Subflexible symplectic manifolds

EMMY MURPHY

KYLER SIEGEL

We introduce a class of Weinstein domains which are sublevel sets of flexible Weinstein manifolds but are not themselves flexible. These manifolds exhibit rather subtle behavior with respect to both holomorphic curve invariants and symplectic flexibility. We construct a large class of examples and prove that every flexible Weinstein manifold can be Weinstein homotoped to have a nonflexible sublevel set.

1 Introduction

It has been known since the work of Gromov [11] that subcritical isotropic submanifolds of symplectic and contact manifolds satisfy an h-principle, meaning they belong to the realm of algebraic topology. In the intervening time, a rich theory of holomorphic curve invariants has shown that Lagrangians and Legendrians are generally quite rigid geometric objects. However, the last few years have seen significant progress in the flexible side of symplectic topology. Murphy's discovery of *loose Legendrians* [18] shows that, at least for isotropics in high-dimensional contact manifolds, symplectic flexibility extends well into the critical dimension. Cieliebak and Eliashberg subsequently used loose Legendrians to define *flexible Weinstein manifolds*, extending this flexibility to the theory of symplectic handlebodies. In a slightly different direction, it was shown

in Eliashberg and Murphy [9] that Lagrangian embeddings with a loose Legendrian negative end also satisfy an h-principle. Applications of this have included Lagrangian immersions with surprisingly few self-intersection points (see Ekholm, Eliashberg, Murphy and Smith [8]), an h-principle for symplectic embeddings of flexible Weinstein manifolds (see [9] and also Theorem 3.7 below), and a complete classification of the smooth topology of polynomially and rationally convex domains in high-dimensional affine space (see Cieliebak and Eliashberg [7]).

On the other hand, certain questions about the precise nature of these flexible objects have been thus far unclear. One question raised by Cieliebak and Eliashberg (see Remark 11.30(3) in [6]) is whether the notion of flexibility for Weinstein manifolds is invariant under Weinstein homotopies, or if it somehow depends on how we chop up the manifold into elementary pieces. A closely related question is whether a subdomain of a flexible Weinstein manifold is necessarily flexible. More specifically, is every sublevel set of a flexible Weinstein structure on $\mathbb{C}^n$ necessarily flexible? By the work of Cieliebak and Eliashberg (see the proof of Theorem 1.5 below), this is equivalent to asking whether every polynomially convex domain in $\mathbb{C}^n$ is necessarily flexible, and the affirmative answer was conjectured in [7].

In this paper, our main result is the following:

Theorem 1.1 *Every flexible Weinstein manifold has, after a Weinstein homotopy, a nonflexible sublevel set.*

Such sublevel sets, which we call *subflexible*, lie close to the interface between flexibility and rigidity. We construct a large class of examples, including many which are sublevel sets of the standard Weinstein structure on $\mathbb{C}^n$ up to Weinstein homotopy. In particular this gives a negative answer to all of the above questions and disproves the conjecture of Cieliebak and Eliashberg.

Our starting observation is that the exotic 6–dimensional Weinstein manifold X first defined by Maydanskiy in [14] becomes flexible after attaching an additional critical Weinstein handle (see Section 4.1). This gives another viewpoint on Maydanskiy's result that X has trivial wrapped Fukaya category (see Section 3 below, along with [2, Proposition 2.3] for the connection with wrapping). On the other hand, Harris [12] observed that X contains a Lagrangian S^3 after arbitrarily small nonexact deformations of the symplectic form. Applying Eliashberg and Murphy's h-principle for embeddings of flexible Weinstein manifolds, we can summarize the discussion so far as:

Theorem 1.2 *In the standard symplectic* $\mathbb{C}^3$, *there is a Liouville subdomain* X *and a* C^∞–*small closed form* $\eta \in \Omega^2 X$ *such that the symplectic manifold* $(X, \omega_{\mathrm{std}}|_X + \eta)$ *contains a Lagrangian* S^3.

In fact, Harris' observation suggests that X might be nonflexible, although deducing this directly from the above theorem appears tricky. The reason is that after deformation X becomes nonexact at infinity, and presently there is not a robust theory of holomorphic curve invariants for such manifolds.

Inspired by the above example and the "homologous recombination" construction of Abouzaid and Seidel [2], we introduce a general "subflexibilization" construction for Weinstein manifolds. Subflexibilization inputs a Weinstein manifold W and outputs a Weinstein manifold $\mathrm{SF}(W)$, and applying the construction to T^*S^3 gives Maydan-skiy's manifold. Topologically this procedure has the effect of adding some subcritical Weinstein handles. Symplectically it renders W subflexible, in particular killing its symplectic cohomology, and yet the symplectic geometry of W is not completely forgotten as we will explain. In fact, Harris' observation can be understood as a manifestation of Seidel's result [22, Proposition 2.6] that squares of two-dimensional Dehn twists are *fragile*, ie are not symplectically isotopic to the identity, but become so after small nonexact deformations of the symplectic form. Correspondingly, for 6–dimensional W we identify the symplectomorphism type of $\mathrm{SF}(W)$ after a small nonexact deformation as the boundary connect sum of W with some simple standard (nonexact) symplectic manifold (see Theorem 4.7).

Detecting nonflexibility among subflexible manifolds is a rather subtle problem. In general the basic tool for detecting nonflexibility of a Weinstein manifold is symplectic cohomology. Indeed, the work of Bourgeois, Ekholm and Eliashberg [3] implies that the symplectic cohomology of a flexible Weinstein manifold must vanish. We give an alternative proof of this in Section 3 which may be of independent interest. However, a standard argument involving the Viterbo transfer map implies that subflexible Weinstein manifolds must have vanishing symplectic cohomology as well, so some other invariant is needed. For this we turn to a twisted variation of symplectic cohomology. Twisted symplectic cohomology also vanishes for *flexible* Weinstein manifolds, but, perhaps surprisingly at first glance, *subflexible* manifolds can have nontrivial twisted symplectic cohomology. In fact, the results of Siegel [25] imply that, for 6–dimensional W, twisted symplectic cohomology of $\mathrm{SF}(W)$ coincides with standard symplectic cohomology of W. Appealing to this computation, we deduce that $\mathrm{SF}(W)$ *is subflexible, yet nonflexible whenever* W *has nonvanishing symplectic cohomology*.

Remark 1.3   In fact, [25] also computes *bulked deformed symplectic cohomology* for an analogous class of examples, and this can be used to distinguish exotic examples for which ordinary or twisted symplectic cohomology cannot. However, for simplicity we focus on twisted symplectic cohomology in this paper, especially since this suffices to prove Theorem 1.1.

Applying the construction with a bit more care and incorporating other recent constructions of exotic Weinstein manifolds, we prove the following. Recall that an *almost symplectomorphism* is a diffeomorphism which furthermore preserves the homotopy class of the symplectic form as a nondegenerate two-form.

Theorem 1.4   *Let X be any Weinstein domain with $\dim X \geq 6$ and $c_1(X) = 0$. Then there is a Weinstein domain X' such that*

- *X' is almost symplectomorphic to the boundary connect sum of X with some standard subcritical Weinstein domain,*

- *X' is a sublevel set of a flexible Weinstein domain almost symplectomorphic to X,*

- *X' has nonvanishing twisted symplectic cohomology.*

In particular, X' is subflexible but not flexible.

Taking X to be the standard Weinstein ball, we get a nonflexible sublevel set X' of a flexible Weinstein domain which is almost symplectomorphic to X, and hence Weinstein deformation equivalent to X by the h-principle for flexible Weinstein domains (see Section 2.2). Since every Weinstein manifold can be homotoped so that it contains the ball as a small sublevel set, this establishes Theorem 1.1.

Incorporating the techniques of Cieliebak and Eliashberg [7], we also prove:

Theorem 1.5   *If X admits a smooth codimension-zero embedding into $\mathbb{C}^n$, and moreover $H_n(X; \mathbb{Z}) = 0$ and $H_{n-1}(X; \mathbb{Z})$ is torsion-free, then X' is a sublevel set of $\mathbb{C}^n$, equipped with the standard Weinstein structure up to deformation. In particular, X' is Weinstein deformation equivalent to a polynomially convex domain in $\mathbb{C}^n$.*

This paper is organized as follows. In Section 2 we review the relevant background on Weinstein and Lefschetz structures and especially the interplay between the two. In Section 3 we discuss symplectic cohomology and its twisted cousin and prove that

these invariants vanish for flexible Weinstein structures. In Section 4 we introduce the subflexibilization construction and establish its main properties, using the groundwork laid in Section 2. Finally, in Section 5 we combine all of our results to produce some exotic subflexible manifolds as in Theorem 1.4.

2 Background

2.1 The geometry of Weinstein structures

We begin with a quick review of open symplectic structures, referring the reader to [6] for more details. Recall that a *Liouville domain* is a pair (W^{2n+2}, λ), where

- W is a compact $(2n+2)$–dimensional manifold with boundary,

- λ is a 1–form on W such that $\omega := d\lambda$ is symplectic,

- the Liouville vector field Z_λ, defined by $Z_\lambda \lrcorner\, d\lambda = \lambda$, is outwardly transverse to ∂W.

There is also a weaker notion of a *compact symplectic manifold with convex boundary* (W, ω), in which ω has a primitive λ defined only near ∂W such that Z_λ is outwardly transverse to ∂W.

A *Weinstein domain* is a triple (W, λ, φ), where

- (W, λ) is a Liouville domain,

- $\phi \colon W \to \mathbb{R}$ is a Morse function with maximal level set ∂W,

- Z_λ is gradient-like for ϕ.

A Weinstein domain has a *completion* $\hat{W} = W \cup ([0, \infty) \times \partial W)$. Let r denote the unique flow coordinate for Z_λ on $\mathcal{O}p(\partial W)$ satisfying $\mathcal{L}_{Z_\lambda}(r) \equiv 1$ and $r|_{\partial W} \equiv 0$. Assuming that $\varphi \equiv r$ on $\mathcal{O}p(\partial W)$, we extend λ to $[0, \infty) \times \partial W$ as $\hat{\lambda} = e^r \lambda|_{\partial W}$ and we extend φ as $\hat{\varphi}(r, x) = r$. Here $\hat{W}$ is an open manifold without boundary, $\hat{\varphi} \colon \hat{W} \to [0, \infty)$ is a proper Morse function, and the flow of the vector field $Z_{\hat{\lambda}}$ is complete. The completion $(\hat{W}, \hat{\lambda}, \hat{\varphi})$ is called a *Weinstein manifold*. In this paper we require Weinstein manifolds to be of *finite type*, meaning the proper Morse function has only finitely many critical points. Any finite-type Weinstein manifold is the completion of a Weinstein domain.

For two Weinstein domains $(W, \lambda_1, \phi_1), (W, \lambda_2, \phi_2)$ with the same underlying smooth manifold, the natural equivalence relation is *Weinstein homotopy*, ie a 1–parameter family of Weinstein structures (W, λ_t, ϕ_t) connecting them, $t \in [1, 2]$, where ϕ_t is additionally allowed to have standard birth–death singularities at finitely many times. For Weinstein manifolds, the definition is similar, but with the added stipulation that the union of the critical points of ϕ_t for all t be contained in some compact subset (this is to prevent critical points from disappearing off to infinity). Two Weinstein structures[1] on a priori different smooth manifolds are *deformation equivalent* if, after pulling back one structure by a diffeomorphism, the two resulting structures on the same smooth manifold are Weinstein homotopic. For Weinstein manifolds, Weinstein homotopy (and hence deformation equivalence) implies exact symplectomorphism. Note however that two homotopic Weinstein *domains* need not be symplectomorphic, since for example their volumes or symplectic capacities could be different. For this reason, regarding questions of symplectomorphisms it is more natural to work with completions.

The definition of a Weinstein domain implies that λ is a contact form when restricted to $Y^c := \varphi^{-1}(c)$ for any regular value c. Furthermore, the descending manifold D_p^k of any critical point $p \in W$ satisfies $\lambda|_{D_p^k} = 0$. Therefore D_p^k is isotropic in the symplectic sense for $d\lambda$, and $\Lambda_p^c := D_p^k \cap Y^c$ is isotropic in the contact sense for $\ker \lambda|_{Y^c}$. In particular $k := \mathrm{ind}(p) \leq n + 1$.

If $c \in \mathbb{R}$ is a critical value of φ with a unique critical point p, then the Weinstein homotopy type of $W^{c+\varepsilon} = \{\varphi < c + \varepsilon\}$ (for $\varepsilon > 0$ sufficiently small) is determined by $W^{c-\varepsilon}$, together with the isotopy type[2] of $\Lambda_p^{c-\varepsilon} \subseteq Y^{c-\varepsilon}$ and a framing of the symplectic

---

[1] Here *structure* means either domain or manifold.

[2] $\Lambda_p^{c-\varepsilon}$ is canonically parametrized as ∂D_p^{k+1}, and its isotopy type through *parametrized* Legendrians affects the symplectomorphism type of $W^{c+\varepsilon}$. We will often suppress this parametrization from the notation.

normal bundle of $\Lambda_p^{c-\varepsilon}$ (which is necessarily trivial). This is also constructive: given a Weinstein domain W and a (parametrized) isotropic sphere $\Lambda^k \subseteq \partial W$, together with a framing of the symplectic normal bundle of Λ (assumed to be trivial), we can construct a new Weinstein domain with one additional critical point, of index $k+1$, whose descending manifold intersects ∂W along precisely Λ. This procedure, called *attaching a Weinstein $(k+1)$–handle along* Λ, depends only on the framed, parametrized isotopy class (through isotropics) of Λ, up to Weinstein homotopy (see [28; 6] for more details).

In particular, for a Legendrian sphere $\Lambda^n \subset \partial W$, we can attach an $(n+1)$–handle $H \cong \mathbb{D}^{n+1} \times \mathbb{D}^{n+1}$ along Λ and extend the Weinstein structure over H. The resulting Morse function agrees with φ on W and has one additional critical point p_H of index $n+1$ in H. The *core* (resp. *cocore*) of H is made up of the Lagrangian disks consisting of all points in H which limit to p_H under the positive (resp. negative) Liouville flow. The boundary of the core (resp. cocore) of H is a Legendrian sphere in ∂W (resp. $\partial(W \cup H)$), which we refer to as the *attaching sphere* (resp. *belt sphere*) of H. We denote the attaching sphere and belt sphere of H by AS(H) and BS(H) respectively.

In general, if a critical point p of $\phi: W^{2n+2} \to \mathbb{R}$ has $\mathrm{ind}(p) = n+1$, we say that p is *critical*; otherwise it is *subcritical*. A Weinstein manifold is called *explicitly subcritical* if all of its critical points are subcritical. Note that explicit subcriticality is not invariant under Weinstein homotopy, since one can easily perform a Weinstein homotopy which creates two canceling critical points, one of index n and one of index $n+1$; see [6, Section 12.6]. We will call a Weinstein manifold *subcritical* if it can be made explicitly subcritical by a Weinstein homotopy.

2.2 Looseness and flexibility

Loose Legendrians were defined in [18], where it was shown that they satisfy an h-principle: two loose Legendrians are Legendrian isotopic if and only if they are *formally isotopic*. Intuitively, two Legendrians are formally isotopic if and only if they are smoothly isotopic in such a way that their normal bundle framings are canonically homotopic. A connected Legendrian of dimension $n > 1$ is loose if it admits a *loose chart*, defined as follows. Let γ_a denote the Legendrian arc in $(B^3_{\mathrm{std}}, \partial B^3_{\mathrm{std}})$, defined up to Legendrian isotopy, with the properties depicted in the left side of Figure 1. Namely, it has a single Reeb chord, of action a, and its front projection has a single transverse
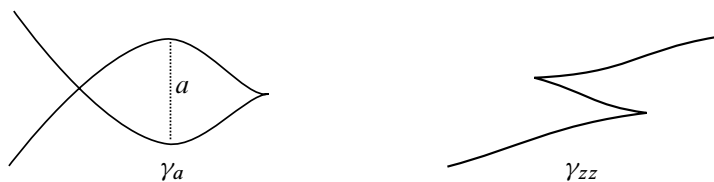
Figure 1: Left: the front picture of the Legendrian arc γ_a with a chord of action a. Right: the front picture of a once-stabilized Legendrian arc.

self-intersection and a single cusp. Set

$$V_\rho^{2n-2} := \{(q, p) \in T^*\mathbb{R}^{n-1} : |q| \leq \rho, \ |p| \leq \rho\},$$
$$Z_\rho^{n-1} := \{(q, p) \in V_\rho^{2n-2} : p = 0\}.$$

Here $T^*\mathbb{R}^{n-1}$ is equipped with the Liouville form $-p_1\, dq_1 - \cdots - p_{n-1}\, dq_{n-1}$, and we equip the product $B_{\text{std}}^3 \times V_\rho$ with the contact form $z - y\, dx - p_1\, dq_1 - \cdots - p_{n-1}\, dq_{n-1}$. A loose chart for a Legendrian $\Lambda \subset N$ is a contact embedding of pairs

$$(B_{\text{std}}^3 \times V_\rho, \gamma_a \times Z_\rho) \hookrightarrow (N, \Lambda)$$

such that $a/\rho^2 < 2$. More generally, we say a Legendrian link is loose if each component admits a loose chart in the complement of the other components.

Since loose Legendrians satisfy an h-principle, one might also expect an h-principle for Weinstein manifolds built by iterative Weinstein handle attachments along loose Legendrian spheres. This indeed turns out to be the case and is explored in depth in [6].

Definition 2.1 For $n > 1$, a Weinstein structure $(W^{2n+2}, \lambda, \phi)$ is *explicitly flexible* if there exist regular values $c_0, c_1, \ldots, c_N$ of ϕ with $c_0 < \min(\phi) < c_1 < \cdots < c_N$ such that:

- All critical points of ϕ are contained in $\{\phi < c_N\}$.

- No gradient trajectories of Z_λ join two critical points in $\{c_i < \phi < c_{i+1}\}$ for $i = 0, \ldots, N-1$, ie $\{c_i \leq \phi \leq c_{i+1}\}$ is an *elementary cobordism* in the language of [6].

- The attaching spheres of all index $n+1$ critical points in $\{c_i < \phi < c_{i+1}\}$ form a loose Legendrian link in $\phi^{-1}(c_i)$ for $i = 1, \ldots, N-1$.

A Weinstein structure is called *flexible* if it is Weinstein homotopic to an explicitly flexible structure.

Remark 2.2 (1) Any (explicitly) subcritical Weinstein domain is (explicitly) flexible.

(2) In a slight change of terminology, our definition of *explicitly flexible* coincides with the definition of *flexible* given in [6].

(3) As explained in [6, Remark 11.30(3)], if $c_0 < c_1 < \cdots < c_N$ is a partition as in Definition 2.1, so is any finer partition of (W, λ, ϕ) into elementary cobordisms. In particular, if the critical points of ϕ have pairwise distinct critical values then (W, λ, ϕ) is explicitly flexible if and only if the attaching Legendrian of each index $n + 1$ critical point p is loose in $\phi^{-1}(\phi(p) - \varepsilon)$ for $\varepsilon > 0$ sufficiently small. If some of the critical points of ϕ share the same critical value q, a similar criterion holds if we consider all the corresponding attaching spheres as a Legendrian link in $\phi^{-1}(q - \varepsilon)$.

To justify the name, Cieliebak and Eliashberg prove the following flexibility results (see [6] for stronger and more precise statements).

Theorem 2.3 (1) *Given an explicitly flexible Weinstein structure $(W^{2n+2}, \lambda, \phi)$ and any Morse function $\widetilde{\phi} \colon W \to \mathbb{R}$ without critical points of index greater than $n + 1$, there is a Weinstein homotopy starting at (W, λ, ϕ) and ending at a Weinstein structure whose Morse function is $\widetilde{\phi}$.*

(2) *Two explicitly flexible Weinstein structures are Weinstein homotopic if and only if their symplectic forms are homotopic as nondegenerate two-forms.*

(3) *Any Weinstein structure (W, λ, ϕ) is **almost symplectomorphic** to a flexible Weinstein structure, ie there is a diffeomorphism respecting the homotopy classes of the symplectic forms as nondegenerate two-forms. We denote this flexible version by $\mathrm{Flex}(W, \lambda, \phi)$. Note that it is well-defined up to Weinstein deformation equivalence.*

One of the main goals of this paper is to show that explicit flexibility is *not* preserved under Weinstein homotopies. To see that this is conceivably possible, imagine a Weinstein domain (W, λ, ϕ) with precisely two critical points of critical index, say p_1 and p_2, such that $\phi(p_1) = 1$ and $\phi(p_2) = 2$. Assume there are no gradient trajectories between p_1 and p_2, and suppose that (W, λ, ϕ) is explicitly flexible, which means that $\mathrm{AS}(p_1) \subset \phi^{-1}\left(\frac{1}{2}\right)$ and $\mathrm{AS}(p_2) \subset \phi^{-1}\left(\frac{3}{2}\right)$ are loose Legendrians. Now suppose we were to homotope (W, λ, ϕ) by lowering the value of p_2 until p_1 and p_2 lie on the same level set of ϕ. The result would be explicitly flexible if

and only if $\mathrm{AS}(p_1) \cup \mathrm{AS}(p_2)$ is a loose link in $\phi^{-1}\left(\frac{1}{2}\right)$. Since we did not assume $\mathrm{AS}(p_2) \subset \phi^{-1}\left(\frac{3}{2}\right)$ is loose in the complement of $\mathrm{BS}(p_1)$, there is no obvious reason why $\mathrm{AS}(p_2) \subset \phi^{-1}\left(\frac{1}{2}\right)$ should be loose in the complement of $\mathrm{AS}(p_1)$.

Note, however, that Theorem 2.3 above is tautologically also true for the more general class of flexible Weinstein manifolds.

Definition 2.4 A Weinstein domain (W, λ, φ) is called *subflexible* if it is Weinstein deformation equivalent to a sublevel set of a flexible Weinstein manifold.

Remark 2.5 A sublevel set of a Weinstein domain is a special case of a *Liouville embedding*, ie a smooth codimension-zero embedding $i\colon (W_0, \lambda_0) \hookrightarrow (W, \lambda)$ of a Liouville domain into a Liouville manifold such that $i^*\lambda - e^\rho \lambda_0$ is exact for some $\rho \in \mathbb{R}$. For Liouville embeddings one can define the Viterbo transfer map on symplectic cohomology [27].

Flexible manifolds are also subflexible by definition, and at first glance it might appear that these are the only examples. For instance, any subflexible Weinstein domain has trivial symplectic cohomology (see Theorem 3.2). On the other hand, as we explain in Section 3, the proof of Theorem 3.2 fails for a twisted version of symplectic cohomology (even though a version of the transfer map still holds in this setting!). In fact, twisted symplectic cohomology is strong enough to detect nonflexibility of the examples we construct in Section 4.

2.3 Lefschetz structures

Roughly speaking, a smooth map $\pi\colon E^{2n} \to B^2$ is a *smooth Lefschetz fibration* if B^2 is a compact surface with boundary, E^{2n} is a compact manifold with corners,[3] and π is a submersion except at finitely many singular points, near which it is modeled on the holomorphic map

$$\mathbb{C}^n \to \mathbb{C}, \quad (z_1, \ldots, z_n) \mapsto z_1^2 + \cdots + z_n^2.$$

If E has additional structure, we would like π to be compatible with this structure in some sense. Roughly speaking, a *symplectic Lefschetz fibration* is a Lefschetz fibration such that the total space is equipped with a symplectic form which restricts to

---

[3]From now on we will not explicitly mention the corners and will assume the corners have been smoothed when needed.

a symplectic form on each fiber, with some additional technical conditions to ensure parallel transport is well-behaved. We now give an actual definition with the caveat that the precise details are not crucial for our purposes and there seems to be no universally agreed upon definition in the literature.

Definition 2.6 A *symplectic Lefschetz fibration* is a smooth map $\pi\colon W \to \mathbb{D}^2$, with (W, ω) a compact symplectic manifold, satisfying the following conditions:

- **Lefschetz-type singularities** π is a submersion whose fibers are smooth manifolds with boundary, except at finitely many critical points. The critical points have distinct critical values lying in $\operatorname{Int} \mathbb{D}^2$, and near each critical point π is modeled on $\mathbb{C}^n \to \mathbb{C}$, $(z_1, \ldots, z_n) \mapsto z_1^2 + \cdots + z_n^2$, with ω identified with the standard Kähler form on $\mathbb{C}^n$.

- **Compatibility with ω** For any regular value $p \in \mathbb{D}^2$, $(\pi^{-1}(p), \omega|_{\pi^{-1}(p)})$ is a compact symplectic manifold with convex boundary.

- **Triviality near vertical boundary** On a neighborhood of the vertical boundary $\partial_v W := \pi^{-1}(\partial \mathbb{D}^2)$, the map π is equivalent (for some $\varepsilon > 0$) to the map

$$\pi \times \operatorname{Id}\colon \partial_v W \times (1 - \varepsilon, 1] \to S^1 \times (1 - \varepsilon, 1] \subset \mathbb{D}^2,$$

  with ω identified with $\omega_v + \pi^* \omega_b$, where ω_v is the pullback of $\omega|_{\partial_v W}$ under the projection $\partial_v W \times (1 - \varepsilon, 1] \to \partial_v W$ and ω_b is some symplectic two-form on $\mathbb{D}^2$.

- **Triviality near horizontal boundary** On a neighborhood of the horizontal boundary $\partial_h W := \partial W \setminus \operatorname{Int} \partial_v W$, the map π is equivalent to the projection $\mathcal{O}p(\partial M) \times \mathbb{D}^2 \to \mathbb{D}^2$, with ω identified with a split symplectic form, where $M := \pi^{-1}(1)$ is the fiber.

The key feature of symplectic Lefschetz fibrations is that the symplectic orthogonals to the vertical tangent spaces define a symplectic connection, meaning any path $\gamma\colon [0, 1] \to \mathbb{D}^2$ which avoids the critical values of π induces a parallel transport symplectomorphism $\pi^{-1}(\gamma(0)) \cong \pi^{-1}(\gamma(1))$. The symplectic Picard–Lefschetz theorem identifies the holonomy around a critical value of π with a symplectic Dehn twist along the corresponding vanishing cycle (see below). The notion of a *Liouville Lefschetz fibration* is similar, with $\omega = d\lambda$ a Liouville structure on W and $(\pi^{-1}(p), \lambda|_{\pi^{-1}(p)})$ a Liouville domain for each regular value $p \in \mathbb{D}^2$. We refer the reader to [23] for a comprehensive treatment.

Let (W, ω) be a symplectic manifold equipped with a symplectic Lefschetz fibration $\pi\colon W \to \mathbb{D}^2$, and let $z_1, \ldots, z_k \in \mathbb{D}^2$ denote the critical values of π. A *basis of vanishing paths* is a collection of embedded paths $\gamma_1, \ldots, \gamma_k \colon [0,1] \hookrightarrow \mathbb{D}^2$ such that $\gamma_i(0) = 1 \in \partial \mathbb{D}^2$, $\gamma_i(1) = z_i$ and $\gamma_i|_{(0,1)}$ is an embedding in $\mathrm{Int}\, \mathbb{D}^2 \setminus \{z_1, \ldots, z_k\}$, for each $i = 1, \ldots, k$. We associate to a basis of vanishing paths the *vanishing cycles* $V_1, \ldots, V_k \subset \pi^{-1}(1)$, where V_i is defined as the set of all points in $M := \pi^{-1}(1)$ which are parallel transported along γ_i to the critical point in $\pi^{-1}(z_i)$. The vanishing cycles $V_1, \ldots, V_k$ are embedded parametrized[4] Lagrangian spheres in $(M, \omega|_M)$.

Now suppose that $(V_1, \ldots, V_k)$ are the vanishing cycles associated to a certain basis of vanishing paths, and $(V_1', \ldots, V_k')$ are those associated to some other basis of vanishing paths. It follows from the symplectic Picard–Lefschetz theorem that $(V_1, \ldots, V_k)$ and $(V_1', \ldots, V_k')$ differ by a sequence of *Hurwitz moves*. A *Hurwitz move* is one of the following two operations:

$$(V_1, \ldots, V_k) \rightsquigarrow (\tau_{V_1} V_2, V_1, V_3, \ldots, V_k),$$
$$(V_1, \ldots, V_k) \rightsquigarrow (V_2, \tau_{V_2}^{-1} V_1, V_3, \ldots, V_k).$$

(Here and throughout the paper, we denote by τ_V the symplectic Dehn twist around the Lagrangian sphere V.) Note that the two moves are inverses of each other, and as usual the vanishing cycles are only cyclically ordered, so the moves can be applied on any two consecutive vanishing cycles.

In general, we call $(M, \omega; V_1, \ldots, V_k)$ a *symplectic Lefschetz datum* if (M, ω) is a compact symplectic manifold with convex boundary and $V_1, \ldots, V_k \subset M$ is a cyclically ordered collection of embedded parametrized Lagrangian spheres. Given a symplectic Lefschetz datum $(M, \omega; V_1, \ldots, V_k)$ and a basis of vanishing paths, one can construct a compact symplectic manifold $\mathcal{S}(M, \omega; V_1, \ldots, V_k)$, equipped with a symplectic Lefschetz fibration

$$\pi_{(M, \omega; V_1, \ldots, V_k)}\colon \mathcal{S}(M, \omega; V_1, \ldots, V_k) \to \mathbb{D}^2.$$

Moreover, the fiber is symplectomorphic to M and the vanishing cycles are identified with $V_1, \ldots, V_k$. Similarly, a Liouville Lefschetz datum is $(M, \lambda; V_1, \ldots, V_k)$ with

---

[4]Just as in the case of Weinstein manifolds and Legendrian attaching maps, vanishing cycles have a canonical parametrization (modulo isotopies) since they are the boundaries of the *Lagrangian thimbles*. Similar to the Weinstein case, this data is necessary to recover the symplectic topology of the total space from the data of the fiber and the vanishing cycles.

(M, λ) a Liouville domain, and one can construct a Liouville Lefschetz fibration

$$\pi_{(M,\lambda;V_1,\dots,V_k)} \colon \mathcal{L}(M, \lambda; V_1, \dots, V_k) \to \mathbb{D}^2$$

which recovers the initial datum $(M, \lambda; V_1, \dots, V_k)$.

The construction of $\pi_{(M,\omega;V_1,\dots,V_k)} \colon \mathcal{S}(M, \omega; V_1, \dots, V_k) \to \mathbb{D}^2$ proceeds as follows. If $k = 0$ we can simply take $M \times \mathbb{D}^2$ with the split symplectic form $\omega + \omega_{\mathrm{std}}$. Now suppose $k = 1$. For any $s > 0$ there is a "model" symplectic Lefschetz fibration $\pi \colon E_s \to \mathbb{D}^2$ whose fiber is symplectomorphic to $D_s^* S^n$, the disk cotangent bundle of S^n of radius s with respect to some Riemannian metric (see [23, Example 15.4]). Near $\partial_h E_s$, the map π is equivalent to the projection $\mathcal{O}p(\partial D_s^* S^n) \times \mathbb{D}^2 \to \mathbb{D}^2$. Here $\mathcal{O}p(\partial D_s^* S^n) \times \mathbb{D}^2$ is equipped with the symplectic form $\omega_{\mathrm{std}} + \omega_b$, where ω_{std} is the canonical symplectic form on $D_s^* S^n$ and ω_b is some symplectic form on $\mathbb{D}^2$. Now consider $M \times \mathbb{D}^2$ with the split symplectic form $\omega + \omega_b$. By the Weinstein neighborhood theorem, a neighborhood U of $V_1 \times \mathbb{D}^2$ is symplectomorphic to $(D_s^* S^n \times \mathbb{D}^2, \omega_{\mathrm{std}} + \omega_b)$ for some $s > 0$. We can therefore remove U from $M \times \mathbb{D}^2$ and symplectically glue in E_s, and this gives the desired symplectic Lefschetz fibration. More generally, if $k \geq 2$ we perform the above construction for each vanishing cycle individually and then glue together the resulting symplectic Lefschetz fibrations along a fiber, reidentifying the base with $\mathbb{D}^2$. The Liouville Lefschetz fibration $\pi_{(M,\lambda;V_1,\dots,V_k)} \colon \mathcal{L}(M, \lambda; V_1, \dots, V_k) \to \mathbb{D}^2$ can be constructed in a similar fashion (see [23, Section 16e] for more details).

There is also a notion of *completion* for a Lefschetz fibration over $\mathbb{D}^2$, resulting in a Lefschetz fibration over $\mathbb{C}$. If $\pi \colon W \to \mathbb{D}^2$ is a symplectic Lefschetz fibration, the completion $\hat{\pi} \colon \hat{W} \to \mathbb{C}$ is obtained as follows. First, by triviality near the horizontal boundary we can symplectically glue $(\hat{M} \setminus M) \times \mathbb{D}^2$ to W, where $\hat{M}$ denotes the completion of $M = \pi^{-1}(1)$ as a compact symplectic manifold with convex boundary. Denote the resulting symplectic manifold by (W_1, ω_1), and note that π has a natural extension $\pi_1 \colon W_1 \to \mathbb{D}^2$. Now by triviality near the vertical boundary we can identify the restriction of π_1 to $\mathcal{O}p(\pi_1^{-1}(\partial \mathbb{D}^2))$ with the map

$$\pi \times \mathrm{Id} \colon \partial_v W_1 \times (1 - \varepsilon, 1] \to \mathbb{D}^2.$$

Here the product $\partial_v W_1 \times (1 - \varepsilon, 1]$ is equipped with the symplectic form $\hat{\omega}_v + \pi^* \omega_b$, where $\hat{\omega}_v := \omega_1|_{\partial_v W_1}$. Let $(\mathbb{C}, \hat{\omega}_b)$ denote the completion of $(\mathbb{D}^2, \omega_b)$. Finally, we complete the horizontal boundary of W_1 by gluing in $\partial_v W_1 \times [1, \infty)$, equipped with the symplectic form $\hat{\omega}_v + \hat{\omega}_b$. One can similarly complete a Liouville Lefschetz fibration,

and in this case the completion as a Lefschetz fibration agrees with the completion as a Liouville domain, up to Liouville deformation equivalence. We denote the completions of $\mathcal{S}$ and $\mathcal{L}$ by $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{L}}$, respectively.

We compile a few basic facts that will be needed later:

- The symplectomorphism type of $\widehat{\mathcal{S}}(M, \omega; V_1, \ldots, V_k)$ is well-defined and invariant under symplectomorphisms of M and Hamiltonian isotopies and Hurwitz moves of the vanishing cycles.

- $\mathcal{L}(M, \lambda; V_1, \ldots, V_k)$ is well-defined up to Liouville deformation equivalence and invariant under Liouville deformation equivalences of (M, λ) and Hamiltonian isotopies and Hurwitz moves of the vanishing cycles.

- $\widehat{\mathcal{L}}(M, \lambda; V_1, \ldots, V_k)$ is symplectomorphic to $\widehat{\mathcal{S}}(M, d\lambda; V_1, \ldots, V_k)$.

- Let $(M, \omega; V_1, \ldots, V_k)$ be a symplectic Lefschetz datum and let Ω be a closed two-form on M with support disjoint from $V_1 \cup \cdots \cup V_k$. Then there is a closed two-form $\widetilde{\Omega}$ on $\widehat{\mathcal{S}}(M, \omega; V_1, \ldots, V_k)$ such that $\widehat{\mathcal{S}}(M, \omega + s\Omega; V_1, \ldots, V_k)$ is symplectomorphic to the result of adding $s\widetilde{\Omega}$ to the symplectic form of $\widehat{\mathcal{S}}(M, \omega; V_1, \ldots, V_k)$ (provided $(M, \omega + s\Omega)$ is symplectic and has convex boundary).

As explained in [3, Section 8.2], the Liouville domain $\mathcal{L}(M, \lambda; V_1, \ldots, V_k)$ can also be constructed (up to Liouville deformation equivalence) by attaching Weinstein handles to $M \times \mathbb{D}^2$ along Legendrian lifts of $V_1, \ldots, V_k$. If (M, λ, ϕ) is a Weinstein domain, this can be slightly refined to construct a Weinstein domain $\mathcal{W}(M, \lambda, \phi; V_1, \ldots, V_k)$, as we now explain. Let (M, λ, ϕ) be a Weinstein domain, and let $(\mathbb{D}^2, \lambda_{\mathrm{std}}, \phi_{\mathrm{std}})$ be the standard Weinstein structure on the unit disk with $\lambda_{\mathrm{std}} = \frac{1}{2}x \, dy - \frac{1}{2}y \, dx$ and $\phi_{\mathrm{std}}(x, y) = x^2 + y^2$. We would like to view the product $(M, \lambda, \phi) \times (\mathbb{D}^2, \lambda_{\mathrm{std}}, \phi_{\mathrm{std}})$ as a Weinstein domain in such a way that part of the boundary is identified with $M \times S^1$. To accomplish this, we Weinstein homotope the completion $(\widehat{M}, \widehat{\lambda}, \widehat{\phi})$ to $(\widehat{M}, \widehat{\lambda}, \widetilde{\phi})$, where $\widetilde{\phi} \colon \widehat{M} \to \mathbb{R}$ is C^∞–small on M and is of the form $h(s)$ on $[0, \infty) \times \partial M$, where s is the coordinate on $[0, \infty)$ and $h'(s) > 0$. Now consider the Weinstein manifold $(\widehat{M} \times \mathbb{C}, \widehat{\lambda} + \frac{1}{2}x \, dy - \frac{1}{2}y \, dx, \widetilde{\phi} + x^2 + y^2)$. Denote the Weinstein domain given by the sublevel set $\{\widetilde{\phi} + x^2 + y^2 \leq 1\}$ as $(M, \lambda, \phi) \boxtimes (\mathbb{D}^2, \lambda_{\mathrm{std}}, \phi_{\mathrm{std}})$ (or just $M \boxtimes \mathbb{D}^2$ when the rest of the data is implicit).

We can decompose the contact boundary $Y := \partial(M \boxtimes \mathbb{D}^2)$ into two parts:

$$Y_1 := Y \cap (M \times \mathbb{C}), \quad Y_2 := Y \setminus \mathrm{Int}\, Y_1.$$

Observe that Y_1 is diffeomorphic to $M \times S^1$, and the contact structure is given by

$$\lambda + \tfrac{1}{2}(1 - \widetilde{\phi})dt,$$

where t is the coordinate on $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. We would like to remove the $\widetilde{\phi}$ term. To this effect, using the smallness assumption on $\widetilde{\phi}|_M$ we can find a contact form on $\partial(M \boxtimes \mathbb{D}^2)$ which is given by $\lambda + \tfrac{1}{2}dt$ on Y_1 and agrees outside $\mathcal{O}p(Y_1)$ with $\left(\widehat{\lambda} + \tfrac{1}{2}x\,dy - \tfrac{1}{2}y\,dx\right)\big|_{\partial(M \boxtimes \mathbb{D}^2)}$. Gray's stability theorem gives a contact embedding

$$\left(M \times S^1, \lambda + \tfrac{1}{2}dt\right) \hookrightarrow \left(\mathcal{O}p(Y_1), \left(\widehat{\lambda} + \tfrac{1}{2}x\,dy - \tfrac{1}{2}y\,dx\right)\big|_{\mathcal{O}p(Y_1)}\right).$$

Using this embedding, we can assume without loss of generality that the contact form on Y_1 is given by $\lambda + \tfrac{1}{2}dt$.

Now suppose that $V_1, \ldots, V_k \subset (M, \lambda)$ are a collection of exact Lagrangians with $\lambda|_{V_i} = dF_i$ for $i = 1, \ldots, k$. If $\|F_i\| < \tfrac{\pi}{2k}$ for $i = 1, \ldots, k$, we can lift $V_1, \ldots, V_k$ to disjoint Legendrian spheres in Y_1. More precisely, we can find Legendrian spheres $\Lambda(V_1), \ldots, \Lambda(V_k) \subset Y_1$ such that, for $i = 1, \ldots, k$:

- $\Lambda(V_i)$ projects diffeomorphically onto V_i under the projection $Y_1 \to M$.
- The projection of $\Lambda(V_i)$ to the S^1 factor is contained in an arc $\Delta_i \subset S^1$ such that $\Delta_i \cap \Delta_j = \varnothing$ if $i \neq j$ and $\Delta_1, \ldots, \Delta_k$ are cyclically ordered in the counterclockwise S^1 direction.

Indeed, we take the Legendrian lift of V_i to be

$$\Lambda(V_i) := \left\{ (p, t) \in Y_1 : p \in V_i, \ t = -2F_i(p) + (i - 1)\frac{2\pi}{k} \right\}.$$

Finally, if the condition $\|F_i\| < \tfrac{\pi}{2k}$ is not satisfied, we can simply replace V_i by its image under the Liouville flow of W for large backwards time, which is Hamiltonian isotopic to V_i. We now set $\mathcal{W}(M, \lambda, \phi; V_1, \ldots, V_k)$ to be the Weinstein domain given by attaching Weinstein handles to $M \boxtimes \mathbb{D}^2$ along the Legendrian lifts $\Lambda(V_1), \ldots, \Lambda(V_k) \subset \partial(M \boxtimes \mathbb{D}^2)$.

Definition 2.7 A *Lefschetz presentation* for a Weinstein domain $(X^{2n+2}, \lambda, \phi)$ is a Weinstein deformation equivalence between it and $\mathcal{W}(M^{2n}, \theta, \psi; V_1, \ldots, V_k)$ for some *Weinstein Lefschetz datum* $(M^{2n}, \theta, \psi; V_1, \ldots, V_k)$.

Remark 2.8 The cyclic ordering of the vanishing cycles is an essential part of the Lefschetz data; changing the cyclic ordering often completely changes the symplectic topology (and even the smooth topology) of the total space.

Definition 2.9 Let $\mathcal{W}(M^{2n}, \theta, \psi; V_1, \ldots, V_k)$ be a Weinstein Lefschetz fibration, and let $D^n \subseteq M$ be a Lagrangian disk whose boundary is a Legendrian sphere in ∂M. Let H be a Weinstein handle attached to M along ∂D, and let $S \subseteq M \cup H$ be the Lagrangian sphere formed by the union of D^n and the core of H. The Weinstein Lefschetz fibration $\mathcal{W}(M \cup H, \theta, \psi; V_1, \ldots, V_k, S)$ is called the *stabilization* of $\mathcal{W}(M, \theta, \psi; V_1, \ldots, V_k)$ along D, which is called a *stabilizing disk* (see eg [13]).

Remark 2.10 As is well known, the Weinstein deformation type of the total space of a Lefschetz fibration does not change with stabilization. One can see this by viewing S and ∂D as the attaching maps of a canceling pair of Weinstein handles. Alternatively, it can be seen as boundary connect summing with $\mathbb{D}^{2n+2}_{\text{std}}$, which has the Lefschetz presentation $\mathcal{W}(T^*S^n, \lambda_{\text{std}}, \psi; Z)$, where Z is the zero section, and ψ is a Morse perturbation of the function $|p|^2$.

To apply subflexibilization to a Lefschetz fibration, we will need to make the following additional assumption on the vanishing cycles $V_1, \ldots, V_k$.

Assumption 2.11 *There are Lagrangian disks $T_1, \ldots, T_k \subset M$ with disjoint Legendrian boundaries in ∂M such that, for each $1 \le i \le k$, the disk T_i intersects V_i transversely in a single point (however T_i is allowed to intersect the other vanishing cycles arbitrarily).*

This is the same as Assumption 3.6 in [2]. In fact, in order to apply the results from [25], we will make a slightly stronger assumption as follows. Consider an embedded path μ in the base of a Lefschetz fibration, which intersects the critical values precisely at its endpoints. In the fiber above the midpoint of the path we get two vanishing cycles corresponding to the critical values at the two endpoints. If these two (parametrized) Lagrangian spheres are Hamiltonian isotopic, we call μ a *matching path*, and we can construct a corresponding *matching cycle*, a Lagrangian sphere in the total space which projects to μ (see [23, Section 16g] for details).

Definition 2.12 A Weinstein Lefschetz datum $(M, \theta, \psi; V_1, \ldots, V_k)$ is of *matching type* if (M, θ, ψ) itself admits Lefschetz presentation such that each V_i is a matching cycle with respect to this auxiliary Lefschetz fibration.

It is explained in Example 3.13 of [2] that every smooth complex affine variety (viewed as a Weinstein domain) admits a Lefschetz presentation of matching type, and any matching type Lefschetz fibration satisfies Assumption 2.11.

Remark 2.13 By recent work of Giroux and Pardon [10], every Weinstein domain admits a Lefschetz presentation.

3 Holomorphic curve invariants and (sub)flexibility

Our main goal in this section is to prove that (twisted) symplectic cohomology vanishes for any flexible Weinstein domain. To review symplectic cohomology and its twisted variant, we refer the reader to [25] and the references cited therein.

Let (W, λ) be a Liouville domain and let $(\widehat{W}, \widehat{\lambda})$ denote its completion. Working over any ground ring $\mathbb{K}$, we have $\mathrm{SH}(W, \lambda)$, the symplectic cohomology of (W, λ). Among other things, this is a unital $\mathbb{K}$–algebra which is invariant under symplectomorphisms (not necessarily exact) of $(\widehat{W}, \widehat{\lambda})$ (see the discussion in [2, Section 2c]). Informally, the symplectic cohomology of (W, λ) is defined as the Hamiltonian Floer cohomology of $(\widehat{W}, \widehat{\lambda})$ with respect to a Hamiltonian which grows sufficiently rapidly at infinity. In particular, the underlying chain complex is generated by 1–periodic Hamiltonian orbits in $\widehat{W}$, and the differential counts isolated solutions to Floer's equation. Symbolically, the differential applied to an orbit γ_+ is of the form

$$\delta(\gamma_+) := \sum_{u \in \mathcal{M}(\gamma_-, \gamma_+)^{\mathrm{o}}} s(u) \gamma_-,$$

where the sum is over all orbits γ_-. Here $\mathcal{M}(\gamma_-, \gamma_+)^{\mathrm{o}}$ denotes the moduli space of isolated (unparametrized) Floer trajectories asymptotic to γ_-, γ_+, and $s(u) \in \{\pm 1\}$ is a certain associated sign. Of course, one must work carefully to ensure that the Gromov's compactness theorem applies, and this necessitates picking Hamiltonians and almost complex structures of a special form at infinity.

From now on we assume for simplicity that $\mathbb{K}$ is a field. In order to apply the twisting construction, we further assume there is an injective group homomorphism $T \colon \mathbb{R} \to \mathbb{K}^*$ from the additive group $\mathbb{R}$ to the group of invertible elements in $\mathbb{K}$. We set $t := T(1)$ and more generally $t^r := T(r)$ for any $r \in \mathbb{R}$. For example, we could take $\mathbb{K}$ to be simply $\mathbb{R}$ with $T(r) := e^r$. Or, more formally, we could take $\mathbb{K}$ to be the field of rational functions in a formal variable t with real exponents and coefficients in say $\mathbb{Z}/2$. Note that in this paper there is no need to take a Novikov completion since we are working in an exact setting.

Remark 3.1 In the case $c_1(W, \omega) = 0$, we can trivialize the canonical bundle of (W, ω), in which case $\mathrm{SH}(W, \lambda)$ inherits a $\mathbb{Z}$–grading via the Conley–Zehnder index. This grading plays a role in Proposition 5.3 below.

Now suppose that Ω is a closed two-form on W. In this case we can define $\mathrm{SH}_\Omega(W, \lambda)$, the symplectic cohomology of (W, λ) twisted by Ω. This is again a unital $\mathbb{K}$–algebra, depending only on the cohomology class of Ω and invariant under symplectomorphisms of $(\hat{W}, \hat{\lambda})$ which respect $[\Omega] \in H^2(\hat{W}; \mathbb{R})$. The definition of $\mathrm{SH}_\Omega(W, \lambda)$ is almost identical to that of $\mathrm{SH}(W, \lambda)$, except that in defining the differential and other relevant structure maps we weight each curve $u \colon \mathbb{R} \times S^1 \to \hat{W}$ by the factor $t^{\int u^* \Omega}$. For example, the twisted differential is of the form

$$\delta_\Omega(\gamma_+) := \sum_{u \in \mathcal{M}(\gamma_-, \gamma_+)^\circ} t^{\int u^* \Omega} s(u) \gamma_-.$$

Our present goal is to prove the following:

Theorem 3.2 For (W, λ, ϕ) a flexible Weinstein domain, we have $\mathrm{SH}_\Omega(W, \lambda) = 0$ for any closed two-form Ω on W.

At least for untwisted symplectic cohomology, this result is already well known to experts. One argument uses the surgery results of [3], together with the triviality of Legendrian contact homology for loose Legendrians. It would be straightforward to extend the techniques of [3] to make this argument work in the twisted case as well. Instead, we give a proof of Theorem 3.2 based on an embedding h-principle for flexible Weinstein domains (see Theorem 3.7 below).

Firstly, we will need the Künneth theorem for symplectic cohomology, adapted to the twisted case. An inspection of Oancea's proof [19] in the untwisted setting shows that the twisting two-forms benignly come along for the ride, yielding the following.

Theorem 3.3 Let Ω_1 and Ω_2 be closed two-forms on Liouville manifolds (W_1, λ_1) and (W_2, λ_2) respectively. There is an isomorphism

$$\mathrm{SH}_{\Omega_1}(W_1, \lambda_1) \otimes \mathrm{SH}_{\Omega_2}(W_2, \lambda_2) \cong \mathrm{SH}_{\Omega_1 + \Omega_2}(W_1 \times W_2, \lambda_1 + \lambda_2).$$

Recall that Oancea's basic idea is to consider a split Hamiltonian and almost complex structure on $\hat{W}_1 \times \hat{W}_2$, for which the resulting Floer cohomology is as expected via the algebraic Künneth theorem. However, this does not compute symplectic cohomology

since this Floer data is not cylindrical at infinity with respect to the Liouville form on $\widehat{W}_1 \times \widehat{W}_2$. To rectify this, Oancea carefully modifies the split data to make it cylindrical at infinity and argues via action considerations that this procedure does not change the resulting homology. Alternatively, an unpublished argument due to Mark McLean starts with cylindrical Floer data and modifies this to the product data. In this case one can appeal to the more robust integrated maximum principle of Abouzaid to rule out undesired Floer trajectories.

Next, we need to understand how flexibility behaves under products.

Proposition 3.4 Let (W, λ, ϕ) be an explicitly flexible Weinstein domain and let (W', λ', ϕ') be any Weinstein domain. Then the product $(\widehat{W} \times \widehat{W}', \widehat{\lambda} + \widehat{\lambda}', \widehat{\phi} + \widehat{\phi}')$ is an explicitly flexible Weinstein manifold.

Proof The critical values of $\widehat{\phi} + \widehat{\phi}'$ are of the form $q + q'$ for q a critical value of ϕ and q' a critical value of ϕ'. Consider some pair q, q' of critical values of ϕ and ϕ' respectively. For sufficiently small $\varepsilon > 0$, set

- $N := (\widehat{\phi} + \widehat{\phi}')^{-1}(q + q' - \varepsilon)$, equipped with the contact form $\alpha_N := (\widehat{\lambda} + \widehat{\lambda}')|_N$,
- $Q := \phi^{-1}(q - \varepsilon)$, equipped with the contact form $\alpha_Q := \lambda|_Q$,
- $Q' := (\phi')^{-1}(q' - \varepsilon)$, equipped with the contact form $\alpha_{Q'} := \lambda'|_{Q'}$.

Let $D^{\mathrm{top}}(q) \subset W$ denote the union of the descending manifolds of all critical points of ϕ with critical value q and index equal to half the dimension of W. Define $D^{\mathrm{top}}(q') \subset W'$ and $D^{\mathrm{top}}(q + q') \subset \widehat{W} \times \widehat{W}'$ similarly. We have Legendrian links

- $\Lambda_N := D^{\mathrm{top}}(q + q') \cap N$ in $(N, \ker \alpha_N)$,
- $\Lambda_Q := D^{\mathrm{top}}(q) \cap Q$ in $(Q, \ker \alpha_Q)$,
- $\Lambda_{Q'} := D^{\mathrm{top}}(q') \cap Q'$ in $(Q', \ker \alpha_{Q'})$.

As explained in Remark 2.2(3), explicit flexibility of (W, λ, ϕ) means that Λ_Q is a loose Legendrian link for each critical value q of ϕ, and our goal is to prove that Λ_N is loose for each critical value $q + q'$ of $\widehat{\phi} + \widehat{\phi}'$.

Using the Liouville flow of $\widehat{\lambda}$, we get a smooth embedding $\Phi \colon \mathbb{R}_s \times Q \hookrightarrow \widehat{W}$ such that $\Phi^*(\widehat{\lambda}) = e^s \alpha_Q$. Here we are using the fact that the Liouville vector field $Z_{\widehat{\lambda}}$ on $\widehat{W}$ is complete and exponentially expands the Liouville one-form $\widehat{\lambda}$. Similarly, we get an embedding $\Phi' \colon \mathbb{R}_{s'} \times Q' \hookrightarrow \widehat{W}'$ such that $(\Phi')^*(\widehat{\lambda}') = e^{s'} \alpha_{Q'}$. Consider the hypersurface

$$H := \{s + s' = 0\} \subset \mathbb{R}_s \times Q \times \mathbb{R}_{s'} \times Q',$$

which comes with a codimension-one embedding

$$i := (\Phi \times \Phi')|_H \colon H \hookrightarrow \widehat{W} \times \widehat{W}'$$

transverse to the Liouville vector field $Z_{\widehat{\lambda} + \widehat{\lambda}'}$. Identifying H with $\mathbb{R}_s \times Q \times Q'$, where s' is determined by the relation $s' = -s$, the hypersurface H naturally inherits the contact one-form

$$\alpha_H := i^*(\widehat{\lambda} + \widehat{\lambda}') = e^s \alpha_Q + e^{-s} \alpha_{Q'},$$

along with the Legendrian submanifold

$$\Lambda_H := i^{-1}(D^{\text{top}}(q) \times D^{\text{top}}(q')) = \mathbb{R}_s \times \Lambda_Q \times \Lambda_{Q'}.$$

By [Lemma 3.5](#) below, Λ_H is loose. Indeed, the ambient contact structure on H is of the form $\ker(\alpha_Q + e^{-2s}\alpha_{Q'})$. In particular, the factor $\mathbb{R}_s \times Q'$ is equipped with the one-form $e^{-2s}\alpha_{Q'}$, for which the dual vector field $-\frac{1}{2}\partial_s$ is everywhere nonzero and tangent to the Lagrangian submanifold $\mathbb{R}_s \times \Lambda_{Q'}$. This means we can find loose charts for Λ_H in $(H, \ker \alpha_H)$, one for each component in the complement of the other components.

We claim that these loose charts for Λ_H can be pushed forward under the Liouville flow of $\widehat{\lambda} + \widehat{\lambda}'$ to loose charts for Λ_N in $(N, \ker \alpha_N)$. Indeed, let us denote by $F_t \colon \widehat{W} \times \widehat{W}' \to \widehat{W} \times \widehat{W}'$ the time-t Liouville flow for $t \in \mathbb{R}$. Since there are no critical values of ϕ in $[q - \varepsilon, q)$ or of ϕ' in $[q' - \varepsilon, q')$, for each $x \in H$ we have

$$\lim_{t \to -\infty} (\widehat{\phi} + \widehat{\phi}')(F_t(i(x))) < q + q' - \varepsilon,$$
$$\lim_{t \to +\infty} (\widehat{\phi} + \widehat{\phi}')(F_t(i(x))) \geq q + q'.$$

Since the function $\widehat{\phi} + \widehat{\phi}'$ is increasing under the Liouville flow, there is a uniquely determined "flow time" function $T \colon H \to \mathbb{R}$ such that the map sending $x \in H$ to $F_{T(x)}(i(x))$ defines a smooth embedding $F_T \colon H \hookrightarrow N$ with the property that $F_T^*(\alpha_N) = e^T \alpha_H$. In particular, F_T is a contact embedding and maps Λ_H to Λ_N since the Liouville vector field is tangent to $D^{\text{top}}(q + q')$. Therefore it sends the loose charts for Λ_H in $(H, \ker \alpha_H)$ to corresponding loose charts for Λ_N in $(N, \ker \alpha_N)$.

Strictly speaking, the above shows that $F_T(\Lambda_H)$ is a loose link in $(N, \ker \alpha_N)$, and in principle there could be duplicates in the collection $\{q + q'\}$ where q and q' vary over the critical values of ϕ and ϕ' respectively. Therefore we need to check that the loose charts we produce for Λ_N in $(N, \ker \alpha_N)$ for different pairs (q, q') are disjoint. To see this, suppose that r, r' is a distinct pair of critical values of ϕ, ϕ' respectively,

with $r + r' = q + q'$. We claim that $D^{\text{top}}(r) \times D^{\text{top}}(r')$ is disjoint from $i(H)$. Since $D^{\text{top}}(r) \times D^{\text{top}}(r')$ is Liouville flow invariant, this will imply that it is also disjoint from $F_T(H)$, and hence from the above loose charts for Λ_H in H corresponding to (q, q'). In fact, observe that $D^{\text{top}}(r) \times D^{\text{top}}(r')$ is disjoint from the image of $\Phi \times \Phi'$. Indeed, if $r < q$, we must have that $D^{\text{top}}(r)$ is disjoint from Q, since upward flow trajectories ending at level r will never reach level $q - \varepsilon$. Otherwise, if $r' < q'$, we must have that $D^{\text{top}}(r')$ is disjoint from Q'. Either way, by invariance with respect to the flows of $Z_{\widehat{\lambda}}$ and $Z_{\widehat{\lambda'}}$, we have that $D^{\text{top}}(r) \times D^{\text{top}}(r')$ is disjoint from $\Phi(\{0\} \times Q) \times \Phi'(\{0\} \times Q')$, and the claim follows. $\qquad \square$

Lemma 3.5 *Let Λ be a loose Legendrian link in a closed contact manifold $(Q, \ker \alpha)$, and let L be a Lagrangian in an exact symplectic manifold (M, θ). Assume that θ vanishes when restricted to L and that the dual vector field Z_θ has flow defined for all time and is nonvanishing along L and tangent to L. Then $\Lambda \times L$ is a loose Legendrian in the contact manifold $(Q \times M, \ker(\alpha + \theta))$.*

Proof Recall from Section 2.2 that $V_r \subset T^* \mathbb{R}^n$ is the box of radius r, typically equipped with the canonical Liouville one-form $\lambda_{\text{can}} = -\sum_{i=1}^n p_i \, dq_i$. Observe that the Liouville vector field $Z_{\lambda_{\text{can}}} = \sum_{i=1}^n p_i \partial_{p_i}$ vanishes identically on the set $Z_r = \{p_1 = \cdots = p_n = 0\} \subset V_r$, whereas by assumption this is not the case for Z_θ on L, and therefore some preparation is needed before we can appeal to Lemma 3.6 below.[5]

Consider the modified Liouville one-form on V_r, given by $\lambda_{\text{mod}} = dp_n - \sum_{i=1}^n p_i \, dq_i$. Notice that the corresponding Liouville vector field $Z_{\lambda_{\text{mod}}} = \partial_{q_n} + \sum_{i=1}^n p_i \partial_{p_i}$ is tangent and nonvanishing along Z_r. We claim we can find an embedding $F: V_r \hookrightarrow M$ such that $F^* \theta = \lambda_{\text{mod}}$ and $F(Z_r) \subset L$, for some sufficiently small $r > 0$. Indeed, note that λ_{mod} restricts to a contact form on $\{q_n = 0\} \subset V_r$, with respect to which $\{q_n = 0\} \cap Z_r$ is Legendrian. Similarly, for a point $x \in L$, we can find a small codimension-one submanifold $H \subset \mathcal{O}p(x)$ which is transverse to Z_θ. Then $\theta|_H$ is a contact form on H, with respect to which $L \cap H$ is Legendrian. By the contact Darboux theorem (see [6, Proposition 6.19]), we can find an embedding $f: \{q_n = 0\} \cap V_r \hookrightarrow H$ such that $f(\{q_n = 0\} \cap Z_r) \subset L \cap H$ and $f^*(\theta|_H) = \lambda_{\text{mod}}|_{\{q_n=0\} \cap V_r}$. Then the map F is obtained by extending f to be equivariant with respect to the flows of $Z_{\lambda_{\text{mod}}}$ and Z_θ and possibly restricting to a smaller $r > 0$.

---

[5]We thank the anonymous referee for pointing out a mistake related to this point in an earlier draft and suggesting a fix.

Similarly, consider a collection of points $x_1, \ldots, x_k \in L$, one on each connected component L_i of L. Applying the argument above, we can find embeddings $F_i \colon V_r \hookrightarrow M$ for $i = 1, \ldots, k$ such that $F_i^* \theta = \lambda_{\mathrm{mod}}$ and $F_i(Z_r) \subset L_i$. We also arrange that the image of F_i is disjoint from $L \setminus L_i$.

Fix some $C > 0$ sufficiently large relative to r. The flow of Z_θ for time $\log(C)$ gives a diffeomorphism $M \to M$ which pulls back θ to $C\theta$. Then the composition given by F_i followed by this flow is a map $\widetilde{F}_i \colon V_r \hookrightarrow M$ such that $\widetilde{F}_i^* \theta = C\lambda_{\mathrm{mod}}$. Since Z_θ is tangent to L, we also have $\widetilde{F}_i(Z_r) \subset L_i$.

Next, consider a function $T \colon V_r \to \mathbb{R}$, and let $G_T \colon Q \times V_r \to Q \times V_r$ be the map given by the time-T Reeb flow of α on the first factor and the identity on the second factor. Then we have $G_T^*(\alpha + C\lambda_{\mathrm{mod}}) = \alpha + C\lambda_{\mathrm{mod}} + dT$. In particular, setting $T = -Cp_n$, we get a map $G := G_T$ which satisfies $G^*(\alpha + C\lambda_{\mathrm{mod}}) = \alpha + C\lambda_{\mathrm{mod}} - C\,dp_n = \alpha + C\lambda_{\mathrm{can}}$. Since T vanishes on Λ, we also have $G(\Lambda \times Z_r) = \Lambda \times Z_r$.

Finally, the composition $(\mathrm{Id} \times \widetilde{F}_i) \circ G \colon Q \times V_r \hookrightarrow Q \times M$ maps $\Lambda \times Z_r$ to $\Lambda \times L_i$ and pulls back $\alpha + \theta$ to $\alpha + C\lambda_{\mathrm{can}}$. By rescaling, we can identify $(V_r, C\lambda_{\mathrm{can}})$ with $(V_R, \lambda_{\mathrm{can}})$ for $R = \sqrt{C}r$. In particular, since R can be arbitrarily large, it follows from Lemma 3.6 below that $\Lambda \times L$ is loose. $\qquad \square$

Lemma 3.6 *Let Λ be a loose Legendrian link in a contact manifold $(N^{2l+1}, \ker \alpha)$. Then, provided r is sufficiently large, $\Lambda \times Z_r^n$ is a loose Legendrian link in the contact manifold $(N^{2l+1} \times V_r^{2n}, \ker(\alpha - p_1\,dq_1 - \cdots - p_n\,dq_n))$.*

Proof Since Λ is loose, for each connected component there is a contact embedding of pairs
$$G \colon (B_{\mathrm{std}}^3 \times V_\rho^{2l-2}, \gamma_a \times Z_\rho^{l-1}) \hookrightarrow (N^{2l+1}, \Lambda)$$
for some $\rho > 0$ with $a < 2\rho^2$, with image disjoint from the other components of Λ. This means that $G^*\alpha$ is equal to $z - y\,dx - p_1\,dq_1 - \cdots - p_{l-1}\,dq_{l-1}$ times e^f for some smooth function f. Then for each component of Λ we can define a contact embedding of pairs
$$(B_{\mathrm{std}}^3 \times V_\rho^{2l-2} \times V_\rho^{2n}, \gamma_a \times Z_\rho^{l-1} \times Z_\rho^n) \hookrightarrow (N^{2l+1} \times V_r^{2n}, \Lambda \times Z_r^n)$$
given by G on the $B_{\mathrm{std}}^3 \times V_\rho^{2l-2}$ factor and the time f Liouville flow on the remaining factor. Note that this map is well-defined for r sufficiently large, and the image is disjoint from the other components of $\Lambda \times Z_r^n$. This restricts to a loose chart
$$(B_{\mathrm{std}}^3 \times V_\rho^{2l-2+2n}, \gamma_a \times Z_\rho^{l-1+n}) \hookrightarrow (N^{2l+1} \times V_r^{2n}, \Lambda \times Z_r^n). \qquad \square$$

Finally, we have the following corollary of the Lagrangian caps h-principle, amounting to an h-principle for symplectic embeddings of flexible Weinstein domains.

Theorem 3.7 (Eliashberg and Murphy [9, Section 6]) *Let $(W^{2n}, \lambda, \varphi)$ be a flexible Weinstein domain, and let $F \colon W \hookrightarrow (X^{2n}, \omega_X)$ be any smooth codimension-zero embedding into a symplectic manifold. If $n = 3$, assume X has infinite Gromov width.[6] Suppose that $F^*\omega_X$ is an exact two-form and is homotopic to $d\lambda$ through nondegenerate two-forms. Then F is isotopic to a symplectic embedding $f \colon (W, \varepsilon d\lambda) \hookrightarrow (X, \omega_X)$ for some sufficiently small $\varepsilon > 0$. If (X, ω_X) is an exact symplectic manifold, then we can also arrange that f is an exact symplectic embedding.*

Proof of Theorem 3.2 Let (W, λ, ϕ) be a flexible Weinstein domain, and let the pair $(D^*S^1, \lambda_{\mathrm{can}})$ denote the unit disk cotangent bundle of S^1, equipped with its canonical Liouville form. By Theorem 3.3, we have

$$\mathrm{SH}_\Omega(W, \lambda) \otimes \mathrm{SH}(D^*S^1, \lambda_{\mathrm{can}}) \cong \mathrm{SH}_\Omega(W \times D^*S^1, \lambda + \lambda_{\mathrm{can}}).$$

(Here Ω is being abusively used to also denote its pullback to $W \times D^*S^1$, and the twisting two-form on D^*S^1 is trivial.) Recall that $\mathrm{SH}(D^*S^1, \lambda_{\mathrm{can}})$ is isomorphic to the homology of the free loop space of S^1 (see [1]) and in particular is nontrivial. Therefore it suffices to prove that $\mathrm{SH}_\Omega(W \times D^*S^1, \lambda + \lambda_{\mathrm{can}})$ is trivial.

Let $(\mathbb{D}_r^2, \lambda_{\mathrm{std}})$ denote the standard Liouville disk of radius $r > 0$. Consider a split symplectic embedding

$$i \colon W \times D^*S^1 \hookrightarrow W \times \mathbb{D}_r^2$$

given by

- the identity on the first factor,
- any area-preserving embedding on the second factor.

Note that this embedding is not exact. Nevertheless, since the product is flexible by Proposition 3.4, it satisfies the hypotheses of Theorem 3.7. Therefore i is isotopic to a Liouville embedding

$$i' \colon (W \times D^*S^1, \lambda + \lambda_{\mathrm{can}}) \hookrightarrow (W \times \mathbb{D}_r^2, \lambda + \lambda_{\mathrm{std}}).$$

Moreover, $i'_*\Omega$ extends to a closed two-form $\widetilde{\Omega}$ on $W \times \mathbb{D}_r^2$. Note that without changing the cohomology class, we can assume that $\widetilde{\Omega}$ is pullback of a closed

---

[6]This extra assumption on the $n = 3$ case proved to be unnecessary, as was shown by Yoshiyasu [29].

two-form on W. Another application of Theorem 3.3, together with the vanishing of $\mathrm{SH}(\mathbb{D}_r^2, \lambda_{\mathrm{std}})$, implies that $\mathrm{SH}_{\widetilde{\Omega}}(W \times \mathbb{D}_r^2, \lambda + \lambda_{\mathrm{std}})$ vanishes. Finally, a well-known argument involving the Viterbo transfer map (see Theorem 3.8 below) shows that $\mathrm{SH}_{\Omega}(W \times D^*S^1, \lambda + \lambda_{\mathrm{can}})$ is trivial. Namely, the transfer map

$$\mathrm{SH}_{\Omega}(W \times \mathbb{D}_r^2, \lambda + \lambda_{\mathrm{std}}) \to \mathrm{SH}_{\Omega}(W \times D^*S^1, \lambda + \lambda_{\mathrm{can}})$$

is a unital ring map, which forces $\mathrm{SH}_{\Omega}(W \times D^*S^1, \lambda + \lambda_{\mathrm{can}})$ to be the trivial ring with unit equal to zero.   □

One very useful feature of symplectic cohomology is the Viterbo transfer map. This also holds for twisted symplectic cohomology, provided we take care in how we twist.

Theorem 3.8 ([27]; see also [21]) *Let (W_0, λ_0) and (W, λ) be Liouville domains, and assume there is a Liouville embedding $i\colon (W_0, \lambda_0) \hookrightarrow (W, \lambda)$. For Ω any closed two-form on W, there is map of unital $\mathbb{K}$–algebras*

$$\mathrm{SH}_{\Omega}(W, \lambda) \to \mathrm{SH}_{i^*\Omega}(W_0, \lambda_0).$$

Corollary 3.9 $\mathrm{SH}(W, \lambda) = 0$ *for any subflexible Weinstein domain (W, λ, ϕ).*

However, we wish to point out that there could be a closed two-form Ω_0 on W_0 which makes $\mathrm{SH}_{\Omega_0}(W_0, \lambda_0)$ nontrivial, but such that Ω_0 does not extend to W as a closed two-form. Indeed, this is precisely what happens for the examples we construct in Section 4.

4 Subflexibilization

4.1 Maydanskiy's manifold revisited

Before giving the general subflexibilization construction we illustrate the main ideas with an important example. We consider the two 6–dimensional Weinstein domains X_1^6 and X_2^6 from [12]. Each is diffeomorphic to $D^*S^3 \natural (D^*S^2 \times \mathbb{D}^2)$ and is represented by a Weinstein Lefschetz presentation. Here $M_1 \natural M_2$ denotes the *boundary connect sum* of two equidimensional Weinstein domains M_1 and M_2, ie the result of attaching a Weinstein 1–handle to the disjoint union of M_1 and M_2 with one endpoint on ∂M_1 and the other on ∂M_2. In the case of X_1, the fiber of the Lefschetz fibration is the Milnor fiber A_2^4, ie a plumbing of two copies of D^*S^2. The fiber A_2^4 itself

admits a secondary Lefschetz fibration with fiber D^*S^1 and three vanishing cycles. The Lefschetz presentation for X_1 has two vanishing cycles $V_1, V_2 \subset A_2^4$, given by the matching cycle construction with the matching paths in the top left of Figure 2. Similarly, the Lefschetz presentation for X_2 has fiber A_4^4 (a plumbing of four copies of D^*S^2), which admits a secondary Lefschetz fibration with fiber D^*S^1 and five vanishing cycles. X_2 has four vanishing cycles $W_1, W_2, W_3, W_4 \subset A_4^4$ specified by the matching paths in the bottom left of Figure 2. We refer the reader to [14; 12] for more details.
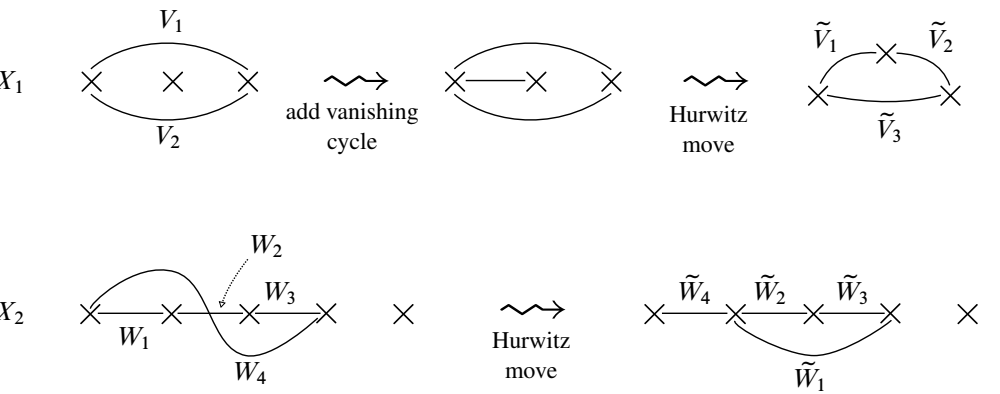

Figure 2: Top: X_1 becomes flexible after adding a vanishing cycle and applying a Hurwitz move. Bottom: X_2 becomes flexible after applying a Hurwitz move.

The manifold X_1 was first considered by Maydanskiy in [14], where he applies Seidel's long exact sequence to show that X_1 has vanishing wrapped Fukaya category (and hence vanishing symplectic cohomology). This also follows from the fact that X_1 is *subflexible*, which can be seen as follows. Adding a vanishing cycle to X_1 and applying a Hurwitz move according to the top of Figure 2, we arrive at the Weinstein domain $\mathcal{W}(A_2^4; \widetilde{V}_1, \widetilde{V}_2, \tau_{\widetilde{V}_2} \widetilde{V}_1)$, which we show is flexible below (Theorem 4.3). Incidentally, $\mathcal{W}(A_2^4; \widetilde{V}_1, \widetilde{V}_2, \tau_{\widetilde{V}_2} \widetilde{V}_1)$ is precisely the exotic D^*S^3 illustrated in [15, Figure 2], which gives another way of seeing that its symplectic cohomology vanishes.

On the other hand, X_2 is actually flexible. To see this, apply the Hurwitz move illustrated in the bottom of Figure 2. The result is Weinstein deformation equivalent to the same flexible D^*S^3 as before, with an extra pair of canceling handles ($\widetilde{W}_4$ is in canceling position) and an extra subcritical handle attached.

The main theorem of [12] states that X_1 and X_2 are not symplectomorphic. Harris argues that an embedded Lagrangian sphere appears in X_1 after an arbitrarily small

deformation of the symplectic form, whereas no such Lagrangian can appear for X_2. From the perspective of this paper, X_1 and X_2 can be distinguished by the fact that X_1 has nontrivial twisted symplectic cohomology by Theorem 4.5, whereas X_2 is flexible and thus has trivial twisted symplectic cohomology.

The fact that X_1 has nontrivial twisted symplectic cohomology reflects the symplectomorphism type of X_1 after a small deformation of its symplectic form. Namely, notice that X_1 can be viewed as $\mathcal{W}(A_2^4; S_1, \tau_{S_2}^2 S_1)$, where $S_1, S_2 \subset A_2^4$ represent the plumbed zero sections. According to Seidel [22, Proposition 2.6], a squared Dehn twist around a two-dimensional sphere is a *fragile* symplectomorphism, and hence we expect $\tau_{S_2}^2 S_1$ to be Hamiltonian isotopic to S_1 in the presence of an ambient deformation of the symplectic form. Since $\mathcal{W}(A_2^4; S_1, S_1)$ is just D^*S^3 with an extra subcritical 2–handle, this should imply that after deformation X_1 becomes symplectomorphic to a rather trivial deformation of the standard Weinstein structure on $D^*S^3 \natural (D^*S^2 \times \mathbb{D}^2)$ (this is made precise in Theorem 4.7). By the standard tools in symplectic cohomology we might expect the symplectic cohomology of the latter space to agree with homology of the free loop space of S^3.

However, we point out that there are technical difficulties in making sense of the symplectic cohomology of X_1 and X_2 after deformations. This is because X_1 and X_2 become nonconvex symplectic manifolds, and the standard definition and maximum principle properties of symplectic cohomology do not apply to such manifolds. On the other hand, twisted symplectic cohomology for Liouville domains is well-defined and poses no serious technical difficulties beyond those of ordinary symplectic cohomology (although computations are still in general quite difficult).

4.2 The subflexibilization construction

Let X^{2n+2} be a Weinstein domain with a Lefschetz presentation which satisfies Assumption 2.11. We explain how to use the Lefschetz presentation to modify X and obtain a new Weinstein domain $\mathrm{SF}(X)$.

Construction 4.1 *Assume X^{2n+2} has a Weinstein Lefschetz presentation with Weinstein fiber M^{2n}, vanishing cycles $V_1, \ldots, V_k$ in M and Lagrangian disks $T_1, \ldots, T_k$ in M as in Assumption 2.11. The Weinstein domain $\mathrm{SF}(X)$ is defined to have a Weinstein Lefschetz presentation with*

- *fiber given by attaching a Weinstein n–handle H_i to M along ∂T_i for each $1 \le i \le k$,*

- vanishing cycles $\tau_{S_1}^2 V_1, \tau_{S_2}^2 V_2, \ldots, \tau_{S_k}^2 V_k$, where S_i is the Lagrangian sphere given by the union of T_i and the core of H_i.

(We note that $\mathrm{SF}(X)$ depends on the Lefschetz presentation of X, but we suppress this from the notation.)

In the case $\dim X = 6$, let X_+^6 denote the boundary connect sum of X with k copies of the subcritical Weinstein domain $D^*S^2 \times \mathbb{D}^2$. In fact, as explained in [16], the cycles $\tau_{S_i}^2 V_i$ and V_i are smoothly isotopic through totally real submanifolds (for some choice of compatible almost complex structure) and this implies that $\mathrm{SF}(X^6)$ is almost symplectomorphic to $\mathcal{W}(M^4 \cup H_1 \cup \cdots \cup H_k; V_1, \ldots, V_k)$. The handles $H_1, \ldots, H_k$ are attached away from the vanishing cycles and the latter space can be identified with X_+^6 (recall that the attaching sphere of each H_i bounds an embedded Lagrangian disk in M^4).

Remark 4.2 For $n > 2$ the diffeomorphism type of $\mathrm{SF}(X^{2n+2})$ is not just the straightforward analogue of X_+^6. For n odd, the intersection form on $H_{n+1}(\mathrm{SF}(X^{2n+2}))$ is different from the one on $H_{n+1}(X^{2n+2})$. This reflects the fact, easily observed by the Picard–Lefschetz formula, that squared Dehn twists around odd-dimensional spheres typically act nontrivially on homology. When n is even and greater than two, the subflexibilization process changes the diffeomorphism type of X in a more subtle way (see [16] for more details). Since we only need to understand the case $n = 2$ for our present applications, we leave it to the reader to work out the topological details in higher dimensions.

Theorem 4.3 *For any X^{2n+2} satisfying [Assumption 2.11](#), $\mathrm{SF}(X)$ is subflexible. More specifically, when $n = 2$, the Weinstein domain $\mathrm{SF}(X^6)$ becomes Weinstein deformation equivalent to $\mathrm{Flex}(X^6)$ after attaching k Weinstein 3–handles.*

The main tool we will use is a proposition from [4] (the proposition there is phrased in terms of open books; boundaries of Lefschetz fibrations are just a special case of this).

Proposition 4.4 *Let $\mathcal{W}(M, \theta, \psi; V_1, \ldots, V_k)$ be a Lefschetz fibration, let $L \subseteq M$ be an exact Lagrangian sphere, and let $T \subseteq M$ be a stabilizing Lagrangian disk which intersects L transversely at one point. Let H be a Weinstein handle attached to M along ∂T, and let S be the Lagrangian sphere given by the union of T and the core disk of H. Then the Legendrian lift of $\tau_S L$ from the fiber $\pi^{-1}(1)$*

of the stabilized Lefschetz presentation $\mathcal{W}(M \cup H, \theta, \psi; V_1, \ldots, V_k, S)$ is loose in the boundary of the total space. In particular, if the Weinstein domain defined by $\mathcal{W}(M, \theta, \psi; V_1, \ldots, V_k)$ was flexible, then the Weinstein domain defined by $\mathcal{W}(M \cup H, \theta, \psi; V_1, \ldots, V_k, S, \tau_S L)$ is flexible as well.

Proof of Theorem 4.3 If $X = \mathcal{W}(M, \theta, \psi; V_1, \ldots, V_k)$, then

$$\mathrm{SF}(X) = \mathcal{W}(M \cup H_1 \cup \ldots \cup H_k, \theta, \psi; \tau_{S_1}^2 V_1, \ldots, \tau_{S_k}^2 V_k),$$

where S_i is the Lagrangian sphere given as the union of T_i and the core of the handle H_i (and we continue to use (θ, ψ) to denote the Weinstein structure on $M \cup H_1 \cup \ldots \cup H_k$). We attach handles to $\mathrm{SF}(X)$ to define a new Weinstein manifold

$$\widetilde{X} = \mathcal{W}(M \cup H_1 \cup \cdots \cup H_k, \theta, \psi; \tau_{S_1}^2 V_1, S_1, \tau_{S_2}^2 V_2, S_2, \ldots, \tau_{S_k}^2 V_k, S_k).$$

We will show that $\widetilde{X}$ is flexible in any dimension. In the case $n = 2$, we have that $\widetilde{X}$ is almost symplectomorphic to

$$\mathcal{W}(M \cup H_1 \cup \cdots \cup H_k, \theta, \psi; V_1, S_1, V_2, S_2, \ldots, V_k, S_k),$$

which is just the original manifold X (after stabilizing its Lefschetz fibration k times). Hence $\widetilde{X}$ is almost symplectomorphic to $\mathrm{Flex}(X)$, so once we show that $\widetilde{X}$ is flexible it follows that it is Weinstein deformation equivalent to $\mathrm{Flex}(X)$ (by Theorem 2.3).

By applying k Hurwitz moves, we get

$$\widetilde{X} = \mathcal{W}(M \cup H_1 \cup \cdots \cup H_k, \theta, \psi; S_1, \tau_{S_1} V_1, \ldots, S_k, \tau_{S_k} V_k).$$

The manifold $\widetilde{X}_0 := \mathcal{W}(M, \theta, \psi; \varnothing)$ is subcritical, so in particular it is flexible. The manifold $\mathcal{W}(M \cup H_1, \theta, \psi; S_1)$, being a stabilization of the previous Lefschetz fibration, is another Lefschetz presentation of $\widetilde{X}_0$ (though not explicitly subcritical). $\widetilde{X}_1 := \mathcal{W}(M \cup H_1, \theta, \psi; S_1, \tau_{S_1} V_1)$ is flexible, since it is built from X_0 by attaching a handle to the loose Legendrian $\tau_{S_1} V_1$. Then $\mathcal{W}(M \cup H_1 \cup H_2, \theta, \psi; S_1, \tau_{S_1} V_1, S_2)$, being a stabilization of the previous Lefschetz presentation, is Weinstein equivalent to $\widetilde{X}_1$. Next $\widetilde{X}_2 = \mathcal{W}(M \cup H_1 \cup H_2, \theta, \psi; S_1, \tau_{S_1} V_1, S_2, \tau_{S_2} V_2)$ is flexible, because it is built from the flexible Weinstein manifold $\widetilde{X}_1$ by attaching a handle along the loose Legendrian sphere $\tau_{S_2} V_2$. Continuing in this way, we see that $\widetilde{X} = \widetilde{X}_k$ is flexible. □

On the other hand, at least when $n = 2$, the manifold $\mathrm{SF}(X^6)$ is often not flexible. Indeed, by Theorem 3.2 it suffices to show that $\mathrm{SF}(X^6)$ has nontrivial twisted symplectic cohomology. In [25] the following result is proved for any X which satisfies Assumption 2.11:

Theorem 4.5 *There exists a closed two-form Ω on X^6 such that*

$$\mathrm{SH}_\Omega(\mathrm{SF}(X^6)) \cong \mathrm{SH}(X^6).$$

Corollary 4.6 $\mathrm{SF}(X^6)$ *is not flexible if* $\mathrm{SH}(X^6) \neq 0$.

The next result describes the symplectomorphism type of $\mathrm{SF}(X^6)$ after certain small deformations. Let ω_{sf} and ω_+ denote the symplectic forms on $\widehat{\mathrm{SF}(X^6)}$ and $\widehat{X^6_+}$ respectively.

Theorem 4.7 *There are closed two-forms Ω_{sf} and Ω_+ on $\widehat{\mathrm{SF}(X^6)}$ and $\widehat{X^6_+}$, respectively, such that $(\widehat{\mathrm{SF}(X^6)}, \omega_{\mathrm{sf}} + \varepsilon\Omega_{\mathrm{sf}})$ and $(\widehat{X^6_+}, \omega_+ + \varepsilon\Omega_+)$ are symplectomorphic for $\varepsilon > 0$ sufficiently small.*

Remark 4.8 In [20], Ritter proves the isomorphism $\mathrm{SH}_\Omega(X, \omega) \cong \mathrm{SH}(X, \omega + \Omega)$ in the case that Ω has compact support, provided $\omega + t\Omega$ is symplectic for all $t \in [0, 1]$. Suppose we also knew this to hold when Ω does not have compact support, and furthermore that the symplectic cohomology of $(\widehat{\mathrm{SF}(X)}, \omega_{\mathrm{sf}} + \varepsilon\Omega_{\mathrm{sf}})$ is well-defined and invariant under symplectomorphisms. Then for small $\varepsilon > 0$ we would have

$$\mathrm{SH}_{\varepsilon\Omega_{\mathrm{sf}}}(\mathrm{SF}(X^6)) \cong \mathrm{SH}(\mathrm{SF}(X^6), \omega_{\mathrm{sf}} + \varepsilon\Omega_{\mathrm{sf}})$$
$$\cong \mathrm{SH}(X^6_+, \omega_+ + \varepsilon\Omega_+)$$
$$\cong \mathrm{SH}_{\varepsilon\Omega_+}(X^6_+).$$

Moreover, based on the behavior of symplectic cohomology with respect to subcritical handles, it seems reasonable to guess that that latter term is isomorphic to $\mathrm{SH}(X^6)$. This gives at least a heuristic explanation of Theorem 4.5.

To prove Theorem 4.7, we need the following lemma, which is essentially due to Seidel.

Lemma 4.9 *Let (M^4, ω) be a symplectic manifold, with $S \subset M$ a Lagrangian sphere and Ω a closed two-form on M such that $[\Omega|_S] \neq 0 \in H^2(S; \mathbb{R})$. Consider the deformation of the symplectic form given by $\omega_s := \omega + s\Omega$. Then for any neighborhood U of S, there is a smooth family of symplectomorphisms Φ_s of (M^4, ω_s), defined for all $s \geq 0$ sufficiently small, such that*

- $\Phi_0 = \tau_S^2$,
- Φ_s *is supported in U for all s,*
- Φ_s *is Hamiltonian isotopic to the identity by an isotopy supported in U, for $s > 0$.*

Proof Let $(D_r^* S^2, \omega_{std})$ denote the radius r disk cotangent bundle of S^2 with respect to the round metric, for some r sufficiently small. Let Ω_{std} denote the pullback to $D_r^* S^2$ of an SO(3)–invariant area form on S^2 with total area equal to $\int_S \Omega$. Consider the deformation $(D_r^* S^2, \omega_{std} + s\Omega_{std})$ for $s \geq 0$ small. By the usual Moser–Weinstein technique we can find a smooth family of symplectic embeddings

$$E_s \colon (D_r^* S^2, \omega_{std} + s\Omega_{std}) \hookrightarrow (M^4, \omega_s)$$

for all $s \geq 0$ sufficiently small. By shrinking r if necessary, we can ensure that all of these embeddings have image in U.

Now, following Seidel [24] there is a smooth family of symplectomorphisms Φ_s of $(D_r^* S^2, \omega_{std} + s\Omega_{std})$, defined for all $s \geq 0$ sufficiently small, such that

- $\Phi_0 = \tau_S^2$,
- Φ_s is supported in Int $D_r^* S^2$,
- Φ_s is Hamiltonian isotopic to the identity by an isotopy supported in Int $D_r^* S^2$.

Then the pushforward of Φ_s by E_s has the desired properties. $\square$

Proof of Theorem 4.7 Let (M_{sf}^4, θ, ψ) denote the Weinstein fiber $SF(X^6)$, ie the result of attaching the Weinstein handles $H_1, \ldots, H_k$ to the fiber of X^6. We have symplectomorphisms

$$(\widehat{SF(X)}, \omega_{sf}) \cong \widehat{\mathcal{S}}(M_{sf}, d\theta; \tau_{S_1}^2 V_1, \ldots, \tau_{S_k}^2 V_k),$$
$$(\widehat{X_+}, \omega_+) \cong \widehat{\mathcal{S}}(M_{sf}, d\theta; V_1, \ldots, V_k).$$

Let Ω be a closed two-form on M_{sf} whose support is disjoint from

$$V_1 \cup \cdots \cup V_k \cup \tau_{S_1}^2 V_1 \cup \cdots \cup \tau_{S_k}^2 V_k$$

and such that $[\Omega|_{S_i}] \neq 0 \in H^2(S_i; \mathbb{R})$ for $1 \leq i \leq k$ and Ω is exact near ∂M_{sf} (we can take Ω to be Poincaré dual to the union of the cocores of the handles $H_1, \ldots, H_k$). As in Section 2.3, we can find closed two-forms Ω_{sf} and Ω_+ on $\widehat{SF(X)}$ and $\widehat{X_+}$ respectively such that for small $\varepsilon > 0$ we have symplectomorphisms

$$(\widehat{SF(X)}, \omega_{sf} + \varepsilon\Omega_{sf}) \cong \widehat{\mathcal{S}}(M_{sf}, d\theta + \varepsilon\Omega; \tau_{S_1}^2 V_1, \ldots, \tau_{S_k}^2 V_k),$$
$$(\widehat{X_+}, \omega_+ + \varepsilon\Omega_+) \cong \widehat{\mathcal{S}}(M_{sf}, d\theta + \varepsilon\Omega; V_1, \ldots, V_k).$$

Lemma 4.9 implies that $\tau_{S_i}^2 V_i$ is Hamiltonian isotopic to V_i in the symplectic manifold $(M_{sf}, d\theta + \varepsilon\Omega)$ for $1 \leq i \leq k$. By the Hamiltonian isotopy invariance of $\widehat{\mathcal{S}}$, it follows that $(\widehat{SF(X)}, \omega_{sf} + \varepsilon\Omega_{sf})$ and $(\widehat{X_+}, \omega_+ + \varepsilon\Omega_+)$ are symplectomorphic. $\square$

5 Applications

We now make use of subflexibilization to construct exotic examples. The starting point is Abouzaid and Seidel's [2] construction of an affine variety U^{2n} for any $2n \geq 6$ such that U is diffeomorphic to $\mathbb{C}^n$ and $\mathrm{SH}(U) \neq 0$. Based on work of McLean [17], the idea is to take U to be the Kaliman modification of $(\mathbb{C}^n, H, p)$, where $H \subset \mathbb{C}^n$ is a singular hypersurface given by the zero set of a weighted homogenous polynomial and $p \in H$ is a smooth point. This means that U is obtained by blowing up $\mathbb{C}^n$ at p and then excising the proper transform of H. Choosing the weighted homogenous polynomial carefully, Abouzaid and Seidel show that $\mathbb{C}^n \setminus H$ has nonvanishing symplectic cohomology, and therefore so does U by [17, Theorem 2.31]. Define $l_0 \in \mathbb{Z}$ to be the minimal number of vanishing cycles of a Weinstein Lefschetz presentation for U^6 satisfying Assumption 2.11.

Before proving Theorem 1.4, we briefly consider the effects of products, boundary connect sums, and subcritical handle attachments on subflexibility and twisted symplectic cohomology.

Lemma 5.1 *Let (W_1, λ_1, ϕ_1) be a subflexible Weinstein domain and (W_2, λ_2, ϕ_2) any Weinstein domain. Then $(W_1, \lambda_1, \phi_1) \times (W_2, \lambda_2, \phi_2)$ is subflexible.*

Proof Up to a Weinstein homotopy we can find a flexible Weinstein domain (X, λ, ϕ) of which (W_1, λ_1, ϕ_1) is a sublevel set. Let m_1 and m_2 denote the maximal critical values of ϕ_1 and ϕ_2, respectively. We can arrange that any critical points of ϕ on $X \setminus W_1$ have critical value at least m, for some $m > m_1 + m_2 - \min_{W_2} \phi_2$. By Proposition 3.4, $(\widehat{X} \times \widehat{W_2}, \widehat{\lambda} + \widehat{\lambda}_2, \widehat{\phi} + \widehat{\phi}_2)$ is flexible, and moreover the sublevel set $\{\phi + \phi_2 \leq C\}$ is Weinstein deformation equivalent to $W_1 \times W_2$ for any

$$m_1 + m_2 < C < m + \min_{W_2} \phi_2. \qquad \square$$

Lemma 5.2 *Let W_1 and W_2 be subflexible Weinstein domains. Then $W_1 \natural W_2$ is subflexible.*

Proof We can assume that W_1 and W_2 become flexible after attaching some number of Weinstein handles, whose attaching regions are disjoint from the attaching region of the 1–handle joining W_1 and W_2 in the construction of $W_1 \natural W_2$. Therefore we can attach Weinstein handles to $W_1 \natural W_2$ to obtain a boundary connect sum of two flexible Weinstein domains, and the latter is clearly flexible. $\qquad \square$

Proposition 5.3 Let (W_0, λ_0) *be a Liouville domain, and let* (W, λ) *be the resulting Liouville domain after attaching a subcritical Weinstein handle. Assume* $c_1(W) = c_1(W_0) = 0$. *Then for any closed two-form* Ω *on* W, *we have an isomorphism*

$$\mathrm{SH}^*_\Omega(W, \lambda) \cong \mathrm{SH}^*_{\Omega|_{W_0}}(W_0, \lambda_0).$$

In the case of untwisted symplectic cohomology, this is a well-known result due to Cieliebak [5]. Following [17, Section 10.3], the isomorphism is induced by the transfer map, which can be constructed using a sequence of admissible Hamiltonians in such a way that the subcritical handle introduces a single constant Hamiltonian orbit whose index is unbounded in the direct limit (here we need the Chern class assumption to define a $\mathbb{Z}$–grading). In particular, one can easily check that the same proof applies in the presence of a twisting two-form.

Proof of Theorem 1.4 We take X' to be $\mathrm{Flex}(X^6) \natural \mathrm{SF}(U^6)$ in the case dim $X' = 6$, and for $2n > 6$ we set

$$X' := \mathrm{Flex}(X^{2n}) \natural (\mathrm{SF}(U^6) \times D^* S^{2n-6}).$$

(Many other options are also possible, giving different subcritical topologies.) As explained in Section 4.2, $\mathrm{SF}(U^6)$ is almost symplectomorphic to U_+, which is the result of attaching l_0 two-handles to the six-dimensional ball. In particular, $\mathrm{SF}(U^6)$ is almost symplectomorphic to a subcritical space, and hence so is $\mathrm{SF}(U^6) \times D^* S^{2n-6}$. Since $\mathrm{Flex}(X)$ is almost symplectomorphic to X, this proves the first part of the theorem.

By Theorem 4.3, $\mathrm{SF}(U^6)$ becomes a ball after attaching l_0 Weinstein handles. In fact, since $\mathbb{D}^2 \times D^* S^k$ becomes the ball after attaching a canceling $(k+1)$–handle, we can also attach handles to convert $\mathrm{SF}(U^6) \times D^* S^{2n-6}$ into the ball, and hence X' into $\mathrm{Flex}(X^{2n}) \natural B^{2n} \cong \mathrm{Flex}(X^{2n})$. This proves the second part of the theorem.

Finally, we endow $\mathrm{SF}(U^6)$ with the twisting two-form from Theorem 4.5 which makes the twisted symplectic cohomology of $\mathrm{SF}(U^6)$ nontrivial. Extend this to a closed two-form on X' by pulling it back under the projection $\mathrm{SF}(U^6) \times D^* S^{2n-6} \to \mathrm{SF}(U^6)$ and extending by zero over $\mathrm{Flex}(X^{2n})$. By Proposition 5.3 and Theorem 3.2, we can ignore the $\mathrm{Flex}(X)$ factor for purposes of twisted symplectic cohomology. The third part of the theorem now follows by appealing to the Künneth-type Theorem 3.3, together with the well-known nonvanishing of symplectic cohomology for $D^* S^{2n-6}$. $\qquad\square$

We conclude by discussing how to use Theorem 1.4 to produce nonflexible polynomially convex domains, disproving the conjecture of Cieliebak and Eliashberg [7]. Following the conventions of [7], recall that a polynomially convex domain in $\mathbb{C}^n$ is a compact domain with smooth boundary $K \subset \mathbb{C}^n$ such that K coincides with its *polynomial hull*

$$\widehat{K}_{\mathcal{P}} := \left\{ z \in \mathbb{C}^n : |P(z)| \leq \max_{u \in K} |P(u)| \text{ for all complex polynomials } P \text{ on } \mathbb{C}^n \right\}.$$

(See also [26] for background on polynomial convexity.) According to Criterion 3.2 of [7], an i–convex domain $W \subset \mathbb{C}^n$ is polynomially convex if and only if there exists an exhausting i–convex function $\phi \colon \mathbb{C}^n \to \mathbb{R}$ for which W is a sublevel set. Here $W \subset \mathbb{C}^n$ is i–*convex* if there is an i–convex function on W such that ∂W is its maximal regular level set.

Proof of Theorem 1.5 Since X' is deformation equivalent to a sublevel set of $\mathrm{Flex}(X)$, we focus on $\mathrm{Flex}(X)$. By [7, Lemma 2.1], the smooth hypotheses on X imply that the Morse function on X extends to a Morse function on $\mathbb{C}^n$ without critical points of index greater than n. By [6, Theorem 13.1], we can therefore extend the Weinstein structure on X to a flexible Weinstein structure on $\mathbb{C}^n$, and by flexibility this is automatically Weinstein homotopic to the standard structure. Theorem 1.5(c) of [7] now produces a Weinstein deformation equivalence of X' onto a polynomially convex domain in $\mathbb{C}^n$. □

References

[1] **M Abouzaid**, *Symplectic cohomology and Viterbo's theorem*, from "Free loop spaces in geometry and topology" (J Latschev, A Oancea, editors), IRMA Lect. Math. Theor. Phys. 24, Eur. Math. Soc., Zürich (2015) 271–485 MR

[2] **M Abouzaid**, **P Seidel**, *Altering symplectic manifolds by homologous recombination*, preprint (2010) arXiv

[3] **F Bourgeois**, **T Ekholm**, **Y Eliashberg**, *Effect of Legendrian surgery*, Geom. Topol. 16 (2012) 301–389 MR

[4] **R Casals**, **E Murphy**, **F Presas**, *Geometric criteria for overtwistedness*, preprint (2015) arXiv

[5] **K Cieliebak**, *Handle attaching in symplectic homology and the chord conjecture*, J. Eur. Math. Soc. 4 (2002) 115–142 MR

[6] **K Cieliebak**, **Y Eliashberg**, *From Stein to Weinstein and back: symplectic geometry of affine complex manifolds*, Amer. Math. Soc. Colloq. Publ. 59, Amer. Math. Soc., Providence, RI (2012) MR

[7]   **K Cieliebak**, **Y Eliashberg**, *The topology of rationally and polynomially convex domains*, Invent. Math. 199 (2015) 215–238   MR

[8]   **T Ekholm**, **Y Eliashberg**, **E Murphy**, **I Smith**, *Constructing exact Lagrangian immersions with few double points*, Geom. Funct. Anal. 23 (2013) 1772–1803   MR

[9]   **Y Eliashberg**, **E Murphy**, *Lagrangian caps*, Geom. Funct. Anal. 23 (2013) 1483–1514   MR

[10] **E Giroux**, **J Pardon**, *Existence of Lefschetz fibrations on Stein and Weinstein domains*, Geom. Topol. 21 (2017) 963–997   MR

[11] **M Gromov**, *Partial differential relations*, Ergeb. Math. Grenzgeb. 9, Springer (1986)   MR

[12] **R M Harris**, *Distinguishing between exotic symplectic structures*, J. Topol. 6 (2013) 1–29   MR

[13] **O van Koert**, *Lecture notes on stabilization of contact open books*, preprint (2010)   arXiv

[14] **M Maydanskiy**, *Exotic symplectic manifolds from Lefschetz fibrations*, PhD thesis, Massachusetts Institute of Technology (2009)   MR   Available at `https://dspace.mit.edu/bitstream/handle/1721.1/50268/465218504-MIT.pdf`

[15] **M Maydanskiy**, **P Seidel**, *Lefschetz fibrations and exotic symplectic structures on cotangent bundles of spheres*, J. Topol. 3 (2010) 157–180   MR

[16] **M Maydanskiy**, **P Seidel**, *Corrigendum: Lefschetz fibrations and exotic symplectic structures on cotangent bundles of spheres*, J. Topol. 8 (2015) 884–886   MR

[17] **M McLean**, *Lefschetz fibrations and symplectic homology*, Geom. Topol. 13 (2009) 1877–1944   MR

[18] **E Murphy**, *Loose Legendrian embeddings in high dimensional contact manifolds*, preprint (2012)   arXiv

[19] **A Oancea**, *The Künneth formula in Floer homology for manifolds with restricted contact type boundary*, Math. Ann. 334 (2006) 65–89   MR

[20] **A F Ritter**, *Deformations of symplectic cohomology and exact Lagrangians in ALE spaces*, Geom. Funct. Anal. 20 (2010) 779–816   MR

[21] **A F Ritter**, *Topological quantum field theory structure on symplectic cohomology*, J. Topol. 6 (2013) 391–489   MR

[22] **P Seidel**, *Floer homology and the symplectic isotopy problem*, DPhil thesis, University of Oxford (1997)   Available at `http://www-math.mit.edu/~seidel/thesis.pdf`

[23] **P Seidel**, *Fukaya categories and Picard–Lefschetz theory*, Eur. Math. Soc., Zürich (2008)   MR

[24] **P Seidel**, *Lectures on four-dimensional Dehn twists*, from "Symplectic 4–manifolds and algebraic surfaces" (F Catanese, G Tian, editors), Lecture Notes in Math. 1938, Springer (2008) 231–267 MR

[25] **K Siegel**, *Squared Dehn twists and deformed symplectic invariants*, preprint (2016) arXiv

[26] **E L Stout**, *Polynomial convexity*, Progress in Math. 261, Birkhäuser, Boston (2007) MR

[27] **C Viterbo**, *Functors and computations in Floer homology with applications, I*, Geom. Funct. Anal. 9 (1999) 985–1033 MR

[28] **A Weinstein**, *Contact surgery and symplectic handlebodies*, Hokkaido Math. J. 20 (1991) 241–251 MR

[29] **T Yoshiyasu**, *On Lagrangian embeddings into the complex projective spaces*, Internat. J. Math. 27 (2016) 1650044, 12 MR

Department of Mathematics, Massachusetts Institute of Technology
Cambridge, MA, United States

e_murphy@math.mit.edu,   kyler@mit.edu