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Quasi-asymptotically conical Calabi–Yau manifolds

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We construct new examples of quasi-asymptotically conical (QAC) Calabi–Yau manifolds that are not quasi-asymptotically locally Euclidean (QALE). We do so by first providing a natural compactification of QAC–spaces by manifolds with fibered corners and by giving a definition of QAC–metrics in terms of an associated Lie algebra of smooth vector fields on this compactification. Thanks to this compactification and the Fredholm theory for elliptic operators on QAC–spaces developed by the second author and Mazzeo, we can in many instances obtain Kähler QAC–metrics having Ricci potential decaying sufficiently fast at infinity. This allows us to obtain QAC Calabi–Yau metrics in the Kähler classes of these metrics by solving a corresponding complex Monge–Ampère equation.

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Introduction

A complete Kähler manifold (X, g, J) of complex dimension m is Calabi–Yau if it is Ricci-flat and has a nowhere vanishing parallel holomorphic volume form $\Omega_X \in H^0(X; K_X)$. In this case, the holonomy of (X, g) is contained in $\mathrm{SU}(m)$ and we say that g is a Calabi–Yau metric. Since the resolution of the Calabi conjecture by Yau [43], we know that a compact Kähler manifold admits a Calabi–Yau metric if and only if its canonical line bundle is trivial, in which case every Kähler class contains a unique Calabi–Yau metric obtained by solving a complex Monge–Ampère equation. As subsequently shown by Tian and Yau in [41; 42], on noncompact complete Kähler manifolds, one can also obtain many examples of complete Calabi–Yau metrics by solving a corresponding complex Monge–Ampère equation, but the triviality of the canonical line bundle is not the only required condition. One also needs to take into account the asymptotic behavior of the metric at infinity.

For example, if $\Gamma \subset \mathrm{SU}(m)$ is a finite subgroup acting freely on $\mathbb{C}^m \setminus \{0\}$ and if $X \to \mathbb{C}^m/\Gamma$ is a crepant resolution, then, as pointed out by Joyce [25], the results of Tian and Yau or of Bando and Kobayashi [6; 7], combined with the results of Bando, Kasue and Nakajima [5], allow one to construct examples of Calabi–Yau metrics on X that are asymptotically locally Euclidean (ALE for short). In fact, Joyce, in [25; 24], gave a more direct and self-contained proof of the existence of these metrics by using a Moser iteration with weights which yields better control of the solution of the complex Monge–Ampère equation at infinity. His approach was subsequently generalized by various authors to obtain examples of asymptotically conical (AC for short) Calabi–Yau manifolds; see van Coevering [11; 12], Conlon and Hein [13; 14], and Goto [21].

In [24; 26], Joyce generalized this approach in another direction by considering a crepant resolution $\pi\colon X \to \mathbb{C}^m/\Gamma$ with $\Gamma \subset \mathrm{SU}(m)$ not acting freely on $\mathbb{C}^m \setminus \{0\}$, that is, with $\mathbb{C}^m/\Gamma$ having nonisolated singularities going off to infinity. For that purpose, Joyce introduced the notion of quasi-asymptotically locally Euclidean metrics (QALE–metrics for short). As the name suggests, away from the singularities, these metrics resemble ALE–metrics. However, near rays of singularities going off to infinity, the crepant resolution introduces some topology and we can no longer use the Euclidean metric as a local model. More precisely, if $p \in (\mathbb{C}^m/\Gamma) \setminus \{0\}$ is a singular point, then there is a neighborhood V of p of the form

$$V = \{(z, \lambda) \in \mathbb{C}^{m-1}/\Gamma_p \times \mathbb{C} : |\lambda| > \delta,\ \theta < \arg\lambda < \theta + \delta,\ |z| < \delta|\lambda|\} \subset \mathbb{C}^{m-1}/\Gamma_p \times \mathbb{C},$$

with p corresponding to the point $(0,\zeta) \in \mathbb{C}^{m-1}/\Gamma_p \times \mathbb{C}$ for some $\zeta \in \mathbb{C}^*$, where $\Gamma_p \subset \mathrm{SU}(m-1)$ is the stabilizer of this point in Γ, $\delta > 0$ and $\theta \in \mathbb{R}$. Now, the type of crepant resolutions $\pi\colon X \to \mathbb{C}^m/\Gamma$ that Joyce considers are local product resolutions in the sense that $\pi^{-1}(V)$ corresponds to a subset of

$$Y_p \times \mathbb{C},$$

with Y_p a (local product) crepant resolution of $\mathbb{C}^{m-1}/\Gamma_p$. Suppose now for simplicity that Γ_p acts freely on $\mathbb{C}^{m-1}\setminus\{0\}$, which is automatic if $m=3$. Then, in this case, an example of a QALE–metric on $\pi^{-1}(V)$ is given by the restriction to $\pi^{-1}(V)$ of the Cartesian product of an ALE–metric g_{Y_p} on Y_p and the Euclidean metric g_E on $\mathbb{C}$, that is,

$$g_{\mathrm{QALE}} = g_{Y_p} + g_E. \tag{1}$$

More generally, if Γ_p does not act freely on $\mathbb{C}^{m-1}$, we can assume inductively that QALE–metrics have been defined in dimension $m-1$, so that one can still use (1) as a model of a QALE–metric on V, this time however with g_{Y_p} a QALE–metric on Y_p instead of an ALE–metric. Using local models as in (1), one can then define the quasi-isometric class of QALE–metrics as the class of complete metrics which, outside a compact set, are quasi-isometric to an ALE–metric away from the singularities and quasi-isometric to the model (1) in the neighborhood $\pi^{-1}(V)$ near a given singular point $p \in \mathbb{C}^m/\Gamma$. However, to solve the complex Monge–Ampère equation and construct Calabi–Yau QALE–metrics, it is important to have some control on the derivatives of the metric. For this reason, in his definition of QALE–metrics, Joyce also imposes some control on the asymptotic behavior of the derivatives of a QALE–metric with respect to the local model (1). With these extra assumptions, Joyce [24, Theorem 9.3.3] proves the following theorem.

Theorem (Joyce [24]) *Let Γ be a finite subgroup of $\mathrm{SU}(m)$ and X a crepant resolution of $\mathbb{C}^m/\Gamma$. Then each Kähler class of QALE–metrics on X contains a unique Kähler Ricci-flat QALE–metric.*

Since the complex Monge–Ampère equation is used to obtain these Calabi–Yau metrics, the form of these metrics is not explicit, but Joyce in [24, Section 9.3] expressed the hope that QALE–metrics with holonomy $\mathrm{Sp}(m)$ should also admit an explicit construction using hyper-Kähler quotients. A first example in this direction was obtained by Carron [9], who showed that the Nakajima metric, constructed by Nakajima [38] via hyper-Kähler quotients on the Hilbert scheme of n points on $\mathbb{C}^2$, is a QALE–metric in the sense of Joyce.

Besides the Moser iteration with weights that generalizes almost immediately to QALE–metrics, one of the key ingredients in the proof of [24, Theorem 9.3.3] is the bijectivity of the Laplacian of a QALE–metric when acting on some suitable weighted Hölder space, a result that Joyce [24, Section 9] obtained using the maximum principle and barrier functions. Joyce also made a more general conjecture [24, Conjecture 9.5.16] on the mapping properties of the Laplacian of a QALE–metric. This conjecture has recently been proven by the second author and Mazzeo [17] by obtaining heat kernel estimates via the methods of Grigor'yan and Saloff-Coste [23]. In fact, in [17], the second author and Mazzeo introduce a much wider class of Riemannian metrics for which their results hold, namely the class of quasi-asymptotically conical metrics (QAC–metrics for short) which generalizes the class of QALE–metrics in the same way that the class of AC–metrics generalizes the class of ALE–metrics. For example, a Cartesian product of ALE–metrics is a QALE–metric, and likewise, a Cartesian product of AC–metrics is a QAC–metric (see Example 1.22).

The goal of the present paper is to extend Joyce's program [24, Section 9] to the wider setting of QAC–metrics and exhibit new examples of Calabi–Yau QAC–metrics, in particular, examples of Calabi–Yau QAC–metrics that are neither QALE–metrics nor Cartesian products of AC–metrics. To achieve this, one of the key ingredients is to introduce a natural compactification of QAC–manifolds by manifolds with fibered corners in the sense of Albin, Leichtnam, Mazzeo and Piazza [1] and Debord, Lescure and Rochon [16]. On the one hand, as in several works of Melrose and collaborators for other types of geometries, this allows one to give a simple description of QAC–metrics in terms of a natural Lie algebra of vector fields on the compactification; see Epstein, Mazzeo, Melrose and Mendoza [34; 35; 31; 18; 30; 32]. More importantly, when it comes to solving the complex Monge–Ampère equation, it allows us to solve iteratively the equation on each face of the compactification, which in turn allows us to reduce the equation to a situation where the Ricci potential decays fast enough at infinity so that the methods of Tian and Yau [41; 42] can be applied.

Remark For the Nakajima metric on the Hilbert scheme of n points on $\mathbb{C}^2$, such a compactification has been independently obtained by Melrose [37] using the hyper-Kähler quotient construction of the metric.

Postponing to Section 1 a detailed description of the compactification, let us begin by explaining how to construct it for a QALE–metric on a crepant resolution X of $\mathbb{C}^m/\Gamma$ in the simple case where the singularities going off to infinity are all of complex codimension $k \geq 2$. In this situation, we first radially compactify $\mathbb{C}^m/\Gamma$ to an orbifold with boundary $\overline{\mathbb{C}^m/\Gamma}$ by adding a boundary $\partial\overline{\mathbb{C}^m/\Gamma} \cong \mathbb{S}^{2m-1}/\Gamma$ at infinity as in [35, Section 1.8]. The boundary itself is then an orbifold, but by our simplifying assumption, its singularities correspond to a disjoint union $S = \bigcup S_i$ of singular edges $S_1, \dots, S_\ell$. A first step in constructing the compactification is to blow up in the sense of Melrose [33; 36] each singular edge of the boundary within $\overline{\mathbb{C}^m/\Gamma}$, that is,

$$\tilde{X}_{\mathrm{sc}} := [\overline{\mathbb{C}^m/\Gamma}; S] = [\overline{\mathbb{C}^m/\Gamma}; S_1, \dots, S_\ell], \tag{2}$$

with blow-down map

$$\beta \colon \tilde{X}_{\mathrm{sc}} \to \overline{\mathbb{C}^m/\Gamma}.$$

As illustrated in Figure 1, this yields an orbifold with corners with one boundary hypersurface $H_i := \beta^{-1}(S_i)$ for each singular edge S_i and one boundary hypersurface $H_{\ell+1} = \overline{\beta^{-1}(\partial\overline{\mathbb{C}^m/\Gamma}) \setminus S}$ corresponding to the lift of the boundary of $\overline{\mathbb{C}^m/\Gamma}$.

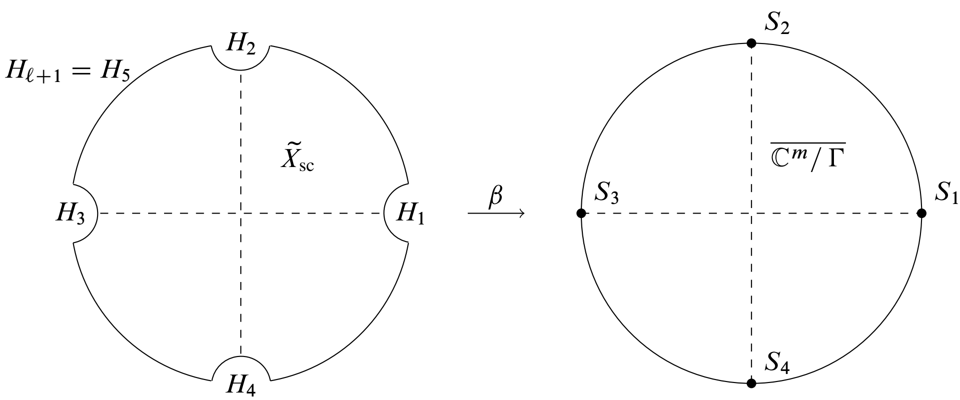

Figure 1: The blow-down map $\beta\colon \tilde{X}_{\mathrm{sc}} \to \overline{\mathbb{C}^m/\Gamma}$ in a case where $\ell = 4$, with the dotted lines corresponding to the singularities of $\mathbb{C}^m/\Gamma$

Moreover, for $i \leq \ell$, the blow-down map β restricts on H_i to induce a fiber bundle structure

$$\begin{tikzcd} \overline{\mathbb{C}^k/\Gamma_i} \arrow[r] & H_i \arrow[d, "\phi_i"] \\ & S_i \end{tikzcd} \tag{3}$$

with base S_i smooth, but with fibers singular at the origin, where $\Gamma_i \subset \mathrm{SU}(k)$ is some finite subgroup acting freely on $\mathbb{C}^k \setminus \{0\}$.

The manifold with corners $\widehat{X}_{\mathrm{QAC}}$ compactifying X is then obtained by observing that the crepant resolution $\pi\colon X \to \mathbb{C}^m/\Gamma$ naturally extends to a resolution $\pi\colon \widehat{X}_{\mathrm{QAC}} \to \overline{\mathbb{C}^m/\Gamma}$ with (3) replaced by

$$\begin{tikzcd} \overline{Y}_i \arrow[r] & \widehat{H}_i \arrow[d, "\widehat{\phi}_i"] \\ & S_i \end{tikzcd} \tag{4}$$

where $\overline{Y}_i$ is the radial compactification of a crepant resolution $\pi_i\colon Y_i \to \mathbb{C}^k/\Gamma_i$. Given a suitable Kähler QAC–metric g on X, our strategy to construct a Calabi–Yau QAC–metric is to first solve the complex Monge–Ampère equation on each fiber $\overline{Y}_i$ of $\widehat{\phi}_i$, which amounts to finding a Calabi–Yau ALE–metric on Y_i. Using these solutions, one can then replace the metric g with another Kähler QAC–metric g' in the same Kähler class, but with the extra property that its Ricci potential, given by

$$r' = \log\left(\frac{(\omega')^m}{c\Omega_{\widehat{X}_{\mathrm{QAC}}} \wedge \overline{\Omega}_{\widehat{X}_{\mathrm{QAC}}}}\right), \tag{5}$$

decays sufficiently fast at infinity, where ω' is the Kähler class of g', $\Omega_{\widehat{X}_{\mathrm{QAC}}}$ is some nowhere vanishing holomorphic volume form on $\widehat{X}_{\mathrm{QAC}}$ and c is a nonzero constant. One can then use standard techniques to solve the complex Monge–Ampère equation

$$\log\left(\frac{(\omega' + \sqrt{-1}\partial\overline{\partial}u)^m}{(\omega')^m}\right) = -r'$$

and obtain a Calabi–Yau QAC–metric with Kähler form $\omega' + \sqrt{-1}\partial\overline{\partial}u$.

Of course, as long as we are considering QALE–metrics, this is essentially the approach of Joyce rephrased in terms of the compactification $\widehat{X}_{\mathrm{QAC}}$. In particular, the compactification is not really needed, since in this simpler setting the bundles (3) and (4) are in fact trivial and to construct a Kähler QALE–metric near $\widehat{H}_i$ with Ricci potential decaying at infinity, one can simply glue directly the Cartesian product

$$g_{E_i} + g_{Y_i} \quad \text{on } (\mathbb{R}^+ \times S_i) \times Y_i \tag{6}$$

to the Euclidean metric on $\mathbb{C}^n/\Gamma$, where

$$g_{E_i} = dr^2 + r^2 g_{S_i}$$

is the Euclidean metric on $\mathbb{R}^+ \times S_i$ and g_{Y_i} is a Calabi–Yau ALE–metric on Y_i.

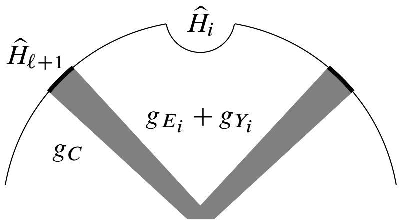

Figure 2: Gluing of $g_{E_i} + g_{Y_i}$ to g_C, with the gluing region in gray

Now, if we replace $\mathbb{C}^m/\Gamma$ with the Euclidean metric by a more general orbifold Calabi–Yau cone (C, g_C), gluing the Cartesian product (6) in a neighborhood of the singularities corresponding to H_i can still be done directly. However, if g_C is not Euclidean, then the QAC–metric introduced in this way will usually fail to have a Ricci potential that decays at the boundary face $\widehat{H}_{\ell+1}$. This is illustrated in Figure 2, where the region in bold on $\widehat{H}_{\ell+1}$ corresponds to the directions in which the Ricci potential fails to decay.

One could instead try to glue $g_{E_i} + g_{Y_i}$ to g_C as illustrated in Figure 3, but then the Ricci potential fails to decay on $\widehat{H}_i$, in this case even if g_C is the Euclidean metric.

Instead, if one uses the compactification, it suffices to put the model (6) on $\widehat{H}_i$ and keep the model given by g_C on $H_{\ell+1}$. Provided that the two models agree on $H_i \cap H_{\ell+1}$, one can then extend the Kähler form in the interior as a closed $(1,1)$–form. By continuity, this form will then be positive definite in a neighborhood of $\widehat{H}_i$ and $\widehat{H}_{\ell+1}$. More precisely, it will be the Kähler form of a QAC–metric with Ricci potential decaying at both $\widehat{H}_i$ and $\widehat{H}_{\ell+1}$.

Thanks to this natural compactification of QAC–manifolds by manifolds with fibered corners, we prove the following result, where we refer the reader to Definition 3.6 for some of the terminology and to Theorem 5.7 and Corollary 5.8 for a more precise and slightly more general statement.

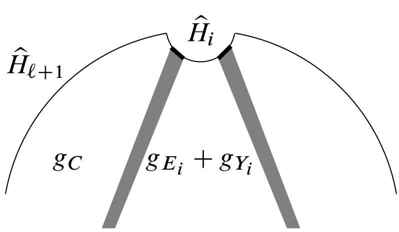

Figure 3: A second way of gluing $g_{E_i} + g_{Y_i}$ to g_C, with the gluing region in gray

Theorem A *Let Z be a compact orbifold of real dimension $2n+1$ and (C, g_C, J_C) a quasiregular Calabi–Yau cone on $C = \mathbb{R}^+ \times Z$ with parallel holomorphic volume form Ω_C. Let X be a Kähler orbifold with nowhere vanishing holomorphic volume form Ω_X and with singular set of complex codimension $\mu \geq 2$. Suppose that there is a compact set $\mathcal{K} \subset X$ such that $X \setminus \mathcal{K}$ is biholomorphic to $((\kappa, \infty) \times Z, J_C)$ for some $\kappa > 0$. Finally, suppose that there is a crepant resolution $\pi\colon \hat{X} \to X$ with Ω_X admitting a lift $\Omega_{\hat{X}} \in H^0(\hat{X}; K_{\hat{X}})$. Then, for any Kähler QAC–metric g asymptotic to g_C with rate α such that $4 \leq \alpha \leq 2\mu$, the complex Monge–Ampère equation*

$$\log\left(\frac{(\omega + \sqrt{-1}\partial\bar{\partial}u)^{n+1}}{(\omega)^{n+1}}\right) = -r$$

has a solution u such that $\omega + \sqrt{-1}\partial\bar{\partial}u$ is the Kähler form of a Calabi–Yau QAC–metric asymptotic to g_C with rate α, where ω is the Kähler form of g and r is its Ricci potential defined in terms of $\Omega_{\hat{X}}$ as in (5).

When one takes $X = \mathbb{C}^{n+1}/\Gamma$ in Theorem A for a choice of finite subgroup $\Gamma \subset \mathrm{SU}(n+1)$ for which X admits a local product Kähler crepant resolution, a subtle point hidden in Definition 3.6 is that Theorem A is not quite the same as the result of Joyce [24, Theorem 9.6.1]. Indeed, in [24, Theorem 9.6.1], one of the hypotheses on the initial QALE–metric g is that its Ricci potential decays at each boundary hypersurface of $\hat{X}_{\mathrm{QAC}}$, whereas in Theorem A, our result is stronger in that the Ricci potential only has to decay at the maximal boundary hypersurface ($\hat{H}_{\ell+1}$ in the simpler setting described above), but weaker in that the rate of decay α at that face has to be at least 4 (Theorem 9.6.1 of [24] only requires that α be strictly larger than 2).

More interestingly, Theorem A applies to situations where the geometry at infinity is QAC, but not QALE. As described in Example 2.4, our main source of examples is given by applying the Calabi ansatz [8; 29] to a Kähler–Einstein Fano orbifold D, which yields an orbifold Calabi–Yau cone metric g_C on $C = K_D \setminus D$. One can then take $X = K_D$ in Theorem A provided that K_D admits a local product Kähler crepant resolution in the sense of [24]. This is the case for instance if D itself admits a local product Kähler crepant resolution $\hat{D}$, in which case $K_{\hat{D}}$ is automatically a local product Kähler crepant resolution of K_D.

Nevertheless, given such a D, one still needs to find a suitable Kähler QAC–metric in order to apply Theorem A. For AC–metrics, it is very easy to produce examples of Kähler AC–metrics thanks to a standard trick comprising cutting off the model metric at infinity by using a convex function. In Lemma 4.1, we do in fact use this trick to construct orbifold Kähler AC–metrics on X. Unfortunately though, this trick does not seem to generalize to QAC–metrics or QALE–metrics, the reason being that the crepant resolution $\pi : \widehat{X} \to X$ introduces some topology at infinity, so that there are typically no model QAC–metrics with Kähler form near infinity given by $\sqrt{-1}\partial\overline{\partial}u$ for some smooth real-valued function u. In particular, just knowing that $\widehat{X}$ is Kähler is not sufficient to conclude that $\widehat{X}$ admits Kähler QAC–metrics.

To overcome this difficulty, we develop a method that allows one to construct a Kähler QAC–metric from an orbifold AC–metric g on X by gluing suitable local models near each singularity, effectively implementing geometrically the crepant resolution. In simple cases, this can be done directly by using cut-off functions. However, in order to be able to tackle situations where the singularities have arbitrary depth relatively easily, we have chosen to proceed with a systematic approach similar to what Kottke and Singer [27] do for gluing monopoles, that is, the gluing is implemented by using a manifold with corners with the various faces describing the metrics that need to be glued.

Deferring to Section 4 a detailed description of this manifold with corners, let us for the moment give the main intuitive idea behind its construction by restricting to the simple setting of (2). To simplify even further, suppose moreover that the crepant resolution of $\mathbb{C}^m/\Gamma$ is obtained by first blowing up the origin (in the sense of algebraic geometry) to obtain K_D with $D := \mathbb{CP}^{m-1}/\Gamma'$ for some finite subgroup $\Gamma' \subset \mathrm{SU}(m)$, so that one can replace (2) with

$$\widetilde{X}_{\mathrm{sc}} := [\overline{K}_D; S] = [\overline{K}_D; S_1, \dots, S_\ell], \tag{7}$$

where $\overline{K}_D$ is the radial compactification of K_D using the Calabi–Yau cone metric. In this case, the singularity S_i on the boundary of $\widetilde{X}_{\mathrm{sc}}$ corresponds to the boundary $\partial\Sigma_i$ of a singular edge Σ_i of complex codimension k in $\widetilde{X}_{\mathrm{sc}}$, and the singular set of $\widetilde{X}_{\mathrm{sc}}$ is given by the disjoint union

$$\Sigma = \bigcup_i \Sigma_i.$$

Introducing a parameter of deformation $\varepsilon \in [0,1)$, one can then consider the orbifold with corners

$$\mathcal{X} := [\widetilde{X}_{\mathrm{sc}} \times [0,1); \Sigma_1 \times \{0\}, \dots, \Sigma_\ell \times \{0\}].$$

This orbifold with corners has one boundary hypersurface $\mathcal{H}_i$ coming from the lift of $H_i \times [0,1)$ to $\mathcal{X}$. For each i, the blow-up of $\Sigma_i \times \{0\}$ also introduces a boundary hypersurface B_i. Notice in particular that the blow-down map $\mathcal{X} \to \widetilde{X}_{\mathrm{sc}} \times [0,1)$

naturally induces a fiber bundle structure,

$$\begin{tikzcd} \overline{\mathbb{C}^k/\Gamma_i} \arrow[r] & B_i \arrow[d, "\varphi_i"] \\ & \Sigma_i \times \{0\} \end{tikzcd} \tag{8}$$

Finally, as illustrated in Figure 4, the lift of the boundary hypersurface $\widetilde{X}_{\mathrm{sc}} \times \{0\}$ induces the boundary hypersurface $B_{\ell+1}$ on $\mathcal{X}$.

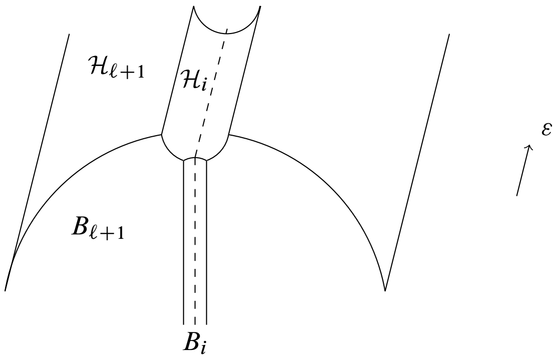

Figure 4: The orbifold with corners $\mathcal{X}$

The manifold with corners implementing the gluing is then obtained by observing that the resolution $\pi\colon \widehat{X}_{\mathrm{QAC}} \to \widetilde{X}_{\mathrm{sc}}$ naturally extends to a resolution $\pi\colon \widehat{\mathcal{X}} \to \mathcal{X}$. In particular, the resolution extends in such a way that the fiber bundle (8) is replaced by

$$\begin{tikzcd} \overline{Y}_i \arrow[r] & \widehat{B}_i \arrow[d, "\widehat{\varphi}_i"] \\ & \Sigma_i \end{tikzcd} \tag{9}$$

with $\overline{Y}_i$ the radial compactification of a crepant resolution Y_i of $\mathbb{C}^k/\Gamma_i$. To see how $\widehat{\mathcal{X}}$ can be used to implement the gluing, notice that an orbifold Kähler AC–metric g on K_D naturally induces a metric on the boundary hypersurface $B_{\ell+1}$ of $\mathcal{X}$. It also induces an orbifold Kähler metric g_i on B_i with Kähler form

$$\omega_i = \varphi_i^* \omega_{\Sigma_i} + \omega_{\varphi_i}, \tag{10}$$

where ω_{Σ_i} is a Kähler form on Σ_i and ω_{φ_i} is a $(1,1)$–form which restricts on each fiber of (8) to a Euclidean metric. On $\widehat{\mathcal{X}}$ and $\widehat{B}_i$, one can therefore replace (10) with

$$\widehat{\omega}_i = \varphi_i^* \omega_{\Sigma_i} + \omega_{\widehat{\varphi}_i}, \tag{11}$$

where $\omega_{\widehat{\varphi}_i}$ is a $(1,1)$–form which restricts on each fiber of (9) to the Kähler form of an ALE–metric asymptotic to the corresponding Euclidean metric induced by ω_{φ_i}. In other words, the metric $\widehat{g}_i$ with Kähler form (11) is the local model that we want to glue to g at the singularity Σ_i. In terms of the manifold $\widehat{\mathcal{X}}$, this gluing can be implemented by considering a closed $(1,1)$–form $\widehat{\omega}$ on (each level set of ε in) $\widehat{\mathcal{X}}$ which restricts to $\widehat{\omega}_i$ on $\widehat{B}_i$ and to the Kähler form of g on $\widehat{B}_{\ell+1} = B_{\ell+1}$. By continuity, for $\delta > 0$ sufficiently small, $\widehat{\omega}|_{\varepsilon=\delta}$ will then be positive definite and will induce the desired Kähler QAC–metric on $\widehat{\mathcal{X}}|_{\varepsilon=\delta} \cong \widehat{X}_{\mathrm{QAC}}$. Strictly speaking, to make this continuity argument rigorous, one must also introduce a suitable vector bundle on $\widehat{\mathcal{X}}$. We refer the reader to Section 4 for its detailed description.

The biggest advantage of gluing using the manifold with corners $\widehat{\mathcal{X}}$ is that it is relatively easy to generalize to singularities of arbitrary depth. Referring to Theorem 4.8 for the precise statement, let us mention some examples of the Kähler QAC–metrics that it produces.

Theorem B (Corollary 4.10) *Let $(D_1, g_1), \dots, (D_q, g_q)$ be Kähler–Einstein orbifolds with at worst isolated singularities of complex codimension at least 2. Suppose that each (D_i, g_i) admits a Kähler crepant resolution $\widehat{D}_i$. Consider the Cartesian products*

$$D := D_1 \times \cdots \times D_q \quad \text{and} \quad \widehat{D} := \widehat{D}_1 \times \cdots \times \widehat{D}_q.$$

Let g_C be the orbifold Calabi–Yau cone metric on $K_D \setminus D$ given by the Calabi ansatz. Then, on $K_{\widehat{D}}$, there exist Kähler QAC–metrics that are asymptotic to g_C at any rate $\alpha > 0$.

Combining with Theorem A yields the following.

Corollary C *On $K_{\widehat{D}}$ as in Theorem B, there are Calabi–Yau QAC–metrics asymptotic to g_C with rate $\alpha = 2\mu$, where μ is the complex codimension of the singular set of D.*

As explained in the appendix, it is possible to push this construction further, giving even more examples of Calabi–Yau QAC–metrics; see (A.1) and the discussion that follows for details.

The paper is organized as follows. In Section 1, we give a definition of QAC–metrics in terms of manifolds with fibered corners and we review some of their properties. In Section 2, we describe how orbifold Calabi–Yau cones can be compactified by an orbifold with fibered corners and how one obtains a corresponding manifold with fibered corners when there is a local product Kähler crepant resolution. In Section 3, we explain how an orbifold Calabi–Yau cone metric can be seen as a QAC–metric on the corresponding orbifold with fibered corners. In Section 4, we introduce the manifold with corners $\widehat{\mathcal{X}}$ and explain how it can be used to produce examples of Kähler QAC–metrics. Finally, in Section 5, we prove our main result, Theorem A, by solving the corresponding complex Monge–Ampère equation.

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1 Manifolds with fibered corners

Let M be a compact manifold with corners in the sense of Melrose [33; 36] with boundary hypersurfaces $H_1, \dots, H_k$. In particular, we assume that each boundary hypersurface of M is embedded in M and we denote by ∂M the union of all the boundary hypersurfaces of M. Suppose that each boundary hypersurface H_i comes endowed with a fiber bundle structure $\phi_i \colon H_i \to S_i$ with base S_i and with each fiber a manifold with corners. We denote by $\phi = (\phi_1, \dots, \phi_k)$ the collection of all fiber bundle maps.

Definition 1.1 [2; 1; 16] We say that (M, ϕ) is a *manifold with fibered corners* if there is a partial order on the boundary hypersurfaces such that:

- Any subset I of boundary hypersurfaces such that $\bigcap_{i \in I} H_i \neq \varnothing$ is totally ordered.

- If $H_i < H_j$, then $H_i \cap H_j \neq \varnothing$, the map $\phi_i|_{H_i \cap H_j}\colon H_i \cap H_j \to S_i$ is a surjective submersion and $S_{ji} := \phi_j(H_i \cap H_j)$ is one of the boundary hypersurfaces of the manifold with corners S_j. Moreover, there is a surjective submersion $\phi_{ji}\colon S_{ji} \to S_i$ such that $\phi_{ji} \circ \phi_j = \phi_i$ on $H_i \cap H_j$.
- The boundary hypersurfaces of S_j are given by the S_{ji} for $H_i < H_j$.

It follows directly from this definition that each base S_j has a natural manifold with fibered corners structure induced by the maps $\phi_{ji}\colon S_{ji} \to S_i$ for each i with $H_i < H_j$. Similarly, each fiber of $\phi_i\colon H_i \to S_i$ has a natural manifold with fibered corners structure. Moreover, S_i is a smooth closed manifold whenever H_i is minimal with respect to the partial order, whereas the fibers of ϕ_i are smooth closed manifolds whenever H_i is maximal. This allows one to prove many assertions by proceeding by induction on the *depth* of (M, ϕ), that is, the largest codimension that a corner of M may have, or by induction on the relative depth of a boundary hypersurface, where we recall that the *relative depth* of a boundary hypersurface H_i in M is the largest integer j such that there exist $j-1$ boundary hypersurfaces $H_{\nu_1}, \dots, H_{\nu_{j-1}}$ with

$$H_i < H_{\nu_1} < \cdots < H_{\nu_{j-1}}.$$

The notion of a manifold with fibered corners is intimately related to the notion of a stratified space, so let us recall briefly what is meant by this latter term.

Definition 1.2 A *stratified space* of dimension n is a locally compact separable metrizable space X together with a *stratification*, which is a locally finite partition $\mathsf{S} = \{s_i\}$ into locally closed subsets of X, called the *strata*, which are smooth manifolds of dimension $\dim s_i \leq n$ such that at least one stratum is of dimension n and

$$s_i \cap \overline{s}_j \neq \varnothing \iff s_i \subset \overline{s}_j.$$

In this case we write that $s_i \leq s_j$ and $s_i < s_j$ if $s_i \neq s_j$. A stratification induces a filtration

$$\varnothing \subset X_0 \subset \cdots \subset X_n = X,$$

where X_j is the union of all strata of dimension at most j. The strata included in $X^\bullet := X \setminus X_{n-1}$ are said to be *regular*, whereas the strata included in X_{n-1} are said to be *singular*. Given a stratified space, notice that the closure of each of its strata is also naturally a stratified space.

Remark 1.3 In the present paper, it is crucial for us to impose no restrictions on the codimension of the singular strata. Notice however that in other situations, it is quite common and natural to require that the singular strata are always at least of codimension 2; see for instance [20].

A good measure of the complexity of a stratified space is given by its depth, a notion which we now recall.

Definition 1.4 The *depth* of a stratified space (X, S) is the largest k such that one can find $k+1$ different strata with

$$s_1 < s_2 < \cdots < s_k < s_{k+1}.$$

On the other hand, the *relative depth* of a stratum s in X is the largest k such that there exists k strata with

$$s < s_1 < \cdots < s_k.$$

More generally, the *relative depth* of a point $p \in X$ is the relative depth of the unique stratum s containing p.

As observed by Melrose and described in [1; 16], a manifold with fibered corners M always arises as a resolution of a stratified space ${}^{\mathsf{S}}M$ given by ${}^{\mathsf{S}}M = M/{\sim}$, where $\sim$ is the relation

$$p \sim q \quad \Longleftrightarrow \quad p = q \quad \text{or} \quad p, q \in H_i \text{ with } \phi_i(p) = \phi_i(q) \text{ for some hypersurface } H_i.$$

In terms of the quotient map

$$\beta\colon M \to {}^{\mathsf{S}}M,$$

which we call the *blow-down map*, the regular stratum is given by $\beta(M \setminus \partial M)$. More importantly, the blow-down map gives a one-to-one correspondence between the boundary hypersurfaces H_i of M and the closure of the singular strata $\overline{s}_i := \beta(H_i)$ of ${}^{\mathsf{S}}M$. The base S_i of the fiber bundle $\phi_i\colon H_i \to S_i$ is in fact itself a resolution of the stratified space $\overline{s}_i$ and we have that $s_i = \beta(\phi_i^{-1}(S_i \setminus \partial S_i))$. Moreover, the correspondence between boundary hypersurfaces and singular strata is compatible with the partial orders, ie

$$H_i < H_j \iff s_i < s_j,$$

so that the relative depth of a boundary hypersurface is equal to the relative depth of the corresponding stratum. Notice also that the depth of ${}^{\mathsf{S}}M$ as a stratified space is equal to the depth of M as a manifold with corners.

Definition 1.5 The stratified space ${}^{\mathsf{S}}M$ is said to be the *blow-down* of the manifold with fibered corners (M,ϕ). Conversely, the manifold with fibered corners (M,ϕ) is said to be a *resolution* of the stratified space ${}^{\mathsf{S}}M$. More generally, a stratified space which admits a resolution by a manifold with fibered corners is said to be *smoothly stratified.*

Remark 1.6 Not all stratified spaces are smoothly stratified, but the property of being smoothly stratified can be described intrinsically on the stratified space without referring to a manifold with fibered corners; see for instance [1; 16].

Remark 1.7 The notion of resolution in Definition 1.5 should not be confused with the notion of crepant resolution discussed in the introduction. In the first case, the singularity is resolved using a manifold with corners and can be done quite generally, while the latter case is much more specific, since the singularity must then be of a particular complex geometric nature and is resolved by a smooth complex manifold with suitable properties.

Example 1.8 An orbifold is naturally a smoothly stratified space — see for instance [40, Section 4.4.10] or [19, pages 210–211] — and the corresponding manifold with fibered corners is obtained by blowing up the singular strata in an order compatible with the partial order.

Let $x_1,\dots,x_k$ be some boundary-defining functions for the boundary hypersurfaces $H_1,\dots,H_k$ of M; that is, x_i is positive on $M\setminus H_i$, $x_i=0$ on H_i and dx_i is nowhere zero on H_i. For each i, we will usually assume, unless otherwise stated, that the boundary-defining function x_i is identically equal to 1 outside some tubular neighborhood of H_i. This assumption is not restrictive, but turns out to be very convenient. We also assume that the boundary-defining functions $x_1,\dots,x_k$ are compatible with ϕ in the following sense.

Definition 1.9 We say that the boundary-defining functions $x_1,\dots,x_k$ are *compatible* with the collection of fiber bundle maps ϕ if for each i and j with $H_i<H_j$, the restriction of x_i to H_j is constant along the fibers of $\phi_j\colon H_j\to S_j$.

The assumption of compatibility with ϕ certainly imposes some restrictions on the choice of boundary-defining functions. However, it imposes no restriction on the type of manifold with fibered corners, since by [16, Lemma 1.4], we know that manifolds with fibered corners always admit compatible boundary-defining functions. One important advantage of this compatibility condition is that for each i, the other boundary-defining functions x_j naturally define boundary-defining functions for S_i that are compatible with the induced manifold with fibered corners structure. Another advantage, as the next lemma shows, is that it gives a nice local product decomposition of the manifold with fibered corners structure near a boundary hypersurface.

Lemma 1.10 *If $x_1, \dots, x_k$ are boundary-defining functions compatible with ϕ, then for each i, there exists a tubular neighborhood*

$$c_i \colon H_i \times [0, \epsilon) \hookrightarrow M$$

such that

(i) $c_i^* x_i = \mathrm{pr}_2$;

(ii) $c_i^* x_j = x_j \circ \mathrm{pr}_1$ *for* $j \neq i$;

(iii) $c_i^{-1} \circ \phi_j \circ c_i(h, t) = \phi_j(h)$ *for* $h \in H_i \cap H_j$ *and* $H_j < H_i$;

(iv) $c_i^{-1} \circ \phi_j \circ c_i(h, t) = (\phi_j(h), t)$ *for* $h \in H_i \cap H_j$ *and* $H_j > H_i$;

where $\mathrm{pr}_1 \colon H_i \times [0, \epsilon) \to H_i$ *and* $\mathrm{pr}_2 \colon H_i \times [0, \epsilon) \to [0, \epsilon)$ *are the projections on the first and second factors.*

Proof Let $\xi \in \mathcal{C}^\infty(TM)$ be a vector field such that

- $dx_i(\xi) > 0$ everywhere on H_i;
- $dx_j(\xi) = 0$ in a neighborhood of H_j in M for $j \neq i$;
- $\xi|_{H_j}$ is tangent to the fibers of $\phi_j : H_j \to S_j$ for $H_j < H_i$.

Then, in a sufficiently small neighborhood $\mathcal{N}_i$ of H_i, the vector field $\eta := \xi / dx_i(\xi)$ is well defined and satisfies the same properties as ξ with the extra feature that $dx_i(\eta) \equiv 1$ in $\mathcal{N}_i$. By construction, the flow of η then generates the desired tubular neighborhood, the last condition being satisfied thanks to the fact that the boundary-defining functions $x_1, \dots, x_k$ are compatible with ϕ. □

Recall from [33] that

$$\mathcal{V}_b(M) := \{\xi \in TM \mid \xi x_i \in x_i \mathcal{C}^\infty(M) \text{ for all } i\}$$

is the Lie algebra of b–vector fields on M, that is, smooth vector fields on M which are tangent to all boundary hypersurfaces. Notice that this definition does not depend on the choice of boundary-defining functions x_i.

Definition 1.11 A *quasi fibered boundary vector field*, or QFB–vector field for short, is a b–vector field ξ such that for each i,

- $\xi|_{H_i}$ is tangent to the fibers of ϕ_i;
- $\xi v_i \in v_i^2 \mathcal{C}^\infty(M)$, where $v_i = \prod_{H_j \geq H_i} x_j$.

We denote the space of QFB–vector fields by $\mathcal{V}_{\mathrm{QFB}}(M)$.

Remark 1.12 This is closely related to the definition of an iterated fibered corners vector field given in [16], the difference being that in [16], one requires that $\xi x_i \in x_i^2 \mathcal{C}^\infty(M)$ for each i instead of asking that

$$\xi v_i \in v_i^2 \mathcal{C}^\infty(M).$$

Example 1.13 If M is a manifold with boundary, $\phi\colon \partial M \to S$ is a fiber bundle and $x \in \mathcal{C}^\infty(M)$ is a boundary-defining function, then $\mathcal{V}_{\mathrm{QFB}}(M)$ is the Lie algebra of fibered boundary vector fields (or ϕ–vector fields) introduced by Mazzeo and Melrose [32]. If in fact $S = \partial M$ and $\phi = \mathrm{Id}$, then $\mathcal{V}_{\mathrm{QFB}}(M)$ is the Lie algebra of scattering vector fields (or asymptotically conical vector fields) introduced by Melrose [35].

When there are corners of codimension 2 and higher, we can give a simple description of QFB–vector fields in terms of coordinates adapted to the fibered corners structure. If $p \in \partial M$ is contained in the corner $H_1 \cap \cdots \cap H_\ell$, we can for simplicity label the boundary hypersurfaces in such a way that

$$H_1 < \cdots < H_\ell.$$

Let $x_1, \dots, x_\ell$ be the corresponding boundary-defining functions. In a neighborhood of p in which each fiber bundle ϕ_i is trivial, consider tuples of functions $y_i = (y_i^1, \dots, y_i^{k_i})$ for $i \in \{1, \dots, \ell\}$ and $z = (z_1, \dots, z_q)$ such that

$$(x_1, y_1, \dots, x_\ell, y_\ell, z) \tag{1.1}$$

defines coordinates near p with the property that on H_i, $(x_1, y_1, \dots, x_{i-1}, y_{i-1}, y_i)$ induces coordinates on the base S_i with ϕ_i corresponding to the map

$$(x_1, y_1, \dots, \widehat{x}_i, y_i, \dots, x_\ell, y_\ell, z) \mapsto (x_1, y_1, \dots, x_{i-1}, y_{i-1}, y_i),$$

where the notation "$\widehat{\ }$" above the variable x_i denotes its omission. In these coordinates, one can check that the Lie algebra $\mathcal{V}_{\mathrm{QFB}}(M)$ is locally spanned over $\mathcal{C}^\infty(M)$ by the vector fields

$$(1.2)\quad v_1x_1\frac{\partial}{\partial x_1},\quad v_1\frac{\partial}{\partial y_1^{n_1}},\quad v_2x_2\frac{\partial}{\partial x_2}-v_1\frac{\partial}{\partial x_1},\quad v_2\frac{\partial}{\partial y_2^{n_2}},\quad \dots,\quad v_\ell x_\ell\frac{\partial}{\partial x_\ell}-v_{\ell-1}\frac{\partial}{\partial x_{\ell-1}},\quad v_\ell\frac{\partial}{\partial y_\ell^{n_\ell}},\quad \frac{\partial}{\partial z_p}$$

for $p\in\{1,\dots,q\}$ and $n_i\in\{1,\dots,k_i\}$, where $x=\prod_{i=1}^{\ell}x_i$ and $v_i=\prod_{m=i}^{\ell}x_m$. In other words, in these coordinates, a QFB–vector field $\xi\in\mathcal{V}_{\mathrm{QFB}}(M)$ is of the form

$$(1.3)\quad a_1v_1x_1\frac{\partial}{\partial x_1}+\sum_{i=2}^{\ell}a_i\left(v_ix_i\frac{\partial}{\partial x_i}-v_{i-1}\frac{\partial}{\partial x_{i-1}}\right)+\sum_{i=1}^{\ell}\sum_{j=1}^{k_i}b_{ij}v_i\frac{\partial}{\partial y_i^j}+\sum_{p=1}^{q}c_p\frac{\partial}{\partial z_p},$$

with $a_i,b_{ij},c_p\in\mathcal{C}^\infty(M)$; cf [16, equation (2.6)].

Definition 1.14 When the manifold with fibered corners (M,ϕ) is such that $S_i=H_i$ and $\phi_i=\mathrm{Id}$ for each maximal boundary hypersurface H_i, we say that a QFB–vector field is a *quasi-asymptotically conical vector field* (QAC–vector field for short) and (M,ϕ) is a QAC–manifold with fibered corners.

The space $\mathcal{V}_{\mathrm{QFB}}(M)$ clearly depends on the fiber bundle structure of each boundary hypersurface H_i. It also depends on the choice of boundary-defining functions.

Definition 1.15 If $H_1,\dots,H_k$ is an exhaustive list of all the boundary hypersurfaces of the manifold with fibered corners (M,ϕ), then two different choices $x_1,\dots,x_k$ and $x'_1,\dots,x'_k$ of boundary-defining functions are said to be QFB*–equivalent* if they yield the same Lie algebra of QFB–vector fields. If (M,ϕ) is a QAC–manifold with fibered corners, then we will also say that they are QAC*–equivalent* when they are QFB–equivalent.

The next lemma gives a criterion to determine when two collections of boundary-defining functions are QFB–equivalent.

Lemma 1.16 *If $H_1,\dots,H_k$ is an exhaustive list of all the boundary hypersurfaces of the manifold with fibered corners (M,ϕ), then two different choices $x_1,\dots,x_k$ and $x'_1,\dots,x'_k$ of boundary-defining functions compatible with ϕ are* QFB*–equivalent if and only if for all i, the function*

$$f_i:=\log\left(\frac{v'_i}{v_i}\right)=\sum_{H_j\geq H_i}\log\left(\frac{x'_j}{x_j}\right)\in\mathcal{C}^\infty(M)$$

is such that for all $H_j\geq H_i$, $f_i|_{H_j}=\phi_j^*h_{ij}$ for some $h_{ij}\in\mathcal{C}^\infty(S_j)$.

Proof Consider the Lie algebra of vector fields

$$\mathcal{V}_{b,\phi}(M) := \{\xi \in \mathcal{V}_b(M) \mid (\phi_i)_*(\xi|_{H_i}) = 0 \text{ for all } i\}.$$

Clearly then, by Definition 1.11, the two choices of boundary-defining functions yield the same space of QFB–vector fields if and only if for all $\xi \in \mathcal{V}_{b,\phi}(M)$,

$$\frac{dv_i}{v_i^2}(\xi) \in \mathcal{C}^\infty(M) \quad \text{for all } i \quad \Longleftrightarrow \quad \frac{dv_i'}{(v_i')^2}(\xi) \in \mathcal{C}^\infty(M) \quad \text{for all } i. \tag{1.4}$$

Now, by definition of $f_i \in \mathcal{C}^\infty(M)$, we have that $v_i' = e^{f_i} v_i$, so that

$$\frac{dv_i'}{(v_i')^2} = e^{-f_i}\left(\frac{dv_i}{v_i^2} + \frac{df_i}{v_i}\right). \tag{1.5}$$

In particular, if for all i and all $H_j \geq H_i$, $f_i|_{H_j} = \phi_j^* h_{ij}$ for some $h_{ij} \in \mathcal{C}^\infty(S_j)$, then we see from (1.5) that (1.4) holds. Conversely, if for some H_i and some $H_j \geq H_i$, $f_i|_{H_j}$ is not the pullback of some element of $\mathcal{C}^\infty(S_j)$, then we can find $\xi \in \mathcal{V}_{b,\phi}(M)$ such that $df_i(\xi)|_{H_j} \neq 0$ and

$$\frac{dv_\ell}{v_\ell^2}(\xi) \in \mathcal{C}^\infty(M) \quad \text{for all } \ell,$$

so that by (1.5), $(dv_i'/(v_i')^2)(\xi)$ is not bounded near H_j, implying in particular that (1.4) does not hold. □

It is clear from the definition that $\mathcal{V}_{\mathrm{QFB}}(M)$ is in fact a Lie subalgebra of $\mathcal{V}_b(M)$. In particular, we can define the space $\mathrm{Diff}^*_{\mathrm{QFB}}(M)$ of QFB–differential operators as the universal enveloping algebra of $\mathcal{V}_{\mathrm{QFB}}(M)$ over $\mathcal{C}^\infty(M)$. Thus, $\mathrm{Diff}^q_{\mathrm{QFB}}(M)$ is the space of operators generated by multiplication by elements of $\mathcal{C}^\infty(M)$ and the action of up to q QFB–vector fields.

Now, as described in (1.2), the Lie algebra $\mathcal{V}_{\mathrm{QFB}}(M)$ is a locally free sheaf of rank $\dim M$ over $\mathcal{C}^\infty(M)$. Hence, by the Serre–Swan theorem, there exists a natural smooth vector bundle, the QFB*–tangent bundle*, which we shall denote by ${}^\phi TM \to M$, and a natural map $\iota_\phi\colon {}^\phi TM \to TM$ restricting to an isomorphism on $M \setminus \partial M$ such that

$$\mathcal{V}_{\mathrm{QFB}}(M) = (\iota_\phi)_* \mathcal{C}^\infty(M; {}^\phi TM).$$

More precisely, at a point $p \in M$, the fiber of ${}^\phi TM$ is given by

$${}^\phi T_p M = \mathcal{V}_{\mathrm{QFB}}(M)/\mathcal{I}_p \cdot \mathcal{V}_{\mathrm{QFB}}(M),$$

where $\mathcal{I}_p$ is the ideal of smooth functions vanishing at p. It is then natural to define the QFB*–cotangent bundle* to be the dual ${}^{\phi}T^*M$ of the QFB–tangent bundle ${}^{\phi}TM$. In terms of the coordinates (1.1) near $H_1 \cap \cdots \cap H_\ell$, a local basis of sections of the QFB–cotangent bundle is given by

$$\frac{dv_1}{v_1^2}, \quad \frac{dy_1^{n_1}}{v_1}, \quad \dots, \quad \frac{dv_\ell}{v_\ell^2}, \quad \frac{dy_\ell^{n_\ell}}{v_\ell}, \quad dz_k, \tag{1.6}$$

for $k \in \{1, \dots, q\}$ and $n_i \in \{1, \dots, k_i\}$.

Definition 1.17 A *quasi fibered boundary metric* (QFB–metric for short) is a choice of Euclidean metric g_ϕ for the vector bundle ${}^{\phi}TM$. A *smooth* QFB*–metric* on $M \setminus \partial M$ is a Riemannian metric on $M \setminus \partial M$ induced by some QFB–metric g_ϕ via the map $\iota_\phi\colon {}^{\phi}TM \to TM$. Hoping this will lead to no confusion, we will also denote by g_ϕ the smooth QFB–metric induced by a QFB–metric $g_\phi \in \mathcal{C}^\infty(M; {}^{\phi}T^*M \otimes {}^{\phi}T^*M)$.

Remark 1.18 Because M is a compact space and all of the Euclidean metrics of a vector bundle on a compact space are quasi-isometric, notice that all smooth QFB–metrics are automatically quasi-isometric among themselves. Thus, more generally, if a Riemannian metric on $M \setminus \partial M$ is quasi-isometric to a smooth QFB–metric g_{QFB} and if all of its derivative taken with respect to the covariant derivative of g_{QFB} are bounded, then we say that it is a QFB*–metric*. Similarly, we say that a QFB–metric g_ϕ is *polyhomogeneous* if g_ϕ is induced by a Euclidean metric on ${}^{\phi}TM$ which is polyhomogeneous on M.

Example 1.19 In the local basis (1.6), an example of a QFB–metric is given by

$$\sum_{i=1}^{\ell} \frac{dv_i^2}{v_i^4} + \sum_{i=1}^{\ell} \sum_{j=1}^{k_i} \frac{(dy_i^j)^2}{v_i^2} + \sum_{k=1}^{q} dz_k^2. \tag{1.7}$$

Example 1.20 If M is a manifold with fibered boundary $\phi\colon \partial M \to S$, then a QFB–metric is a fibered boundary metric (or ϕ–metric) in the sense of [32]. If moreover $S = \partial M$ and $\phi = \mathrm{Id}$, then a QFB–metric is a scattering metric (also called an asymptotically conical metric) in the sense of [35].

Notice that given a smooth QFB–metric g_{QFB}, there is an alternative description of the Lie algebra $\mathcal{V}_{\mathrm{QFB}}(M)$, namely, it is given by the smooth vector fields on M which are uniformly bounded with respect to g_{QFB}:

$$\mathcal{V}_{\mathrm{QFB}}(M) = \{\xi \in \mathcal{C}^\infty(M; TM) \mid \sup_{M \setminus \partial M} g_{\mathrm{QFB}}(\xi, \xi) < \infty\}. \tag{1.8}$$

In this paper, we are interested in the following particular example of a QFB–metric first considered in [17].

Definition 1.21 A *quasi-asymptotically conical metric* (QAC–metric for short) is a QFB–metric on a manifold with fibered corners such that $S_i = H_i$ and $\phi_i = \mathrm{Id}$ for each maximal boundary hypersurface H_i with respect to the partial order.

Example 1.22 (cf [17, Section 2.3.5]) Let M_1 and M_2 be two smooth manifolds with boundary and consider the manifold with corners $M = [M_1 \times M_2; \partial M_1 \times \partial M_2]$ obtained by blowing up the corner of $M_1 \times M_2$ in the sense of Melrose [34]. Let $\beta\colon M \to M_1 \times M_2$ denote the blow-down map. As illustrated in Figure 5, M has three boundary hypersurfaces, H_1 and H_2 coming respectively from the old faces $\partial M_1 \times M_2$ and $M_1 \times \partial M_2$, and H_3 coming from the blown-up corner.

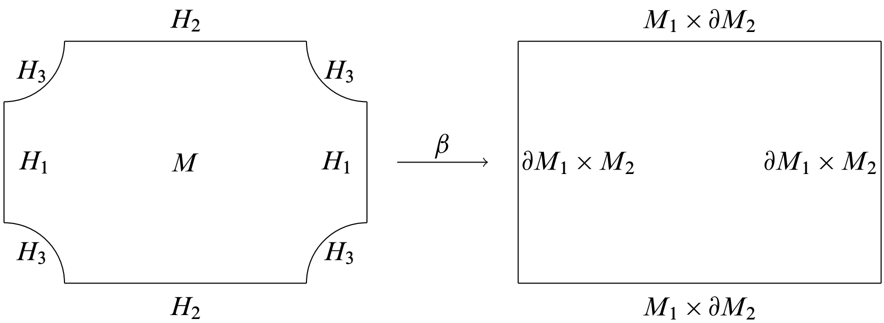

Figure 5: The blow-down map $\beta\colon M \to M_1 \times M_2$

These faces come naturally with fiber bundle structures

$$\phi_1\colon \partial M_1 \times M_2 \to \partial M_1, \quad \phi_2 : M_1 \times \partial M_2 \to \partial M_2, \quad \phi_3 = \mathrm{Id}\colon H_3 \to H_3,$$

given respectively by the projections on the left and right factors for $H_1 = \partial M_1 \times M_2$ and $H_2 = M_1 \times \partial M_2$, and by the identity map on H_3. These fiber bundles $\phi = (\phi_1, \phi_2, \phi_3)$ endow (M, ϕ) with a manifold with fibered corners structure with partial order given by $H_1 < H_3$ and $H_2 < H_3$. Let u_1 and u_2 be boundary-defining functions on M_1 and M_2 respectively and denote also by u_1 and u_2 their pullbacks to $M_1 \times M_2$ via the projections on the left and right factors. On M, consider the polar coordinates $u_1 = r\cos\theta$, $u_2 = r\sin\theta$, so that $x_1 = \cos\theta$, $x_2 = \sin\theta$ and $x_3 = r$ are boundary-defining functions for H_1, H_2 and H_3, respectively. In terms of these choices, the functions v_i of Definition 1.11 are given by

$$v_1 = x_1 x_3 = u_1, \quad v_2 = x_2 x_3 = u_2, \quad v_3 = x_3 = r.$$

In terms of the local basis of sections (1.6) of the QAC–cotangent bundle, we thus see that any pair of asymptotically conical metrics $g_i \in \mathcal{C}^\infty(M_i; {}^{\mathrm{sc}}TM_i)$, for $i = 1, 2$, naturally induces a QAC–metric g_{QAC} on M by taking their Cartesian product:

$$g_{\mathrm{QAC}} = \beta^*(g_1 \times g_2).$$

Remark 1.23 On a manifold with boundary M with fiber bundle structure on ∂M given by the identity map, all choices of boundary-defining functions lead to the same Lie algebra of scattering vector fields. In Example 1.22, this is reflected by the fact that the Lie algebra of QAC–vector fields does not depend on the choice of u_1 and u_2. However, choosing boundary-defining functions x_i not induced by the choices of u_1 and u_2 as in the previous example may change the Lie algebra of QAC–vector fields.

Proposition 1.24 *Definition 1.21 is a particular case of the notion of a QAC–metric introduced in [17].*

Proof Let (M, ϕ) be a manifold with fibered corners such that $S_i = H_i$ and $\phi_i = \mathrm{Id}$ for each maximal hypersurface. The simplest way to see that Definition 1.21 is a special case of [17] is to proceed by recurrence on the depth of M. If the depth of M is one, that is, if M is a manifold with boundary, then Definition 1.21 just corresponds to the notion of a scattering metric in the sense of [35], so that the result is obvious in this case.

Suppose then that the result holds for all QAC–manifolds with fibered corners of depth $k \geq 1$ and smaller and let (M, ϕ) be a QAC–manifold with fibered corners of depth $k+1$. Let H_1 be a boundary hypersurface of relative depth $k+1$ in M. Using a tubular neighborhood $\mathcal{N}_1 \cong H_1 \times [0, \epsilon)_{x_1}$ of H_1 as in Lemma 1.10, we see from Example 1.19 that in $\mathcal{N}_1$, an example of a smooth QAC–metric is given by

$$\frac{dv_1^2}{v_1^4} + \frac{\phi_1^* g_{S_1}}{v_1^2} + h, \tag{1.9}$$

where h is a symmetric 2–tensor which restricts to a smooth QAC–metric on each fiber of $\phi_1 \colon H_1 \to S_1$ and g_{S_1} is a Riemannian metric on the closed manifold S_1. Here, we think of h as being constant in x_1, that is, defined on H_1 and pulled back to the tubular neighborhood of H_1 given by Lemma 1.10.

Alternatively, we can look at the restriction of h to each level set of v_1. For $c > 0$, the region of the level set $v_1 = c$ contained in the tubular neighborhood $\mathcal{N}_1$ corresponds to the open set $\mathcal{U}_c = \{p \in H_1 \mid v_2(p) > c/\epsilon\}$ in H_1 under the retraction $r_1 \colon \mathcal{N}_1 \to H_1$. In fact, we have that

$$H_1 \setminus \partial H_1 = \bigcup_{0<c<\epsilon} \mathcal{U}_c.$$

Thus, in terms of the level sets of v_1, and restricting to a region $\mathcal{W}$ of S_1 where the fiber bundle $\phi_1 \colon H_1 \to S_1$ is trivial, we see that the metric (1.9) can be assumed to be the Cartesian product of the cone metric $dv_1^2/v_1^4 + g_{S_1}/v_1^2$ with a QAC–metric h on Z_1, but restricted to the region

$$\left\{(c, v, z) \in [0, \epsilon)_{v_1} \times \mathcal{W} \times Z_1 \mid v_2(z) > \frac{c}{\epsilon}\right\} \subset [0, \epsilon)_{v_1} \times \mathcal{W} \times Z_1,$$

where Z_1 is a typical fiber of ϕ_1 so that $\phi_1^{-1}(\mathcal{W}) \cong \mathcal{W} \times Z_1$. This is precisely the local inductive model given in [17, Lemma 2.9]. This completes the proof. □

Remark 1.25 Using directly [17, Lemma 2.9], we did not have to deal with the notion of resolution blow-ups introduced in [17]. Since this is intuitively useful, let us nevertheless quickly explain how resolution blow-ups arise in terms of Definition 1.21. Let (M, ϕ) be a QAC–manifold with fibered corners and let D be the (disjoint) union of all maximal boundary hypersurfaces of M. Then D naturally inherits a manifold with fibered corners structure. As such, there is an associated smoothly stratified space Y_0 with singular strata in one-to-one correspondence with the boundary hypersurfaces of D. Let $x = \prod_{H_i \subset \partial M} x_i$ be the product of all boundary-defining functions of M. Then, for $\epsilon > 0$ small, the level set

$$Y_\epsilon := \{p \in M \mid x(p) = \epsilon\}$$

of x corresponds to a smooth desingularization of Y_0 in the sense of [17, Section 2.3]. Let $\iota_\epsilon : Y_\epsilon \to M$ be the natural inclusion and consider the metric $g_\epsilon := \iota_\epsilon^*(x^2 g_{\mathrm{QAC}})$ for g_{QAC} a choice of QAC–metric on (M, ϕ). Then, for $\epsilon > 0$, the family (Y_ϵ, g_ϵ) is a resolution blow-up of (Y_0, g_0) for some incomplete iterated edge metric g_0 on Y_0, the stratified space associated to D. A subtle point however is that g_0 is not equal to $\iota_0^* x^2 g_{\mathrm{QAC}}$ for $\iota_0 \colon D \to M$ the natural inclusion. For instance, if $x_{\max}$ denotes the boundary-defining function for D, then terms of the form $\iota_\epsilon^* x^2\, dx_{\max}^2/x_{\max}^4$ lead to nonzero contributions in the limit $\epsilon \searrow 0$, but these contributions are clearly not captured by $\iota_0^* x^2 g_{\mathrm{QAC}}$, which in fact is not an incomplete iterated edge metric. See Remark 4.6 for a related phenomenon.

Remark 1.26 One advantage of using QAC–manifolds with fibered corners is that the weight functions of [17, Section 2.4] admit a simple description in terms of boundary-defining functions. If $x = \prod_{H_i \subset \partial M} x_i$ is the product of all of the boundary-defining functions of M, then the radial function of [17] can be taken to be $\rho = 1/x$. Moreover, if (M, ϕ) is of depth ℓ, then for $1 \le j \le \ell$, the weight function w_j of [17, Section 2.4] is simply the product of all boundary-defining functions associated to boundary hypersurfaces of relative depth at least j. In particular, in the situation where the boundary hypersurfaces of M are totally ordered, say $H_1 < \cdots < H_\ell$, the weight function w_j is simply given by

$$w_j = \prod_{i \le j} x_i = \frac{x}{v_{j+1}}.$$

Using a similar approach as in the proof of Proposition 1.24, we can obtain the following general result about QFB–metrics.

Proposition 1.27 *Each QFB–metric is a complete metric of infinite volume with bounded geometry.*

Proof A QFB–metric is a particular example of a metric with Lie structure at infinity in the sense of [3]. As such, it is complete and of infinite volume by [3, Proposition 4.1 and Corollary 4.9]. By [3, Corollary 4.3], its curvature is bounded, as well as all of its covariant derivatives. By [3, Corollary 4.20] and Remark 1.18, for a fixed Lie algebra of QFB–vector fields, it suffices to find one example of a QFB–metric with positive injectivity radius to conclude that all QFB–metrics have positive injectivity radius. Now, it is well known that closed Riemannian manifolds have a positive injectivity radius. Thus, proceeding by induction on the depth, we can assume that QFB–metrics on a manifold with fibered corners of depth k have a positive injectivity radius. On a manifold with fibered corners (M, ϕ) of depth $k+1$, let H_1 be one of the boundary hypersurfaces of relative depth $k+1$. Using a tubular neighborhood $\mathcal{N}_1 \cong H_1 \times [0, \epsilon)_{x_1}$ of H_1 as in Lemma 1.10, we can consider a QFB–metric as in (1.9), this time however with h restricting to a QFB–metric on each fiber. In particular, each fiber has positive injectivity radius by our induction hypothesis. Clearly then, by the discussion at the end of the proof of Proposition 1.24, the points contained in the smaller tubular neighborhood $\mathcal{N}_1 \cong H_1 \times [0, \epsilon/2)_{x_1}$ have positive injectivity radii uniformly bounded from below by a positive constant. Considering similar metrics near each boundary hypersurface of relative depth $k+1$, we can then extend in an arbitrary way to obtain a global QFB metric on (M, ϕ), which, thanks to our induction hypothesis, will have positive injectivity radius. Hence, by our earlier observation, all QFB–metrics have positive injectivity radius. Combined with the control on the curvature and on its derivatives mentioned previously, this shows that all QFB–metrics have bounded geometry. □

For QAC–metrics, we also have another important property, namely the Sobolev inequality.

Lemma 1.28 (Sobolev inequality) *Given a QAC–metric g on a manifold with fibered corners M of dimension m, there exists a constant $C>0$ such that*

$$\left(\int_{M\setminus\partial M}|u|^{2m/(m-2)}\,d\mathrm{vol}(g)\right)^{(m-2)/m} \le C\int|\nabla u|_g^2\,d\mathrm{vol}(g) \quad \textit{for all } u\in\mathcal{C}_c^\infty(M\setminus\partial M). \tag{1.10}$$

Proof By [17, (3.20) and (4.16)], the heat kernel of g satisfies a Gaussian bound, which is well known to be equivalent to the existence of a constant $C>0$ such that (1.10) holds; see for instance [22]. □

Given an asymptotically conical metric, one can define corresponding Hölder spaces. However, in some situations — see for instance [13; 15] — it is more convenient to work with the weighted Hölder spaces associated to a conformally related metric, namely a b–metric in the sense of Melrose [34]. The same phenomenon arises for QAC–metrics, since the Hölder spaces introduced and used in [24; 17] are really those associated to a metric conformal to a QAC–metric.

Definition 1.29 Let (M,ϕ) be a QAC–manifold with fibered corners and let $x_{\max}$ be the product of the boundary-defining functions associated to all maximal boundary hypersurfaces of M. A smooth quasi b–metric on M (Qb–metric for short) is a metric g_{Qb} of the form

$$g_{\mathrm{Qb}} = x_{\max}^2 g_{\mathrm{QAC}}$$

for some smooth QAC–metric g_{QAC}.

As for a QAC–metric, the space

$$\mathcal{V}_{\mathrm{Qb}}(M) = \{\xi\in\mathcal{C}^\infty(M;TM) \mid \sup_{M\setminus\partial M} g_{\mathrm{Qb}}(\xi,\xi)<\infty\} \tag{1.11}$$

of smooth vector fields on M uniformly bounded with respect to g_{Qb} is in fact a Lie algebra. Indeed, we see from Definition 1.11 and (1.8) that this space can alternatively be defined as the space of b–vector fields ξ such that for each i,

- $\xi|_{H_i}$ is tangent to the fibers of ϕ_i if H_i is not maximal;
- $\xi v_i \in (v_i^2/x_{\max})\mathcal{C}^\infty(M)$,

which is clearly closed under the Lie bracket. In particular, the Lie algebra $\mathcal{V}_{\mathrm{Qb}}(M)$ does not depend on the choice of the Qb–metric for a fixed QAC–manifold with fibered corners and for a fixed choice of boundary-defining functions. Looking at the corresponding universal enveloping algebra over $\mathcal{C}^\infty(M)$, one can then define the space $\mathrm{Diff}^k_{\mathrm{Qb}}(M)$ of Qb–differential operators of order k to be the space of differential operators generated by $\mathcal{C}^\infty(M)$ and products of up to k elements of $\mathcal{V}_{\mathrm{Qb}}(M)$. In particular, we obtain the following analog of Proposition 1.27.

Proposition 1.30 *Each Qb–metric is a complete metric of infinite volume with bounded geometry.*

Proof A Qb–metric is a particular example of a metric with Lie structure at infinity in the sense of [3], so we can proceed as in the proof of Proposition 1.27 to conclude that it is complete of infinite volume with curvature and all its covariant derivatives bounded. To show that it has positive injectivity radius, we can proceed by induction on the depth as in the proof of Proposition 1.27. □

There are various functional spaces that we can associate to QAC–metrics and to Qb–metrics. We will be particularly interested in Hölder spaces. Recall that to a given complete metric g on $M\setminus\partial M$, a Euclidean vector bundle $E\to M\setminus\partial M$ with a compatible choice of connection and $k\in\mathbb{N}_0$, one can associate the space $\mathcal{C}^k_g(M\setminus\partial M;E)$ comprising continuous sections $f\colon M\setminus\partial M\to E$ such that

$$\nabla^j f\in\mathcal{C}^0(M\setminus\partial M;T^0_j(M\setminus\partial M)\otimes E)\quad\text{and}\quad\sup_{p\in M\setminus\partial M}|\nabla^j f(p)|_g<\infty \tag{1.12}$$

for all $j\in\{0,\dots,k\}$, where ∇ denotes the covariant derivative induced by the Levi-Civita connection of g and the connection on E, $|\cdot|_g$ is the norm induced by the metric g and the Euclidean structure on E, and

$$T^0_j(M\setminus\partial M)=\underbrace{T^*(M\setminus\partial M)\otimes\cdots\otimes T^*(M\setminus\partial M)}_{j\text{ times}}.$$

The space $\mathcal{C}^k_g(M \setminus \partial M; E)$ is in fact a Banach space with norm given by

$$\|f\|_{g,k} := \sum_{j=0}^{k} \sup_{p \in M \setminus \partial M} |\nabla^j f(p)|_g. \tag{1.13}$$

Taking the intersection over all k, we also get the Fréchet space

$$\mathcal{C}^\infty_g(M \setminus \partial M; E) = \bigcap_{k \in \mathbb{N}_0} \mathcal{C}^k_g(M \setminus \partial M; E).$$

For $\alpha \in (0,1]$ and $k \in \mathbb{N}_0$, we can also consider the Hölder space $\mathcal{C}^{k,\alpha}_g(M \setminus \partial M; E)$ of functions $f \in \mathcal{C}^k_g(M \setminus \partial M; E)$ such that

$$\begin{aligned}
&[\nabla^k f]_{g,\alpha} \\
&\quad := \sup\left\{ \frac{|P_\gamma(\nabla^k f(\gamma(0))) - \nabla^k f(\gamma(1))|}{\ell(\gamma)^\alpha} \;\middle|\; \gamma \in \mathcal{C}^\infty([0,1]; M \setminus \partial M),\ \gamma(0) \neq \gamma(1) \right\} \\
&\quad < \infty,
\end{aligned}$$

where $P_\gamma \colon (T^0_k(M \setminus \partial M) \otimes E)|_{\gamma(0)} \to (T^0_k(M \setminus \partial M) \otimes E)|_{\gamma(1)}$ is the parallel transport along γ and $\ell(\gamma)$ is the length of γ with respect to the metric g. This is also a Banach space with norm given by

$$\|f\|_{g,k,\alpha} := \|f\|_{g,k} + [\nabla^k f]_{g,\alpha}. \tag{1.14}$$

For $\rho \in \mathcal{C}^\infty(M \setminus \partial M)$ a positive function, we can also consider the weighted version

$$\begin{aligned}
&\rho\mathcal{C}^{k,\alpha}_g(M \setminus \partial M; E) \\
&\qquad := \left\{ f \;\middle|\; \frac{f}{\rho} \in \mathcal{C}^{k,\alpha}_g(M \setminus \partial M; E) \right\} \quad \text{with norm } \|f\|_{\rho\mathcal{C}^{k,\alpha}_g} := \left\| \frac{f}{\rho} \right\|_{g,k,\alpha}.
\end{aligned} \tag{1.15}$$

By choosing $g = g_{\mathrm{QAC}}$ to be a smooth QAC–metric, we get in particular the Banach spaces

$$\mathcal{C}^k_{\mathrm{QAC}}(M \setminus \partial M; E) \quad \text{and} \quad \mathcal{C}^{k,\alpha}_{\mathrm{QAC}}(M \setminus \partial M; E).$$

Notice however that they do not correspond to the Hölder spaces of [24, Section 9] and [17]. Indeed, [24, Section 9] and [17] consider instead weighted versions of the Banach spaces $\mathcal{C}^k_{\mathrm{Qb}}(M \setminus \partial M; E)$ and $\mathcal{C}^{k,\alpha}_{\mathrm{Qb}}(M \setminus \partial M; E)$ obtained by choosing $g = g_{\mathrm{Qb}} = x_{\max}^2 g_{\mathrm{QAC}}$ to be a smooth Qb–metric on M. The reason for this choice is that one can obtain nicer mapping properties for elliptic QAC–operators when acting on weighted Qb–Hölder spaces. Results stated in terms of Qb–Hölder spaces are also more precise, since, as one can easily check, there are continuous strict inclusions

$$\begin{aligned} \mathcal{C}^{k}_{\mathrm{Qb}}(M\setminus\partial M;E) &\subset \mathcal{C}^{k}_{\mathrm{QAC}}(M\setminus\partial M;E),\\ \mathcal{C}^{k,\alpha}_{\mathrm{Qb}}(M\setminus\partial M;E) &\subset \mathcal{C}^{k,\alpha}_{\mathrm{QAC}}(M\setminus\partial M;E). \end{aligned}\tag{1.16}$$

As in [24, Section 9] and [17], we will mostly deal with Qb–Hölder spaces and their weighted counterparts. Still, we will also have to deal with QAC–Hölder spaces. The following lemma will be helpful in these cases in bringing the discussion back to Qb–Hölder spaces.

Lemma 1.31 *For $0<\delta<1$, there is a continuous inclusion $x^{\delta}_{\max}\mathcal{C}^{0,1}_{\mathrm{QAC}}(M\setminus\partial M;E)\subset \mathcal{C}^{0,\alpha}_{\mathrm{Qb}}(M\setminus\partial M;E)$ for $\alpha\le\delta$.*

Proof Let g_{QAC} be a choice of smooth QAC–metric and consider the conformally related Qb–metric $g_{\mathrm{Qb}}:=x^2_{\max}g_{\mathrm{QAC}}$. Given $\gamma\in\mathcal{C}^\infty([0,1];M\setminus\partial M)$, let $\ell_{\mathrm{QAC}}(\gamma)$ and $\ell_{\mathrm{Qb}}(\gamma)$ denote the length of γ with respect to the metrics g_{QAC} and g_{Qb}. Then, using the Hölder inequality with $p=1/\delta$ and $q=1/(1-\delta)$, observe that

$$\begin{aligned} &\left|\frac{1}{x_{\max}(\gamma(0))^{\delta}}-\frac{1}{x_{\max}(\gamma(1))^{\delta}}\right|\\ &\qquad=\delta\left|\int_\gamma\frac{dx_{\max}}{x^{1+\delta}_{\max}}\right|=\delta\left|\int_\gamma\frac{dx_{\max}}{x^{2\delta}_{\max}x^{1-\delta}_{\max}}\right|\le\delta\left|\int_\gamma\frac{dx_{\max}}{x^{2}_{\max}}\right|^{\delta}\left|\int_\gamma\frac{dx_{\max}}{x_{\max}}\right|^{1-\delta}\\ &\qquad\le\delta(K\ell_{\mathrm{QAC}}(\gamma))^{\delta}(K\ell_{\mathrm{Qb}}(\gamma))^{1-\delta}\\ &\qquad=\delta K\ell_{\mathrm{QAC}}(\gamma)^{\delta}\ell_{\mathrm{Qb}}(\gamma)^{1-\delta} \end{aligned}\tag{1.17}$$

for some constant $K>0$ depending on g_{QAC} and g_{Qb}, but not on the path γ.

For $f\in x^{\delta}_{\max}\mathcal{C}^{0,1}_{\mathrm{QAC}}(M\setminus\partial M;E)$, consider the positive constant $C:=\|f/x^{\delta}_{\max}\|_{g_{\mathrm{QAC}},0,1}$. Then, for any path $\gamma\in\mathcal{C}^\infty([0,1];M\setminus\partial M)$, we have that

$$\begin{aligned} &2C\min\{\ell_{\mathrm{QAC}}(\gamma),1\}\\ &\quad\ge\left|\frac{P_\gamma(f(\gamma(0)))}{x_{\max}(\gamma(0))^{\delta}}-\frac{f(\gamma(1))}{x_{\max}(\gamma(1))^{\delta}}\right|\\ &\quad=\left|\frac{P_\gamma(f(\gamma(0)))}{x_{\max}(\gamma(0))^{\delta}}-\frac{f(\gamma(1))}{x_{\max}(\gamma(0))^{\delta}}+\frac{f(\gamma(1))}{x_{\max}(\gamma(0))^{\delta}}-\frac{f(\gamma(1))}{x_{\max}(\gamma(1))^{\delta}}\right|\\ &\quad\ge\frac{|P_\gamma(f(\gamma(0)))-f(\gamma(1))|}{x_{\max}(\gamma(0))^{\delta}}-\|f\|_{g_{\mathrm{QAC}},0}\left|\frac{1}{x_{\max}(\gamma(0))^{\delta}}-\frac{1}{x_{\max}(\gamma(1))^{\delta}}\right|\\ &\quad\ge\frac{|P_\gamma(f(\gamma(0)))-f(\gamma(1))|}{x_{\max}(\gamma(0))^{\delta}}-CK\delta\ell_{\mathrm{QAC}}(\gamma)^{\delta}\ell_{\mathrm{Qb}}(\gamma)^{1-\delta}, \end{aligned}\tag{1.18}$$

where we have used (1.17) and the inequality $\|f\|_{g_{\mathrm{QAC}},0} \le \|f/x_{\max}^{\delta}\|_{g_{\mathrm{QAC}},0,1} = C$ in the last step. Thus, we infer from (1.18) that for all $\gamma \in \mathcal{C}^{\infty}([0,1]; M \setminus \partial M)$,

$$|P_\gamma(f(\gamma(0))) - f(\gamma(1))| \le 2Cx_{\max}(\gamma(0))^{\delta} \min\{\ell_{\mathrm{QAC}}(\gamma), 1\} + CK\delta x_{\max}(\gamma(0))^{\delta}\ell_{\mathrm{QAC}}(\gamma)^{\delta}\ell_{\mathrm{Qb}}(\gamma)^{1-\delta}. \tag{1.19}$$

Thus, if $\ell_{\mathrm{Qb}}(\gamma) \ge 1$, then

$$\begin{aligned}\frac{|P_\gamma(f(\gamma(0))) - f(\gamma(1))|}{\ell_{\mathrm{Qb}}(\gamma)^{\alpha}} &\le |P_\gamma(f(\gamma(0))) - f(\gamma(1))| \\ &\le 2\|f\|_{g_{\mathrm{QAC}},0} \le 2\left\|\frac{f}{x_{\max}^{\delta}}\right\|_{g_{\mathrm{QAC}},0,1} \\ &= 2C \le 2C + CK\delta.\end{aligned} \tag{1.20}$$

If instead $\ell_{\mathrm{Qb}}(\gamma) \le 1$, let $t_{\min} \in [0,1]$ be such that

$$x_{\max}(\gamma(t_{\min})) = \min\{x_{\max}(\gamma(t)) : t \in [0,1]\},$$

so that

$$\left|\log\left(\frac{x_{\max}(\gamma(t_{\min}))}{x_{\max}(\gamma(0))}\right)\right| \le \left|\int_0^{t_{\min}} \frac{dx_{\max}}{x_{\max}} \circ \gamma\right| \le K\ell_{\mathrm{Qb}}(\gamma) \le K$$
$$\implies \frac{x_{\max}(\gamma(0))}{x_{\max}(\gamma(t_{\min}))} \le e^{\ell_{K\mathrm{Qb}}(\gamma)} \le e^{K},$$

where $K > 0$ is the constant occurring in (1.17). Then (1.19) yields

$$\begin{aligned}&\frac{|P_\gamma(f(\gamma(0))) - f(\gamma(1))|}{\ell_{\mathrm{Qb}}(\gamma)^{\alpha}} \\ &\le \frac{2Cx_{\max}(\gamma(0))^{\delta}\min\{\ell_{\mathrm{QAC}}(\gamma),1\}}{\ell_{\mathrm{Qb}}(\gamma)^{\alpha}} + \frac{CK\delta x_{\max}(\gamma(0))^{\delta}\ell_{\mathrm{QAC}}(\gamma)^{\delta}\ell_{\mathrm{Qb}}(\gamma)^{1-\delta}}{\ell_{\mathrm{Qb}}(\gamma)^{\alpha}} \\ &\le \frac{2Cx_{\max}(\gamma(0))^{\delta}\min\{\ell_{\mathrm{QAC}}(\gamma),1\}}{(x_{\max}(\gamma(t_{\min})))^{\alpha}\ell_{\mathrm{QAC}}(\gamma)^{\alpha}} + \frac{CK\delta x_{\max}(\gamma(0))^{\delta}\ell_{\mathrm{QAC}}(\gamma)^{\delta}\ell_{\mathrm{Qb}}(\gamma)^{1-\alpha}}{(x_{\max}(\gamma(t_{\min})))^{\delta}\ell_{\mathrm{QAC}}(\gamma)^{\delta}} \\ &\le 2Ce^{K\alpha} + CK\delta\ell_{\mathrm{Qb}}(\gamma)^{1-\alpha}e^{K\delta} \le (2+K\delta)Ce^{K\delta},\end{aligned} \tag{1.21}$$

since $\alpha \le \delta$. Hence, combining (1.20) with (1.21) and taking the supremum over γ yields

$$[f]_{g_{\mathrm{Qb}},0,\alpha} \le (2+K\delta)Ce^{K\delta} = (2+K\delta)e^{K\delta}\left\|\frac{f}{x_{\max}^{\delta}}\right\|_{g_{\mathrm{QAC}},0,1},$$

from which the result follows. □

2 Manifolds with fibered corners coming from Sasaki–Einstein orbifolds

Let Z be a closed orbifold of real dimension $2n+1$. A *Ricci-flat Kähler cone metric* on the Cartesian product $C := \mathbb{R}^+ \times Z$ is a cone metric

$$g_C = dr^2 + r^2 g_Z,$$

where g_Z is some Riemannian metric on Z, together with a complex structure J_C on C such that g_C is Ricci-flat and Kähler with respect to J_C with Kähler form given by

$$\omega_C = \tfrac{\sqrt{-1}}{2}\partial\overline{\partial} r^2.$$

In particular, when such a metric g_C and complex structure J_C exist, the corresponding canonical line bundle K_C of C is flat with respect to the Chern connection induced from g_C and J_C.

Notice then that the metric g_Z is Sasakian and that there is a corresponding *Reeb vector field* on Z given by

$$\xi = J_C \frac{\partial}{\partial r}.$$

The orbits of the flow of this vector field induce a foliation called the *Reeb foliation*. In the present paper, we will always assume that the Ricci-flat Kähler cone (C, g_C, J_C) is *quasiregular* in the following sense.

Definition 2.1 The Ricci-flat Kähler cone (C, g_C, J_C) is *quasiregular* if there exists a Kähler–Einstein Fano orbifold (D, g_D) and a holomorphic line bundle L over D with Hermitian metric h_L such that $L \setminus D$ is biholomorphic to (C, J_C) with the properties that on $L \setminus D$:

(1) The radial function r is given by

$$r = \|\cdot\|_{h_L}^{1/q} \quad \text{on } L \setminus D$$

for some $q \in \mathbb{N}$ and $Z = r^{-1}(1)$.

(2) The Reeb foliation corresponds to the fibers of the unit circle bundle of (L, h_L), that is,

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] & Z \arrow[d, "\nu_L"] \\ & D \end{tikzcd} \tag{2.1}$$

(3) The map ν_L in (2.1) is an orbifold Riemannian submersion from (Z, g_Z) to (D, g_D).

(4) For some $p \in \mathbb{N}$, the line bundle K_C^p has a nowhere vanishing holomorphic section $\Omega_C^p \in H^0(C; K_C^p)$ which is parallel with respect to the induced Chern connection from K_C and J_C. In particular, there is a nonzero constant c_p such that

$$(\omega_C^{n+1})^p = c_p \Omega_C^p \wedge \overline{\Omega_C^p}, \tag{2.2}$$

where $(\omega_C^{n+1})^p$ is the tensorial product of ω_C^{n+1} with itself p times seen as a section of $K_C \otimes \overline{K}_C$.

Furthermore, when $p = 1$ in item (4), we say that (C, g_C, J_C) is a *quasiregular Calabi–Yau cone*.

As a simple computation shows, the Kähler–Einstein constant of the metric g_D is completely determined by the previous definition, namely

$$\mathrm{Ric}(g_D) = 2(n+1) g_D.$$

Furthermore, the Kähler form of g_D is such that

$$\sqrt{-1}\Theta_{h_L} = -2q\omega_D, \tag{2.3}$$

where Θ_{h_L} is the curvature of (L, h_L) with respect to the Chern connection.

Remark 2.2 Let us clarify that we assume that L is locally trivial in the sense that in a neighborhood of any point p on D, there are orbifold charts $\psi\colon \mathcal{U} \to \mathbb{C}^n/\Gamma$ and $\psi_L\colon p_L^{-1}(\mathcal{U}) \to (\mathbb{C}^n \times \mathbb{C})/\Gamma$ inducing a commutative diagram

$$\begin{tikzcd} p_L^{-1}(\mathcal{U}) \arrow[r, "\psi_L"] \arrow[d, "p_L"] & (\mathbb{C}^n \times \mathbb{C})/\Gamma \arrow[d, "\mathrm{pr}_1"] \\ \mathcal{U} \arrow[r, "\psi"] & \mathbb{C}^n/\Gamma \end{tikzcd} \tag{2.4}$$

where pr_1 is the projection on the first factor. In particular, although we do assume that C and Z may have orbifold singularities, not all singularities of D correspond to singularities of Z and C. Indeed, even if Z were smooth, as a space of leaves, D could still be singular. Also, notice that a point $p \in Z$ is singular if and only if all points of $\nu_L^{-1}(\nu_L(p))$ are singular.

In fact, a simple way to construct examples of quasiregular Calabi–Yau cone metrics is to start off with a Kähler–Einstein Fano orbifold (D, g_D) and to apply the Calabi ansatz [8; 29], which yields a Calabi–Yau cone $(K_D \setminus D, g_C)$ with the zero section $D \subset K_D$ corresponding to the apex of the cone, where Z is then the total space of the unit circle bundle of K_D over D. The Kähler form ω_C of g_C can then be written explicitly in terms of the Kähler–Einstein metric g_D as

$$\omega_C = \tfrac{\sqrt{-1}}{2}\partial\bar{\partial}|\cdot\|_{g_D}^{2/(n+1)}, \tag{2.5}$$

where $\|\cdot\|_{g_D}$ is the Hermitian metric on K_D induced by g_D seen as a function on $K_D \setminus D$.

To describe the natural holomorphic volume form associated to this Calabi–Yau cone, recall that the space of sections $H^0(K_D; \pi^*(K_D))$ has a tautological element ϖ given by $\varpi_p = \pi^* p$ for $p \in K_D$, where $\pi\colon K_D \to D$ is the canonical projection. For any choice of local coordinates $(z_1, \dots, z_n)$ on D, we have a local section $dz_1 \wedge \cdots \wedge dz_n$ of K_D, and hence local coordinates $(z_1, \dots, z_n, v)$ on K_D with $(z_1, \dots, z_n, v)$ corresponding to $v\, dz_1 \wedge \cdots \wedge dz_n \in K_D|_{(z_1,\dots,z_n)}$. In these local coordinates, ϖ is simply given by

$$\varpi = v\, dz_1 \wedge \cdots \wedge dz_n.$$

Taking the exterior derivative of the tautological element ϖ yields the canonical holomorphic volume form Ω_C of K_D, namely,

$$\Omega_C := d\varpi = \partial\varpi \in H^0(K_D; K_{K_D}).$$

The fact that the cone metric g_C is Calabi–Yau amounts to the fact that

$$\omega_C^{n+1} = c\Omega_C \wedge \bar{\Omega}_C$$

for some fixed nonzero constant c.

More generally, if $p_L\colon L \to D$ is a q^{th} root of K_D, namely if $L^{\otimes q} = K_D$, then the natural map

$$\begin{aligned} Q\colon L \setminus D &\to K_D \setminus D, \\ \sigma &\mapsto \sigma^{\otimes q}, \end{aligned} \tag{2.6}$$

is a $\mathbb{Z}_q$–cover of $K_D \setminus D$, so that pulling back Ω_C and ω_C to $L \setminus D$ endows $L \setminus D$ with an orbifold Calabi–Yau cone structure. Conversely, given such a line bundle L, we have that $\mathbb{Z}_p$, seen as the group of p^{th} roots of unity, naturally acts by isometry on $(L \setminus D, Q^* g_C)$ via the natural $\mathbb{S}^1$–action, and the quotient, which is naturally identified with $L^p \setminus D$, naturally inherits the structure of a quasiregular Ricci-flat Kähler cone since the $\mathbb{Z}_p$–action is trivial on $(Q^*\Omega_C)^p \in H^0(L \setminus D; K^p_{L\setminus D})$.

Now, suppose that (C, g_C) is a quasiregular Ricci-flat Kähler cone with holomorphic parallel section $\Omega^p_C \in H^0(C; K^p_C)$ as above and suppose that X is a Kähler orbifold with a nowhere vanishing holomorphic section $\Omega^p_X \in H^0(X; K^p_X)$ such that for some compact subset $\mathcal{K} \subset X$, there is a biholomorphism

$$X \setminus \mathcal{K} \cong (\kappa, \infty) \times Z \subset C \tag{2.7}$$

for some $\kappa \geq 0$ identifying Ω^p_X with Ω^p_C. Assume further that the singular set of X has complex codimension at least 2 and that X admits a local product Kähler crepant resolution $\hat{X} \to X$ in the sense of [24] such that Ω^p_X lifts to a nowhere vanishing holomorphic section $\Omega^p_{\hat{X}} \in H^0(\hat{X}; K^p_{\hat{X}})$. In the remainder of this section, we will explain how the resolution $\hat{X}$ can then be naturally compactified as a manifold with fibered corners. However, before doing that, let us give some examples of such a space X.

Example 2.3 We can take $X = \mathbb{C}^{n+1}/\Gamma$ for some finite subgroup $\Gamma \subset \mathrm{SU}(n+1)$. It is not automatic that X will admit a local product Kähler crepant resolution, but for many choices of Γ it will [24, Sections 6.4–6.6]. For instance, if $\Gamma \subset \mathrm{SU}(2) \subset \mathrm{SU}(n+1)$, there is automatically a crepant resolution by [28]. On the other hand, we can take $(C, g_C) = ((\mathbb{C}^{n+1}/\Gamma) \setminus \{0\}, g_E)$, where g_E is the Euclidean metric on $\mathbb{C}^{n+1}/\Gamma$ and $\mathcal{K} = \{0\}$, to obtain trivially that $X \setminus \{0\}$ is biholomorphic to C.

Example 2.4 Let (D, g_D) be a Kähler–Einstein Fano orbifold of complex dimension n and assume that D admits a local product (Kähler) crepant resolution $\hat{D}$. This automatically implies that the total space $K_{\hat{D}}$ of the canonical line bundle of $\hat{D}$ is a local product (Kähler) crepant resolution of K_D and that the diagram

$$\begin{tikzcd} K_{\hat{D}} \arrow[r, "\beta_K"] \arrow[d] & K_D \arrow[d] \\ \hat{D} \arrow[r, "\beta"] & D \end{tikzcd} \tag{2.8}$$

commutes, where the vertical maps are the canonical projections and the horizontal maps are the blow-down maps of the local product resolutions. Thus, in this setting, we can take $X = K_D$ and the Calabi–Yau cone $C = K_D \setminus D$ with Kähler metric given by (2.5).

Remark 2.5 In the previous example, we could for instance take $D = \mathbb{CP}^n/\Gamma$ with $\Gamma \subset \mathrm{SU}(n+1)$ a finite subgroup. In this case, the Fubini–Study metric descends to a Kähler–Einstein metric g_D on D. On the other hand, D does not always admit a local product Kähler crepant resolution, but in some cases it does, for instance in Example A.2 in the appendix or when $n=2$ and $\Gamma \subset \mathrm{SU}(3)$ is the subgroup generated by the diagonal matrix with diagonal entries given by $e^{2\pi i/3}$, $e^{-2\pi i/3}$ and 1. We could also take D to be a Kähler–Einstein log del Pezzo surface with canonical singularities of degree at most 4; see [39] for examples. According to [39, page 165], D then has only singularities of type D_4 and A_k for $k \le 7$, so automatically admits a local product Kähler crepant resolution by proceeding as in [28]. Finally, notice that if (D_1, g_{D_1}) and (D_2, g_{D_2}) are two examples of Kähler–Einstein Fano orbifolds admitting local product Kähler crepant resolutions, then after scaling the metrics, we can assume without loss of generality that $\mathrm{Ric}(g_{D_i}) = g_{D_i}$ for $i = 1, 2$, so that the Cartesian product $(D_1 \times D_2, g_{D_1} \times g_{D_2})$ is another example of a Kähler–Einstein Fano orbifold admitting a local Kähler crepant resolution.

To describe the natural compactification of $\widehat{X}$, observe first that we can compactify X into an orbifold with boundary X_{sc} such that the biholomorphism (2.7) extends to a diffeomorphism

$$X_{\mathrm{sc}} \setminus \mathcal{K} \cong (\kappa, \infty] \times Z.$$

Here, we mean that X_{sc} is an *orbifold with boundary* in the sense that it is locally modeled by charts of the form

$$\mathbb{R}^{2n+2}/\Gamma_1 \quad \text{or} \quad (\mathbb{R}^{2n+1}/\Gamma_2) \times [0, \infty),$$

with $\Gamma_1 \subset \mathrm{GL}(2n+2, \mathbb{R})$ and $\Gamma_2 \subset \mathrm{GL}(2n+1, \mathbb{R})$ finite subgroups. Notice in particular that with this definition, the orbifold singularities are always transversal to the boundary. Clearly, we have that $\partial X_{\mathrm{sc}} \cong Z$ and that the function $x := 1/r$ defined near ∂X_{sc} is a natural choice of boundary-defining function.

As an orbifold, ∂X_{sc} has the structure of a smoothly stratified space with strata corresponding to the various types of orbifold singularities. In particular, the strata are partially ordered by the relation

$$s_1 \le s_2 \iff s_1 \subset \overline{s}_2.$$

In fact, by Remark 2.2, each stratum $\overline{s}_i$ of ∂X_{sc} is naturally equipped with an $\mathbb{S}^1$–orbibundle structure

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] & \overline{s}_i \arrow[d, "\nu_i"] \\ & \overline{\sigma}_i \end{tikzcd} \tag{2.9}$$

in which $\overline{\sigma}_i := \nu_i(\overline{s}_i)$. The partial order on the strata of ∂X_{sc} gives us a systematic way of blowing up the strata of ∂X_{sc} in X_{sc} in the sense of [1]. Namely, let $\{s_1, \dots, s_k\}$ be an exhaustive list of all the strata of ∂X_{sc} compatible with the partial order in the sense that

$$s_i \le s_j \implies i \le j.$$

In particular, s_k will be the regular stratum, so that $\partial X_{\mathrm{sc}} = \overline{s}_k$. One constructs a natural space $\widetilde{X}_{\mathrm{sc}}$ out of X_{sc} by blowing up all of the strata of ∂X_{sc} in X_{sc} except the regular one; that is,

$$\widetilde{X}_{\mathrm{sc}} = [X_{\mathrm{sc}}; \overline{s}_1, \overline{s}_2, \dots, \overline{s}_{k-1}] \quad \text{with blow-down map } \beta_{\mathrm{sc}}\colon \widetilde{X}_{\mathrm{sc}} \to X_{\mathrm{sc}}. \tag{2.10}$$

Each of these blow-ups creates a new boundary hypersurface, so that $\widetilde{X}_{\mathrm{sc}}$ is not an orbifold with boundary, but rather an *orbifold with corners*, by which we mean that $\widetilde{X}_{\mathrm{sc}}$ is locally modeled on charts of the form $\mathbb{R}^{2n+2-q}/\,\Gamma \times [0,\infty)^q$ for $q \in \{0, 1, \dots, 2n+2\}$ and $\Gamma \subset \mathrm{GL}(2n+2-q, \mathbb{R})$ a finite subgroup. By construction, $\widetilde{X}_{\mathrm{sc}}$ has a boundary hypersurface H_i for each stratum s_i of ∂X_{sc}. Since we blow up each $\overline{s}_i$ in X_{sc} instead of in ∂X_{sc}, notice that each boundary hypersurface H_i remains an orbifold for each $i < k$, whereas H_k is the manifold with fibered corners that resolves the stratified space ∂X_{sc}, hence contains no orbifold singularities. As we are about to show, there is a natural fiber bundle structure on each boundary hypersurface, making $\widetilde{X}_{\mathrm{sc}}$ an orbifold with fibered corners in the following sense.

Definition 2.6 Let M be an orbifold with corners and suppose that each of its boundary hypersurfaces H_i is a fiber bundle $\phi_i\colon H_i \to S_i$ whose base S_i is a manifold with corners (hence has no orbifold singularities) and whose fibers are all orbifolds with fibered corners. If ϕ denotes the collection of fiber bundle maps ϕ_i, then we say that (M, ϕ) is an *orbifold with fibered corners* if there exists a partial order on the boundary hypersurfaces such that the same conditions as in Definition 1.1 are satisfied, namely:

- Any subset I of boundary hypersurfaces such that $\bigcap_{i \in I} H_i \ne \varnothing$ is totally ordered.

- If $H_i < H_j$, then $H_i \cap H_j \neq \varnothing$, $\phi_i|_{H_i \cap H_j}\colon H_i \cap H_j \to S_i$ is a surjective submersion and $S_{ji} := \phi_j(H_i \cap H_j)$ is one of the boundary hypersurfaces of the manifold with corners S_j. Moreover, there is a surjective submersion $\phi_{ji}\colon S_{ji} \to S_i$ such that $\phi_{ji} \circ \phi_j = \phi_i$ on $H_i \cap H_j$.
- The boundary hypersurfaces of S_j are given by the S_{ji} for $H_i < H_j$.

Proposition 2.7 *The orbifold with corners $\widetilde{X}_{\mathrm{sc}}$ itself has a natural orbifold with fibered corners structure.*

Proof Before we perform the blow-ups for larger strata, the closure of the stratum s_i lifts to a submanifold S_i of $[X_{\mathrm{sc}}; \overline{s}_1, \dots, \overline{s}_{i-1}]$, so that the blow-up face associated to s_i is just the radial compactification $\overline{NS_i}$ of the normal bundle of S_i (as a suborbifold of a boundary hypersurface of $[X_{\mathrm{sc}}; \overline{s}_1, \dots, \overline{s}_{i-1}]$), whose fibers are orbifolds of the form $\overline{V}_i = \overline{\mathbb{C}^{m_i} / \Gamma_i}$ with $m_i = n - \dim_{\mathbb{C}} \sigma_i$ and $\Gamma_i \subset \mathrm{SU}(n_i)$ a finite subgroup, that is,

$$\begin{array}{ccc} \overline{V}_i & \longrightarrow & \overline{NS_i} \\ & & \Big\downarrow{\scriptstyle \phi_i} \\ & & S_i \end{array} \tag{2.11}$$

The subsequent blow-ups of strata modify the face associated to s_i, but only in the fibers of (2.11), so that ultimately H_i comes equipped with a fiber bundle structure

$$\begin{array}{ccc} \widetilde{V}_i & \longrightarrow & H_i \\ & & \Big\downarrow{\scriptstyle \phi_i} \\ & & S_i \end{array} \tag{2.12}$$

with $\widetilde{V}_i$ obtained from $\overline{V}_i$ by blowing up the singular strata of $\partial \overline{V}_i$ in $\overline{V}_i$ in an order compatible with the partial order of the strata of $\partial \overline{V}_i$. In other words, $\overline{V}_i$ is naturally an orbifold with boundary and $\widetilde{V}_i$ is obtained from $\overline{V}_i$ in the same way that $\widetilde{X}_{\mathrm{sc}}$ is obtained from X_{sc}.

Notice that thanks to the order in which we do the blow-ups, S_i is the natural manifold with fibered corners that resolves the closure $\overline{s}_i$ seen as a stratified space. In particular, it has no orbifold singularities and the singularities of the face H_i are all in the fibers of (2.12). Now, clearly, since each boundary hypersurface of $\widetilde{X}_{\mathrm{sc}}$ is associated to a stratum of ∂X_{sc}, the partial order of the strata of ∂X_{sc} induces a partial order on the boundary hypersurfaces of $\widetilde{X}_{\mathrm{sc}}$. Moreover, we clearly have that

$$H_i \cap H_j \neq \varnothing \qquad \Longleftrightarrow \qquad s_i \leq s_j \quad \text{or} \quad s_j \leq s_i,$$

and we see that the partial order on the boundary hypersurfaces of $\widetilde{X}_{\mathrm{sc}}$ totally orders any subset I of the boundary hypersurfaces with $\bigcap_{i\in I} H_i \neq \varnothing$. Finally, if $H_i < H_j$, then it follows from the order in which we blew up the strata that ϕ_j restricts to a surjective submersion on $H_i \cap H_j$ onto the boundary hypersurface S_{ji} of S_j, and that the fiber bundle structure $\phi_{ji}\colon S_{ji} \to S_i$ coming from the orbifold with fibered corners of S_j is such that $\phi_{ji} \circ \phi_j = \phi_i$. □

The space $\widetilde{X}_{\mathrm{sc}}$ provides a compactification of X, but it still has orbifold singularities. On the other hand, since we only blew up strata of ∂X_{sc}, we still have a natural identification $\widetilde{X}_{\mathrm{sc}} \setminus \partial \widetilde{X}_{\mathrm{sc}} = X$, so we can still use the local product Kähler crepant resolution of X to remove the orbifold singularities.

Theorem 2.8 *The local product Kähler crepant resolution $\beta_X\colon \widehat{X} \to X$ naturally extends to give a resolution $\beta_{\mathrm{QAC}}\colon \widehat{X}_{\mathrm{QAC}} \to \widetilde{X}_{\mathrm{sc}}$ of the orbifold with fibered corners $\widetilde{X}_{\mathrm{sc}}$ by a manifold with fibered corners $\widehat{X}_{\mathrm{QAC}}$, inducing the commutative diagram*

$$\begin{tikzcd}
\widehat{X} \arrow[rr, hook] \arrow[d, "\beta_X"] & & \widehat{X}_{\mathrm{QAC}} \arrow[d, "\beta_{\mathrm{QAC}}"] \\
X \arrow[r, hook] & X_{\mathrm{sc}} & \widetilde{X}_{\mathrm{sc}} \arrow[l, "\beta_{\mathrm{sc}}"']
\end{tikzcd}$$

The space $\widehat{X}_{\mathrm{QAC}}$ is called the ***QAC–compactification of*** $\widehat{X}$ *and we say that it is a* ***QAC–resolution of*** $\widetilde{X}_{\mathrm{sc}}$.

Proof If $\sigma_1 = \nu_1(s_1)$ is a point, then $S_1 = s_1 = \mathbb{S}^1$ and $H_1 = \overline{V}_1 \times \mathbb{S}^1$ with $\phi_1\colon \overline{V}_1 \times \mathbb{S}^1 \to \mathbb{S}^1$ given by the projection on the right factor and $\overline{V}_1 = \overline{\mathbb{C}^n/\Gamma_1}$ for a certain finite subgroup $\Gamma_1 \subset \mathrm{SU}(n)$ acting freely on $\mathbb{C}^n \setminus \{0\}$. Here, $\mathbb{C}^n/\Gamma_1$ can be seen as an orbifold chart for the corresponding stratum σ_1 in D. Clearly then, the crepant resolution of X extends to one for $\widetilde{X}_{\mathrm{sc}}$ and H_1. If $\widehat{H}_1$ denotes the induced resolution of H_1, then $\widehat{H}_1 = \widehat{Y}_1 \times \mathbb{S}^1$, where $\widehat{Y}_1$ is the radial compactification of the crepant resolution Y_1 of $\mathbb{C}^n/\Gamma_1$, so that we still have a natural fiber bundle $\widehat{\phi}_1\colon \widehat{H}_1 \to S_1$ which is just the projection $\widehat{Y}_1 \times \mathbb{S}^1 \to \mathbb{S}^1$ on the right factor. If σ_1 is not a point but $\overline{s}_1$ is still a singularity of relative depth one, then we can proceed in the same way, that is, the crepant resolution of X clearly extends to give a resolution of $\widetilde{X}_{\mathrm{sc}}$ near H_1 with resolution $\widehat{H}_1$ of H_1 obtained from (2.11) by replacing each fiber $\overline{V}_1$ by its crepant resolution $\widehat{Y}_1$, that is,

$$\begin{tikzcd} \widehat{Y}_1 \arrow[r] & \widehat{H}_1 \arrow[d, "\widehat{\phi}_1"] \\ & S_1 \end{tikzcd} \tag{2.13}$$

More generally, using the fact that the resolution $\widehat{X} \to X$ is a local product resolution, we see by induction on the depth of X_{sc} that $\widehat{H}_i$ will be the total space of a fiber bundle

$$\begin{tikzcd} \widehat{Y}_i \arrow[r] & \widehat{H}_i \arrow[d, "\widehat{\phi}_i"] \\ & S_i \end{tikzcd} \tag{2.14}$$

with $\widehat{Y}_i$ the QAC–resolution of $\widetilde{V}_i$. Indeed, the local product Kähler crepant resolution of X naturally induces one on V_i, and since $\widetilde{V}_i$ is an orbifold with fibered corners of smaller depth than X_{sc}, we can assume by induction that the theorem already holds true for $\widetilde{V}_i$.

Since at each step the local product resolution $\widehat{X} \to X$ is only used fiberwise in the fiber orbibundle $\phi_i \colon H_i \to S_i$, we see that $\widehat{X}_{\mathrm{QAC}}$ is naturally a manifold with fibered corners with fiber bundle structure on the boundary hypersurface $\widehat{H}_i$ given by (2.14) and with partial order on the boundary hypersurfaces of $\widehat{X}_{\mathrm{QAC}}$ induced by the one on the boundary hypersurface of $\widetilde{X}_{\mathrm{sc}}$. □

Notice that for the face $\widehat{H}_k$ associated to the regular stratum s_k, we have that $S_k = H_k = \widehat{H}_k$ and that the fiber bundle $\widehat{\phi}_k \colon \widehat{H}_k \to S_k$ is just the identity map. Since the only maximal stratum with respect to the partial order is the regular stratum, we see that $\widehat{H}_k$ is the only maximal boundary hypersurface of $\widehat{X}_{\mathrm{QAC}}$.

Using Lemma 1.16, we can also specify a natural Lie algebra of QAC–vector fields on $\widetilde{X}_{\mathrm{sc}}$, that is, a natural choice of QAC–equivalence class of boundary-defining functions. Indeed, let $\beta_0 := \beta_{\mathrm{sc}} \colon \widetilde{X}_{\mathrm{sc}} \to X_{\mathrm{sc}}$ be the blow-down map and for each $i \geq 1$, consider the partial blow-down maps

$$\beta_i \colon \widetilde{X}_{\mathrm{sc}} \to [X_{\mathrm{sc}}; \overline{s}_1, \ldots, \overline{s}_i] \quad \text{and} \quad \beta_i' \colon [X_{\mathrm{sc}}; \overline{s}_1, \ldots, \overline{s}_i] \to X_{\mathrm{sc}}, \tag{2.15}$$

so that $\beta_0 = \beta_i' \circ \beta_i$ for each i. For each $i \geq 1$, consider the lift of ∂X_{sc} to $[X_{\mathrm{sc}}; \overline{s}_1, \ldots, \overline{s}_i]$, namely

$$B_i := \overline{(\beta_i')^{-1}(\partial X_{\mathrm{sc}} \setminus (\overline{s}_1 \cup \cdots \cup \overline{s}_i))}.$$

Let also W_i be the boundary hypersurface of $[X_{\mathrm{sc}}; \overline{s}_1, \dots, \overline{s}_i]$ coming from the blow-up of $\overline{s}_i$ and let ρ_0 be any choice of boundary-defining function for $B_0 := \partial X_{\mathrm{sc}}$ in X_{sc}. More generally, proceeding recursively on i, choose a boundary-defining function $\rho_i \in \mathcal{C}^\infty([X_{\mathrm{sc}}; \overline{s}_1, \dots, \overline{s}_i])$ of B_i such that ρ_i is identically equal to the pullback of $\beta^*_{i,i-1}\rho_{i-1}$ outside a small neighborhood $\mathcal{U}_i$ of W_i not intersecting the boundary hypersurfaces of $[X_{\mathrm{sc}}; \overline{s}_1, \dots, \overline{s}_i]$ disjoint from W_i, where

$$\beta_{i,i-1}\colon [X_{\mathrm{sc}}; \overline{s}_1, \dots, \overline{s}_i] \to [X_{\mathrm{sc}}; \overline{s}_1, \dots, \overline{s}_{i-1}]$$

is the blow-down map. Then, on $\widetilde{X}_{\mathrm{sc}}$, the functions

$$x_i = \frac{\beta^*_{i-1}\rho_{i-1}}{\beta^*_i \rho_i} \quad \text{for } i < k \qquad \text{and} \qquad x_k := \rho_k \tag{2.16}$$

are such that $x_i \in \mathcal{C}^\infty(\widetilde{X}_{\mathrm{sc}})$ is a boundary-defining function for H_i for each i.

Lemma 2.9 *On $(\widetilde{X}_{\mathrm{sc}}, \phi)$, the Lie algebra of QAC–vector fields specified by the choice of the QAC–equivalence class of the boundary-defining functions (2.16) does not depend on the choice of the functions ρ_i, hence yields a natural Lie algebra of QAC–vector fields.*

Proof Suppose that for each i, ρ_i and ρ'_i are two different choices of boundary-defining functions and suppose recursively that ρ_i, respectively ρ'_i, is identically equal to $\beta^*_{i,i-1}\rho_{i-1}$, respectively $\beta^*_{i,i-1}\rho'_{i-1}$, outside a small neighborhood $\mathcal{U}_i$ of W_i not intersecting the boundary hypersurfaces of $[X_{\mathrm{sc}}; \overline{s}_1, \dots, \overline{s}_i]$ disjoint from W_i. In this case, essentially by definition of the blow-down map, we have that for $j > i$,

$$\left.\frac{\beta^*_i \rho_i}{\beta^*_i \rho'_i}\right|_{H_j} = \phi^*_j h_{ij} \quad \text{for some } h_{ij} \in \mathcal{C}^\infty(S_j). \tag{2.17}$$

On the other hand, thanks to the identification of ρ_i with $\beta^*_{i,i-1}\rho_{i-1}$ outside $\mathcal{U}_i$, we also have that for $H_j \geq H_i$,

$$\left.\frac{\beta^*_{i-1}\rho_{i-1}}{v_i}\right|_{H_j} = \phi^*_j f_{ij} \quad \text{for some } f_{ij} \in \mathcal{C}^\infty(S_j), \text{ where } v_i = \prod_{H_j \geq H_i} x_i. \tag{2.18}$$

Of course, there is a similar statement for ρ'_i, so the combination of (2.17) and (2.18) allows us to apply Lemma 1.16, from which the result follows. □

Provided that we can choose the functions ρ_i so that their lifts $\beta_{\mathrm{QAC}}^*\beta_i^*\rho_i$ to $\widehat{X}_{\mathrm{QAC}}$ are smooth, we obtain corresponding boundary-defining functions on $\widehat{X}_{\mathrm{QAC}}$,

$$(2.19)\quad \hat{x}_i = \beta_{\mathrm{QAC}}^* x_i = \frac{\beta_{\mathrm{QAC}}^*\beta_{i-1}^*\rho_{i-1}}{\beta_{\mathrm{QAC}}^*\beta_i^*\rho_i} \quad \text{for } i<k, \qquad \hat{x}_k := \beta_{\mathrm{QAC}}^* x_k = \beta_{\mathrm{QAC}}^*\rho_k.$$

Lemma 2.10 *The QAC–equivalence class of the boundary-defining functions (2.19) does not depend on the choice of the functions ρ_i that lift to be smooth on $\widehat{X}_{\mathrm{QAC}}$.*

Proof Given two different choices ρ_i and ρ_i', we can proceed as in the proof of Lemma 2.9 with (2.17) and (2.18) replaced respectively by

$$(2.20)\qquad \left.\frac{\beta_{\mathrm{QAC}}^*\beta_i^*\rho_i}{\beta_{\mathrm{QAC}}^*\beta_i^*\rho_i'}\right|_{\widehat{H}_j} = \hat{\phi}_j^* h_{ij} \quad \text{for some } h_{ij}\in\mathcal{C}^\infty(S_j)$$

and

$$(2.21)\quad \left.\frac{\beta_{\mathrm{QAC}}^*\beta_{i-1}^*\rho_{i-1}}{\hat{v}_i}\right|_{\widehat{H}_j} = \hat{\phi}_j^* f_{ij} \quad \text{for some } f_{ij}\in\mathcal{C}^\infty(S_j), \text{ where } \hat{v}_i = \prod_{\widehat{H}_j\ge\widehat{H}_i}\hat{x}_i. \qquad \square$$

Thus, to see that $\widehat{X}_{\mathrm{QAC}}$ comes endowed with a natural Lie algebra of QAC–vector fields, we need to show that the functions ρ_i can be chosen in such a way that they lift to be smooth on $\widehat{X}_{\mathrm{QAC}}$, a discussion that we postpone until Lemma 3.4.

3 The Ricci-flat Kähler cone metric seen as a QAC–metric

Continuing with the setup of the previous section, we will show in this section how an orbifold Ricci-flat Kähler cone metric can be seen as a QAC–metric on $\widetilde{X}_{\mathrm{sc}}$ in a neighborhood of $\partial\widetilde{X}_{\mathrm{sc}}$. Let $p\in C$ be a singular point. Then, by Remark 2.2, $C = L\setminus D$, where L is an orbifold holomorphic line bundle over a Kähler–Einstein Fano orbifold D, and we can find orbifold charts as in (2.4). However, since p and the points of the entire fiber of L containing p are singular, we know by averaging that the holomorphic line bundle $\mathrm{pr}_1\colon \mathbb{C}^n\times\mathbb{C}\to\mathbb{C}^n$ in (2.4) has a Γ–invariant holomorphic section that does not vanish near $0\in\mathbb{C}^n$. Thus, this means that without loss of generality, we can assume that there is a finite subgroup $\Gamma_1\subset\mathrm{SU}(n)$ acting linearly on $\mathbb{C}^n$ together with orbifold charts $\psi\colon\mathcal{U}\to\mathbb{C}^n/\Gamma_1$ and $\psi_L\colon p_L^{-1}(\mathcal{U})\to\mathbb{C}^n/\Gamma_1\times\mathbb{C}$ inducing the commutative diagram

$$
\begin{tikzcd}
p_L^{-1}(\mathcal{U}) \arrow[r, "\psi_L"] \arrow[d, "p_L"] & \mathbb{C}^n/\Gamma_1 \times \mathbb{C} \arrow[d, "\mathrm{pr}_1"] \\
\mathcal{U} \arrow[r, "\psi"] & \mathbb{C}^n/\Gamma_1
\end{tikzcd}
\tag{3.1}
$$

and such that $\psi_L(p) = (0,1)$. Let $z = (z_1, \dots, z_n)$ be the coordinates on $\mathbb{C}^n \setminus \Gamma_1$ and v the coordinate on L. Then the Kähler form of the Ricci-flat Kähler cone metric g_C takes the form

$$\omega_C = \frac{\sqrt{-1}}{2}\partial\overline{\partial}|v|^{2/q} f(z,\overline{z}) \tag{3.2}$$

for some positive smooth function f and some positive $q \in \mathbb{Q}$. Let $W_1 \subset \mathbb{C}^n$ be the subspace of points fixed by each element of Γ_1 and suppose without loss of generality that it corresponds to the subspace $z_1 = \cdots = z_{m_1} = 0$, so that we have the decomposition $\mathbb{C}^n/\Gamma_1 = \mathbb{C}^{m_1}/\Gamma_1 \times W_1$ with $W_1 = \mathbb{C}^{n-m_1}$. In terms of the orbifold chart (3.1), $\psi^{-1}(W_1)$ corresponds to the singular stratum in which $p_L(p)$ lies.

Lemma 3.1 *The differential of f is such that $(\partial f/\partial z^i)|_{W_1} = (\partial f/\partial \overline{z}^i)|_{W_1} = 0$ for $i \in \{1, \dots, m_1\}$.*

Proof The function f is invariant under the action of Γ_1. In particular, $df|_{W_1}$ is invariant under the action of Γ_1, which, by definition of W_1, holds if and only if the statement of the lemma holds. □

To distinguish between the factors $\mathbb{C}^{m_1}/\Gamma_1$ and W_1, let us set $u^i = z^i$ for $i \le m_1$ and $\eta_1^j = z^j$ for $j > m_1$. Then, by the previous lemma, the Taylor expansion of f at W_1 is of the form

$$f(u,\overline{u},\eta_1,\overline{\eta}_1) = f_0(\eta_1,\overline{\eta}_1) + \mathrm{Hess}(f)_{W_1}(u,\overline{u}) + \mathcal{O}(|u|^3), \tag{3.3}$$

where $\mathrm{Hess}(f)_{W_1}$ is the Hessian of f restricted to W_1 and only applied to the normal bundle of W_1.

Instead of the coordinates (z,v), one can then consider the holomorphic coordinates (ζ,λ) related to (z,v) by

$$v = \lambda_1^q, \qquad z^i = \frac{\zeta_1^i}{\lambda_1} \quad \text{for } i \le m_1, \qquad \eta_1^j = z^j \quad \text{for } j > m_1. \tag{3.4}$$

Away from $\lambda_1 = 0$, this is a valid change of coordinates for $\theta < \arg \zeta_1^i < \theta + 2\pi/q$ for all i for some fixed choice of θ.

In these new coordinates, the Kähler form of the Ricci-flat Kähler cone metric takes the form

$$\begin{aligned}\omega_C &= \frac{\sqrt{-1}}{2}\partial\overline{\partial}\left(|\lambda_1|^2 f\left(\frac{\zeta_1}{\lambda_1},\frac{\overline{\zeta}_1}{\overline{\lambda}_1},\eta_1,\overline{\eta}_1\right)\right)\\ &= \frac{\sqrt{-1}}{2}\left(\partial\overline{\partial}|\lambda_1|^2 f_E\left(\frac{\zeta_1}{\lambda_1},\frac{\overline{\zeta}_1}{\overline{\lambda}_1},\eta_1,\overline{\eta}_1\right)+\partial\overline{\partial}\mathcal{P}\right),\end{aligned}\tag{3.5}$$

where $f_E(u,\overline{u},\eta_1,\overline{\eta}_1):=f_0(\eta_1,\overline{\eta}_1)+\mathrm{Hess}(f)_{W_1}(u,\overline{u})$ and

$$\begin{aligned}\mathcal{P} &= |\lambda_1|^2 f\left(\frac{\zeta_1}{\lambda_1},\frac{\overline{\zeta}_1}{\overline{\lambda}_1},\eta_1,\overline{\eta}_1\right)-|\lambda_1|^2 f_0(\eta_1,\overline{\eta}_1)-\mathrm{Hess}(f)_{W_1}(\zeta_1,\overline{\zeta}_1)\\ &= \mathcal{O}\left(\frac{|\zeta|^3}{|\lambda_1|}\right)\quad \text{as } |\lambda_1|\to\infty \text{ and } \frac{|\zeta|}{|\lambda_1|}+|\eta_1|\le C.\end{aligned}\tag{3.6}$$

3.1 The Ricci-flat Kähler cone metric seen as a QAC–metric when $\widetilde{X}_{\mathrm{sc}}$ is of depth one

We now suppose that $\widetilde{X}_{\mathrm{sc}}$ is an orbifold with fibered corners of depth one. In this case, if p lies in a singularity of relative depth 1, then the action of Γ_1 on $\mathbb{C}^{m_1}\setminus\{0\}$ is free. Let H_1 denote the boundary hypersurface of $\widetilde{X}_{\mathrm{sc}}$ corresponding to p and $H_{\max}$ the boundary hypersurface corresponding to the maximal stratum. In terms of the coordinates (3.4), notice that $x_{\max}=1/\sqrt{1+|\zeta_1|^2}$ is a boundary-defining function for $H_{\max}$ and $x_1=\sqrt{1+|\zeta_1|^2}/|\lambda_1|$ is a boundary-defining function for H_1. Moreover, $\zeta_1^1,\dots,\zeta_1^{m_1}$ are holomorphic coordinates in the interior of the fibers of $\phi_1: H_1\to S_1$, and $\arg\lambda_1$, η_1, $\overline{\eta}_1$ are coordinates on the interior of S_1.

With this interpretation, we can replace (3.6) with the more precise estimate

$$\mathcal{P}\in x_{\max}^{-2}x_1\mathcal{C}^\infty(\widetilde{X}_{\mathrm{sc}})\quad\Longrightarrow\quad \partial\overline{\partial}\mathcal{P}\in x_1\mathcal{C}^\infty(\widetilde{X}_{\mathrm{sc}};{}^{\phi}T^*\widetilde{X}_{\mathrm{sc}}\wedge{}^{\phi}T^*\widetilde{X}_{\mathrm{sc}}).\tag{3.7}$$

In particular, we deduce that $\omega_E:=\frac{\sqrt{-1}}{2}\partial\overline{\partial}|\lambda_1|^2 f_E(\zeta_1/\lambda_1,\overline{\zeta}_1/\overline{\lambda}_1,\eta_1,\overline{\eta}_1)$ is also a Kähler form near H_1.

Proposition 3.2 *The Kähler metric g_E associated to the Kähler form ω_E is a QAC–metric with respect to the Lie algebra of QAC–vector fields induced by the choice of x_1 and $x_{\max}$.*

Proof Notice first that in terms of the boundary-defining functions x_1 and $x_{\max}$,

$$d\lambda_1,\quad d\overline{\lambda}_1,\quad \lambda_1\,d\eta_1,\quad \overline{\lambda}_1\,d\overline{\eta}_1,\quad d\zeta_1,\quad d\overline{\zeta}_1\tag{3.8}$$

is a local basis of QAC–forms. Now, the Kähler form ω_E is of the form

$$\begin{aligned}(3.9)\quad \omega_E = \frac{\sqrt{-1}}{2}\Bigg(f_E d\lambda_1 \wedge d\bar{\lambda}_1 &+ \sum_{i=1}^{m_1}\sum_{j=1}^{m_1} \frac{\partial^2 f}{\partial u^i \partial \bar{u}^j}\bigg|_{W_1} d\zeta_1^i \wedge d\bar{\zeta}_1^j \\ &+ |\lambda_1|^2 \sum_{i=m_1+1}^{n}\sum_{j=m_1+1}^{n} \frac{\partial^2 f_0}{\partial \eta_1^i \partial \bar{\eta}_1^j} d\eta_1^i \wedge d\bar{\eta}_1^j \\ &+ \sum_{i=1}^{m_1}\bigg(\lambda_1 \frac{\partial f_0}{\partial \eta_1^i} d\eta_1^i \wedge d\bar{\lambda}_1 + \bar{\lambda}_1 \frac{\partial f_0}{\partial \bar{\eta}_1^i} d\lambda_1 \wedge d\bar{\eta}_1^i\bigg) + \nu\Bigg),\end{aligned}$$

where ν is

$$\sum_{i=1}^{m_1}\sum_{j=m_1+1}^{n} (a_{ij}\, d\zeta_1^i \wedge \bar{\lambda}_1\, d\bar{\eta}_1^j + b_{ij}\lambda_1\, d\eta_1^j \wedge d\bar{\zeta}_1^i) + \sum_{i=m_1}^{n}\sum_{j=m_1+1}^{n} c_{ij}|\lambda_1|^2\, d\eta_1^i \wedge d\bar{\eta}_1^j,$$

with

$$a_{ij}, b_{ij}, c_{ij} \in x_1 \mathcal{C}^\infty(\widetilde{X}_{\mathrm{sc}})$$

and with the convention that $d\eta_1^{m_1} := d\log\lambda_1$. By Example 1.19 and the local basis of QAC–forms (3.8), we therefore see that the metric g_E is a QAC–metric with respect to the boundary-defining functions x_1 and $x_{\max}$. □

In particular, we see from (3.9) that $(\mathrm{Hess}(f)_{W_1})_{ij} = (\partial^2 f/(\partial u^i \partial \bar{u}^j))|_{W_1}$ is positive definite. In fact,

$$g_{\phi_1} := \mathrm{Hess}(f)_{W_1}$$

is a family of Euclidean metrics on the interior of the fibers of the fiber bundle $\phi_1\colon H_1 \to S_1$, that is, a Euclidean metric for the vector bundle $\phi_1\colon H_1 \setminus \partial H_1 \to S_1$.

Corollary 3.3 *The metric g_C is a QAC–metric which has the same restriction as g_E on H_1, namely*

$$\begin{aligned}(3.10)\quad f_E\, d\lambda_1 \otimes d\bar{\lambda}_1 &+ \sum_{i=1}^{m_1}\sum_{j=1}^{m_1} \frac{\partial^2 f}{\partial u^i \partial \bar{u}^j}\bigg|_{W_1} d\zeta_1^i \otimes d\bar{\zeta}_1^j \\ &+ |\lambda_1|^2 \sum_{i=m_1+1}^{n}\sum_{j=m_1+1}^{n} \frac{\partial^2 f_0}{\partial \eta_1^i \partial \bar{\eta}_1^j} d\eta_1^i \otimes d\bar{\eta}_1^j \\ &+ \sum_{i=1}^{m_1}\bigg(\lambda_1 \frac{\partial f_0}{\partial \eta_1^i} d\eta_1^i \otimes d\bar{\lambda}_1 + \bar{\lambda}_1 \frac{\partial f_0}{\partial \bar{\eta}_1^i} d\lambda_1 \otimes d\bar{\eta}_1^i\bigg).\end{aligned}$$

Proof This is a direct consequence of (3.7) and Proposition 3.2. □

3.2 The Ricci-flat Kähler cone metric seen as a QAC–metric when $\widetilde{X}_{\mathrm{sc}}$ is of arbitrary depth

More generally, if $\widetilde{X}_{\mathrm{sc}}$ is of arbitrary depth, then given a singular point p, we can still use the coordinates (3.4), but with the difference that this time the action of Γ_1 on $\mathbb{C}^{m_1}\setminus\{0\}$ is not necessarily free. However, to see that this model agrees with models at boundary hypersurfaces of lower relative depth in $\widetilde{X}_{\mathrm{sc}}$, we will need to introduce more refined coordinates.

Let us first describe these more refined coordinate systems. Suppose that the action of Γ_1 on $\mathbb{C}^{m_1}\setminus\{0\}$ is not free. Let $p_2 \in (\mathbb{C}^{m_1}/\Gamma_1)\setminus\{0\}$ be given and let ℓ_2 be the corresponding complex line passing through p_2 and the origin. If p_2 is not singular, then it suffices to use the coordinates (3.4) to describe the behavior at infinity of the metric g_C in the direction of ℓ_2. In this case, the discussion is essentially as before. If instead p_2 is singular, then the coordinates (3.4) are no longer appropriate to describe the metric g_C at infinity in the direction of ℓ_2. Near $p_2 \in \mathbb{C}^{m_1}/\Gamma_1$, where it is understood that we are using the coordinates $\zeta_1^i = \lambda_1 z^i$, we can introduce an orbifold chart of the form

$$\varphi_2\colon \mathbb{C}^{m_2}/\Gamma_2 \times \mathbb{C} \times \mathbb{C}^{\nu_2-1} \to \mathcal{U}_2 \subset \mathbb{C}^{m_1}/\Gamma_1, \quad \text{where } \nu_2 = m_1 - m_2,$$

with $p_2 = \varphi_2(0,1,0)$ and with $\Gamma_2 \subset \mathrm{SU}(m_2)$ a finite subgroup such that $\mathrm{Fix}(\Gamma_2) = \{0\}$, where

$$\mathrm{Fix}(\Gamma_2) = \{q \in \mathbb{C}^{m_2} \mid \gamma\cdot q = q \text{ for all } \gamma \in \Gamma_2\}.$$

In particular, notice that Γ_2 can be identified with a subgroup of Γ_1, namely, with the stabilizer of a lift $\tilde{p}_2$ of p_2 to $\mathbb{C}^{m_1}$ under the action of Γ_1. We denote by $(\zeta_2, \lambda_2, z_2)$ the complex linear coordinates on each factor of $\mathbb{C}^{m_2}\times\mathbb{C}\times\mathbb{C}^{\nu_2-1}$. However, in terms of QAC–geometry, the coordinates that will actually be useful are instead given by

$$\zeta_2, \quad \lambda_2, \quad \eta_2 := \frac{z_2}{\lambda_2}.$$

If the action of Γ_2 on $\mathbb{C}^{m_2}\setminus\{0\}$ is free, then these coordinates combined with λ_1 and η_1 in (3.4) will be what we need. If the action is not free, then these coordinates will still suffice away from the singularities of $(\mathbb{C}^{m_2}/\Gamma_2)\setminus\{0\}$, but for $p_3 \in (\mathbb{C}^{m_2}/\Gamma_2)\setminus\{0\}$ a singular point, the coordinates must be refined again in the direction of the complex line ℓ_3 passing through p_3 and the origin. Even once these coordinates have been introduced, we may still have to repeat this step finitely many times before we have an adequate description of g_C in all directions at infinity. More precisely, if ℓ is the relative depth of the point p, then we might have to apply this procedure up to $\ell-1$ times to the coordinates (3.4) depending on the direction at infinity in which we wish to look.

Thus, in general, we will have a finite sequence of subgroups

$$\Gamma_\ell \subset \Gamma_{\ell-1} \subset \cdots \subset \Gamma_2 \subset \Gamma_1,$$

with $\Gamma_i \subset \mathrm{SU}(m_i)$ acting linearly on $\mathbb{C}^{m_i}$ in such a way that $\mathrm{Fix}(\Gamma_i)=\{0\}$, and orbifold charts

$$\varphi_i\colon \mathbb{C}^{m_i}/\Gamma_i \times \mathbb{C} \times \mathbb{C}^{\nu_i-1} \to \mathcal{U}_i \subset \mathbb{C}^{m_{i-1}}/\Gamma_{i-1}.$$

For such a chart $(\zeta_i,\lambda_i,z_i) \in \mathbb{C}^{m_i}/\Gamma_i \times \mathbb{C} \times \mathbb{C}^{\nu_i-1}$, we consider instead the projectivized coordinates

$$\zeta_i, \quad \lambda_i, \quad \eta_i := \frac{z_i}{\lambda_i},$$

with the recursive relation $\zeta_{i-1}=\varphi_i(\zeta_i,\lambda_i,\lambda_i\eta_i)$. Combining these yields the holomorphic coordinates

$$(3.11)\quad (\zeta_\ell,\lambda_\ell,\eta_\ell,\lambda_{\ell-1},\eta_{\ell-1},\dots,\lambda_1,\eta_1) \in \mathbb{C}^{m_\ell}/\Gamma_\ell \times (\mathbb{C}\times\mathbb{C}^{\nu_{\ell-1}}) \times \cdots \times (\mathbb{C}\times\mathbb{C}^{\nu_1-1}).$$

Suppressing the reference to the map φ_i of the orbifold chart to lighten notation, we relate these new coordinates to the coordinates $(\zeta_1,\lambda_1,\eta_1)$ by

$$\zeta_1=(\zeta_\ell,\lambda_\ell,\lambda_\ell\eta_\ell,\lambda_{\ell-1},\lambda_{\ell-1}\eta_{\ell-1},\dots,\lambda_2,\lambda_2\eta_2), \quad \lambda_1=\lambda_1, \quad \eta_1=\eta_1.$$

This should be compared with the real coordinate system of (1.1) with $1/|\lambda_i|$ playing the role of v_i; $\arg(\lambda_i),\eta_i$ playing the role of y_i; and $\zeta_\ell/|\zeta_\ell|$ playing the role of $y_{\ell+1}$, since the function $1/|\zeta_\ell|$ can be used as a boundary-defining function $x_{\ell+1}$ of the maximal boundary hypersurface $H_{\ell+1} := H_{\max}$. Notice that this labeling of the boundary hypersurfaces is compatible with the partial order given by the orbifold with fibered corners structure in that

$$H_i < H_j \implies i<j.$$

We can thus pick

$$x_\ell = \frac{v_\ell}{x_{\ell+1}} = \frac{|\zeta_\ell|}{|\lambda_\ell|} \qquad \text{and} \qquad x_i=\frac{v_i}{v_{i+1}}=\frac{|\lambda_{i+1}|}{|\lambda_i|} \quad \text{for } i<\ell,$$

as boundary-defining functions for the other boundary hypersurfaces $H_\ell,\dots,H_1$ involved in this coordinate system.

Moreover, the coordinate system (3.11) induces the coordinates

$$\left(\zeta_\ell, \lambda_\ell, \eta_\ell, \dots, \lambda_{i+1}, \eta_{i+1}, \arg\lambda_i, \eta_i, \frac{\lambda_{i-1}}{|\lambda_i|}, \eta_{i-1}, \dots, \frac{\lambda_1}{|\lambda_i|}, \eta_1\right)$$

on H_i, and in terms of these coordinates the projection $\phi_i\colon H_i \to S_i$ is given by

$$\phi_i\left(\zeta_\ell, \lambda_\ell, \eta_\ell, \dots, \lambda_{i+1}, \eta_{i+1}, \arg\lambda_i, \eta_i, \frac{\lambda_{i-1}}{|\lambda_i|}, \eta_{i-1}, \dots, \frac{\lambda_1}{|\lambda_i|}, \eta_1\right) = \left(\arg\lambda_i, \eta_i, \frac{\lambda_{i-1}}{|\lambda_i|}, \eta_{i-1}, \dots, \frac{\lambda_1}{|\lambda_i|}, \eta_1\right) \in S_i.$$

Notice moreover that in the coordinates (3.11), the crepant resolution $\beta_X\colon \widehat{X} \to X$ is a local product in the sense that it is described entirely in terms of the coordinates ζ_ℓ, so that the other coordinates of (3.11) naturally lift to holomorphic coordinates on the crepant resolution $\widehat{X}$. This allows us to complete the discussion of Section 2 about the existence of a natural Lie algebra of QAC–vector fields on $\widehat{X}_{\mathrm{QAC}}$.

Lemma 3.4 *The functions ρ_i introduced in (2.16) can be chosen in such a way that they lift to be smooth on $\widehat{X}_{\mathrm{QAC}}$. In particular, by Lemma 2.10, the boundary-defining functions (2.19) induce a well-defined Lie algebra of QAC–vector fields on $\widehat{X}_{\mathrm{QAC}}$.*

Proof Using a suitable partition of unity on $\widehat{X}_{\mathrm{QAC}}$, the problem becomes local on $\widetilde{X}_{\mathrm{sc}}$, so we can work with the coordinate chart (3.11). But in this chart, the choice of ρ_i is equivalent to the choice of v_i, so we can set

$$\rho_i = v_i = \frac{1}{|\lambda_i|},$$

which clearly lifts to be smooth on $\widehat{X}_{\mathrm{QAC}}$. □

Now, to describe the metric g_C near H_i in the coordinates (3.11), we can, as in Lemma 3.1, use the local invariance of f under the action of Γ_i to deduce that, at $W_i = \mathrm{Fix}(\Gamma_i)$, the function f has a Taylor expansion of the form

$$f\left(\frac{\zeta}{\lambda_1}, \frac{\overline{\zeta}}{\overline{\lambda}_1}, \eta_1, \overline{\eta}_1\right) = f_{S_i}\left(\frac{\lambda_i}{\lambda_1}, \frac{\overline{\lambda}_i}{\overline{\lambda}_1}, \frac{\lambda_i\eta_i}{\lambda_1}, \frac{\overline{\lambda}_i\overline{\eta}_i}{\overline{\lambda}_1}, \dots, \frac{\lambda_2}{\lambda_1}, \frac{\overline{\lambda}_2}{\overline{\lambda}_1}, \frac{\lambda_2\eta_2}{\lambda_1}, \frac{\overline{\lambda}_2\overline{\eta}_2}{\overline{\lambda}_1}, \eta_1\right) + \mathrm{Hess}(f)_{W_i}\left(\frac{u_i}{\lambda_1}, \frac{\overline{u}_i}{\overline{\lambda}_1}\right) + \mathcal{O}\left(\left|\frac{u_i}{\lambda_1}\right|^3\right), \tag{3.12}$$

with $u_i = (\zeta_\ell, \lambda_\ell, \lambda_\ell \eta_\ell, \dots, \lambda_{i+1}, \lambda_{i+1}\eta_{i+1})$. In particular,

$$|\lambda_1|^2 f\left(\frac{\zeta}{\lambda_1}, \frac{\overline{\zeta}}{\overline{\lambda}_1}, \eta_1, \overline{\eta}_1\right) = f_{E_i} + \mathcal{O}\left(\frac{|u_i|^3}{|\lambda_1|}\right), \tag{3.13}$$

with

$$f_{E_i} = |\lambda_1|^2 f_{S_i}\left(\frac{\lambda_i}{\lambda_1}, \frac{\overline{\lambda}_i}{\overline{\lambda}_1}, \frac{\lambda_i\eta_i}{\lambda_1}, \frac{\overline{\lambda}_i\overline{\eta}_i}{\overline{\lambda}_1}, \dots, \frac{\lambda_2}{\lambda_1}, \frac{\overline{\lambda}_2}{\overline{\lambda}_1}, \frac{\lambda_2\eta_2}{\lambda_1}, \frac{\overline{\lambda}_2\overline{\eta}_2}{\overline{\lambda}_1}, \eta_1, \overline{\eta}_1\right) + \operatorname{Hess}(f)_{W_i}(u_i, \overline{u}_i). \tag{3.14}$$

Since $|u_i/\lambda_1| = \mathcal{O}(w_i)$ with $w_i = \prod_{j \le i} x_i$, we see that

$$\omega_{E_i} - \omega_C \in w_i \mathcal{C}^\infty(\widetilde{X}_{\mathrm{sc}}; \Lambda^2({}^\phi T\widetilde{X}_{\mathrm{sc}})), \tag{3.15}$$

where

$$\omega_{E_i} := \tfrac{\sqrt{-1}}{2}\partial\overline{\partial}|\lambda_1|^2 f_{E_i} = \omega_{\phi_i} + \phi_i^*\omega_{S_i} + \nu_i, \tag{3.16}$$

with $\nu_i \in w_i\mathcal{C}^\infty(\widetilde{X}_{\mathrm{sc}}; \Lambda^2({}^\phi T\widetilde{X}_{\mathrm{sc}}))$ and

$$\omega_{\phi_i} := \tfrac{\sqrt{-1}}{2}\partial\overline{\partial}\operatorname{Hess}(f)_{W_i}(u_i, \overline{u}_i), \tag{3.17}$$

$$\omega_{S_i} := \tfrac{\sqrt{-1}}{2}\partial\overline{\partial}\left(|\lambda_1|^2 f_{S_i}\left(\frac{\lambda_i}{\lambda_1}, \frac{\overline{\lambda}_i}{\overline{\lambda}_1}, \frac{\lambda_i\eta_i}{\lambda_1}, \frac{\overline{\lambda}_i\overline{\eta}_i}{\overline{\lambda}_1}, \dots, \frac{\lambda_2}{\lambda_1}, \frac{\overline{\lambda}_2}{\overline{\lambda}_1}, \frac{\lambda_2\eta_2}{\lambda_1}, \frac{\overline{\lambda}_2\overline{\eta}_2}{\overline{\lambda}_1}, \eta_1, \overline{\eta}_1\right)\right). \tag{3.18}$$

In particular, since ω_C is a Kähler form, this implies that near H_i, ω_{E_i} is the Kähler form of a Kähler metric g_{E_i} and that ω_{ϕ_i} is a Kähler form in each fiber of $\phi_i\colon H_i \to S_i$ of a corresponding family of Kähler metrics g_{ϕ_i} which are in fact Euclidean. This also implies that

$$\omega_C|_{H_i} = \omega_{E_i}|_{H_i}. \tag{3.19}$$

Moreover, since $w_j/w_i \in \mathcal{C}^\infty(\widetilde{X}_{\mathrm{sc}})$ for $i < j$, we also have that in this case

$$\omega_{E_i} - \omega_{E_j} = \omega_{E_i} - \omega_C - (\omega_{E_j} - \omega_C) \in w_i\mathcal{C}^\infty(\widetilde{X}_{\mathrm{sc}}; \Lambda^2({}^\phi T\widetilde{X}_{\mathrm{sc}})),$$

which implies that

$$\omega_{E_i}|_{H_i\cap H_j} = \omega_{E_j}|_{H_i\cap H_j}.$$

Lemma 3.5 *The metrics g_{E_i} and the Ricci-flat Kähler cone metric g_C are QAC–metrics on $\widetilde{X}_{\mathrm{sc}}$.*

Proof In terms of the holomorphic coordinates (3.11), a local basis of the complexification of ${}^{\phi}T^*\widetilde{X}_{\mathrm{sc}}$ (cf (1.6)) is given by

$$d\lambda_1,\quad d\overline{\lambda}_1,\quad \lambda_1\,d\eta_1,\quad \overline{\lambda}_1\,d\overline{\eta}_1,\quad \dots,\quad d\lambda_\ell,\quad d\overline{\lambda}_\ell,\quad \lambda_\ell\,d\eta_\ell,\quad \overline{\lambda}_\ell\,d\overline{\eta}_\ell,\quad d\zeta_\ell,\quad d\overline{\zeta}_\ell.$$

Thus, we see from (3.15) and the local description (3.16) that g_{E_i} and g_C are QAC–metrics. $\square$

Recall that the interior of the fibers of $\phi_i\colon H_i \to S_i$ are modeled on $\mathbb{C}^{m_i}/\Gamma_i$ for some finite subgroup $\Gamma_i \subset \mathrm{SU}(m_i)$, while the fibers of $\widehat{\phi}_i\colon \widehat{H}_i \to S_i$ are modeled on a Kähler crepant resolution Y_i of $\mathbb{C}^{m_i}/\Gamma_i$. This suggests the following definition.

Definition 3.6 A Kähler QAC–metric $g \in \mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X}; {}^{\widehat{\phi}}T^*\widehat{X}_{\mathrm{QAC}} \otimes {}^{\widehat{\phi}}T^*\widehat{X}_{\mathrm{QAC}})$ on $\widehat{X}$ with Kähler form ω is said to be *asymptotic with rate* δ to the Ricci-flat Kähler cone metric g_C if:

1. $g - g_C \in \hat{x}^{\delta}_{\max}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X}; {}^{\widehat{\phi}}T^*\widehat{X}_{\mathrm{QAC}} \otimes {}^{\widehat{\phi}}T^*\widehat{X}_{\mathrm{QAC}})$ near $\widehat{H}_{\max}$;
2. $\omega - (\omega_i + \widehat{\phi}_i^*\omega_{S_i}) \in \hat{x}^{\delta}_{\max}\hat{x}_i\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X}; {}^{\widehat{\phi}}T^*\widehat{X}_{\mathrm{QAC}} \otimes {}^{\widehat{\phi}}T^*\widehat{X}_{\mathrm{QAC}})$ near $\widehat{H}_i$, with ω_{S_i} as in (3.18) and with ω_i a closed $(1,1)$–form on $\widehat{H}_i$ which restricts on each fiber of $\widehat{\phi}_i\colon \widehat{H}_i \to S_i$ to the Kähler form of a Kähler QAC–metric asymptotic to g_{ϕ_i} with rate δ. Moreover, as a family of $(1,1)$–forms parametrized by S_i, ω_i is smooth up to ∂S_i.

Remark 3.7 This definition is not circular. Since the fibers of $\widehat{\phi}_i\colon \widehat{H}_i \to S_i$ are of depth lower than those of $\widehat{X}_{\mathrm{QAC}}$, we can assume, proceeding by induction on the depth of $\widehat{X}_{\mathrm{QAC}}$, that the notion of a Kähler QAC–metric asymptotic to g_{ϕ_i} with rate δ has already been defined.

4 Existence of Kähler QAC–metrics asymptotic to the Ricci-flat Kähler cone metric

Before discussing examples of Kähler QAC–metrics, we need to provide examples of orbifolds equipped with asymptotically conical Kähler metrics. Let g_C be a Ricci-flat Kähler cone metric defined on $C = L \setminus D$ as in Section 2, where L is some holomorphic line bundle over a Kähler–Einstein Fano orbifold. For $N > 0$, set

$$D_N = \{\ell \in L \mid \|\ell\|_{h_L} \le N\}.$$

Suppose that X is a complex orbifold such that for some compact set $\mathcal{K} \subset X$, $X \setminus \mathcal{K}$ is biholomorphic to $L \setminus D_N$ for some $N > 0$.

Lemma 4.1 *Suppose that ω is a compactly supported closed $(1,1)$–form on X which is positive on $\mathcal{K}$. Then there exists a Kähler form $\widetilde{\omega}$ on X such that $(X \setminus \mathcal{K}', \widetilde{\omega})$ is isometric and biholomorphic to $(L \setminus D_{N'}, \omega_C)$ for some compact set $\mathcal{K}' \subset X$ and some $N' > 0$.*

Proof By continuity, ω is positive in a small neighborhood $\mathcal{U}$ of $\mathcal{K}$. Using the identification $X \setminus \mathcal{K} \cong L \setminus D_N$, we will work on $L \setminus D_N$, so that ω will be positive on $\mathcal{U} \cap (L \setminus D_N)$. Now, let $\eta \in \mathcal{C}^\infty(\mathbb{R})$ be a nondecreasing convex function such that

$$\eta(t) = \begin{cases} t & \text{if } t \geq 2, \\ \frac{3}{2} & \text{if } t \leq 1, \end{cases}$$

and set $\eta_{\delta,a}(t) = \eta((t-a)/\delta)$ for $\delta > 0$ and $a \in \mathbb{R}$, so that $\eta_{\delta,a}$ is also a nondecreasing convex function. On $L \setminus D_N$, one computes that

$$\begin{aligned} (4.1) \quad & \frac{\sqrt{-1}}{2} \partial\overline{\partial} \eta_{\delta,a}(\|\cdot\|_{h_L}^{2/q}) \\ &= \frac{\sqrt{-1}}{2} \eta''_{\delta,a}(\|\cdot\|_{h_L}^{2/q}) \partial\|\cdot\|_{h_L}^{2/q} \wedge \overline{\partial}\|\cdot\|_{h_L}^{2/q} + \frac{\sqrt{-1}}{2} \eta'_{\delta,a}(\|\cdot\|_{h_L}^{2/q}) \partial\overline{\partial}\|\cdot\|_{h_L}^{2/q} \\ &\geq \frac{\sqrt{-1}}{2} \eta'_{\delta,a}(\|\cdot\|_{h_L}^{2/q}) \partial\overline{\partial}\|\cdot\|_{h_L}^{2/q} \geq 0, \end{aligned}$$

so that this $(1,1)$–form is nonnegative. Moreover, in the region where $\|\cdot\|_{h_L}^{2/q} \geq 2\delta + a$, it is equal to

$$\frac{\sqrt{-1}}{2} \partial\overline{\partial}\|\cdot\|_{h_L}^{2/q} = \omega_C,$$

the Kähler form of the Ricci-flat Kähler cone metric g_C, whereas it vanishes in the region where $\|\cdot\|_{h_L}^{2/q} \leq \delta + a$. In particular, choosing $\delta > 0$ sufficiently small and $a = N^{2/q}$, we have that

$$\nu = \frac{\sqrt{-1}}{2} \partial\overline{\partial} \eta_{\delta,a}(\|\cdot\|_{h_L}^{2/q})$$

is a nonnegative closed $(1,1)$–form which vanishes on D_N, is strictly positive on $L \setminus (\mathcal{U} \cap (L \setminus D_N))$, and is equal to ω_C outside a compact set. Hence, it suffices to take

$$\widetilde{\omega} = \nu + \epsilon\omega$$

with $\epsilon > 0$ sufficiently small. □

We are in particular interested in the case $X = L$. Notice then that g_C is not smooth in the orbifold sense since it has a singularity along D, so we cannot simply take g_C itself to obtain the conclusion of Lemma 4.1.

Proposition 4.2 *On L, there exists a Kähler metric smooth in the orbifold sense which is equal to g_C outside a compact set.*

Proof Let $p_L\colon L \to D$ denote the bundle projection, let $\varphi \in H^0(L; p_L^* L)$ denote the tautological section and equip $p_L^* L$ with the Hermitian metric

$$h_{p_L} := e^{\|\varphi\|^2_{h_L}}\, p_L^* h_L.$$

The curvature of $(p_L^* L, h_{p_L})$ is then given by

$$\sqrt{-1}\Theta_{h_{p_L}} = -\sqrt{-1}\partial\bar{\partial}\|\varphi\|^2_{h_L} - 2q p_L^* \omega_D,$$

where we use formula (2.3) for the curvature of L. Clearly, the restriction of the $(1,1)$–form $\sqrt{-1}\partial\bar{\partial}\|\varphi\|^2_{h_L}$ is positive on each fiber of $p_L\colon L \to D$. Moreover, $p_L^*\omega_D$ is positive in the directions transverse to the fibers of $p_L\colon L \to D$. Because the nonvertical part of $\sqrt{-1}\partial\bar{\partial}\|\varphi\|^2_{h_L}$ is $\mathcal{O}(\|\varphi\|_{h_L})$, we see that $-\sqrt{-1}\Theta_{h_{p_L}}$ is positive in D_N for $N>0$ sufficiently small. Replacing h_{p_L} with $\hat{h}_{p_L}$ such that $\hat{h}_{p_L} = h_{p_L}$ on D_N and $h_{p_L}(\varphi,\varphi) \equiv 1$ outside a compact set, we see that

$$\omega := -\sqrt{-1}\Theta_{\hat{h}_{p_L}}$$

is a compactly supported closed $(1,1)$–form which is positive on D_N. It then suffices to apply the previous lemma with ω to obtain the desired Kähler form. $\square$

Suppose now that X is a complex orbifold and g is a complete Kähler metric on X with Kähler form ω such that there is a biholomorphism $X \setminus \mathcal{K} \cong L \setminus D_N$ for some $N>0$ and some compact set $\mathcal{K} \subset X$ inducing at the same time an isometry between g and g_C. Let $\widetilde{X}_{\mathrm{sc}}$ be the corresponding orbifold with fibered corners given by (2.10) and let $H_1, \dots, H_k$ be an exhaustive list of the boundary hypersurfaces of $\widetilde{X}_{\mathrm{sc}}$ compatible with the partial order in the sense that $H_i < H_j \implies i < j$. The goal of this section is to construct examples of Kähler QAC–metrics on the QAC–resolution $\widehat{X}_{\mathrm{QAC}}$ of $\widetilde{X}_{\mathrm{sc}}$. Our strategy will consist of gluing local models of Kähler metrics to the Kähler metric g at places where the singularities of $\widetilde{X}_{\mathrm{sc}}$ are resolved by a local product Kähler crepant resolution. To do this in a systematic way, we introduce a natural space $\widehat{\mathcal{X}}$ on which this gluing can be performed. This is in fact where most of the effort will be put, for once this space is defined, the gluing construction becomes very simple; see the proof of Theorem 4.8.

To introduce this space $\widehat{\mathcal{X}}$, we need to work with the orbifold with fibered corners $\widetilde{X}_{\mathrm{sc}}$. As an orbifold, $\widetilde{X}_{\mathrm{sc}}$ is automatically a stratified space. Let $\Sigma_{k+1}, \dots, \Sigma_\ell$ be an exhaustive list of its strata compatible with the inclusion in the sense that

$$\Sigma_i \subset \overline{\Sigma}_j \implies i < j.$$

In particular, Σ_ℓ is the regular stratum. Consider then the orbifold with corners

$$\widetilde{X}_{\mathrm{sc}} \times [0,1) \tag{4.2}$$

and set

$$\mathcal{X} = [\widetilde{X}_{\mathrm{sc}} \times [0,1); \overline{\Sigma}_{k+1} \times \{0\}, \dots, \overline{\Sigma}_{\ell-1} \times \{0\}]. \tag{4.3}$$

Clearly, $\mathcal{X}$ is an orbifold with corners. Some of the boundary hypersurfaces come from the lift of old hypersurfaces $H_i \times [0,1)$ to $\mathcal{X}$, namely

$$\mathcal{H}_i = \overline{\beta^{-1}(\mathring{H}_i \times (0,1))} \quad \text{for } i \in \{1, \dots, k\},$$

where $\beta\colon \mathcal{X} \to \widetilde{X}_{\mathrm{sc}} \times [0,1)$ is the blow-down map. As is clear from the definition, $\mathcal{H}_i$ is also naturally equipped with a fiber bundle structure

$$\begin{array}{ccc} \mathcal{V}_i & \longrightarrow & \mathcal{H}_i \\ & & \big\downarrow{\scriptstyle \varphi_i} \\ & & S_i \end{array} \qquad \text{for } i \in \{1, \dots, k\}, \tag{4.4}$$

where $\mathcal{V}_i$ is obtained from $\widetilde{V}_i \times [0,1)$ in the same way that $\mathcal{X}$ was obtained from $\widetilde{X}_{\mathrm{sc}} \times [0,1)$, namely by blowing up the strata of $\widetilde{V}_i \times \{0\}$ in $\widetilde{V}_i \times [0,1)$ in order of decreasing relative depth.

The other boundary hypersurfaces of $\mathcal{X}$ arise from the blow-up of $\overline{\Sigma}_i \times \{0\}$ in $\widetilde{X}_{\mathrm{sc}} \times [0,1)$, as well as the lift of the hypersurface $\widetilde{X}_{\mathrm{sc}} \times \{0\}$. Thus, for each $i \in \{k+1, \dots, \ell\}$, $\mathcal{X}$ has a boundary hypersurface $\mathcal{H}_i$ associated to the stratum Σ_i. As in the proof of Proposition 2.7, this boundary hypersurface $\mathcal{H}_i$ is naturally equipped with a fiber bundle structure

$$\begin{array}{ccc} \widetilde{V}_i & \longrightarrow & \mathcal{H}_i \\ & & \big\downarrow{\scriptstyle \varphi_i} \\ & & \widetilde{\Sigma}_i \end{array} \qquad \text{for } i \in \{k+1, \dots, \ell\}, \tag{4.5}$$

where $\widetilde{\Sigma}_i$ is the manifold with fibered corners that resolved the stratified space $\overline{\Sigma}_i$ and $\widetilde{V}_i$ is obtained from $\overline{V}_i = \overline{\mathbb{C}^{n+1-\dim_{\mathbb{C}} \Sigma_i}/\Delta_i}$ by blowing up the singular strata of $\partial \overline{V}_i$ in $\overline{V}_i$ in an order compatible with the partial order of the strata of $\partial \overline{V}_i$, where $\Delta_i \subset \mathrm{GL}(n+1-\dim_{\mathbb{C}} \Sigma_i, \mathbb{C})$ is some finite subgroup. In particular, the base $\widetilde{\Sigma}_i$

is smooth as a manifold with fibered corners and only the fibers $\widetilde{V}_i$ remain with orbifold singularities; cf the fiber bundle (2.12). Notice also that in the case that $i=\ell$, we have that $\widetilde{\Sigma}_\ell = \mathcal{H}_\ell$ and φ_i is just the identity map.

Lemma 4.3 *The orbifold with corners $\mathcal{X}$ is in fact an orbifold with fibered corners $(\mathcal{X},\varphi)$, where $\varphi=(\varphi_1,\dots,\varphi_\ell)$ is the collection of fiber bundle maps given by (4.4) and (4.5).*

Proof It suffices to observe that for the partial order on the boundary hypersurfaces given by

$$\mathcal{H}_i < \mathcal{H}_j \quad \iff \quad i<j \quad \text{and} \quad \mathcal{H}_i \cap \mathcal{H}_j \neq \varnothing, \tag{4.6}$$

all the compatibility conditions of Definition 1.1 are satisfied for the collection of bundle maps $\varphi=(\varphi_1,\dots,\varphi_\ell)$. □

As for $\widetilde{X}_{\mathrm{sc}}$, there are natural choices of boundary-defining functions for the boundary hypersurfaces of $\mathcal{X}$. Indeed, for $\mathcal{H}_i$ with $i\le k$, we simply take

$$r_i := \beta^* \operatorname{pr}_1^* x_i,$$

where $\operatorname{pr}_1\colon \widetilde{X}_{\mathrm{sc}}\times[0,1)\to\widetilde{X}_{\mathrm{sc}}$ is the projection on the first factor and $x_i\in\mathcal{C}^\infty(\widetilde{X}_{\mathrm{sc}})$ is a boundary-defining function for H_i as specified in Section 2. For $i>k$, we proceed as follows. Consider the partial blow-down map

$$\beta_i\colon \mathcal{X}\to[\widetilde{X}_{\mathrm{sc}}\times[0,1);\overline{\Sigma}_{k+1}\times\{0\},\dots,\overline{\Sigma}_i\times\{0\}]$$

and denote by B_i the boundary hypersurface of $[\widetilde{X}_{\mathrm{sc}}\times[0,1);\overline{\Sigma}_{k+1},\dots,\overline{\Sigma}_i]$ given by the lift of $\widetilde{X}_{\mathrm{sc}}\times\{0\}$, with the convention that $B_k:=\widetilde{X}_{\mathrm{sc}}\times\{0\}$ in $\widetilde{X}_{\mathrm{sc}}\times[0,1)$. Let also W_i denote the boundary hypersurface of $[\widetilde{X}_{\mathrm{sc}}\times[0,1);\overline{\Sigma}_{k+1}\times\{0\},\dots,\overline{\Sigma}_i\times\{0\}]$ corresponding to the blow-up of $\overline{\Sigma}_i\times\{0\}$. Set

$$\rho_k := \operatorname{pr}_2\in\mathcal{C}^\infty(\widetilde{X}_{\mathrm{sc}}\times[0,1)), \tag{4.7}$$

where $\operatorname{pr}_2\colon \widetilde{X}_{\mathrm{sc}}\times[0,1)\to[0,1)$ is the projection on the second factor. Starting with $i=k+1$, recursively choose a boundary-defining function ρ_i for B_i such that, outside a small neighborhood $\mathcal{U}_i$ of W_i disjoint from the boundary hypersurfaces not intersecting W_i, the function ρ_i is identified with the lift of ρ_{i-1}. Then, for $i>k$, we can take for $\mathcal{H}_i$ the boundary-defining function

$$r_i := \begin{cases} (\beta_{i-1}^*\rho_{i-1})/(\beta_i^*\rho_i) & \text{if } k<i<\ell, \\ \rho_i & \text{if } i=\ell, \end{cases} \tag{4.8}$$

with the convention that $\beta_k:=\beta$.

Lemma 4.4 *The* QAC*–equivalence class of the boundary-defining functions* $r_1,\dots,r_\ell$ *does not depend on the choice of the functions* ρ_i *for* $i>k$*. Hence the canonical choice (4.7) yields a natural Lie algebra of QAC–vector fields* $\mathcal{V}_{\mathrm{QAC}}(\mathcal{X})$ *on* $\mathcal{X}$*.*

Proof One proceeds as in the proof of Lemma 2.9. The details are left to the reader. □

Using the function $\varepsilon:=\beta^*\rho_k=\beta^*\operatorname{pr}_2\in\mathcal{C}^\infty(\mathcal{X})$, we consider the Lie subalgebra of $\mathcal{V}_{\mathrm{QAC}}(\mathcal{X})$

$$\mathcal{V}_{\mathrm{QAC},\varepsilon}(\mathcal{X}):=\{\xi\in\mathcal{V}_{\mathrm{QAC}}(\mathcal{X})\mid \xi\varepsilon\equiv 0\},\tag{4.9}$$

which corresponds to the Lie algebra of QAC–vector fields tangent to the level sets of ε. As for $\mathcal{V}_{\mathrm{QAC}}(\mathcal{X})$, there exists a natural vector bundle $\mathcal{E}\to\mathcal{X}$ and a natural map $\iota_\varepsilon\colon\mathcal{E}\to T\mathcal{X}$ such that there is a canonical identification

$$\mathcal{V}_{\mathrm{QAC},\varepsilon}(\mathcal{X})=(\iota_\varepsilon)_*\mathcal{C}^\infty(\mathcal{X};\mathcal{E}).$$

In fact, $\mathcal{E}$ is naturally a vector subbundle of ${}^{\varphi}T\mathcal{X}$, which induces a natural map

$${}^{\varphi}T^*\mathcal{X}\to\mathcal{E}^*.\tag{4.10}$$

This means in particular that a smooth QAC–metric naturally restricts to define an element of $\mathcal{C}^\infty(\mathcal{X};\mathcal{E}^*\otimes\mathcal{E}^*)$.

We are interested in the pullback $g_\varepsilon:=\beta^*\operatorname{pr}_1^*g$ to $\mathcal{X}$ of the smooth Kähler QAC–metric g on $\widetilde{X}_{\mathrm{sc}}$.

Lemma 4.5 *The pullback* $g_\varepsilon:=\iota_\varepsilon^*\beta^*\operatorname{pr}_1^*g$ *is such that* $g_\varepsilon/\varepsilon^2\in\mathcal{C}^\infty(\mathcal{X};\mathcal{E}^*\otimes\mathcal{E}^*)$*.*

Proof It suffices to check that given a section $s\in\mathcal{C}^\infty(\widetilde{X}_{\mathrm{sc}};{}^{\phi}T^*\widetilde{X}_{\mathrm{sc}})$, its pullback $s_\varepsilon:=\iota_\varepsilon^*\beta^*\operatorname{pr}_1^*s$ is such that

$$\frac{s_\varepsilon}{\varepsilon}\in\mathcal{C}^\infty(\mathcal{X};\mathcal{E}^*).$$

This can be seen by using the local basis of sections (1.6) of ${}^{\phi}T^*\widetilde{X}_{\mathrm{sc}}$ on $\widetilde{X}_{\mathrm{sc}}$. Indeed, let us denote by

$$\hat{v}_i:=\beta^*\operatorname{pr}_1^*v_i,\quad \hat{y}_i^{n_i}=\beta^*\operatorname{pr}_1^*y_i^{n_i},\quad \hat{z}_q=\beta^*\operatorname{pr}_1^*z_q$$

the pullbacks of the functions appearing in (1.6). Now, the function $\hat{v}_i$ should be compared with its analog on $\mathcal{X}$, namely

$$w_i=\prod_{\mathcal{H}_j\ge\mathcal{H}_i}r_j=\hat{v}_it_i\quad\text{with } t_i=\prod_{\mathcal{H}_j\ge H_i,\,j>k}r_j.$$

By the choice of the functions ρ_i above, notice that we can assume that $t_i = \varepsilon$. In this case, we compute that

$$(4.11)\qquad \frac{dw_i}{w_i^2} = \frac{d\hat{v}_i}{\varepsilon\hat{v}_i^2} + \frac{d\varepsilon}{\hat{v}_i\varepsilon^2} \quad\Longrightarrow\quad \iota_\varepsilon^*\left(\frac{d\hat{v}_i}{\varepsilon\hat{v}_i^2}\right) = \iota_\varepsilon^*\left(\frac{dw_i}{w_i^2}\right) \in \mathcal{C}^\infty(\mathcal{X};\mathcal{E}^*).$$

Similarly,

$$\varepsilon^{-1}\iota_\varepsilon^*\beta^*\,\frac{dy_i^{n_i}}{v_i} = \iota_\varepsilon^*\,\frac{d\hat{y}_i^{n_i}}{w_i} \in \mathcal{C}^\infty(\mathcal{X};\mathcal{E}^*)$$

and

$$\varepsilon^{-1}\iota_\varepsilon^*\beta^*\,dz_q = \iota_\varepsilon^*\,\frac{d\hat{z}_q}{\varepsilon} \in \mathcal{C}^\infty(\mathcal{X};\mathcal{E}^*).$$

Thus, this local computation shows that $\varepsilon^{-1}s_\varepsilon \in \mathcal{C}^\infty(\mathcal{X};\mathcal{E}^*)$, as desired. □

Remark 4.6 Because of the term $d\varepsilon/(\hat{v}_i\varepsilon^2)$ in (4.11), $\varepsilon^{-2}g$ is not an element of

$$\mathcal{C}^\infty(\mathcal{X};{}^{\varphi}T^*\mathcal{X}\otimes{}^{\varphi}T^*\mathcal{X}),$$

namely, it is singular as a section of ${}^{\varphi}T^*\mathcal{X}\otimes{}^{\varphi}T^*\mathcal{X}$ near $\beta^{-1}(\partial\widetilde{X}_{\mathrm{sc}}\times\{0\})\subset\mathcal{X}$.

To give a description of the restriction of $\varepsilon^{-2}g_\varepsilon$ to $\mathcal{H}_i$ for $i>k$, we need first to give a description of the restriction of $\mathcal{E}$ to $\mathcal{H}_i$. First, observe that the orbifold with fibered corners structure of $\mathcal{X}$ naturally induces an orbifold with fibered corners structure on $\mathcal{H}_i$ by considering the fiber bundle $\varphi_j\colon \mathcal{H}_j\cap\mathcal{H}_i\to\widetilde{\Sigma}_{ji}$ on $\mathcal{H}_j\cap\mathcal{H}_i$ for $\mathcal{H}_j>\mathcal{H}_i$, $\varphi_j\colon \mathcal{H}_j\cap\mathcal{H}_i\to\widetilde{\Sigma}_j$ on $\mathcal{H}_j\cap\mathcal{H}_i$ with $j>k$ and $\mathcal{H}_j<\mathcal{H}_i$, and $\varphi_j\colon \mathcal{H}_j\cap\mathcal{H}_i\to S_j$ on $\mathcal{H}_j\cap\mathcal{H}_i$ with $j\le k$ and $\mathcal{H}_j<\mathcal{H}_i$.

Restricting the boundary-defining functions of $\mathcal{X}$ to $\mathcal{H}_i$, we also obtain a Lie algebra of QFB–vector fields and a corresponding QFB–tangent bundle that we will denote by ${}^{\varphi}T\mathcal{H}_i$. Clearly, there is a natural map $\mathcal{E}|_{\mathcal{H}_i}\to{}^{\varphi}T\mathcal{H}_i$ and its kernel

$$N_i\mathcal{E} := \ker(\mathcal{E}|_{\mathcal{H}_i}\to{}^{\varphi}T\mathcal{H}_i)$$

is a vector bundle on $\mathcal{H}_i$. On the other hand, $\mathrm{Im}(\mathcal{E}|_{\mathcal{H}_i}\to{}^{\varphi}T\mathcal{H}_i)$ is also a vector bundle, namely the vertical tangent bundle ${}^{\varphi}T(\mathcal{H}_i/\widetilde{\Sigma}_i)$ which, on each fiber $\varphi_i^{-1}(p)$ of (4.5), restricts to define the QFB–tangent bundle of that fiber. Consequently, there is a natural short exact sequence of vector bundles

$$(4.12)\qquad 0\to N_i\mathcal{E}\to\mathcal{E}|_{\mathcal{H}_i}\to{}^{\varphi}T(\mathcal{H}_i/\widetilde{\Sigma}_i)\to 0.$$

Since there is a natural inclusion ${}^{\varphi}T(\mathcal{H}_i/\widetilde{\Sigma}_i) \subset \mathcal{E}|_{\mathcal{H}_i}$, this short exact sequence splits and yields a natural decomposition

$$\mathcal{E}|_{\mathcal{H}_i} = N_i\mathcal{E} \oplus {}^{\varphi}T(\mathcal{H}_i/\widetilde{\Sigma}_i). \tag{4.13}$$

The vertical tangent bundle ${}^{\varphi}T(\mathcal{H}_i/\widetilde{\Sigma}_i)$ also naturally fits into another natural short exact sequence, namely

$$0 \to {}^{\varphi}T(\mathcal{H}_i/\widetilde{\Sigma}_i) \to {}^{\varphi}T\mathcal{H}_i \to \varphi_i^*({}^{\varphi}T\widetilde{\Sigma}_i) \to 0, \tag{4.14}$$

where ${}^{\varphi}T\widetilde{\Sigma}_i$ is the QFB–tangent bundle of $\widetilde{\Sigma}_i$. Notice that this tangent bundle is well defined thanks to the fact that the boundary-defining functions of $\mathcal{X}$ are compatible with the collection of fiber bundle maps φ; see the discussion just below Definition 1.9. In particular, this yields the natural identification

$$\varphi_i^*({}^{\varphi}T\widetilde{\Sigma}_i) = {}^{\varphi}T\mathcal{H}_i/{}^{\varphi}T(\mathcal{H}_i/\widetilde{\Sigma}_i).$$

On the other hand, multiplication by the boundary-defining function r_i induces the identification

$${}^{\varphi}T\mathcal{H}_i/{}^{\varphi}T(\mathcal{H}_i/\widetilde{\Sigma}_i) \cong N_i\mathcal{E}|_{\mathcal{H}_i},$$

so that there is a canonical identification

$$N_i\mathcal{E}|_{\mathcal{H}_i} \cong \varphi_i^*({}^{\varphi}T\widetilde{\Sigma}_i). \tag{4.15}$$

Summing up, we have a canonical decomposition

$$\mathcal{E}|_{\mathcal{H}_i} = \varphi_i^*({}^{\varphi}T\widetilde{\Sigma}_i) \oplus {}^{\varphi}T(\mathcal{H}_i/\widetilde{\Sigma}_i). \tag{4.16}$$

Now, proceeding as in Section 3 but replacing $|\lambda_1|$ with ε, we can show that in terms of this decomposition, the restriction of $\varepsilon^{-2}g_\varepsilon$ to $\mathcal{H}_i$ takes the form

$$\left.\frac{g_\varepsilon}{\varepsilon^2}\right|_{\mathcal{H}_i} = g_{\varphi_i} + \varphi_i^* g_{\widetilde{\Sigma}_i}, \tag{4.17}$$

where $g_{\widetilde{\Sigma}_i} \in \mathcal{C}^\infty(\widetilde{\Sigma}_i; N^*\widetilde{\Sigma}_i \otimes N^*\widetilde{\Sigma}_i)$ and where g_{φ_i} is on each fiber $\widetilde{V}_i$ of (4.5) a QAC–metric induced by a corresponding Euclidean metric on $\overline{V}_i$. In terms of the form $\omega_\varepsilon = \beta^*\operatorname{pr}_1^*\omega$, we have a corresponding decomposition

$$\left.\frac{\omega_\varepsilon}{\varepsilon^2}\right|_{\mathcal{H}_i} = \omega_{\varphi_i} + \varphi_i^*\omega_{\widetilde{\Sigma}_i}, \tag{4.18}$$

where ω_{φ_i} and $\omega_{\widetilde{\Sigma}_i}$ are closed $(1,1)$–forms.

We can finally introduce the space of deformations that will allow us to obtain Kähler QAC–metrics on $\widehat{X}_{\mathrm{QAC}}$.

Lemma 4.7 *The local product Kähler crepant resolution $\widehat{X}_{\mathrm{QAC}}$ of $\widetilde{X}_{\mathrm{sc}}$ extends to give a resolution of $\mathcal{X}$ by a manifold with fibered corners $\widehat{\mathcal{X}}$.*

Proof The proof is similar to the proof of Theorem 2.8 and is left as an exercise. □

On the resolution $\widehat{\mathcal{X}}$, the boundary hypersurface $\mathcal{H}_i$ is replaced by a boundary hypersurface $\widehat{\mathcal{H}}_i$ that is a resolution of $\mathcal{H}_i$. Moreover, the fiber bundles (4.4) and (4.5) are replaced by

$$\begin{array}{ccc} \mathcal{Y}_i & \longrightarrow & \widehat{\mathcal{H}}_i \\ & & \downarrow{\scriptstyle \widehat{\varphi}_i} \\ & & S_i \end{array} \qquad \text{for } i \in \{1,\dots,k\}, \tag{4.19}$$

where $\mathcal{Y}_i$ is a local product crepant resolution of $\mathcal{V}_i$, and

$$\begin{array}{ccc} \mathcal{Y}_i & \longrightarrow & \widehat{\mathcal{H}}_i \\ & & \downarrow{\scriptstyle \widehat{\varphi}_i} \\ & & \widetilde{\Sigma}_i \end{array} \qquad \text{for } i \in \{k+1,\dots,\ell\}, \tag{4.20}$$

with $\mathcal{Y}_i$ a local product crepant resolution of $\widetilde{V}_i$. Notice also that the function ε on $\mathcal{X}$ naturally extends to a smooth function on $\widehat{\mathcal{X}}$, which we also denote by ε. Similarly, as in Lemma 3.4, the boundary-defining functions r_i defined in (4.8) can be chosen to lift to smooth boundary-defining functions on $\widehat{\mathcal{X}}$, yielding a natural Lie algebra $\mathcal{V}_{\mathrm{QAC}}(\widehat{\mathcal{X}})$ of QAC–vector fields. Hence, we can introduce a Lie subalgebra of $\mathcal{V}_{\mathrm{QAC}}(\widehat{\mathcal{X}})$, namely,

$$\mathcal{V}_{\mathrm{QAC},\varepsilon}(\widehat{\mathcal{X}}) := \{\xi \in \mathcal{V}_{\mathrm{QAC}}(\widehat{\mathcal{X}}) \mid \xi\varepsilon \equiv 0\}, \tag{4.21}$$

and a corresponding vector bundle $\widehat{\mathcal{E}} \to \widehat{\mathcal{X}}$ with a map $\iota\colon \widehat{\mathcal{E}} \to T\widehat{\mathcal{X}}$ inducing a canonical identification

$$\iota_* \mathcal{C}^\infty(\widehat{\mathcal{X}}; \widehat{\mathcal{E}}) = \mathcal{V}_{\mathrm{QAC},\varepsilon}(\widehat{\mathcal{X}}).$$

As the next theorem shows, the deformation space allows us to formulate a criterion for the existence of Kähler QAC–metrics on $\widehat{X}_{\mathrm{QAC}}$.

Theorem 4.8 *Suppose that for each $i \in \{k+1,\dots,\ell\}$, we can find a smooth closed $(1,1)$–form $\omega_{\widehat{\varphi}_i}$ on $\widehat{\mathcal{H}}_i$ that restricts on each fiber of $\widehat{\varphi}_i\colon \widehat{\mathcal{H}}_i \to \widetilde{\Sigma}_i$ to the Kähler form of a QAC–metric asymptotic to ω_{φ_i} with rate $\delta > 0$ in the sense of Definition 3.6.* (Notice that this is trivial for $i = \ell$ since the fibers of $\widehat{\varphi}_i = \varphi_i$ are points.) *Suppose moreover that the forms*

$$\omega_i := \omega_{\widehat{\varphi}_i} + \widehat{\varphi}_i^* \omega_{\widetilde{\Sigma}_i}$$

on $\widehat{\mathcal{H}}_i$ are compatible in the sense that $\omega_i|_{\widehat{\mathcal{H}}_i\cap\widehat{\mathcal{H}}_j} = \omega_j|_{\widehat{\mathcal{H}}_i\cap\widehat{\mathcal{H}}_j}$ for every i and j in $\{k+1,\dots,\ell\}$. Then $\widehat{X}_{\mathrm{QAC}}$ admits a smooth Kähler QAC–metric asymptotic to g_C with rate δ.

Proof Thanks to the compatibility condition, we can find $\widehat{\omega} \in \mathcal{C}^\infty(\widehat{\mathcal{X}}; \widehat{\mathcal{E}}^* \wedge \widehat{\mathcal{E}}^*)$ such that $\widehat{\omega}$ is a closed $(1,1)$–form on each level set of the function ε whose restriction to $\mathcal{H}_i$ is ω_i for each $i \in \{k+1,\dots,\ell\}$. By continuity, this means that for $c > 0$ sufficiently small, the restriction of $\widehat{\omega}$ to the level set $\{\varepsilon = c\} \cong \widehat{X}_{\mathrm{QAC}}$ is positive definite as a section of $\widehat{\mathcal{E}}^* \wedge \widehat{\mathcal{E}}^*$. Since $\widehat{\mathcal{E}}|_{\{\varepsilon=c\}} \cong {}^{\widehat{\phi}}T\widehat{X}_{\mathrm{QAC}}$, we see that $\widehat{\omega}|_{\{\varepsilon=c\}}$ is the desired Kähler form. □

Corollary 4.9 *If the compact Kähler–Einstein Fano orbifold (D, g_D) has only isolated singularities of complex codimension at least two with each locally admitting a Kähler crepant resolution, then D admits a Kähler crepant resolution $\widehat{D}$ and the QAC–compactification $\widehat{X}_{\mathrm{QAC}}$ of $K_{\widehat{D}}$ admits a Kähler QAC–metric equal to g_C in a neighborhood of the maximal boundary hypersurface of $\widehat{X}_{\mathrm{QAC}}$, in particular, asymptotic to g_C with rate δ for any $\delta > 0$.*

Proof By Proposition 4.2, K_D admits a smooth Kähler metric g which is equal to g_C outside a compact set, so we have a corresponding metric $g_\varepsilon = \iota_\varepsilon^* \beta^* \mathrm{pr}_1^* g$ on $\mathcal{X}$ to which we can hope to apply Theorem 4.8. Now, since the singularities of D are isolated, the fiber bundles $\widehat{\varphi}_i : \widehat{\mathcal{H}}_i \to \widetilde{\Sigma}_i$ are trivial and $\widetilde{\Sigma}_i \cong \overline{\mathbb{C}}$. Thus, using Lemma 4.1, we can construct for each i the forms $\omega_{\widehat{\varphi}_i}$ in the statement of Theorem 4.8 which are equal to ω_{φ_i} in a neighborhood of $\widehat{\mathcal{H}}_i \cap \widehat{\mathcal{H}}_\ell$, where $\widehat{\mathcal{H}}_\ell$ is the maximal boundary hypersurface of $\widehat{\mathcal{X}}$. The result then follows by applying Theorem 4.8. Notice in particular that without loss of generality, the deformation $\widehat{\omega}|_{\{\varepsilon=c\}}$ can be chosen to be equal to ω_C in a neighborhood of the maximal hypersurface of $\widehat{X}_{\mathrm{QAC}}$. Moreover, by restricting to the zero section of $K_{\widehat{D}}$, we also obtain a Kähler metric on $\widehat{D}$, as claimed. □

Corollary 4.10 *If the compact Kähler–Einstein Fano orbifold (D, g_D) is of the form*

$$(D, g_D) = (D_1 \times \cdots \times D_q, g_1 \times \cdots \times g_q),$$

with each D_i an orbifold as in the previous corollary, then the QAC–compactification $\widehat{X}_{\mathrm{QAC}}$ of $K_{\widehat{D}}$ admits a Kähler QAC–metric equal to g_C in a neighborhood of the maximal boundary hypersurface of $\widehat{X}_{\mathrm{QAC}}$, in particular, asymptotic to g_C with rate δ for any $\delta > 0$.

Proof Again, we can use Proposition 4.2 to construct a smooth Kähler metric g on the orbifold K_D equal to g_C outside a compact set, so we have a corresponding metric g_ε on $\mathcal{X}$ to which we can hope to apply Theorem 4.8. We still also know that the fiber bundles $\hat{\varphi}_i\colon \hat{\mathcal{H}}_i \to \tilde{\Sigma}_i$ are all trivial. For those for which the fibers are manifolds with boundary, we can proceed as before using Lemma 4.1 to construct the form $\omega_{\hat{\varphi}_i}$. For the other fiber bundles, the fibers are QAC–resolutions of spaces of the form

$$\mathbb{C}^{n_1}/\Gamma_1 \times \cdots \times \mathbb{C}^{n_r}/\Gamma_r,$$

with $\Gamma_j \subset \mathrm{SU}(n_j)$ a finite subgroup acting freely on $\mathbb{C}^{n_j} \setminus \{0\}$, hence we can apply Lemma 4.1 on the crepant resolution of each factor to obtain the form $\omega_{\hat{\varphi}_i}$. It suffices to make consistent choices to ensure that the compatibility conditions of Theorem 4.8 are satisfied. Hence the result follows by applying Theorem 4.8. Again, we can do this in such a way that the resulting Kähler form is equal to ω_C in a neighborhood of the maximal boundary hypersurface of $\hat{X}_{\mathrm{QAC}}$. □

5 Solving the complex Monge–Ampère equation

Let X be a Kähler orbifold as in Section 2 and let μ be the complex codimension of the singular set of X_{sc}. In other words,

$$\mu = \min_{H_i < H_{\max}} m_i, \tag{5.1}$$

with m_i the complex dimension of the fibers of the fiber bundle $\hat{\phi}_i\colon \hat{H}_i \to S_i$. Notice that by the hypotheses of Section 2, we are assuming in particular that

$$2 \le \mu \le n. \tag{5.2}$$

Now let ω_0 be the Kähler form of a Kähler QAC–metric g_0 on $\hat{X}_{\mathrm{QAC}}$ asymptotic to the Ricci-flat Kähler cone metric g_C with rate $\epsilon > 0$ for some $\epsilon > 0$. Notice that the examples of Kähler QAC–metrics of Corollary 4.10 are in fact asymptotic to g_C with a rate of α as large as we want, but their Ricci potentials do not necessarily decay near $\partial \hat{X}_{\mathrm{QAC}} \setminus \hat{H}_{\max}$. In fact, knowing that g_0 is asymptotic to g_C at rate ϵ implies that its Ricci potential

$$r_0 := \log\left(\frac{(\omega_0^{n+1})^p}{c_p \Omega^p_{\hat{X}} \wedge \overline{\Omega^p_{\hat{X}}}}\right) \tag{5.3}$$

is in $x_{\max}^{\epsilon} \mathcal{C}^{\infty}_{\mathrm{Qb}}(\hat{X})$, where $\Omega^p_{\hat{X}}$ is the lift of Ω^p_X to the local product Kähler crepant resolution $\hat{X}$ of X. Hence, the Ricci potential decays near $H_{\max}$, but not necessarily near the other boundary hypersurfaces of $\widehat{X}_{\mathrm{QAC}}$. However, by Definition 3.6, we see that r_0 has well-defined restrictions at the other boundary hypersurfaces in the following sense.

Definition 5.1 For $\widehat{H}_i < \widehat{H}_{\max}$, let $\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{H}_i/S_i)$ be the space of smooth functions on

$$\widehat{H}_i \setminus \Big(\bigcup_{\widehat{H}_j > \widehat{H}_i} \widehat{H}_j \cap \widehat{H}_i \Big)$$

which restrict on each fiber $\phi_i^{-1}(s)$ of $\widehat{\phi}_i \colon \widehat{H}_i \setminus \big(\bigcup_{\widehat{H}_j > H_i} \widehat{H}_j \cap \widehat{H}_i\big) \to S_i$ to a function in $\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{\phi}_i^{-1}(s))$. A function $f \in x^\alpha_{\max}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X})$ is said to *restrict to* $\partial\widehat{X}_{\mathrm{QAC}}$ if for each $\widehat{H}_i < \widehat{H}_{\max}$, there is an $f_i \in x^\alpha_{\max}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{H}_i/S_i)$ such that

$$f - f_i \in x^\alpha_{\max} x_i \mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X}).$$

We denote by $x^\alpha_{\max}\mathcal{C}^\infty_{\mathrm{Qb},r}(\widehat{X})$ the space of functions in $x^\alpha_{\max}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X})$ that restrict to $\partial\widehat{X}_{\mathrm{QAC}}$.

To obtain a Ricci-flat QAC–metric on $\widehat{X}_{\mathrm{QAC}}$, we then need to solve the complex Monge–Ampère equation

$$\log\left(\frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial}u)^{n+1}}{\omega_0^{n+1}}\right) = -r_0. \tag{5.4}$$

In fact, this is a particular case of the more general complex Monge–Ampère equation

$$\log\left(\frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial}u)^{n+1}}{\omega_0^{n+1}}\right) = f, \quad f \in x^\alpha_{\max}\mathcal{C}^\infty_{\mathrm{Qb},r}(\widehat{X}). \tag{5.5}$$

Since it does not require any further work, we will solve (5.5) with $f \in x^\alpha_{\max}\mathcal{C}^\infty_{\mathrm{Qb},r}(\widehat{X})$ not necessarily equal to $-r_0$.

To begin, we make the simplifying assumptions that

$$4 \le \alpha \le 2\mu \tag{5.6}$$

and that the restriction of f to $\partial\widehat{X}_{\mathrm{QAC}}$ is zero, so that in fact $f \in x^\alpha_{\max}x_{\mathrm{sing}}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X})$, where

$$x_{\mathrm{sing}} = \prod_{H_i < H_{\max}} x_i = \frac{x}{x_{\max}}$$

is the product of all of the boundary-defining functions except the one of the maximal boundary hypersurface. As we shall soon see, it is always possible to reduce the problem to this simpler setting. Now, to solve (5.5) for $f \in x^\alpha_{\max}x_{\mathrm{sing}}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X})$, our strategy is to modify the metric so that (5.5) is replaced with a complex Monge–Ampère equation with a new f decaying faster at infinity. In order to do this, we follow the strategy of [13, Lemma 2.12] using the following Fredholm theory result.

Theorem 5.2 [17] *For all $s \in \mathbb{N}_0$ and $\gamma \in (0,1)$, the Laplacian Δ of a QAC–metric on $\widehat{X}_{\mathrm{QAC}}$ induces an isomorphism*

$$\Delta\colon x^{-\delta} x_{\mathrm{sing}}^{\tau} \mathcal{C}_{\mathrm{Qb}}^{s+2,\gamma}(\widehat{X}) \to x^{2-\delta} x_{\mathrm{sing}}^{\tau-2} \mathcal{C}_{\mathrm{Qb}}^{s,\gamma}(\widehat{X}) \tag{5.7}$$

provided that $-2n<\delta<0$ and $2-2\mu<\tau<0$, where we recall that $\widehat{X}=\widehat{X}_{\mathrm{QAC}} \setminus \partial \widehat{X}_{\mathrm{QAC}}$.

Proof Recall that in terms of the notation of [17], $x=\rho^{-1}$ and $x_{\mathrm{sing}}=w_1$ is the function defined in Remark 1.26. Thus, by [17, Theorem 7.6], we know that the map (5.7) is Fredholm. Since $\partial\widehat{X}_{\mathrm{QAC}}$ is connected, we can deduce from [17, Theorem 6.10] and the proof of [17, Theorem 7.6] that the map (5.7) is in fact an isomorphism. □

Lemma 5.3 *If $f \in x_{\max}^{\alpha} x_{\mathrm{sing}} \mathcal{C}_{\mathrm{Qb}}^{\infty}(\widehat{X})$, with α as in (5.6), then there exists $v \in x_{\max}^{\alpha-2} x_{\mathrm{sing}} \mathcal{C}_{\mathrm{Qb}}^{\infty}(\widehat{X})$ such that $\widetilde{\omega}_0=\omega_0+\sqrt{-1}\partial\bar{\partial}v$ is the Kähler form of a QAC–metric $\widetilde{g}_0$ asymptotic to the Ricci-flat Kähler cone metric g_C with rate α and*

$$\widetilde{f} := f - \log\left(\frac{\widetilde{\omega}_0^{n+1}}{\omega_0^{n+1}}\right) \in x_{\max}^{2\alpha} x_{\mathrm{sing}}^{3} \mathcal{C}_{\mathrm{Qb}}^{\infty}(\widehat{X}) \subset x_{\max}^{\alpha+1} x_{\mathrm{sing}}^{3} \mathcal{C}_{\mathrm{Qb}}^{\infty}(\widehat{X}).$$

Proof We follow the strategy of [13, Lemma 2.12] using Theorem 5.2. As the reader will see, the inequality (5.2) will be used in an essential way in the proof. Since

$$f \in x_{\mathrm{sing}} x_{\max}^{\alpha} \mathcal{C}_{\mathrm{Qb}}^{\infty}(\widehat{X}) = x^{\alpha} x_{\mathrm{sing}}^{1-\alpha} \mathcal{C}_{\mathrm{Qb}}^{\infty}(\widehat{X}),$$

we see, by taking $\tau=3-\alpha$ and $\delta=2-\alpha\le -2$ in Theorem 5.2, that there exists a unique $u \in x^{-\delta} x_{\mathrm{sing}}^{\tau} \mathcal{C}_{\mathrm{Qb}}^{\infty}(\widehat{X})$ such that

$$\Delta_{g_0} u = 2f,$$

where $\Delta_{g_0}=g_0^{ij}\nabla_i\nabla_j$ is the Laplacian associated to g_0. Since $\delta<0$ and $\tau-\delta=1>0$, u is decaying at infinity, hence $\omega_0+\sqrt{-1}\partial\bar{\partial}u$ is still positive definite outside a compact set. Moreover, we can truncate u to obtain a new function v_1 equal to u outside a compact set such that $\omega_1 := \omega_0+\sqrt{-1}\partial\bar{\partial}v_1$ is positive definite everywhere. Now, one computes that

$$(5.8)\quad \begin{aligned}(\omega_0+\sqrt{-1}\partial\overline{\partial}v_1)^{n+1} &= \bigl(1+\tfrac{1}{2}\Delta_{g_0}v_1\bigr)\omega_0^{n+1}+\frac{(n+1)!}{2!(n+1-2)!}\omega_0^{n+1-2}(\sqrt{-1}\partial\overline{\partial}v_1)^2 \\ &\qquad\qquad +\cdots+(\sqrt{-1}\partial\overline{\partial}v_1)^{n+1} \\ &= (1+f)\omega_0^{n+1}+x^{4-2\delta}x_{\mathrm{sing}}^{2\tau-4}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X};\Lambda^{2n+2}({}^{\hat{\phi}}T^*\widehat{X}_{\mathrm{QAC}})),\end{aligned}$$

which implies that

$$f_1 := f-\log\left(\frac{\omega_1^{n+1}}{\omega_0^{n+1}}\right)\in x^{4-2\delta}x_{\mathrm{sing}}^{2\tau-4}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X}) = x^{2\alpha}x_{\mathrm{sing}}^{2-2\alpha}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X})\subset x^{\alpha}x_{\mathrm{sing}}^{2-\alpha}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X}).$$

In particular, we see that f_1 decays faster than f at infinity. Repeating the above argument with ω_1 and f_1 in place of ω_0 and f, this time using the isomorphism (5.7) with g_1 instead of g_0 and with $\delta = 2-\alpha$ as before, but with $\tau = \frac{7}{2}-\alpha$, we can find $v_2 \in x^{-\delta}x_{\mathrm{sing}}^{\tau}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X})$ such that $\omega_2 := \omega_1+\sqrt{-1}\partial\overline{\partial}v_2$ is positive definite, with

$$\begin{aligned} f_2 := f-\log\left(\frac{\omega_2^{n+1}}{\omega_0^{n+1}}\right) &= f_1-\log\left(\frac{\omega_2^{n+1}}{\omega_1^{n+1}}\right) \\ &\in x^{4-2\delta}x_{\mathrm{sing}}^{2\tau-4}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X}) = x^{2\alpha}x_{\mathrm{sing}}^{3-2\alpha}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X}) = x_{\max}^{2\alpha}x_{\mathrm{sing}}^{3}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X}).\end{aligned}$$

Thus, it suffices again to take $v = v_1+v_2$ to obtain the result. □

For the Kähler metric $\widetilde{\omega}_0$ and the function $\widetilde{f}$, we can now appeal to the result of Tian and Yau [42] or its parabolic version [10] to solve the complex Monge–Ampère equation.

Theorem 5.4 *For the Kähler form $\widetilde{\omega}_0$ and the function $\widetilde{f}$ given by Lemma 5.3, the complex Monge–Ampère equation*

$$(5.9)\qquad \log\left(\frac{(\widetilde{\omega}_0+\sqrt{-1}\partial\overline{\partial}u)^{n+1}}{\widetilde{\omega}_0^{n+1}}\right) = -\widetilde{f}$$

has a unique solution u in $x_{\max}^{\alpha-1}x_{\mathrm{sing}}^{2}\mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X})$.

Proof This is very similar to what has been done for asymptotically conical metrics in [24; 21]. We will therefore go over the argument putting emphasis on the new features. The idea is to apply the continuity method to

$$(5.10)\qquad \log\left(\frac{(\widetilde{\omega}_0+\sqrt{-1}\partial\overline{\partial}u_t)^{n+1}}{\widetilde{\omega}_0^{n+1}}\right) = t\widetilde{f}$$

for $t \in [0,1]$. That is, we will show that the set

$$S = \{s \in [0,1] \mid \text{there is a solution } u_s \in x_{\max}^{\alpha-1} x_{\mathrm{sing}}^3 \mathcal{C}_{\mathrm{Qb}}^\infty(\hat{X}) \text{ of (5.10) for } t = s\}$$

is in fact all of $[0,1]$ by showing that it is nonempty, open and closed. Clearly, $u_0 = 0$ is a solution of (5.9) for $t = 0$, so that S is nonempty. The openness of S follows from Theorem 5.2.

For closedness, suppose that $[0,\tau) \subset S$ for some $0 < \tau \le 1$. We need to show that (5.10) has a solution for $t = \tau$. For this, we need to derive a priori estimates for solutions of (5.10). We do this as follows. First of all, thanks to the Sobolev inequality (1.10), we can apply a Moser iteration to obtain an a priori $\mathcal{C}^0$–bound on a solution u_t of (5.10). Yau's method then provides a uniform bound on $\sqrt{-1}\partial\bar{\partial}u_t$. By the Evans–Krylov theorem, this yields an a priori $\mathcal{C}^{2,\gamma}$–bound on solutions, where the Hölder norm is defined in term of $\tilde{g}_0$. If $\{t_i\}$ is a strictly increasing sequence with $t_i \nearrow \tau$ and $\{u_{t_i}\}$ is a corresponding sequence of solutions of (5.10) for $t = t_1, t_2, \dots$, then using the Arzelà –Ascoli theorem, one can extract a subsequence that converges in $\mathcal{C}^2_{\mathrm{QAC}}(\hat{X})$ to some function u. Clearly then, u is solution of (5.10) for $t = \tau$. Bootstrapping, we thus see that $u \in \mathcal{C}^\infty_{\mathrm{QAC}}(\hat{X})$.

To see that u is in fact in $x_{\max}^{\alpha-1} x_{\mathrm{sing}}^3 \mathcal{C}_{\mathrm{Qb}}^\infty(\hat{X})$, we need to work slightly harder. First, using Moser iteration with weights as in [24, Section 8.6.2] and the fact that $\tilde{f} \in x_{\max}^{\alpha+1} x_{\mathrm{sing}}^3 \mathcal{C}_{\mathrm{Qb}}^\infty(\hat{X}) \subset x^3 \mathcal{C}_{\mathrm{Qb}}^\infty(\hat{X})$, we obtain an a priori bound in $x^\nu \mathcal{C}^0(\hat{X})$ for some $0 < \nu < 3-2 = 1$ for the solutions u_{t_i}. The argument in [24, Section 8.6.2] is, strictly speaking, written for ALE–metrics, but as subsequently explained in [24, Section 9.6.2], because we have the Sobolev inequality (1.10), the argument also works for the QAC–metric $\tilde{g}_0$ and only involves minor notational changes. This a priori bound thus implies that $u \in x^\nu \mathcal{C}^0(\hat{X}) \cap \mathcal{C}^\infty_{\mathrm{QAC}}(\hat{X})$.

To improve the statement about the regularity of u, we now work directly with (5.10) for $t = \tau$. Notice first that the equation can be rewritten as

$$\tau\tilde{f} = \int_0^1 \frac{\partial}{\partial t} \log\left(\frac{(\tilde{\omega}_0 + t\sqrt{-1}\partial\bar{\partial}u)^{n+1}}{\tilde{\omega}_0^{n+1}}\right) dt = \int_0^1 \left(\frac{(n+1)\tilde{\omega}_{u,t}^n \wedge \sqrt{-1}\partial\bar{\partial}u}{\tilde{\omega}_{u,t}^{n+1}}\right) dt,$$

where $\tilde{\omega}_{u,t} = \tilde{\omega}_0 + t\sqrt{-1}\partial\bar{\partial}u$. In other words, the complex Monge–Ampère equation can be rewritten as

$$\Delta_u u = \tau\tilde{f}, \tag{5.11}$$

where

$$\Delta_u v = \int_0^1 \left(\frac{(n+1)\widetilde{\omega}_{u,t}^n \wedge \sqrt{-1}\partial\bar{\partial} v}{\widetilde{\omega}_{u,t}^{n+1}} \right) dt = \frac{1}{2} \int_0^1 (\Delta_{\widetilde{\omega}_{u,t}} v)\, dt,$$

with $\Delta_{\widetilde{\omega}_{u,t}}$ the Laplacian associated to the Kähler form $\widetilde{\omega}_{u,t}$.

Since a QAC–metric has bounded geometry by Proposition 1.27 or [17, Remark 2.20], applying the Schauder estimate to (5.11), we find that in fact $u \in x^\nu \mathcal{C}^\infty_{\mathrm{QAC}}(\widehat{X})$. Since, by Lemma 1.31, $x^\nu \mathcal{C}^1_{\mathrm{QAC}}(\widehat{X}) \subset \mathcal{C}^{0,\gamma}_{\mathrm{Qb}}(\widehat{X})$ for $\gamma \le \nu$, we see in particular that $\|\partial\bar{\partial} u\|_{g_0} \in \mathcal{C}^{0,\gamma}_{\mathrm{Qb}}(\widehat{X})$. Rewriting (5.11) in terms of an elliptic Qb–operator, that is,

$$(x_{\max}^{-2}\Delta_u)u = x_{\max}^{-2}\tau \widetilde{f}, \tag{5.12}$$

we can, thanks to Proposition 1.30, apply the Schauder estimate once again and bootstrap to see that $u \in x^\nu \mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X})$. Finally, using the inclusion $x_{\max}^{\alpha-1} x_{\mathrm{sing}}^2 \mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X}) \subset x^\nu x_{\mathrm{sing}}^{-\nu} \mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X})$, we can apply the isomorphism (5.7) with (δ, τ) equal to $(-\nu, -\nu)$ and $(1-\alpha, 3-\alpha)$ to conclude that $u \in x_{\max}^{\alpha-1} x_{\mathrm{sing}}^2 \mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X})$. This shows that the set S is closed and completes the proof of existence.

For uniqueness, we can proceed as in [4, Proposition 7.13], but using the isomorphism (5.7) instead of the maximum principle. □

This means that for $f \in x_{\max}^\alpha x_{\mathrm{sing}} \mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X})$, we can solve the original equation (5.5).

Corollary 5.5 *For $f \in x_{\max}^\alpha x_{\mathrm{sing}} \mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X})$, the complex Monge–Ampère equation (5.5) has a unique solution $u \in x_{\max}^{\alpha-2} x_{\mathrm{sing}} \mathcal{C}^\infty_{\mathrm{Qb}}(\widehat{X})$.*

Proof Applying Lemma 5.3, this amounts to solving the complex Monge–Ampère equation (5.9), so existence follows from Theorem 5.4. On the other hand, uniqueness follows again by proceeding as in [4, Proposition 7.13], but using the isomorphism (5.7) instead of the maximum principle. □

The decay of the Ricci potential at infinity, and more generally of the function f, is a strong assumption. Still, since it is satisfied by the examples of the previous section in the case that $\widehat{X}_{\mathrm{QAC}}$ is a manifold with boundary, we will be able to relax this assumption by proceeding by induction on the depth of $\widehat{X}_{\mathrm{QAC}}$ and using the existence result of Corollary 5.5.

Proposition 5.6 *Given $f \in x_{\max}^\beta \mathcal{C}^\infty_{\mathrm{Qb},r}(\widehat{X})$ with $\beta \ge 4$ and $\beta \ne 2\mu$, there exists a function $u \in x_{\max}^{\alpha-2} \mathcal{C}^\infty_{\mathrm{Qb},r}(\widehat{X})$ with $\alpha = \min\{2\mu, \beta\}$ such that*

(1) $\omega := \omega_0 + \frac{\sqrt{-1}}{2}\partial\overline{\partial}u$ *is the Kähler form of a Kähler* QAC*–metric* $\widetilde{g}$ *asymptotic to* g_C *with rate* α *;*

(2) $f - \log(\omega^{n+1}/\omega_0^{n+1}) \in x_{\max}^{\alpha} x_{\mathrm{sing}} \mathcal{C}^{\infty}_{\mathrm{Qb}}(\widehat{X})$.

Proof Let $\widehat{H}_1, \dots, \widehat{H}_{\ell+1}$ be an exhaustive list of the boundary hypersurfaces of $\widehat{X}_{\mathrm{QAC}}$ compatible with the partial order in the sense that

$$\widehat{H}_i < \widehat{H}_j \implies i < j.$$

To construct ω, we will recursively construct the restrictions $u|_{\widehat{H}_i}$ proceeding in order of increasing relative depth, that is, in decreasing order with respect to the index i. More precisely, for each $i \in \{1, \dots, \ell+1\}$, we will show that there exists a function $u_i \in x_{\max}^{\alpha-2}\mathcal{C}^{\infty}_{\mathrm{Qb},r}(\widehat{X})$ such that $\omega + \sqrt{-1}\partial\overline{\partial}u_i$ is the Kähler form of a QAC–metric asymptotic to g_C with rate α and with

$$f - \log\left(\frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial}u_i)^{n+1}}{\omega^{n+1}}\right) \in x_{\max}^{\alpha}\left(\prod_{i \le j < \ell+1} x_j\right)\mathcal{C}^{\infty}_{\mathrm{Qb}}(\widehat{X}). \tag{5.13}$$

For the maximal boundary hypersurface $\widehat{H}_{\ell+1}$, we simply take $u_{\ell+1} = 0$. Suppose now that for some i, we can find $u_{i+1} \in x_{\max}^{\alpha-2}\mathcal{C}^{\infty}_{\mathrm{Qb},r}(\widehat{X}_{\mathrm{QAC}} \setminus \partial\widehat{X}_{\mathrm{QAC}})$ such that $\omega := \omega_0 + \sqrt{-1}\partial\overline{\partial}u_{i+1}$ is the Kähler form of a QAC–metric asymptotic to g_C with rate α such that

$$f - \log\left(\frac{(\omega + \sqrt{-1}\partial\overline{\partial}u_{i+1})^{n+1}}{\omega^{n+1}}\right) \in x_{\max}^{\alpha}\left(\prod_{i < j < \ell+1} x_j\right)\mathcal{C}^{\infty}_{\mathrm{Qb}}(\widehat{X}).$$

In terms of $f_i := f|_{\widehat{H}_i}$ and the restrictions of ω_0 and ω to $\widehat{H}_i$ respectively given by

$$\begin{aligned} \omega_0|_{\widehat{H}_i} &= \frac{\sqrt{-1}}{2} f_{E_i}\, d\lambda_1 \wedge d\overline{\lambda}_1 + \omega_{0,i} + \phi_i^*\omega_{S_i}, \\ \omega|_{\widehat{H}_i} &= \frac{\sqrt{-1}}{2} f_{E_i}\, d\lambda_1 \wedge d\overline{\lambda}_1 + \omega_i + \phi_i^*\omega_{S_i}, \end{aligned} \tag{5.14}$$

this means that

$$\widetilde{f}_i := f_i - \log\left(\frac{\omega_i^{m_i}}{\omega_{0,i}^{m_i}}\right) \in x_{\max}^{\alpha}\left(\prod_{i < j < \ell+1} x_j\right)\mathcal{C}^{\infty}_{\mathrm{Qb}}(\widehat{\phi}_i^{-1}(s))$$

in each fiber $\widehat{\phi}_i^{-1}(s)$ of $\widehat{\phi}_i \colon \widehat{H}_i \to S_i$, where m_i is the complex dimension of the fibers of $\widehat{\phi}_i$. Moreover, ω_i is a closed $(1,1)$–form which restricts on each fiber of $\widehat{\phi}_i \colon \widehat{H}_i \to S_i$ to the Kähler form of a Kähler QAC–metric asymptotic to the Euclidean metric $g_{\widehat{\phi}_i}$ with rate α. We then need to distinguish two cases.

Case 1: The relative depth of $\widehat{H}_i$ is strictly bigger than 1 In this case, we apply Corollary 5.5 to each fiber $\widehat{Y}_i$ of $\widehat{\phi}_i\colon \widehat{H}_i \to S_i$ to obtain a unique function

$$v_i \in \Big(\prod_{i<j\leq \ell} x_j\Big) x_{\max}^{\alpha-2} \mathcal{C}^{\infty}_{\mathrm{Qb}}(\widehat{Y}_i)$$

such that

$$\log\bigg(\frac{(\omega_i + \sqrt{-1}\partial\overline{\partial} v_i)^{n_i}}{\omega_i^{n_i}}\bigg) = \widetilde{f}_i. \tag{5.15}$$

On $\widehat{H}_i$, this yields a function that we will also denote by v_i. Notice that for all $t \in [0,1]$,

$$\omega_i + t\sqrt{-1}\partial\overline{\partial}(v_i) = (1-t)\omega_i + t(\omega_i + \sqrt{-1}\partial\overline{\partial} v_i) \tag{5.16}$$

is a convex sum of two Kähler forms, hence is itself a Kähler form. We can use this to extend v_i to a function $v \in \big(\prod_{i<j\leq \ell} x_j\big) x_{\max}^{\alpha-2}\mathcal{C}^{\infty}_{\mathrm{Qb}}(\widehat{X})$ such that $\widetilde{\omega} + \sqrt{-1}\partial\overline{\partial} v$ is the Kähler form of a QAC–metric. To see this, let $c_i\colon \widehat{H}_i \times [0,\epsilon) \to \widehat{X}_{\mathrm{QAC}}$ be a collar neighborhood of $\widehat{H}_i$ in $\widehat{X}_{\mathrm{QAC}}$ as in Lemma 1.10, so that $x_i \circ c_i\colon \widehat{H}_1 \times [0,\epsilon) \to [0,\epsilon)$ is the projection on the second factor. If $\psi \in \mathcal{C}^{\infty}(\mathbb{R})$ is a cut-off function that takes values in $[0,1]$ with $\psi(t) \equiv 1$ for $t < 1$ and $\psi(t) \equiv 0$ for $t > 2$, then, given the local descriptions (5.14) and (1.6), it suffices to take

$$v = (c_i)_*(\psi(Nx_i)v_i)$$

for $N > 0$ a constant chosen sufficiently large so that the $(1,1)$–form $\widetilde{\omega} + \sqrt{-1}\partial\overline{\partial} v$ remains positive definite. Indeed, set $\rho_i = \prod_{i<j\leq \ell+1} x_j$. Then, using local coordinates as in (1.2), one computes that

$$dx_i \in x_i\rho_i \mathcal{C}^{\infty}_{\mathrm{Qb}}(\widehat{X}; {}^{\phi}T^*\widehat{X}_{\mathrm{QAC}}) \quad \text{and} \quad \partial\overline{\partial} x_i \in x_i\rho_i\mathcal{C}^{\infty}_{\mathrm{Qb}}(\widehat{X}; \Lambda^2({}^{\phi}T^*\widehat{X}_{\mathrm{QAC}})).$$

On the other hand,

$$\sqrt{-1}\partial\overline{\partial}(\psi(Nx_i)v_i) = \sqrt{-1}\psi(Nx_i)\partial\overline{\partial} v_i + Q,$$

with

$$\begin{aligned} Q = \sqrt{-1} v_i\big(N^2\psi''(Nx_i)\partial x_i \wedge \overline{\partial} x_i + N\psi'(Nx_i)\partial\overline{\partial} x_i\big) \\ + \sqrt{-1}N\psi'(Nx_i)\big(\partial v_i \wedge \overline{\partial} x_i + \partial x_i \wedge \overline{\partial} v_i\big). \end{aligned}$$

Since $x_i \leq 2N^{-1}$ on the support of $\psi(Nx_i)$, for $\nu > 0$ to be taken small, we see that in the region $\rho_i < \nu$,

$$\|\sqrt{-1}\partial\overline{\partial}(\psi(Nx_i)v_i)\|_{g_0} \leq C\nu$$

for a constant C independent of $N \geq 1$. Taking ν small enough, we can thus ensure that $\omega + \sqrt{-1}\partial\bar{\partial}v > 0$ in the region $\rho_i < \nu$ whatever the choice of $N \geq 1$. Keeping $\nu > 0$ fixed, we now adjust the choice of $N \geq 1$ to ensure that $\omega + \sqrt{-1}\partial\bar{\partial}v$ is also positive definite in the region $\rho_i \geq \nu$. In this region, one easily computes that

$$dx_i \in x_i^2 \mathcal{C}^\infty_{\mathrm{Qb}}(\hat{X}; {}^{\hat{\phi}}T^*\hat{X}_{\mathrm{QAC}}) \quad \text{and} \quad \partial\bar{\partial}x_i \in x_i^2 \mathcal{C}^\infty_{\mathrm{Qb}}(\hat{X}; \Lambda^2({}^{\hat{\phi}}T^*\hat{X}_{\mathrm{QAC}})).$$

Hence, since again $x_i \leq 2N^{-1}$ on the support of $\psi(Nx_i)$, we see that

$$\|Q\|_g \leq \frac{C_\nu}{N} \quad \text{in the region } \rho_i \geq \nu$$

for some constant $C_\nu > 0$ depending on ν. Thus, Q can be taken as small as we want by taking N sufficiently large. On the other hand, by construction, the term $\sqrt{-1}\psi(Nx_i)\partial\bar{\partial}v_i$ is not expected to be small in the region $\rho_i \geq \nu$, but by the convexity property (5.16), we can still ensure that

$$\omega + \sqrt{-1}\psi(Nx_i)\partial\bar{\partial}v_i > 0 \quad \text{in the region } \rho_i \geq \nu$$

provided that $N \geq 1$ is taken large enough. Summing up, we can therefore ensure that $\omega + \sqrt{-1}\partial\bar{\partial}v > 0$ provided that $N \geq 1$ is large enough, as claimed. Notice then that by (5.15), it suffices to take $u_i = u_{i+1} + v \in x_{\max}^\alpha \mathcal{C}^\infty_{\mathrm{Qb},r}(\hat{X})$ to obtain the desired function for which $\omega + \sqrt{-1}\partial\bar{\partial}u_i$ is the Kähler form of a QAC–metric asymptotic to g_C with rate α and

$$f - \log\left(\frac{(\omega + \sqrt{-1}\partial\bar{\partial}u_i)^{n+1}}{\omega^{n+1}}\right) \in x_{\max}^\alpha \left(\prod_{i \leq j < \ell+1} x_j\right) \mathcal{C}^\infty_{\mathrm{Qb}}(\hat{X}),$$

thereby completing the inductive step.

Case 2: The relative depth of $\hat{H}_i$ is equal to 1 In this case, the fibers of $\hat{\phi}_i\colon \hat{H}_i \to S_i$ are necessarily manifolds with boundary and the corresponding fiberwise Kähler metrics are ALE. Instead of applying Corollary 5.5, we then apply standard results about ALE–metrics to find v_i in (5.15); see [24, Section 8.5] and also [13, Theorem 2.1]. In particular, when $\beta > 2\mu$, it is in this step that β is replaced by $\alpha = 2\mu$. We then proceed as in Case 1 to extend v_i to a function on $\hat{X}$ and obtain u_i. □

This leads to the main result of this section.

Theorem 5.7 *Let g_0 be a Kähler QAC–metric on $\hat{X}_{\mathrm{QAC}}$ asymptotic to the Ricci-flat Kähler cone metric g_C with rate $\epsilon > 0$. Let ω_0 be its Kähler form. Given $f \in x_{\max}^\beta \mathcal{C}^\infty_{\mathrm{Qb},r}(\hat{X})$ with $\beta \geq 4$ such that $\beta \neq 2\mu$, there exists a unique $u \in x_{\max}^{\alpha-2} \mathcal{C}^\infty_{\mathrm{Qb},r}(\hat{X})$*

with $\alpha = \min\{2\mu, \beta\}$ *solving the complex Monge–Ampère equation*

$$\log\left(\frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial}u)^{n+1}}{\omega_0^{n+1}}\right) = f.$$

Proof The existence is obtained by combining Proposition 5.6 and Corollary 5.5. Uniqueness can be seen by proceeding as in [4, Proposition 7.13], but using the isomorphism (5.7) instead of the maximum principle. □

Taking f to be the Ricci potential of g_0 gives the following result.

Corollary 5.8 *Let* g_0 *with Kähler form* ω_0 *be a QAC–metric on* $\widehat{X}_{\mathrm{QAC}}$ *asymptotic to the Ricci-flat Kähler cone metric* g_C *with rate* $\epsilon \geq 4$ *such that* $\epsilon \neq 2\mu$. *Then there exists a unique* $u \in x_{\max}^{\alpha-2}\mathcal{C}^{\infty}_{\mathrm{Qb},r}(\widehat{X})$ *with* $\alpha = \min\{2\mu, \epsilon\}$ *solving the complex Monge–Ampère equation*

$$\log\left(\frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial}u)^{n+1}}{\omega_0^{n+1}}\right) = -r_0,$$

where r_0 *is the Ricci potential defined in* (5.3). *In particular,* $\omega_0 + \sqrt{-1}\partial\overline{\partial}u$ *is the Kähler form of a Ricci-flat Kähler QAC–metric.*

Remark 5.9 If we can take $p = 1$ in the definition of X and $\widehat{X}$ near (2.7), then $\Omega^1_{\widehat{X}}$ is a nowhere vanishing parallel holomorphic volume form on $\widehat{X}$ and the Ricci-flat Kähler QAC–metric of Corollary 5.8 is in fact Calabi–Yau.

Appendix More examples of Kähler–Einstein orbifolds admitting a crepant resolution

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It is possible to slightly widen the situations where we can apply Theorem 4.8, yielding in turn more examples to which Theorem 5.7 can be applied. In this appendix, we will be interested in the case where (D, g_D) is a Kähler–Einstein Fano orbifold with nonisolated singularities of depth 1 having a nontrivial normal bundle, a situation not covered by Corollary 4.10. In fact, we will be very specific and only consider orbifold singularities that are locally modeled on

$$\mathbb{C}^{n-m_i} \times (\mathbb{C}^{m_i}/\mathbb{Z}_{m_i}), \tag{A.1}$$

with the generator $e^{2\pi\sqrt{-1}/m_i}$ of $\mathbb{Z}_{m_i}$ acting on $\mathbb{C}^{m_i}$ via complex multiplication. For singularities of this kind, it is well known that $\pi_i\colon K_{\mathbb{CP}^{m_i-1}} \to \mathbb{C}^{m_i}/\mathbb{Z}_{m_i}$ is a crepant resolution.

Theorem A.1 *Let (D, g_D) be a compact Kähler–Einstein Fano orbifold with at most depth-one singularities. Assume that the isolated singularities can be resolved by a Kähler crepant resolution and that the nonisolated singularities are locally of the form (A.1). Then D admits a Kähler crepant resolution $\widehat{D}$ and the QAC–compactification $\widehat{X}_{\mathrm{QAC}}$ of $K_{\widehat{D}}$ admits a Calabi–Yau QAC–metric asymptotic to g_C with rate 2μ, where μ is the complex codimension of the singular set of D.*

Proof The idea is to first construct a Kähler QAC–metric on $\widehat{X}_{\mathrm{QAC}}$ by applying Theorem 4.8. By Proposition 4.2, K_D admits a smooth Kähler metric g which is equal to g_C outside a compact set, so we have a corresponding metric $g_\varepsilon = \iota_\varepsilon^* \beta^* \operatorname{pr}_1^* g$ on $\mathcal{X}$ to which we can hope to apply Theorem 4.8. In order to do that, we need to also provide a smooth closed $(1,1)$–form $\omega_{\widehat{\varphi}_i}$ on $\widehat{\mathcal{H}}_i$ as in the statement of Theorem 4.8. For a boundary hypersurface $\widehat{\mathcal{H}}_i$ associated to an isolated singularity of D, we can proceed exactly as in the proof of Corollary 4.9 to construct the form $\omega_{\widehat{\varphi}_i}$. For a singular stratum of D locally modeled on $\mathbb{C}^{n-m_i} \times (\mathbb{C}^{m_i}/\mathbb{Z}_{m_i})$ as in (A.1), the corresponding crepant resolution is locally of the form $\mathbb{C}^{n-m_i} \times K_{\mathbb{CP}^{m_i-1}}$, so that the corresponding fiber bundle $\widehat{\varphi}_i\colon \widehat{\mathcal{H}}_i \to \widetilde{\Sigma}_i$ is a $\overline{K}_{\mathbb{CP}^{m_i-1}}$–bundle, where $\overline{K}_{\mathbb{CP}^{m_i-1}}$ is the radial compactification (or equivalently in this case the QAC–compactification) of $K_{\mathbb{CP}^{m_i-1}}$ with respect to the corresponding Calabi–Yau cone metric. In fact, $\widetilde{\Sigma}_i$ itself is a fiber bundle over the corresponding singular stratum of D. For convenience, we will denote this singular stratum by σ_i and the corresponding fiber bundle by

$$\nu_i\colon \widetilde{\Sigma}_i \to \sigma_i, \quad \text{where } \dim_{\mathbb{C}} \sigma_i = n - m_i.$$

Now, what is important for us is that $\widehat{\varphi}_i\colon \widehat{\mathcal{H}}_i \to \widetilde{\Sigma}_i$ is in fact the pullback of a $\overline{K}_{\mathbb{CP}^{m_i-1}}$–bundle over σ_i,

$$\overline{h}_i \to \sigma_i,$$

for $h_i \to \sigma_i$ a $K_{\mathbb{CP}^{m_i-1}}$–bundle with $h_i = \overline{h}_i \setminus \partial \overline{h}_i$.

Alternatively, we can regard h_i as a complex line bundle L_i over a $\mathbb{CP}^{m_i-1}$–bundle $\mathcal{P}_i \to \sigma_i$. By the proof of Proposition 4.2, we can therefore find a smooth closed $(1,1)$–form ω_i on the total space of h_i which has compact support and is positive definite in a neighborhood of $\mathcal{P}_i$ in h_i. Pulling back this form to $\widehat{\mathcal{H}}_i$, one can then apply Lemma 4.1 fiberwise to $\widehat{\varphi}_i\colon \widehat{\mathcal{H}}_i \to \widetilde{\Sigma}_i$ to obtain the closed $(1,1)$–form required in Theorem 4.8. This theorem thus yields a Kähler QAC–metric on $\widehat{X}_{\mathrm{QAC}}$ asymptotic to g_C with rate α for any $\alpha > 0$. Restricting this metric to $\widehat{D} \subset K_{\widehat{D}}$ shows in particular that $\widehat{D}$ is a Kähler crepant resolution of D. Finally, applying Theorem 5.7, we obtain the desired Calabi–Yau QAC–metric. □

Example A.2 In the previous theorem, one can take $D = \mathbb{CP}^{2m-1}/\mathbb{Z}_m$, with the generator $e^{2\pi\sqrt{-1}/m}$ of $\mathbb{Z}_m$ acting on $\mathbb{CP}^{2m-1}$ by

$$e^{2\pi\sqrt{-1}/m}\cdot[z_0 : \cdots : z_{2m-1}] = [e^{2\pi\sqrt{-1}/m} z_0 : \cdots : e^{2\pi\sqrt{-1}/m} z_{m-1} : z_m : \cdots : z_{2m-1}].$$

The fixed points of this action are given by the disjoint union of

$$\sigma_1 := \{[z_0 : \cdots : z_{2m-1}] \in \mathbb{CP}^{2m-1} \mid z_0 = \cdots = z_{m-1} = 0\}$$

and

$$\sigma_2 := \{[z_0 : \cdots : z_{2m-1}] \in \mathbb{CP}^{2m-1} \mid z_m = \cdots = z_{2m-1} = 0\},$$

so that D has two disjoint singular strata σ_1 and σ_2 which are each locally modeled on $\mathbb{C}^{m-1} \times (\mathbb{C}^m/\mathbb{Z}_m)$. Since $\mathbb{Z}_m$ acts isometrically on $\mathbb{CP}^{2m-1}$ with respect to the Fubini–Study metric, the orbifold D is naturally a Kähler–Einstein Fano orbifold.

More generally, proceeding as in Corollary 4.10, we can extend Theorem A.1 to apply to a Kähler–Einstein Fano orbifold (D, g_D) of the form

$$(D, g_D) = (D_1 \times \cdots \times D_q, g_1 \times \cdots g_q),$$

where each (D_i, g_i) a Kähler–Einstein Fano orbifold as in the statement of Theorem A.1.

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