded by a constant times $e_T w_{\widetilde{\mathcal{S}}} w_0$, which is inequality (11) from the proof of Lemma 2.15. Therefore the inequality (35) holds.

We shall now bound terms involving H in (34):

$$
\begin{aligned}
\| w_{\mathcal{S}}^{-\delta} (\Delta_I H)(e_I \Delta_{(\widetilde{\mathcal{S}} - I)} d_{\mathrm{vert}} \phi) \|_{\delta \& \delta'} &\leq c \| w_{\widetilde{\mathcal{S}} \cup \mathcal{S}_0}^{-\delta} (\Delta_I H)(e_I \Delta_{(\widetilde{\mathcal{S}} - I)} d_{\mathrm{vert}} \phi) \|_{\delta'} \\
&\leq c' \| (w_I^{-\delta} \Delta_I H)(e_I w_{(\widetilde{\mathcal{S}} \cup \mathcal{S}_0) - I}^{-\delta} \Delta_{(\widetilde{\mathcal{S}} - I)} d_{\mathrm{vert}} \phi) \|_{\delta'} \\
&\leq c'' \| e_I w_{(\widetilde{\mathcal{S}} \cup \mathcal{S}_0) - I}^{-\delta} \Delta_{(\widetilde{\mathcal{S}} - I)} d_{\mathrm{vert}} \phi \|_{\delta'}.
\end{aligned}
$$

We wish to use our inductive hypothesis to estimate this last term, but $\widetilde{\mathcal{S}} - I$ may not be a lifted set of strata. To remedy this, note that

$$
\tag{36} e_I \Delta_{\widetilde{\mathcal{S}} - I} \Delta_{\mathcal{S}_0} = e_I \Delta_{\widetilde{\mathcal{S}}'} \Delta_{\mathcal{S}_0},
$$

where $\widetilde{\mathcal{S}}'$ is the largest lifted collection of strata which is a subset of $\widetilde{\mathcal{S}} - I$. Therefore $e_I \Delta_{\widetilde{\mathcal{S}} - I} d_{\mathrm{vert}} \phi = e_I \Delta_{\widetilde{\mathcal{S}}'} d_{\mathrm{vert}} \phi$. We also have that

$$
\tag{37} e_I w_{(\widetilde{\mathcal{S}} \cup \mathcal{S}_0) - I}^{-\delta} \leq c e_I w_{\widetilde{\mathcal{S}}'}^{-\delta} w_0^{-\delta},
$$

because if ζ is a smooth monomial from $w_{\widetilde{\mathcal{S}}'} w_0$, then $e_I \zeta$ must vanish on I^c, and therefore must vanish on $(\widetilde{\mathcal{S}} \cup \mathcal{S}_0) - I$ and be dominated by $w_{(\widetilde{\mathcal{S}} \cup \mathcal{S}_0) - I}$. Therefore,

$$
\| e_I w_{(\widetilde{\mathcal{S}} \cup \mathcal{S}_0) - I}^{-\delta} \Delta_{(\widetilde{\mathcal{S}} - I)} d_{\mathrm{vert}} \phi \|_{\delta'} \leq c \| e_I w_{\widetilde{\mathcal{S}}'}^{-\delta} \Delta_{\widetilde{\mathcal{S}}'} d_{\mathrm{vert}} \phi \|_{\delta + \delta'}.
$$

If I is not a lifted set of strata, then the right-hand side of the above inequality is bounded by a constant times $\| w_{\widetilde{\mathcal{S}}'}^{-\delta} \Delta_{(\widetilde{\mathcal{S}}')} d_{\mathrm{vert}} \phi \|_{\delta \& \delta'}$. So if I is not a lifted set of strata,

$$
\tag{38} \| w_{\mathcal{S}}^{-\delta} (\Delta_I H)(e_I \Delta_{(\widetilde{\mathcal{S}} - I)} d_{\mathrm{vert}} \phi) \|_{\delta \& \delta'} \leq c \| w_{\widetilde{\mathcal{S}}'}^{-\delta} \Delta_{\widetilde{\mathcal{S}}'} d_{\mathrm{vert}} \phi \|_{\delta \& \delta'}.
$$

On the other hand, if I is a lifted set of strata, then $\widetilde{\mathcal{S}} - I$ is lifted, so $\widetilde{\mathcal{S}}' = \widetilde{\mathcal{S}} - I$, and we can derive the above inequality (38) using Lemma 2.8 as follows:

$$
\begin{aligned}
\| w_{\mathcal{S}}^{-\delta} (\Delta_I H)(e_I \Delta_{(\widetilde{\mathcal{S}} - I)} d_{\mathrm{vert}} \phi) \|_{\delta \& \delta'} &\leq c \| (w_I^{-\delta} \Delta_I H)(e_I w_{\widetilde{\mathcal{S}} - I}^{-\delta} \Delta_{(\widetilde{\mathcal{S}} - I)} d_{\mathrm{vert}} \phi) \|_{\delta \& \delta'} \\
&\leq c' \| e_I w_{\widetilde{\mathcal{S}}'}^{-\delta} \Delta_{\widetilde{\mathcal{S}}'} d_{\mathrm{vert}} \phi \|_{\delta \& \delta'} \\
&\leq c' \| w_{\widetilde{\mathcal{S}}'}^{-\delta} \Delta_{\widetilde{\mathcal{S}}'} d_{\mathrm{vert}} \phi \|_{\delta \& \delta'}.
\end{aligned}
$$

Now, we need to bound the terms involving E' in (34):

$$
\begin{aligned}
\| w_{\widetilde{\mathcal{S}}}^{-\delta} (\Delta_I E')(e_I \Delta_{(\widetilde{\mathcal{S}} - I)} \phi) \|_{\delta \& \delta'} &\leq c \| w_{\widetilde{\mathcal{S}} \cup \mathcal{S}_0}^{-\delta} (\Delta_I E')(e_I \Delta_{(\widetilde{\mathcal{S}} - I)} \phi) \|_{\delta'} \quad \text{(using (35))} \\
&\leq c' \| (w_{I \cup \mathcal{S}_0}^{-\delta} \Delta_{I \cup \mathcal{S}_0} E')(e_I w_{\widetilde{\mathcal{S}} - I}^{-\delta} \Delta_{(\widetilde{\mathcal{S}} - I)} \phi) \|_{\delta'} \\
&\leq c'' \sup | e_I w_{(\widetilde{\mathcal{S}} - I)}^{-\delta} \Delta_{(\widetilde{\mathcal{S}} - I)} \phi |.
\end{aligned}
$$

We must bound $\sup|e_I w_{(\tilde{S}-I)}^{-\delta}\Delta_{(\tilde{S}-I)}\phi|$. If I is a lifted set of strata and $\tilde{S}' = \tilde{S} - I$, then the above inequality immediately gives

$$(39) \qquad \|w_{\tilde{S}}^{-\delta}(\Delta_I E')(e_I \Delta_{(\tilde{S}-I)}\phi)\|_{\delta\&\delta'} \le c\|w_{\tilde{S}'}^{-\delta}\Delta_{\tilde{S}'}\phi\|_{\delta\&\delta'}^1.$$

On the other hand, if I is not a lifted set of strata, there exists some stratum $S \in \tilde{S} - I$ and $\bar{S} \in S^c$ such that $\bar{S} \in I$. Then we can use Lemma 2.13 and the observation that $e_I w_{\tilde{S}-I}^{-\delta}$ is bounded by a constant times $e_I w_{\tilde{S}'}^{-\delta} w_0^{-\delta}$ (which follows from (37)) to get

$$
\begin{aligned}
\sup|e_I w_{(\tilde{S}-I)}^{-\delta}\Delta_{(\tilde{S}-I)}\phi| &\le c\,\sup|e_I w_{\tilde{S}'}^{-\delta} w_0^{-\delta}\Delta_{(\tilde{S}-I)}\phi| \\
&= c\,\sup e_I w_0^{-\delta}|e_{\bar{S}}\Delta_S(e_I w_{\tilde{S}'}^{-\delta}\Delta_{(\tilde{S}-I)}\phi)| \\
&\le c'\|e_{\bar{S}}d_{\mathrm{vert}}(e_I w_{\tilde{S}'}^{-\delta}\Delta_{(\tilde{S}-I)}\phi)\|_{\delta+\delta'} \\
&= c'\|e_I w_{\tilde{S}'}^{-\delta}\Delta_{(\tilde{S}-I)}d_{\mathrm{vert}}\phi\|_{\delta+\delta'} \\
&= c'\|e_I w_{\tilde{S}'}^{-\delta}\Delta_{\tilde{S}'}d_{\mathrm{vert}}\phi\|_{\delta+\delta'} \qquad \text{(using (36))} \\
&\le c''\|w_{\tilde{S}'}^{-\delta}\Delta_{\tilde{S}'}d_{\mathrm{vert}}\phi\|_{\delta\&\delta'} \qquad \text{(as } I^c \ne \varnothing).
\end{aligned}
$$

Therefore, inequality (39) holds for all nonempty sets of strata $I \subset \tilde{S}$. Using the inequalities (38) and (39) along with Equation (34) gives

$$(40) \qquad \|D\bar{\partial}w_{\tilde{S}}^{-\delta}\Delta_{\tilde{S}}\phi - w_{\tilde{S}}^{-\delta}\Delta_{\tilde{S}}D\bar{\partial}\phi\|_{\delta\&\delta'} \le c\sum_{\tilde{S}'\subsetneq\tilde{S}}\|w_{\tilde{S}'}^{-\delta}\Delta_{\tilde{S}'}\phi\|_{\delta\&\delta}^1.$$

If $\tilde{S}$ consists entirely of strata S such that $S^c = \varnothing$, then $\tilde{\Delta}_S = \Delta_{\tilde{S}}$, and we obtain the inequality below. Otherwise, we may apply the estimates of Lemma 2.21 to the above inequality to exchange $\Delta_{\tilde{S}}$ with $\tilde{\Delta}_S$ and get error terms involving $\tilde{\Delta}_{S'}$, where $S' \subsetneq S$, and also apply the estimates of Lemma 2.21 to the right-hand side of (40) to get

$$(41) \qquad \|D\bar{\partial}w_S^{-\delta}\tilde{\Delta}_S\phi - w_{\tilde{S}}^{-\delta}\tilde{\Delta}_S D\bar{\partial}\phi\|_{\delta\&\delta'} \le c\sum_{S'\subsetneq S}\|w_{S'}^{-\delta}\tilde{\Delta}_{S'}\phi\|_{\delta\&\delta}^1.$$

The right-hand side of (41) is bounded by $c\|\pi_V D\bar{\partial}\phi\|_{0,\delta}$ by inductive assumption (33). As (41) applies on all the coordinate charts U_i over U, we can apply π_V to the left-hand term in (41), use the triangle inequality, and rearrange to get

$$(42) \qquad \|\pi_V D\bar{\partial}w_S^{-\delta}\tilde{\Delta}_S\phi\|_{\delta\&\delta'} \le \|\pi_V w_{\tilde{S}}^{-\delta}\tilde{\Delta}_S D\bar{\partial}\phi\|_{\delta\&\delta'} + c\|\pi_V D\bar{\partial}\phi\|_{0,\delta}.$$

We now estimate the right-hand side of this inequality by a constant times $\|\pi_V D\bar{\partial}\phi\|_{0,\delta}$. (This estimation is trivial if V has rank 0, but it will take us over two pages.) After that, the inequality (32) can be applied to complete the inductive argument.

Let v' be any section of V with $\|v'\|_{0,\delta}$ finite, such that

$$(43) \qquad 2\|\pi_V D\bar{\partial}\phi\|_{0,\delta} \geq \|D\bar{\partial}\phi - v'\|_{0,\delta}.$$

Then, the following holds:

$$(44) \quad \|\pi_V w_S^{-\delta}\widetilde{\Delta}_S D\bar{\partial}\phi\|_{\delta\&\delta'} := \inf_v \|w_S^{-\delta}\widetilde{\Delta}_S D\bar{\partial}\phi - v\|_{\delta\&\delta'}$$

$$\leq \|w_S^{-\delta}\widetilde{\Delta}_S(D\bar{\partial}\phi - v')\|_{\delta\&\delta'} + \inf_v \|w_S^{-\delta}\widetilde{\Delta}_S v' - v\|_{\delta\&\delta'}$$

$$\leq c\|\pi_V D\bar{\partial}\phi\|_{0,\delta} + \inf_v \|w_S^{-\delta}\widetilde{\Delta}_S v' - v\|_{\delta\&\delta'}$$

$$= c\|\pi_V D\bar{\partial}\phi\|_{0,\delta} + \|\pi_V w_S^{-\delta}\widetilde{\Delta}_S v'\|_{\delta\&\delta'}.$$

We must show that $\|\pi_V w_S^{-\delta}\widetilde{\Delta}_S v'\|_{\delta\&\delta'}$ can be bounded in terms of $\|\pi_V D\bar{\partial}\phi\|_{0,\delta}$.

In our coordinate chart U, choose a trivialization of V using a basis $\{v_i\}$ of $C^{\infty,\underline{1}}$ sections. We then have

$$v' = \sum_i f_i v_i,$$

where $\{f_i\}$ is some collection of fiberwise-constant real-valued functions. Choose v_i so that for some constant c,

$$(45) \qquad |g_i| \leq c\Big\|\sum_i g_i v_i\Big\|_{\delta'}$$

for any collection of functions g_i on U.

Using the identity $\Delta_S fg = (\Delta_S f)g + (e_S f)\Delta_S g$ repeatedly, we get

$$(46) \qquad \Delta_{\widetilde{S}} v' = \sum_{I \subset \widetilde{S}, i} (\Delta_I e_{(\widetilde{S}-I)} f_i)(\Delta_{(\widetilde{S}-I)} v_i).$$

As f_i is fiberwise-constant, $\Delta_S e_{\bar{S}} f_i = 0$ when $\bar{S} \in S^c$. Therefore, we can rewrite the above expression using only lifted sets of strata:

$$(47) \qquad \Delta_{\widetilde{S}} v' = \sum_{I \subset S, i} (\Delta_{\widetilde{I}} e_{(\widetilde{S}-\widetilde{I})} f_i)(\Delta_{(\widetilde{S}-\widetilde{I})} v_i).$$

We need to estimate $\widetilde{\Delta}_S v'$; this equals $\Delta_{\widetilde{S}} v'$ unless we're in a coordinate chart of type (I) with coordinates including $|\tilde{z}_1| < c$ and $|\tilde{z}^\beta| < c$. Using notation from Definition 2.19, write I^+ for the subset of $\widetilde{I}$ where $\underline{\tilde{z}_1} = \mathfrak{t}^0$, and I^- for the subset where $\underline{\tilde{z}^\beta} = \mathfrak{t}^0$. Then

$$\widetilde{\Delta}_S = \Delta_{\widetilde{S}} - \check{\rho}^+(\Delta_{\widetilde{S}} - \Delta_{S^+}) - \check{\rho}^-(\Delta_{\widetilde{S}} - \Delta_{S^-}),$$

where $\breve{\rho}^+$ and $\breve{\rho}^-$ are some cutoff functions. As f_i is constant on fibers,

$$e_{I+}f = e_{\tilde{I}}f_i = e_I f_i,$$

where in $e_I f_i$, we consider f_i as a function on U. Similarly,

$$\Delta_{\tilde{I}} f_i = \Delta_{I+} f_i = \Delta_{I-} f_i = \Delta_I f_i,$$

so $\Delta_{\tilde{I}} f_i = \tilde{\Delta}_I f$. As every stratum in $\mathcal{S}$ has a unique lift to $\mathcal{S}^+$, (46) implies that

$$(48) \qquad \Delta_{\mathcal{S}^+} v' = \sum_{I \subset \mathcal{S}, i} (\Delta_I e_{(\mathcal{S}-I)} f_i)(\Delta_{(\mathcal{S}-I)+} v_i)$$

and similarly,

$$(49) \qquad \Delta_{\mathcal{S}^-} v' = \sum_{I \subset \mathcal{S}, i} (\Delta_I e_{(\mathcal{S}-I)} f_i)(\Delta_{(\mathcal{S}-I)-} v_i),$$

so using equations (47)–(49) on a coordinate chart of type (I), we get

$$(50) \qquad \tilde{\Delta}_{\mathcal{S}} v' = \sum_{I \subset \mathcal{S}, i} (\Delta_I e_{(\mathcal{S}-I)} f_i)(\tilde{\Delta}_{(\mathcal{S}-I)} v_i).$$

We've proved the above equation (50) for coordinate charts of type (I), and on all other coordinate charts, (50) is equivalent to (47), so (50) is valid for all our coordinate charts over U.

We can apply equation (50) along with inequality (45) to bound $|w_{\mathcal{S}}^{-\delta} \Delta_{\mathcal{S}} f_i|$ as follows:

$$\sup |w_{\mathcal{S}}^{-\delta} \Delta_{\mathcal{S}} f_i| \le c \|\sum_i (w_{\mathcal{S}}^{-\delta} \Delta_{\mathcal{S}} f_i) v_i\|_{\delta'}$$

$$\le c \|w_{\mathcal{S}}^{-\delta} \tilde{\Delta}_{\mathcal{S}} v'\|_{\delta'} + c \sum_{I \subsetneqq \mathcal{S}, j} \|w_{\mathcal{S}}^{-\delta} (\Delta_I e_{(\mathcal{S}-I)} f_j)(\tilde{\Delta}_{(\mathcal{S}-I)} v_j)\|_{\delta'}$$

$$\le c \|w_{\mathcal{S}}^{-\delta} \tilde{\Delta}_{\mathcal{S}} v'\|_{\delta'} + c' \sum_{I \subsetneqq \mathcal{S}, j} \|(w_I^{-\delta} \Delta_I e_{(\mathcal{S}-I)} f_j)(w_{\mathcal{S}-I}^{-\delta} \tilde{\Delta}_{(\mathcal{S}-I)} v_j)\|_{\delta'}$$

$$\le c \|w_{\mathcal{S}}^{-\delta} \tilde{\Delta}_{\mathcal{S}} v'\|_{\delta'} + c'' \sum_{I \subsetneqq \mathcal{S}, j} \sup |w_I^{-\delta} \Delta_I e_{(\mathcal{S}-I)} f_j|.$$

Using the above inequality again on the terms with fewer strata gives

$$(51) \qquad \sup |w_{\mathcal{S}}^{-\delta} \Delta_{\mathcal{S}} f_i| \le c \sum_{I \subseteq \mathcal{S}} \|w_I^{-\delta} \tilde{\Delta}_I v'\|_{\delta'}.$$

As the sections v_i are $C^{\infty,1}$, the norm $\|w_I^{-\delta}\widetilde{\Delta}_I v_i\|_{\delta\&\delta'}$ is bounded. Now bound terms on the right-hand side of equation (50) as follows:

$$
\begin{aligned}
\|\pi_V w_{\mathcal{S}}^{-\delta}(\Delta_I e_{(\mathcal{S}-I)} f_i)(\widetilde{\Delta}_{(\mathcal{S}-I)} v_i)\|_{\delta\&\delta'} & \\
\le c\|\pi_V(w_I^{-\delta}\Delta_I e_{(\mathcal{S}-I)} f_i)(w_{(\mathcal{S}-I)}^{-\delta}\widetilde{\Delta}_{(\mathcal{S}-I)} v_i)\|_{\delta\&\delta'} & \\
\le c\sup|w_I^{-\delta}\Delta_I e_{(\mathcal{S}-I)} f_i|\|\pi_V w_{(\mathcal{S}-I)}^{-\delta}\widetilde{\Delta}_{(\mathcal{S}-I)} v_i\|_{\delta\&\delta'} & \\
\le c'\sum_{I'\subseteq I}\|w_{I'}^{-\delta}\widetilde{\Delta}_{I'} v'\|_{\delta'} & \quad \text{(using (51)).}
\end{aligned}
$$

When considering π_V of equation (50), we can remove the term where $I=\mathcal{S}$, because $(\Delta_{\mathcal{S}} f_i)v_i \in V$. In the remaining terms, f_i is always being acted on by Δ_I where $|I| < |\mathcal{S}|$. Therefore, using (50) and the above inequality,

$$
\begin{aligned}
\|\pi_V w_{\mathcal{S}}^{-\delta} v'\|_{\delta\&\delta'} &\le c\sum_{I\subsetneq\mathcal{S}}\|w_I^{-\delta}\widetilde{\Delta}_I v'\|_{\delta'} \\
&\le c'\left(\sum_{I\subsetneq\mathcal{S}}\|w_I^{-\delta}\widetilde{\Delta}_I D\overline{\partial}\phi\|_{\delta'} + \|\pi_V D\overline{\partial}\phi\|_{0,\delta}\right) \quad \text{(using (43))} \\
&\le c''\left(\sum_{I\subsetneq\mathcal{S}}\|w_I^{-\delta}\widetilde{\Delta}_I \phi\|_{\delta\&\delta'}^1 + \|\pi_V D\overline{\partial}\phi\|_{0,\delta}\right) \quad \text{(using (41))} \\
&\le c'''\|\pi_V D\overline{\partial}\phi\|_{0,\delta} \quad \text{(using (33)).}
\end{aligned}
$$

Using the above inequality with equations (42) and (44), we get

$$
\|\pi_V D\overline{\partial} w_{\mathcal{S}}^{-\delta}\widetilde{\Delta}_{\mathcal{S}}\phi\|_{\delta\&\delta'} \le c\|\pi_V D\overline{\partial}\phi\|_{0,\delta}.
$$

Applying inequality (32) to the left-hand side of the above gives

$$
\|w_{\mathcal{S}}^{-\delta}\widetilde{\Delta}_{\mathcal{S}}\phi\|_{\delta\&\delta'}^1 \le c\|\pi_V D\overline{\partial}\phi\|_{0,\delta}.
$$

This completes the inductive argument that the above expression holds for all sets of strata $\mathcal{S}$ in $\underline{U}$. Therefore, using the equivalent norm from Lemma 2.20, we've proved the required inequality on the set of coordinate charts over U,

$$
(52) \qquad\qquad\qquad \|\phi\|_{0,\delta}^1 \le c\|\pi_V D\overline{\partial}\phi\|_{0,\delta}.
$$

It follows that inequality (52) holds on our entire family, as our entire family is covered by a finite collection of allowable coordinate charts. We shall use the above inequality (52) to see that $\pi_V D\overline{\partial}\colon X_{0,\delta} \to Y/V_{0,\delta}$ has a bounded inverse, determined

by the individual inverses to $\pi_V D\overline{\partial}(f)\colon X_{0,\delta}(f) \to Y/V_{0,\delta}(f)$ by

$$((\pi_V D\overline{\partial})^{-1}(\theta))(f) := (\pi_V D\overline{\partial}(f))^{-1}(\theta(f)).$$

Let us check that $(\pi_V D\overline{\partial})^{-1}(\theta)$ actually is in $X_{0,\delta}$. Approximate θ by a $C^{\infty,\underline{1}}$ section θ'. Then, for each f, [Lemma 3.4](#) implies that $(\pi_V D\overline{\partial}(f))^{-1}(\theta(f))$ is $C^{\infty,\underline{1}}$. We can then choose a $C^{\infty,\underline{1}}$ extension ϕ_f of $(\pi_V D\overline{\partial}(f))^{-1}(\theta(f))$ to a neighborhood of f (using, for example, the map G from [Lemma 3.9](#)). So, $\pi_V D\overline{\partial}\phi_f$ is $C^{\infty,\underline{1}}$, and agrees with θ' over f. This implies that $\|\pi_V D\overline{\partial}\phi_f - \theta'\|_{0,\delta}$ can be made arbitrarily small by restricting to a suitably small neighborhood of f. We can patch together such ϕ_f using a partition of unity (on F) to define a section ϕ' in $X^{\infty,\underline{1}}(\hat{f})$ with $\|\pi_V D\overline{\partial}(\phi') - \theta'\|_{0,\delta}$ as small as we like. There therefore exists a sequence of sections ϕ' in $X^{\infty,\underline{1}}$ with $\pi_V D\overline{\partial}(\phi')$ converging to θ in $Y/V_{0,\delta}$. The bound [(52)](#) then implies that this sequence ϕ' converges in $X_{0,\delta}$, and converges to $(\pi_V D\overline{\partial})^{-1}(\theta)$. $\square$

Proposition 3.12 *If $\pi_V D\overline{\partial}\colon X_{0,\delta} \to Y/V_{0,\delta}$ has a bounded inverse, then*

$$\pi_V D\overline{\partial}\colon X_{k,\delta} \to Y/V_{k,\delta}$$

has a bounded inverse.

Proof This proposition is a tautology for $k = 0$. Assume for induction that the corresponding inequality for the case $k-1$ holds, so

$$\|\phi\|_{k-1,\delta}^1 \le c\|\pi_V D\overline{\partial}\phi\|_{k-1,\delta}.$$

Suppose that w is an extendable $C^{\infty,\underline{1}}$ vectorfield on the total space of our family $\hat{C}$, and that w is the lift of some vectorfield on the base F. In a coordinate chart,

$$
\begin{aligned}
(53) \quad \nabla_w(D\overline{\partial}\phi) &= \nabla_w(E'(\phi) + H(d_{\mathrm{vert}}\phi)) \\
&= D\overline{\partial}(\nabla_w\phi) + (\nabla_w E')(\phi) + (\nabla_w H)(d_{\mathrm{vert}}\phi) + H([d_{\mathrm{vert}}, \nabla_w]\phi),
\end{aligned}
$$

where E' and H are $C^{\infty,\underline{1}}$ tensors such that $\Delta_{S_0} E' = E'$. Note that $[d_{\mathrm{vert}}, \nabla_w]$ is an extendable $C^{\infty,\underline{1}}$ first-order operator that involves derivatives only in the vertical direction, because w is the lift of a vectorfield on F. Therefore, there exists a constant c, depending on w, such that

$$\|D\overline{\partial}\nabla_w\phi - \nabla_w(D\overline{\partial}\phi)\|_{k-1,\delta} \le c\|\phi\|_{k-1,\delta}^1.$$

We have proved the above inequality for a single coordinate chart. As our family is covered by a finite collection of such coordinate charts, the above inequality holds

globally. Therefore, taking π_V of the left-hand side gives the following inequality, where the constant c depends on w:

$$(54) \qquad \|\pi_V D\overline{\partial}\nabla_w\phi\|_{k-1,\delta} \leq \|\pi_V\nabla_w(D\overline{\partial}\phi)\|_{k-1,\delta} + c\|\phi\|_{k-1,\delta}^1.$$

We wish now to bound $\|\pi_V\nabla_w(D\overline{\partial}\phi)\|_{k-1,\delta}$ by $\|\pi_V D\overline{\partial}\phi\|_{k,\delta} + \|\phi\|_{k-1,\delta}^1$ times something depending on w (a particularly easy task if V is a rank-zero vectorbundle). Choose some section v of V with $\|v\|_{k,\delta}$ finite so that

$$(55) \qquad 2\|\pi_V D\overline{\partial}\phi\|_{k,\delta} \geq \|D\overline{\partial}\phi - v\|_{k,\delta}.$$

Therefore there exists some constant c depending on w such that

$$(56) \qquad \begin{aligned} c\|\pi_V D\overline{\partial}\phi\|_{k,\delta} &\geq \|\nabla_w(D\overline{\partial}\phi - v)\|_{k-1,\delta} \\ &\geq \|\pi_V\nabla_w(D\overline{\partial}\phi)\|_{k-1,\delta} - \|\pi_V\nabla_w v\|_{k-1,\delta}. \end{aligned}$$

Work in some finite collection of extendable coordinate patches on $\boldsymbol{F}$ on which V is a trivial vectorbundle, and choose some basis $v_1, \ldots, v_n$ of $C^{\infty,\underline{1}}$ sections of V so that we can write any section of V as $v = \sum f_i v_i$, where $\|f_i\|_{k-1,\delta} \leq \|v\|_{k-1,\delta}$. Note that

$$\pi_V\nabla_w(f_i v_i) = \pi_V f_i \nabla_w v_i.$$

Therefore we can bound the rightmost term in (56) using Lemma 2.8 as follows:

$$(57) \qquad \begin{aligned} \|\pi_V\nabla_w v\|_{k-1,\delta} &\leq \left\|\sum_i f_i\nabla_w v_i\right\|_{k-1,\delta} \\ &\leq c\sum_i \|f_i\|_{k-1,\delta}\|\nabla_w v_i\|_{k-1,\delta}^1 \\ &\leq c'\|v\|_{k-1,\delta} \\ &\leq c'(\|D\overline{\partial}\phi\|_{k-1,\delta} + 2\|\pi_V D\overline{\partial}\phi\|_{k,\delta}) \quad \text{(using (55))} \\ &\leq c''(\|\phi\|_{k-1,\delta}^1 + \|\pi_V D\overline{\partial}\phi\|_{k,\delta}). \end{aligned}$$

Using the inequality (57) for the rightmost term in (56) and rearranging gives

$$\|\pi_V D\overline{\partial}\nabla_w\phi\|_{k-1,\delta} \leq c(\|\pi_V D\overline{\partial}\phi\|_{k,\delta} + \|\phi\|_{k-1,\delta}^1).$$

Then, using our inductive hypothesis on the leftmost and rightmost terms above,

$$\|\nabla_w\phi\|_{k-1,\delta}^1 \leq c\|\pi_V D\overline{\partial}\phi\|_{k,\delta}.$$

In the above inequality, c depends on w, which is any extendable $C^{\infty,\underline{1}}$ vectorfield lifted from a vectorfield on $\boldsymbol{F}$. As we can span all vectorfields on $\widehat{\boldsymbol{C}}$ by real functions

times a finite collection of such w, it follows that

$$(58) \qquad \|\phi\|^1_{k,\delta} \leq c\|\pi_V D\overline{\partial}\phi\|_{k,\delta},$$

where c is independent of ϕ. By induction on k, the bound (58) holds for all k (with different constants for different k).

We still need to prove that the bound (58) implies that the restriction to $Y/V_{k,\delta}$ of our low-regularity inverse is the inverse we want. Suppose that θ is in $Y^{\infty,\underline{1}}$. Let us argue that $(\pi_V D\overline{\partial})^{-1}(\pi_V\theta)$ is in $X^{\infty,\underline{1}}$, by first arguing that all its derivatives exist. For each f, Lemma 3.9 implies that $(\pi_V D\overline{\partial}(f))^{-1}(\pi_V\theta(f))$ is $C^{\infty,\underline{1}}$. Using the map G from Lemma 3.9, we can extend $(\pi_V D\overline{\partial}(f))^{-1}(\pi_V\theta(f))$ to a $C^{\infty,\underline{1}}$ section ϕ_0, so $D\overline{\partial}\phi_0$ plus some $C^{\infty,\underline{1}}$ section v_0 of V agrees with θ over f. For showing that derivatives exist, we can assume without losing generality that $F = \mathbb{R}^n$, with coordinates x_i, and f at 0. We can approximate θ to order k over 0 by

$$D\overline{\partial}\left(\sum_{|\alpha|\leq k}(x^\alpha\phi_\alpha)\right) + \sum x^\alpha v_\alpha = \sum_{|\alpha|\leq k} x^k(D\overline{\partial}\phi_\alpha + v_\alpha),$$

where, for each multi-index α, ϕ_α is a section of $X^{\infty,\underline{1}}$ and v_α is a $C^{\infty,\underline{1}}$ section of V such that over 0,

$$D^\alpha\left(\theta - \sum_{|\alpha'|<|\alpha|} x^k(D\overline{\partial}\phi_{\alpha'} + v_{\alpha'})\right) = |\alpha|!(D\overline{\partial}\phi_\alpha + v_\alpha).$$

Above, D^α indicates the partial derivative corresponding to the multi-index α — this partial derivative is well-defined and independent of the lifts of $\partial/\partial x_i$ or connection, because D^α is acting on something that vanishes to order $(|\alpha|-1)$ over 0. We can make $\|\pi_V(\theta - D\overline{\partial}\sum x^\alpha\phi_\alpha)\|_{k,\delta}$, and therefore $\|(\pi_V D\overline{\partial})^{-1}(\pi_V\theta) - \sum x^\alpha\phi_\alpha\|^1_{k,\delta}$, as small as we like by restricting to a sufficiently small neighborhood of 0. It follows that at 0, the derivatives of $(\pi_V D\overline{\partial})^{-1}(\pi_V\theta)$ exist up to order k. Repeating the argument for different k and at different points gives that all derivatives of $(\pi_V D\overline{\partial})^{-1}(\pi_V\theta)$ exist. The bounds (58) together with Lemma 7.8 of [18] then imply that $(\pi_V D\overline{\partial})^{-1}(\pi_V\theta)$ is $C^{\infty,\underline{\delta}}$. Using Theorem 3.10 and Proposition 3.11, we can locally repeat this argument for any $0 < \delta < 1$, to get that $(\pi_V D\overline{\partial})^{-1}(\pi_V\theta)$ is in fact $C^{\infty,\underline{1}}$.

Now we know that $(\pi_V D\overline{\partial})^{-1}$ sends $C^{\infty,\underline{1}}$ sections to $C^{\infty,\underline{1}}$ sections, it follows that $(\pi_V D\overline{\partial})^{-1}$ is our required inverse. For any $\theta \in Y/V_{k,\delta}$, we can choose a sequence of $C^{\infty,\underline{1}}$ sections θ' converging to θ, then $(\pi_V D\overline{\partial})^{-1}(\theta')$ is a sequence of $C^{\infty,\underline{1}}$ sections converging in $X_{k,\delta}$ to $(\pi_V D\overline{\partial})^{-1}\theta$. $\qquad\square$

Corollary 3.13   *Suppose that $V \subset Y^{\infty,1}$ is a finite-rank subvectorbundle such that for every curve f in our family, $D\overline{\partial}(f): X^{\infty,1}(f) \to Y^{\infty,1}(f)$ is injective and complementary to $V(f)$. Then, for each $\theta \in Y^{\infty,1}$, there exists a unique $v \in X^{\infty,1}$ and $C^{\infty,1}$ section v of V such that*

$$D\overline{\partial}v = \theta + v.$$

Proof   Lemmas 3.3 and 3.4 imply that $\pi_V \circ D\overline{\partial}(f): X_\delta \to Y/V_\delta$ is invertible for all f, so there is a unique $v(f) \in X_\delta(f)$ and $v(f) \in V(f)$ such that $D\overline{\partial}v(f) = \theta(f) + v(f)$. Lemma 3.4 implies that $v(f)$ is $C^{\infty,1}$, and it remains to verify that the corresponding global section v of X is $C^{\infty,1}$; this may be done locally. Theorem 3.10 and Propositions 3.11 and 3.12 imply that locally, $\pi_V D\overline{\partial}: X_{k,\delta} \to Y/V_{k,\delta}$ has a bounded inverse for all $k \in \mathbb{N}$ and $\delta < 1$. Because $v(f)$ is the unique solution to $\pi_V D\overline{\partial}(f)v(f) = \pi_V(\theta(f))$, the inverse image of $\pi_V(\theta)$ must coincide with v, so v is locally in $X_{k,\delta}$ for all $k \in \mathbb{N}$ and $\delta < 1$. Therefore v is locally $C^{\infty,1}$. $\square$

We can now prove Theorem 1.2.

Proof of Theorem 1.2   First, $D\overline{\partial}(f): X^{\infty,1}(f) \to Y^{\infty,1}(f)$ has closed image, finite-dimensional kernel and cokernel, and the required index. This follows from Lemmas 3.3, 3.4 and 3.5. It remains to show that, if $D\overline{\partial}(f)$ is transverse to $V(f) \subset Y^{\infty,1}(f)$, there exists a neighborhood $\boldsymbol{F}'$ of $f \subset \boldsymbol{F}$ such that $D\overline{\partial}$ surjects onto $Y^{\infty,1}(\boldsymbol{F}')/V$ and $D\overline{\partial}^{-1}(V)$ restricted to $X^{\infty,1}(\boldsymbol{F}')$ is a finite-rank subvectorbundle in the sense of Definition 1.1.

Choose sections $\{s_i\}$ of $\widehat{C} \to \boldsymbol{F}$ so that $D\overline{\partial}(f)^{-1}(V(f)) = 0$ when the domain of $D\overline{\partial}$ is restricted to the subspace of $X^{\infty,1}$ vanishing at the image of all s_i. On some neighborhood $\boldsymbol{F}'$ of $f \in \boldsymbol{F}$, extend V to a finite-rank subvectorbundle $V' \subset Y^{\infty,1}(\boldsymbol{F}')$ so that $V'(f)$ is complementary to the image of (the restricted) $D\overline{\partial}(f)$. Then Proposition 3.11 implies that, by shrinking our neighborhood $\boldsymbol{F}'$, we may assume that the same holds for all f' in $\boldsymbol{F}'$.

Choose a finite-rank subvectorbundle $K' \subset X^{\infty,1}(\boldsymbol{F}')$ that is a complement to the subspace of sections vanishing at s_i. We can parametrize K' by the value of its sections at s_i. Corollary 3.13 allows us to parametrize $D\overline{\partial}^{-1}(V)$ by K' as follows: for any section v of K', Corollary 3.13 implies that there exists a unique section $v_0 \in X^{\infty,1}(\boldsymbol{F}')$ that vanishes at the image of all s_i so that $D\overline{\partial}v = \overline{\partial}v_0 \bmod V'$. We may modify K' to K'' by replacing sections v of K' with sections $v - v_0$ of K''. Such

sections are still parametrized by their values at s_i, and the map $\nu \mapsto \nu - \nu_0$ is linear with respect to multiplication by $C^{\infty,\underline{1}}(F',\mathbb{R})$, so K'' is again a finite-rank subvectorbundle of $X^{\infty,\underline{1}}(\hat{f}')$. The uniqueness part of Corollary 3.13 implies that $K'' = D\overline{\partial}^{-1}V'$.

Now $D\overline{\partial}$ restricted to K'' is a $C^{\infty,\underline{1}}$ map of vectorbundles $K'' \to V'$. By assumption, this map is transverse to $V \subset V'$ at f, therefore by shrinking F', we may assume that it is transverse to V everywhere in F', so $D\overline{\partial}^{-1}(V)$ is a finite-rank $C^{\infty,\underline{1}}$ subvector-bundle of K''. Corollary 3.13 has already told us that $D\overline{\partial}$ surjects onto $Y^{\infty,\underline{1}}(F')/V'$, and we have now assumed that $D\overline{\partial}$ restricted to K'' surjects onto V'/V. Therefore, $D\overline{\partial}$ surjects onto $Y^{\infty,\underline{1}}(F')/V$. This completes the proof of Theorem 1.2. $\qquad\square$

3.5 Regularity for the $\overline{\partial}$ equation on variations of a family

Theorem 3.14 *Suppose that $V \subset Y^{\infty,\underline{1}}(F)$ is a finite-rank subvectorbundle (see Definition 1.1) and at $f \in F$, we have that $\overline{\partial}f \in V(f)$ and*

$$D\overline{\partial}(f): X^{\infty,\underline{1}}(f) \to Y^{\infty,\underline{1}}(f)$$

is injective and has image complementary to $V(f)$. Then there exists a neighborhood F' of $f \in F$, and a neighborhood O of 0 in $X_{0,\delta}(F')$, such that:

(i) *Given any $f' \in F'$ and $\nu \in O$,*

$$\pi_V D\overline{\partial}(\nu(f')): X_{\delta'}(f') \to Y/V_{\delta'}(f')$$

   is invertible.

(ii) *The map*

$$\pi_V\overline{\partial}: O \to Y/V_{0,\delta}$$

   is a homeomorphism onto a neighborhood of $0 \in Y/V_{0,\delta}$.

(iii) *Given any $f' \in F'$, let $O(f')$ be the restriction of O to f'. Then the map*

$$\pi_V\overline{\partial}(f'): O(f') \to Y/V_{0,\delta}(f')$$

   is also a homeomorphism onto a neighborhood of $0 \in Y/V_{0,\delta}(f')$.

(iv) *For any $\nu \in O$, if $\pi_V\overline{\partial}\nu$ has regularity $C^{\infty,\underline{1}}$ then ν has regularity $C^{\infty,\underline{1}}$. In particular, there is a unique $C^{\infty,\underline{1}}$ solution to the equation $\pi_V\overline{\partial}\nu = 0$ over F'.*

Proof Apply Theorem 3.10 and Proposition 3.11 to see that there exists an allowable collection of coordinate charts covering an open neighborhood of f such that, in these coordinates, there is a bounded inverse to $\pi_V \circ D\overline{\partial}: X_{0,\delta} \to Y/V_{0,\delta}$.

Theorem 3.2 implies that $\pi_V \circ \bar{\partial}\colon X_{0,\delta} \to Y/V_{0,\delta}$ is C^1, so we may choose a neighborhood O of $0 \in X_{0,\delta}$ so that for $v \in O$,

$$(59) \qquad \|(\pi_V \circ D\bar{\partial})^{-1} \circ \pi_V \circ D\bar{\partial}(v)(\cdot) - \mathrm{id}\| < \tfrac{1}{2}.$$

Therefore, $\pi_V \circ \bar{\partial}$ is a homeomorphism from O to an open subset of $Y/V_{0,\delta}$. As $\pi_V \bar{\partial}$ of the zero-section is a $C^{\infty,\underline{1}}$ section which vanishes at f, it follows that $\pi_V \bar{\partial}(O)$ contains some section that vanishes on a neighborhood of f, so, by restricting to a small enough neighborhood of f, the image under $\pi_V \bar{\partial}$ of O will contain $0 \in Y/V_{0,\delta}$. We have therefore proved item (ii) of our theorem. As the construction may be restricted to any curve f', item (iii) also holds, and restricting our estimate on the inverse of $\pi_V D\bar{\partial}$ to $v(f')$ gives item (i).

It remains to prove item (iv). We must show that if $v \in O$ and $\pi_V \bar{\partial}v = \theta$ is $C^{\infty,\underline{1}}$, then v is $C^{\infty,\underline{1}}$. Do this locally around any curve g in F' as follows: First, $\pi_V \bar{\partial}v(g)$ is $C^{\infty,\underline{1}}$, so $\bar{\partial}v(g) \in C^{\infty,\underline{1}}$ and Proposition 3.7 implies that $v(g)$ is $C^{\infty,\underline{1}}$. Choose any $C^{\infty,\underline{1}}$ extension of $v(g)$ to an open neighborhood of g (such an extension can be constructed using the map G from Lemma 3.9) and call this extension v_0.

Consider the following Newton iteration scheme:

$$v_{n+1} := v_n - (\pi_V D\bar{\partial}(v_n))^{-1}(\pi_V \bar{\partial}v_n - \theta).$$

On a sufficiently small neighborhood U of g, the estimate (59) implies $(\pi_V D\bar{\partial}(v_0))^{-1}$ exists and is a bounded operator. Moreover, by choosing U small, we may ensure that $\|\pi_V \bar{\partial}v_0 - \theta\|_{0,\delta}$ is as small as we like. Therefore, as $D\bar{\partial}\colon X_{0,\delta} \to Y_{0,\delta}$ is C^1, for a sufficiently small U the above Newton iteration scheme will converge in $X_{0,\delta}(U)$ and remain in (the restriction to U of) our chosen neighborhood O.

We shall now prove that if v_n is $C^{\infty,\underline{1}}$, then v_{n+1} is $C^{\infty,\underline{1}}$ as well. As the map $\pi_V D\bar{\partial}(v_n)\colon X_{0,\delta} \to Y/V_{0,\delta}$ is invertible, it follows that for all f' in F', also the map $D\bar{\partial}(v_n(f'))\colon X_\delta(f') \to Y/V_\delta(f')$ is invertible. Hence $D\bar{\partial}(v_n)(f')\colon X^{\infty,\underline{1}}(f') \to Y^{\infty,\underline{1}}(f')$ is injective and complementary to $V(f)$. Corollary 3.13 then implies that v_{n+1} is $C^{\infty,\underline{1}}$, so by induction v_n is $C^{\infty,\underline{1}}$ for all n.

Now we shall see that this Newton iteration scheme will converge in $X_{k,\delta}$ for all k and all $\delta < 1$ when restricted to a small enough open neighborhood of g (dependent on k and δ).

Claim *The size of $\|\pi_V \bar{\partial}v_n - \theta\|_{n,\delta}$ can be made arbitrarily small by restricting to a suitably small open neighborhood of the curve over g. More precisely, $\|\pi_V \bar{\partial}v_n - \theta\|_{n,\delta}$*

can be made arbitrarily small by restricting to a small enough neighborhood of the curve over g while using the same metric and coordinate chart choices in the definition of $\|\cdot\|_{n,\delta}$.

Consider $\bar{\partial} v_n$ as a section of the bundle Y. As v_n and θ are $C^{\infty,\underline{1}}$, the above claim is equivalent to $\bar{\partial} v_n$ being tangent to order n over the domain of g to some $C^{\infty,\underline{1}}$ section θ' with $\pi_V \theta' = \theta$.

We shall prove the above claim by induction: it holds for $n = 0$, now assume that it holds for some n. Therefore, $\|v_{n+1} - v_n\|_{n,\delta}^1$ can be made arbitrarily small by restricting to a suitably small open neighborhood of the domain of g; ie v_n and v_{n+1} are tangent to order n on the domain of g. Define an operator (D^{n+1}) as follows: The domain of (D^{n+1}) consists of $C^{\infty,\underline{1}}$ sections of Y tangent to $\bar{\partial} v_n$ to order n over g. Define $(D^{n+1})\theta$ to be the derivative to order $n + 1$ of the section θ, restricted to the domain of g. With this domain, (D^{n+1}) is an affine operator, so $(D^{n+1})(\lambda\theta_1 + (1-\lambda)\theta_2) = \lambda(D^{n+1})(\theta_1) + (1-\lambda)(D^{n+1})(\theta_2)$ for any constant λ, and θ_i in the domain of (D^{n+1}). The operator $(D^{n+1})\bar{\partial}$ is also an affine operator when restricted to $v \in C^{\infty,\underline{1}}$, so that v is tangent to v_n to order n over the domain of g. Therefore, for such v, using that $(D^{n+1})\bar{\partial}$ is affine gives

$$(D^{n+1})\bar{\partial}(v_n + \lambda(v - v_n)) = (D^{n+1})\bar{\partial} v_n + \lambda((D^{n+1})\bar{\partial} v - (D^{n+1})\bar{\partial} v_n).$$

Then, as (D^{n+1}) is affine, it commutes with taking first-order approximations, so taking (D^{n+1}) of the approximation of $\bar{\partial}(v_n + \lambda(v - v_n))$ at $\lambda = 0$ gives

$$(D^{n+1})(\bar{\partial} v_n + \lambda D\bar{\partial}(v_n)(v - v_n)) = (D^{n+1})\bar{\partial} v_n + \lambda((D^{n+1})\bar{\partial} v - (D^{n+1})\bar{\partial} v_n).$$

Setting $\lambda = 1$ in the above expression gives

$$(D^{n+1})\bar{\partial} v = (D^{n+1})(\bar{\partial} v_n + D\bar{\partial}(v_n)(v - v_n)),$$

so

$$\begin{aligned}
(D^{n+1})\bar{\partial} v_{n+1} &= (D^{n+1})(\bar{\partial} v_n + D\bar{\partial}(v_n)(v_{n+1} - v_n)) \\
&= (D^{n+1})(\bar{\partial} v_n - D\bar{\partial}(v_n)(\pi_V D\bar{\partial}(v_n))^{-1}(\pi_V \bar{\partial} v_n - \theta)) \\
&= (D^{n+1})(\theta'),
\end{aligned}$$

where θ' is a $C^{\infty,\underline{1}}$ section of Y such that

$$\pi_V \theta' = \pi_V \bar{\partial} v_n - \pi_V \bar{\partial} v_n + \theta = \theta.$$

In other words, $\bar{\partial} v_{n+1}$ will be tangent to order $n + 1$ over the domain of g to some θ' such that $\pi_V \theta' = \theta$. By induction, this is true for all n, and our claim has been proven.

Apply Theorem 3.10, Propositions 3.11 and 3.12, Theorem 3.2 and the above claim to infer that for any $\delta < 1$, restricted to a small enough neighborhood of the curve over g, the above Newton iteration scheme converges in $X_{k,\delta}$ to our solution ν to $\pi_V \overline{\partial} \nu = \theta$. This implies that, for all k and $\delta < 1$, there exists some open neighborhood of g such that our solution ν is in $C^{k,\delta}$ restricted to this neighborhood. Repeating the argument around any point gives that ν is $C^{\infty,\underline{1}}$. $\qquad\Box$

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