Virtual fundamental classes via dg–manifolds

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We construct virtual fundamental classes for dg–manifolds whose tangent sheaves have cohomology only in degrees 0 and 1. This condition is analogous to the existence of a perfect obstruction theory in the approach of Behrend and Fantechi [3] or Li and Tian [11]. Our class is initially defined in K–theory as the class of the structure sheaf of the dg–manifold. We compare our construction with that of [3] as well as with the original proposal of Kontsevich. We prove a Riemann–Roch type result for dg–manifolds which involves integration over the virtual class. We prove a localization theorem for our virtual classes. We also associate to any dg–manifold of our type a cobordism class of almost complex (smooth) manifolds. This supports the intuition that working with dg–manifolds is the correct algebro-geometric replacement of the analytic technique of "deforming to transversal intersection".

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Introduction

In many moduli problems in algebraic geometry there is a difference between the actual dimension of the moduli space and the expected, or virtual, dimension. When this happens, the moduli problem is said to be obstructed. The actual dimension, at the level of tangent spaces, is typically the dimension of H^0 or H^1 of some coherent sheaf $\mathcal{F}$, while the virtual dimension is the Euler characteristic of $\mathcal{F}$. Over $\mathbb{C}$, one can often represent the moduli space as a possibly nontransversal intersection inside an infinite-dimensional ambient space, and by analogy with the finite-dimension intersection theory (see Fulton [7]) one expects a "virtual fundamental class" of the expected dimension, associated to the moduli space. Such classes were constructed by Behrend and Fantechi [3] and by Li and Tian [11], for the case when the obstruction is simple, or "perfect" (typically, $\mathcal{F}$ has one more cohomology group). In this case the expected dimension is less or equal than the actual one, and the class lies in the Chow group of the moduli space.

M Kontsevich suggested in [9] that all such problems can be handled by working with appropriate derived versions of moduli spaces. Following this suggestion, the authors

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developed in [5; 6] the basic theory of such derived objects, called dg–manifolds, and constructed the derived versions of Grothendieck's $Quot$ and Hilbert schemes as well as of Kontsevich's moduli spaces of stable maps.

The goal of the present paper is to define virtual classes in the context of "simply obstructed" dg–manifolds. By simply obstructed we mean that the tangent dg–spaces have cohomology only in degrees 0 and 1. Some of the features of our approach are similar to those of [3]. In particular, it is clear that whenever both approaches are applicable, they give the same result. On the other hand, the language of dg–manifolds exhibits all the necessary constructions as analogs of the most standard procedures of usual algebraic geometry. In particular, the structure sheaf of a dg–manifold gives rise to the K–theoretic virtual class, and we prove (Theorem 3.3.1) that it lies in the right level of the dimension filtration and gives the homological class after passing to the quotient. Further, we prove a Riemann–Roch-type result for dg–manifolds (Theorem 4.5.1) which involves integration over the virtual class. In a similar way, applying the Bott–Thomason localization theorem to the structure sheaf of a dg–manifold with a torus action gives at once the localization theorem for virtual classes proved by Graber and Pandharipande [8].

The intuitive point of view behind the language of dg–manifolds is that they provide an algebro-geometric analog of "deformation to transversal intersection" which often cannot be achieved within pure algebraic geometry. We prove a result confirming this intuition in a new way. Namely, we associate, in Theorem 4.6.4, to each dg–manifold X of our type a cobordism class of almost complex (smooth) manifolds.

One of the most attractive features of our approach is that it suggests a definition of the virtual class also in the case when the obstruction is no longer simple. In this case it is not even clear a priori where the virtual class should lie, as the expected dimension can well be greater that the actual one (due to many alternating summands in the Euler characteristic). The language of dg–manifolds suggests that it should lie in the Chow group (of the expected dimension) of a certain natural fiber bundle Π over the moduli space. To be precise (see Section 1.1 below), a dg–manifold X consists of a smooth algebraic variety X^0 and a sheaf $\mathcal{O}_X^\bullet$ of dg–algebras on X^0. The role of the moduli space is played by the subscheme $\pi_0(X) \subset X^0$ which is the spectrum of $\underline{H}^0(\mathcal{O}_X^\bullet)$. The fiber bundle Π is the spectrum of $\underline{H}^{\mathrm{even}}(\mathcal{O}_X^\bullet)$, the ring of even cohomology, and the odd cohomology gives a coherent sheaf $\mathcal{H}$ on it. The virtual class should lie in the Chow group of Π and come from the class $1-[\mathcal{H}]$ in its K–theory. Further, since $\underline{H}^{\mathrm{even}}(\mathcal{O}_X^\bullet)$ is graded, Π is a cone with apex $\pi_0(X)$, so it has an action of $\mathbb{G}_m$ with fixed locus $\pi_0(X)$. This allows one to localize all the data back to $\pi_0(X)$. This program will be developed in a future paper.

Here is the outline of the paper. In Section 1 we develop the formalism of deformation to the normal cone in the context of dg–manifolds. This allows us to replace, in enumerative arguments, the underlying variety X^0 of a dg–manifold X by the normal cone to $\pi_0(X)$ in X^0. In Section 2 we introduce the class of [0, 1]–manifolds which formalize the concept of a simply obstructed moduli space. An important property of such manifolds is that the cohomology $\underline{H}^\bullet(\mathcal{O}_X^\bullet)$ is bounded, so one can speak about its class in the Grothendieck group of $\pi_0(X)$. This is exactly the K–theoretic virtual class as defined in Section 3. We also introduce in Section 3 the homological class and compare it to the K–theoretic one. In Section 4 we give a different definition of the homological virtual class in terms of the Chern character. This was the initial proposal of Kontsevich [9]. Therefore our paper connects the approaches of [9] and [3]. This equivalence of the two definitions can be seen as a particular case of a Riemann–Roch theorem for dg–manifolds which we also prove in Section 4. Finally, Section 5 is devoted to the Bott localization for dg–manifolds.

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1 Deformation to the normal cone for dg–manifolds

1.1 Notation

We recall briefly here some definitions and basic facts about dg–schemes; the reader is referred to [5, Section 2] for more background and details. Fix a base field k of characteristic 0. A dg–scheme is a dg–ringed space $X = (X^0, \mathcal{O}_X^\bullet)$, where X^0 is a k–scheme and $\mathcal{O}_X^\bullet$ is a sheaf of (graded-commutative) dg–algebras on X^0, situated in degrees ≤ 0, such that $\mathcal{O}_X^0 = \mathcal{O}_{X^0}$ and quasicoherent as a module over $\mathcal{O}_{X^0}$. We denote the differential in $\mathcal{O}_X^\bullet$ by d. Because of the grading condition, d is linear over $\mathcal{O}_{X^0}$. Further, $\underline{H}^0(\mathcal{O}_X^\bullet) = \mathcal{O}_{X^0}/d(\mathcal{O}_X^{-1})$ is a quotient of $\mathcal{O}_{X^0}$, so $\pi_0(X) := \operatorname{Spec}\underline{H}^0(\mathcal{O}_X^\bullet)$ is a closed subscheme of X^0. A dg–sheaf on a dg–scheme X is a sheaf $\mathcal{F}^\bullet$ of dg–modules over $\mathcal{O}_X^\bullet$ which is quasicoherent over $\mathcal{O}_{X^0}$. A dg–sheaf is called a dg–vector bundle, if it is bounded from above, and, considered as a sheaf of graded modules over $\mathcal{O}_X^\bullet$, is locally free with finitely many generators in each degree.

By a dg–manifold we mean a dg–scheme X such that X^0 is a smooth algebraic variety over k, and $\mathcal{O}_X^\bullet$, considered as a sheaf of graded $\mathcal{O}_{X^0}$–algebras, is locally free with finitely many generators in each degree. In other words, locally in the Zariski topology of X^0, we have $\mathcal{O}^\bullet_{X,\#} \cong S^\bullet_{\mathcal{O}_{X^0}}(Q^{-1} \oplus Q^{-2} \oplus \dots)$, with Q^{-i} vector bundles of finite rank on X^0. The graded vector bundle $Q^\bullet$ can be defined *globally* as the bundle $Q^\bullet := \mathcal{O}_X^{\leq -1}/(\mathcal{O}_X^{\leq -1})^2$ of indecomposable elements, but it does not come with a global embedding in $\mathcal{O}^\bullet_X$. The dg–cotangent sheaf $\Omega^{1,\bullet}_X$ of a dg–manifold X is defined as the target of the universal k–derivation $\partial\colon \mathcal{O}^\bullet_X \to \overline{\Omega}^{1,\bullet}_X$. It is easy to see that it is a dg–vector bundle, which is identified as a graded sheaf with the cotangent bundle $\Omega^1_{X^0}$ of X^0 in degree zero, and with $\mathcal{O}^\bullet_X \otimes_{\mathcal{O}_{X^0}} Q^\bullet$ in degrees ≤ -1. We also have the dg–tangent bundle

$$T^\bullet_X = \mathcal{D}er_k(\mathcal{O}^\bullet_X, \mathcal{O}^\bullet_X) = \mathcal{H}om_{\mathcal{O}^\bullet_X}(\Omega^{1,\bullet}_X, \mathcal{O}^\bullet_X).$$

Let X be a dg–manifold. For any dg–vector bundle $E^\bullet$ on X we denote by $E^\bullet|_{\pi_0(X)}$ the restriction of $E^\bullet$ (as a complex of vector bundles on X^0) to $\pi_0(X) \subset X^0$. The restriction $\mathcal{O}^\bullet_X|_{\pi_0(X)}$ will be denoted by $\mathbb{O}^\bullet_X$ or simply $\mathbb{O}^\bullet$. This is a sheaf of dg–algebras on $\pi_0(X)$ with $d\colon \mathbb{O}^{-1} \to \mathbb{O}^0$ vanishing. Thus $\mathbb{O}^{\leq -1}$ is a dg–ideal in $\mathbb{O}^\bullet$. For any $E^\bullet$ as above the restriction $E^\bullet|_{\pi_0(X)}$ is a dg–module over $\mathbb{O}^\bullet$.

We denote $\bar{E}^\bullet = E^\bullet \otimes_{\mathcal{O}^\bullet_X} \mathcal{O}_{\pi_0(X)}$ the restriction of $E^\bullet$ to $\pi_0(X)$ in the sense of dg–manifolds. This is a complex of vector bundles on $\pi_0(X)$. It is clear that

$$\bar{E}^\bullet = E^\bullet|_{\pi_0(X)} \big/ \mathbb{O}^{\leq -1} \cdot E^\bullet|_{\pi_0(X)}.$$

We also denote

$$\omega^\bullet = \omega^\bullet_X = \overline{\Omega}^{1,\bullet}_X, \quad \mathbf{t}^\bullet = \mathbf{t}^\bullet_X = \bar{T}^\bullet_X.$$

These are complexes of vector bundles on $\pi_0(X)$ situated in degrees ≤ 0, ≥ 0 respectively, and dual to each other. Note that we have $\omega^0 = \Omega^1_{X^0}|_{\pi_0(X)}$ and $\omega^{\leq -1} = \mathbb{O}^{\leq -1}/(\mathbb{O}^{\leq -1})^2$ as graded vector bundles. Dually, $\mathbf{t}^0 = T_{X^0}|_{\pi_0(X)}$, while for $n > 0$

$$\mathbf{t}^n = \operatorname{Ker}\left\{ \mathbb{O}^{-n*} \to \bigoplus_{\substack{i+j=n \\ i,j>0}} \mathbb{O}^{-i*} \otimes \mathbb{O}^{-j*} \right\}$$

is the space of primitive elements in $\mathbb{O}^{-n*}$. In particular, $\omega^{-1} = \mathbb{O}^{-1}$ and $\mathbf{t}^1 = \mathbb{O}^{-1*}$.

For a dg–bundle $E^\bullet$ on X we have the decomposability filtration $\mathcal{D}$ in $E^\bullet|_{\pi_0(X)}$

$$\mathcal{D}^n E^\bullet|_{\pi_0(X)} = (\mathbb{O}^{\leq -1})^n \cdot E^\bullet|_{\pi_0(X)}.$$

The above discussion shows the following:

1.1.1 Proposition *We have*

$$\operatorname{gr}^n_{\mathcal{D}}\big(E^\bullet|_{\pi_0(X)}\big) = \bar{E}^\bullet \otimes S^n(\omega^{\leq -1}).$$

1.2 The J–adic filtration and the normal cone

Let $J = d(\mathcal{O}_X^{-1}) \subset \mathcal{O}_X^0 = \mathcal{O}_{X^0}$ be the ideal of the subscheme $\pi_0(X)$. We denote by

$$N = N_{\pi_0(X)/X^0} = \operatorname{Spec} \bigoplus_n J^n/J^{n+1}$$

the normal cone of $\pi_0(X)$ in X^0. Let also

$$K = \operatorname{Ker}\{d^1\colon \mathbf{t}_X^1 \to \mathbf{t}_X^2\}.$$

This is a coherent sheaf on $\pi_0(X)$. Since it is defined as the kernel of a morphism of vector bundles, we can associate to it its total space $\underline{K} \subset \underline{\mathbf{t}}^1$ (which is a cone).

1.2.1 Proposition *There is a natural closed embedding $N \subset \underline{K}$ of cones over $\pi_0(X)$.*

Proof We have

$$\omega^{-1} = \mathbb{O}^{-1} = \mathcal{O}_X^{-1}|_{\pi_0(X)} = \mathcal{O}_X^{-1}/(d\mathcal{O}_X^{-1})\cdot\mathcal{O}_X^{-1}.$$

Therefore

$$\underline{\mathbf{t}}^1 = \operatorname{Spec} S(\mathbb{O}^{-1}), \quad \underline{K} = \operatorname{Spec}\big(S(\mathbb{O}^{-1})/(d\mathbb{O}^{-2})\cdot S(\mathbb{O}^{-1})\big)$$

The differential $d^{-1}\colon \mathcal{O}_X^{-1} \twoheadrightarrow J$ induces, after passing to the n–th symmetric power and restricting to $\pi_0(X)$, a surjective map

$$\delta_n\colon S^n(\mathbb{O}^{-1}) \to J^n/J^{n+1}.$$

Explicitly, let $\varphi_1,\dots,\varphi_n$ be local sections of $\mathbb{O}^{-1}$. Then

$$\delta_n(\varphi_1\cdots\varphi_n) = d^{-1}(\widetilde{\varphi}_1)\cdots d^{-1}(\widetilde{\varphi}_n) \bmod J^{n+1}$$

where $\widetilde{\varphi}_i$ is a local section of $\mathcal{O}_X^{-1}$ extending φ_i. Therefore we get a surjective homomorphism of sheaves of algebras

$$\delta = \bigoplus \delta_n\colon S(\mathbb{O}^{-1}) \to \bigoplus J^n/J^{n+1}$$

which induces a closed embedding $\delta^*\colon N \subset \underline{\mathbf{t}}^1$. To show that $\operatorname{Im}(\delta^*) \subset \underline{K}$, it is enough to show that $\delta_n((d\varphi)\cdot\varphi_1\cdots\varphi_{n-1}) = 0$ for any local sections $\varphi \in \mathbb{O}^{-2}$, $\varphi_i \in \mathbb{O}^{-1}$. Let $\widetilde{\varphi} \in \mathcal{O}_X^{-2}$, $\widetilde{\varphi}_i \in \mathcal{O}_X^{-1}$ be local sections that extend φ, φ_i. Then,

$$\delta_n\big((d\varphi)\cdot\varphi_1\cdots\varphi_n\big) = d^{-1}(d^{-2}\widetilde{\varphi})\cdot d^{-1}(\widetilde{\varphi}_1)\cdots d^{-1}(\widetilde{\varphi}_n) \bmod J^{n+1}$$

which is clearly 0. □

1.3 Deformation to the normal cone

Let V be any vector bundle on X^0. We equip it with the J–adic filtration by setting $J^n V = J^n \cdot V$, so that V becomes a filtered module over the filtered algebra $(\mathcal{O}_{X^0}, \{J^n\})$. Hence $\mathrm{gr}_J V$ is a graded module over the graded algebra $\mathrm{gr}_J \mathcal{O}_{X^0}$ and gives, by localization, a coherent sheaf $\widetilde{\mathrm{gr}}_J V$ on $N = \mathrm{Spec}\, \mathrm{gr}_J \mathcal{O}_{X^0}$. Let $p\colon N \to \pi_0(X)$ be the projection. The following is well known, with proof supplied for completeness.

1.3.1 Proposition *The sheaf $\widetilde{\mathrm{gr}}_J V$ is identified with $p^*(V|_{\pi_0(X)})$. If $f\colon V \to W$ is any morphism of vector bundles on X^0, then $\widetilde{\mathrm{gr}}_J(f)$ is identified with $p^*(f|_{\pi_0(X)})$.*

Proof Denote for short $Z = \pi_0(X)$. The surjective homomorphism $V \otimes_{\mathcal{O}_{X^0}} J^n \to J^n V$ induces, after restricting to Z, a surjective homomorphism

$$h_n\colon (V/JV) \otimes_{\mathcal{O}_Z} (J^n/J^{n+1}) \to J^n V/J^{n+1} V.$$

We claim that it is an isomorphism. Indeed, if $V = \mathcal{O}_{X^0}$, the statement is tautological. Hence it is true for a trivial bundle $V = \mathcal{O}_{X^0}^r$. In general, the fact that h_n is an isomorphism can be verified locally on the Zariski topology, so it follows from local triviality of V. Now, notice that $V/JV = V|_Z$ and tensoring with $\bigoplus J^n/J^{n+1}$ over $\mathcal{O}_Z$ is geometrically the pullback p^*, so $\bigoplus h_n$ gives the required identification. The statement about morphisms follows from the naturality of maps h_n. $\square$

Next, we extend the J–adic filtration to $\mathcal{O}_X^\bullet$ by setting $J^n \mathcal{O}_X^\bullet$ to have components $J^n \cdot \mathcal{O}_X^i$, $i \le 0$. Then J^n is a multiplicative filtration on the sheaf of dg–algebras $\mathcal{O}_X^\bullet$, so $\mathrm{gr}_J^\bullet \mathcal{O}_X^\bullet$ is a graded sheaf of dg–algebras on X^0 supported on $\pi_0(X)$. We have therefore a dg–scheme

$$\mathrm{Spec}\,(\mathrm{gr}_J^\bullet \mathcal{O}_X^\bullet) \longrightarrow \pi_0(X).$$

The underlying ordinary scheme of this dg–scheme is $\mathrm{Spec}\,(\mathrm{gr}_J^\bullet \mathcal{O}_X^0) = N$.

Further, let $E^\bullet$ be a dg–vector bundle on X. Then we have the J–adic filtration $J^n E^\bullet$ similarly to the above. The associated graded object $\mathrm{gr}_J^\bullet E^\bullet$ is then a sheaf of dg–modules over $\mathrm{gr}_J^\bullet \mathcal{O}_X^\bullet$ and as such localizes to a sheaf of dg–modules $\widetilde{\mathrm{gr}}_J E^\bullet$ on the dg–scheme $\mathrm{Spec}\,(\mathrm{gr}_J \mathcal{O}_X^\bullet)$. The following is an immediate consequence of Proposition 1.3.1.

1.3.2 Proposition (1) *The structure sheaf of the dg–scheme* $\mathrm{Spec}\,(\mathrm{gr}_J^\bullet \mathcal{O}_X^\bullet)$ *is isomorphic, as a sheaf of dg–algebras, to $p^* \mathbb{O}_X^\bullet$, where p^* means the usual pullback of coherent sheaves on schemes.*

(2) *With respect to the identification of (1), the sheaf of dg–modules* $\widetilde{\mathrm{gr}}_J E^\bullet$ *is isomorphic to* $p^*(E^\bullet|_{\pi_0(X)})$.

1.3.3 Proposition (1) *The pullback to* $p^*(E^\bullet|_{\pi_0(X)})$ *of the filtration* $\mathcal{D}$ *is compatible with the differential.*

(2) *The sheaf of dg–algebras* $\mathrm{gr}_{p^*\mathcal{D}}(p^*\mathbb{O}^\bullet, p^*d_{\mathbb{O}})$ *is isomorphic to* $p^*S(\omega^{\leq -1})$, *the restriction of the Koszul complex* $q^*S(\omega^{\leq -1})$ *to* $N \subset \underline{K}$ *(here* $q\colon \underline{K} \to \pi_0(X)$ *is the projection).*

(3) *The sheaf of dg–modules* $\mathrm{gr}_{p^*\mathcal{D}}\big(p^*(E^\bullet|_{\pi_0(X)})\big)$ *is isomorphic to the pullback* $p^*(\bar{E} \otimes S(\omega^{\leq -1}))$.

Proof (1) It is enough to prove that the differential in $E^\bullet|_{\pi_0(X)}$ (denote it δ) is compatible with the filtration $\mathcal{D}$, ie,

$$\delta\big((\mathbb{O}^{\leq -1})^n \cdot E^\bullet|_{\pi_0(X)}\big) \subset (\mathbb{O}^{\leq -1})^n \cdot E^\bullet|_{\pi_0(X)}.$$

This follows from the Leibniz rule

$$\delta(fe) = d_{\mathbb{O}} f \cdot e + (-1)^{\deg(f)\cdot\deg(e)} f \cdot \delta(e), \qquad f \in \mathbb{O}^\bullet, e \in E^\bullet|_{\pi_0(X)}$$

and the fact that $d_{\mathbb{O}}(\mathbb{O}^{-1}) = 0$.

Parts (2) and (3) follow from (1) and Proposition 1.1.1. □

2 Bounded dg–manifolds and [0, 1]–manifolds

2.1 $[0, n]$–manifolds

Let X be a dg–manifold, and $n \geq 0$.

2.1.1 Proposition *The following are equivalent:*

(i) *For all* $x \in \pi_0(X)(\mathbb{C})$ *the tangent dg–space* $T_x^\bullet X$ *is exact outside the degrees in* $[0, n]$.

(ii) *The complex* $\mathbf{t}_X^\bullet$ *is exact outside the degrees in* $[0, n]$.

Proof (i) $\Rightarrow$ (ii) A fiberwise exact sequence of vector bundles is exact at the level of sheaves of sections.

(ii) $\Rightarrow$ (i) This follows from the spectral sequence

$$\mathrm{Tor}_i^{\mathcal{O}_{\pi_0(X)}}(\underline{H}^j(\mathbf{t}_X^\bullet), \mathbb{C}_x) \quad \Rightarrow \quad H^{j-i}(T_x^\bullet X)$$

and the fact that $T_x^\bullet X$ is situated in degrees ≥ 0. □

2.1.2 Definition We say that X is a $[0,n]$–manifold if the conditions of Proposition 2.1.1 are satisfied.

2.1.3 Examples (a) If Y is a projective variety of dimension n, then the dg–manifold $\mathrm{RQuot}_h(\mathcal{F})$ constructed in [5], is a $[0,n]$–manifold for any coherent sheaf $\mathcal{F}$ on Y and any polynomial h.

(b) The dg–manifold $\mathrm{RHilb}_h^{\mathrm{LCI}}(Y)$ constructed in [6], is a $[0,d]$–manifold, where $d=\deg(h)$.

(c) Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be a diagram of smooth algebraic varieties (trivial dg–structure). Then the derived fiber product $X \times_Z^R Y$, constructed in [5], is a $[0,1]$–manifold. Indeed, let (x,y) be a point of

$$\pi_0(X \times_Z^R Y) = X \times_Z Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}.$$

and $z = f(x) = g(y)$. Then $T^\bullet_{(x,y)}\big(X \times_Z^R Y\big)$ is, up to quasi-isomorphism, the derived functor of the fiber product in the category of vector spaces evaluated on the diagram

$$T_x X \xrightarrow{d_x f} T_z Z \xleftarrow{d_y g} T_y Y.$$

This derived functor is represented by the 2–term complex

$$T_x X \oplus T_y Y \xrightarrow{d_x f - d_y g} T_z Z.$$

In particular, when f,g are closed embeddings, the derived fiber product is the derived intersection $X \cap_Z^R Y$ which is, therefore, a $[0,1]$–manifold.

2.1.4 Remark An affine $[0,n]$–manifold is the spectrum of a perfect resolving algebra in the sense of Behrend [2].

2.2 Boundedness and [0, 1]–manifolds

2.2.1 Definition A dg–manifold X is called bounded, if $\underline{H}^i(\mathcal{O}_X^\bullet) = 0$ for $i \ll 0$.

2.2.2 Theorem *Any* $[0,1]$*–manifold is bounded.*

Proof Let $\mu = \max_{x \in \pi_0(X)} \dim H^1(T_x^\bullet X)$. We will prove that $\underline{H}^i(\mathcal{O}_X^\bullet) = 0$ for $i < -\mu$. Since taking cohomology sheaves commutes with completion, it is enough to prove that $\forall x \in \pi_0(X)(\mathbb{C})$ the complete local dg–ring

$$\widehat{\mathcal{O}}^\bullet_{X,x} = \mathcal{O}_X^\bullet \otimes_{\mathcal{O}_X^0} \widehat{\mathcal{O}}_{X^0,x}$$

is exact in degrees $< -\mu$.

2.2.3 Proposition *There is a spectral sequence*

$$E_2 = S^\bullet\big(H^\bullet(T_x^* X)\big) \quad\Rightarrow\quad H^\bullet(\widehat{\mathcal{O}}^\bullet_{X,x}).$$

Proof Let $M \subset \widehat{\mathcal{O}}^\bullet_{X,x}$ be the maximal dg–ideal corresponding to x, ie, $M = \mathbf{m} + \widehat{\mathcal{O}}^{<0}_{X,x}$ where $\mathbf{m} \subset \widehat{\mathcal{O}}_{X^0,x}$ is the usual maximal ideal in the completed local ring. Then

$$M^n/M^{n+1} \simeq S^n(T_x^* X)$$

as dg–vector spaces, so

$$H^\bullet(M^n/M^{n+1}) = S^n\big(H^\bullet(T_x^* X)\big).$$

Our spectral sequence is therefore associated to the filtration $\{M^n\}$. □

To finish the proof of Theorem 2.2.2, note that $S^\bullet(H^\bullet(T_x^* X))$ is isomorphic to the tensor product of the symmetric algebra of $H^0(T_x^\bullet X)^*$ (situated in degree 0) and the exterior algebra of $H^1(T_x^\bullet X)^*$ with the grading being the negative of the usual grading by the degree of exterior powers. So it clearly vanishes in degrees $< -\mu$. □

2.2.4 Remark The converse to Theorem 2.2.2 is not true. For example, if E is a vector bundle on a manifold X^0, then $\big(X^0, \Lambda^\bullet(E)\big)$ with $\deg(E) = -3$ is bounded but is not a [0, 1]–manifold.

3 The virtual fundamental class of a [0, 1]–manifold

3.1 Reminder on Grothendieck and Chow groups

For any quasiprojective scheme Y we denote by $K_\circ(Y)$ the Grothendieck group of coherent sheaves on Y. For such a sheaf $\mathcal{F}$ we denote by $[\mathcal{F}]$ its class in $K_\circ(Y)$. We also denote by $K^\circ(Y)$ the Grothendieck ring of vector bundles. As well known, $K_\circ(Y)$ is a module over $K^\circ(Y)$. We denote by $A_r(Y)$ the Chow group of r–dimensional cycles on Y. Let $F_r K_\circ(Y)$ be the subgroup generated by $[\mathcal{F}]$ with $\dim \operatorname{supp} \mathcal{F} \le r$. Let

$$\mathrm{cl}_r\colon F_r K_\circ(Y) \to A_r(Y) \otimes \mathbf{Q}, \quad [\mathcal{F}] \mapsto \sum_{Z \subset \operatorname{supp}(\mathcal{F}) \,:\, \dim(Z) = r} \operatorname{mult}_Z(\mathcal{F}) \cdot Z$$

be the class map. See Fulton [7, Example 18.3.11].

Let $i\colon Z \to Y$ be a regular embedding of codimension d such that $\mathcal{O}_Z$ has a finite locally free resolution by $\mathcal{O}_Y$–modules. We denote by

$$i_A^*\colon A_r(Y) \to A_{r-d}(Z), \quad i_K^*\colon K_\circ(Y) \to K_\circ(Z)$$

the Gysin maps on the Chow and Grothendieck groups. Recall that

$$i_K^*([\mathcal{F}]) = \sum_i (-1)^i [\underline{\mathrm{Tor}}_i^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Z)]. \tag{3.1.1}$$

Recall also the following [7, Example 18.3.15].

3.1.2 Proposition *We have*

$$i_K^*(F_r K_\circ(Y)) \subset F_{r-d} K_\circ(Z)$$

and $\mathrm{cl}_{r-d} i_K^* = i_A^* \mathrm{cl}_r$ *.*

3.2 The virtual classes

3.2.1 Definition Let X be a bounded dg–manifold. Its K–theoretic virtual fundamental class is defined to be $[X]_K^{\mathrm{vir}} = [\underline{H}^\bullet(\mathcal{O}_X^\bullet)] \in K_\circ\big(\pi_0(X)\big)$.

From now on we assume that X is a $[0,1]$–manifold and use the notation of Section 1.

3.2.2 Proposition *The sheaf* K *(defined in Section 1.2) is locally free.*

Proof This is a consequence of the following lemma.

3.2.3 Lemma *Let* A *be a Noetherian local ring with residue field* k *and*

$$Q^1 \xrightarrow{d_1} Q^2 \xrightarrow{d_2} Q^3$$

an exact sequence of finitely generated free A*–modules, which also remains exact after tensoring with* k*. Then* $M = \mathrm{Ker}(d_1)$ *is free.*

Proof A finitely generated A–module M is free $\Longleftrightarrow \mathrm{Tor}_1^A(M,k) = 0$.

In our case, the resolution $M \sim \{Q^1 \xrightarrow{d_1} Q^2 \xrightarrow{d_2} \mathrm{Im}\ d_2\}$ and the fact that $\mathrm{Ker}(d_2 \otimes k) = \mathrm{Im}\ (d_1 \otimes k)$ implies that $\mathrm{Tor}_1(M,k) = \mathrm{Tor}_{-1}(\mathrm{Im}\ d_2, k) = 0$. □

Since X is a $[0,1]$–manifold, the truncation

$$\tau_{\leq 1} \mathbf{t}^\bullet = \{\mathbf{t}^0 \to K\}$$

is quasi-isomorphic to $\mathbf{t}^\bullet$.

3.2.4 Proposition *The dual complex* $\{K^* \to \omega_X^0\}$ *is a perfect obstruction theory on* $\pi_0(X)$ *in the sense of [3].*

Proof The embedding of dg–schemes $\pi_0(X) \hookrightarrow X$ induces the morphism of tangent complexes

$$RT^\bullet\big(\pi_0(X)\big) \to T^\bullet X \otimes_{\mathcal{O}_X^\bullet} \mathcal{O}_{\pi_0(X)} = \mathbf{t}^\bullet.$$

Dualizing and passing to truncations, we get a morphism of 2–term complexes

$$\{K^* \to \omega_X^0\} = \tau_{\geq -1}\omega_X^\bullet \to \tau_{\geq -1} L\Omega^{1\bullet}(\pi_0(X)) \cong \{J/J^2 \to \Omega^1_{X^0}|_{\pi_0(X)} = \omega^0\}$$

which is clearly an isomorphism on $\underline{H}^0$.

Explicitly, this morphism is identical on the 0–th terms and on the (-1)–st terms is induced by the surjective map

$$d\colon \mathcal{O}^{-1}/d\mathcal{O}^{-2} \longrightarrow J = d\mathcal{O}^{-1}$$

after restricting to π_0. So we have a morphism of 2–term complexes which is an isomorphism in degree 0 and a surjection in degree (-1), inducing an isomorphism on $\underline{H}^0$ and surjection on $\underline{H}^{-1}$. This is precisely the definition of a perfect obstruction theory. □

Following [3], we give:

3.2.5 Definition Let $i\colon \pi_0(X) \to \underline{K}$ be the embedding of the zero section. The homological virtual fundamental class of X is the element

$$[X]^{\mathrm{vir}} = i_A^*[N] \in A_r(\pi_0(X)).$$

Here $r = \mathrm{vdim}(X) = \mathrm{rk}(\mathbf{t}^0) - \mathrm{rk}(K)$ is the virtual dimension of X.

3.3 Main theorem

3.3.1 Theorem *The K–theoretic fundamental class $[X]_K^{\mathrm{vir}}$ lies in $F_r K\big(\pi_0(X)\big)$ and*

$$\mathrm{cl}_r\big([X]_K^{\mathrm{vir}}\big) = [X]^{\mathrm{vir}}.$$

Proof Associated to the perfect obstruction theory in Proposition 3.2.4, we have the K–theory class (cf [3, Remark 5.4])

$$\sum (-1)^i [\underline{\mathrm{Tor}}_i^{\mathcal{O}_{\underline{K}}}(\mathcal{O}_N, \mathcal{O}_{\pi_0(X)})] \in K_\circ(\pi_0(X)),$$

the sum being finite since $\mathcal{O}_{\pi_0(X)}$ has a finite locally free resolution over $\mathcal{O}_{\underline{K}}$, namely the Koszul complex. By (3.1.1), this class (which was studied first in some detail by Lee in [10]) is simply the K–theoretic Gysin pullback of $\mathcal{O}_N$ to $\pi_0(X)$ via the embedding i, and so it follows from 3.1.2 that it satisfies the conclusions of our theorem. Therefore it is enough to show the following:

3.3.2 Proposition

$$[X]^{\mathrm{vir}}_K = \sum (-1)^i [\underline{\mathrm{Tor}}_i^{\mathcal{O}_{\underline{K}}}(\mathcal{O}_N, \mathcal{O}_{\pi_0(X)})].$$

Denoting $q\colon \underline{K} \to \pi_0(X)$ the projection, we can write the Koszul resolution as $\Lambda^\bullet(q^*K^*) \sim \mathcal{O}_{\pi_0(X)}$, with the differential induced by the tautological section $\xi \in \Gamma(\underline{K}, q^*K)$. The embedding $K \subset \mathbf{t}^1$ defines a quasi-isomorphism

$$\varphi\colon K \to \mathbf{t}^{\geq 1}[1] = \{\mathbf{t}^1 \to \mathbf{t}^2 \to \cdots\}, \quad \deg(\mathbf{t}^i) = i-1.$$

In particular, we have a section $q^*(\varphi)(\xi)$ of the dg–bundle $q^*(\mathbf{t}^{\geq 1}[1])$ on $\underline{K}$ and the induced Koszul complex $q^*\big(S(\omega_X^{\leq -1})\big)$ is a resolution of $\mathcal{O}_{\pi_0(X)}$ on $\underline{K}$.

The direct image map $i_*\colon K_\circ\big(\pi_0(X)\big) \to K_\circ(\underline{K})$ preserves the dimension filtration. By the above discussion, if i_* were injective, 3.3.2, and hence our theorem, would follow from the equality

(3.3.3) $$i_*[\underline{H}^\bullet(\mathcal{O}_X^\bullet)] = [\underline{H}^\bullet(q^*S(\omega^{\leq -1}) \otimes_{\mathcal{O}_{\underline{K}}} \mathcal{O}_N)]$$

in $K_\circ(\underline{K})$. While, in general, i_* is not injective, it becomes so after passing to *equivariant* K–theory. Specifically, consider the $\mathbb{G}_m$–action on $\underline{K}$ given by dilations on the fibers. By a slight abuse of notation, let us denote by

$$i_*\colon K_\circ^{\mathbb{G}_m}\big(\pi_0(X)\big) \to K_\circ^{\mathbb{G}_m}(\underline{K})$$

the direct image map in $\mathbb{G}_m$–equivariant K–theory. Recall that

$$K_\circ^{\mathbb{G}_m}(\mathrm{pt}) = \mathbb{C}[\mu, \mu^{-1}].$$

The section i embeds $\pi_0(X)$ into $\underline{K}$ as the fixed point locus of the action. Therefore, it follows from the localization theorem [15, Theorem 2.1], that i_* becomes an isomorphism after tensoring with the quotient field $\mathbb{C}(\mu)$ of $\mathbb{C}[\mu, \mu^{-1}]$. Since

$$K_\circ^{\mathbb{G}_m}\big(\pi_0(X)\big) \cong K_\circ\big(\pi_0(X)\big) \otimes \mathbb{C}[\mu, \mu^{-1}]$$

has no $\mathbb{C}[\mu, \mu^{-1}]$–torsion, we conclude that (the equivariant version of) i_* is injective. Further, if we consider $K, \mathbf{t}^1, \mathbf{t}^2, \dots$ as equivariant bundles on $\pi_0(X)$ (with $\mathbb{G}_m$ acting by dilations in the fibers) and use the equivariant flat pullback q^*, then the Koszul complexes $\Lambda^\bullet(q^*K^*)$ and $q^*\big(S(\omega_X^{\leq -1})\big)$ are equivariant resolutions of $\mathcal{O}_{\pi_0(X)}$. Finally, we have the $\mathbb{G}_m$–equivariant Gysin map i^* (defined by the same formula with Tor's) which satisfies

$$i_*(i^*(\mathcal{F})) = \Lambda^\bullet(q^*K^*) \otimes_{\mathcal{O}_{\underline{K}}} \mathcal{F}, \quad \mathcal{F} \in K_\circ^{\mathbb{G}_m}(\underline{K}).$$

We conclude that it is indeed sufficient to prove the equality (3.3.3), but in an upgraded form, in which all maps and sheaves are considered in $\mathbb{G}_m$–equivariant K–theory. So in the rest of the proof we will deal with equivariant theory.

Let us factor i into the composition of two embeddings

$$\pi_0(X) \overset{i_3}{\hookrightarrow} N \overset{i_2}{\hookrightarrow} \underline{K}. \tag{3.3.4}$$

The inclusions i_2, i_3, as well as the projection $p\colon N \to \pi_0(X)$, are $\mathbb{G}_m$–equivariant and we use the corresponding equivariant pushforward or pullback maps.

Note that the right-hand side of (3.3.3) is equal to

$$i_{2*}[\underline{H}^\bullet(p^* S(\omega^{\leq 1}))]. \tag{3.3.5}$$

Next, recall (Proposition 1.3.2) that the sheaf of dg–algebras $p^*\mathbb{O}^\bullet$ on N is the localization on $N = \operatorname{Spec} \operatorname{gr}_J \mathcal{O}_{X^0}$ of the sheaf of graded dg–algebras $\operatorname{gr}_J \mathcal{O}^\bullet_X$. Proposition 1.3.3(2) implies that

$$[\underline{H}^\bullet(p^*\mathbb{O}, \delta)] = [\underline{H}^\bullet\big(p^* S(\omega^{\leq -1})\big)] \in K_\circ(N) \tag{3.3.6}$$

by virtue of the spectral sequence of the filtered complex $\big((p^*\mathbb{O}, \delta), p^*\mathcal{D}\big)$.

3.3.7 Lemma *We have a convergent spectral sequence of $\mathbb{G}_m$–equivariant coherent sheaves on N:*

$$E_1 = \underline{H}^\bullet(p^*\mathbb{O}, \delta) \quad \Rightarrow \quad i_{3*}\underline{H}^\bullet(\mathcal{O}^\bullet_X).$$

Proof The spectral sequence, together with the identification of the E_1 and E_∞–terms, is obtained from Proposition 1.3.2 by localizing the spectral sequence of the sheaf of filtered dg–algebras $(\mathcal{O}^\bullet_X, J)$ over $N = \operatorname{Spec} \operatorname{gr}_J \mathcal{O}_{X^0}$.

Denote the r–th term of this spectral sequence by E_r, so it is a $\mathbb{G}_m$–equivariant coherent sheaf on N equipped with an endomorphism d_r of square zero such that

$$H(E_r, d_r) = \operatorname{Ker}(d_r)/\operatorname{Im}(d_r) = E_{r+1}.$$

Let us show that the spectral sequence converges, ie, for $r \gg 0$ we have $d_r = 0$ and $E_r = E_{r+1}$. Indeed, define subsheaves $Z_r, B_r \subset E_1$ as in [4, page 317], for example $Z_1 = \operatorname{Ker}(d_1)$, while Z_2 is the preimage of $\operatorname{Ker}(d_2)$ etc. Then we have inclusions

$$B_1 \subset B_2 \subset \cdots, \quad Z_1 \supset Z_2 \supset \cdots,$$

and isomorphisms

$$E_r = Z_r/B_r, \quad Z_r/Z_{r+1} = B_{r+1}/B_r$$

(see Cartan and Eilenberg [4, page 317]). Since E_1 is a Noetherian sheaf, the sequence (B_r) stabilizes. The last isomorphism above then implies that the (E_r) stabilize as well which implies convergence. The lemma is proved. □

Now the spectral sequence of the lemma implies the equality

(3.3.8) $$[\underline{H}^\bullet(p^*\mathbb{O}, \delta)] = i_{3*}[\underline{H}^\bullet(\mathcal{O}_X^\bullet)] \in K_\circ^{\mathbb{G}_m}(N).$$

From equations (3.3.5), (3.3.6) and (3.3.8), we get (3.3.3). □

4 The virtual class via the Chern character

4.1 Reminder on local Chern character and Riemann–Roch

Let $Z \to Y$ be a closed embedding of schemes of finite type over k. We denote by $A^m(Z \to Y)$, $m \in \mathbb{Z}$, the m–th operational Chow group [7]. Its elements act by homomorphisms $A_p(Y) \to A_{p-m}(Z)$, and $A_p(Z) = A^{-p}(Z \to \mathrm{pt})$. When Y is smooth, $A^m(Z \to Y) = A_{\dim Z - m}(Z)$.

If $F^\bullet$ is a finite complex of vector bundles on Y exact outside of Z, one has the localized Chern character

$$\mathrm{ch}_Z^Y(F^\bullet) \in A^\bullet(Z \to Y) \otimes \mathbb{Q}.$$

We denote by

$$\tau_Z\colon K_\circ(Z) \to A_\bullet(Z)$$

the Riemann–Roch map of Baum–Fulton–McPherson [7]. If Y is a smooth quasiprojective variety containing Z as a closed subscheme, and $\mathcal{F}$ is a coherent sheaf on Z, then

(4.1.1) $$\tau_Z[\mathcal{F}] = \mathrm{ch}_Z^Y(F^\bullet) \cdot \mathrm{Td}(T_Y)$$

where $F^\bullet$ is a locally free resolution of $\mathcal{F}$ on Y.

For any proper morphism $f\colon Z \to W$ of quasiprojective schemes we denote

$$f_*^A\colon A_\bullet(Z) \to A_\bullet(W)$$

the direct image map on the Chow groups. The Riemann–Roch theorem in the form of Baum–Fulton–McPherson (see Fulton [7, Theorem 18.2]) says that

(4.1.2) $$\tau_W\big(f_*(z)\big) = f_*^A\big(\tau_Z(z)\big), \quad z \in K_\circ(Z).$$

Let $Z \to Y$ be a closed embedding of quasiprojective schemes and $F^\bullet$ a finite complex of vector bundles on Y, exact outside Z. Then for any coherent sheaf $\mathcal{G}$ on Y we have the Riemann–Roch formula [7, Example 18.3.12]:

(4.1.3) $$\tau_Z[\underline{H}^\bullet(F^\bullet \otimes \mathcal{G}^\bullet)] = \mathrm{ch}_Z^Y(F^\bullet) \cdot \tau_Y(\mathcal{G}).$$

Let now Z be proper. Combining (4.1.3) for $\mathcal{G} = \mathcal{O}_Y$ with the formula (4.1.2) for $W = \mathrm{pt}$, we get the following form of the Riemann–Roch theorem:

(4.1.4) $$\chi\big(Y, \underline{H}^\bullet(F^\bullet)\big) = \int_Z \mathrm{ch}_Z^Y(F^\bullet) \cdot \mathrm{Td}(T_Y).$$

4.2 Kontsevich's definition of the homological virtual class

Let X be a $[0,1]$–manifold. In [9], M Kontsevich proposed to consider the element

(4.2.1) $$\kappa_X = \tau_{\pi_0(X)}[\underline{H}^\bullet(\mathcal{O}_X^\bullet)] \cdot \mathrm{Td}^{-1}(\mathbf{t}_X^\bullet) \in A_\bullet(\pi_0(X)) \otimes \mathbb{Q}$$

as the virtual fundamental class of X. Since we use the embedding $\pi_0(X) \subset X^0$ for the definition of $\tau_{\pi_0(X)}$, we have, applying (4.1.3) to $F^\bullet = \mathcal{O}_X^\bullet$, $\mathcal{G} = \mathcal{O}_{X^0}$, that

(4.2.2) $$\kappa_X = \mathrm{ch}_{\pi_0(X)}^{X^0}(\mathcal{O}_X^\bullet) \cdot \mathrm{Td}(\mathbf{t}_X^{\geq 1}[1].)$$

4.2.3 Theorem $\kappa_X = [X]^{\mathrm{vir}}$ *(equality in* $A_\bullet(\pi_0(X)) \otimes \mathbb{Q}$*). In particular,* κ_X *is homogeneous of degree* $\mathrm{vdim}(X)$*.*

4.3 Proof of Theorem 4.2.3

We use the notation introduced in the proof of Theorem 3.3.1, in particular, the embeddings i_2, i_3, their composition i and the projections p, q. We need to show that $\kappa_X = i_A^*([N])$.

Using the quasi-isomorphism $\mathbf{t}_X^\bullet \sim \{\mathbf{t}_X^0 \to K\}$, we have

$$\kappa_X = \tau_{\pi_0(X)}\big([\underline{H}^\bullet(\mathcal{O}_X^\bullet)]\big) \cdot \mathrm{Td}^{-1}(\mathbf{t}_X^0) \cdot \mathrm{Td}(K).$$

We have shown in Theorem 3.3.1 that $[\underline{H}^\bullet(\mathcal{O}_X^\bullet)] = i^*(\mathcal{O}_N)$. On the other hand, since i is a regular embedding with normal bundle K, we have by [7, Theorem 18.2(3)]

$$\tau_{\pi_0(X)}\big(i^*(\mathcal{O}_N)\big) = \mathrm{Td}^{-1}(K) \cdot i_A^*\big(\tau_{\underline{K}}(\mathcal{O}_N)\big),$$

hence

$$\kappa_X = i_A^*\big(\tau_{\underline{K}}(\mathcal{O}_N)\big) \cdot \mathrm{Td}^{-1}(\mathbf{t}_X^0).$$

Now use the Riemann–Roch formula (4.1.2) for the embedding i_2 to get

(4.3.1) $$\kappa_X = i_A^*\big(i_{2*}^A \tau_N(\mathcal{O}_N)\big)\cdot \mathrm{Td}^{-1}(\mathbf{t}_X^0).$$

Our proof is then a consequence of the following:

4.3.2 Lemma *Let $i\colon Z \subset Y$ be a closed embedding of quasiprojective schemes with Y smooth and $p\colon N \to Z$ be the projection of the normal cone $N = N_{Z/Y}$. Then*

$$\tau_N(\mathcal{O}_N) = p^*\mathrm{Td}(T_Y\big|_Z)\cdot[N].$$

Specifically, we apply the lemma to $Z=\pi_0(X)$, $Y = X^0$, so that $\mathbf{t}_X^0 = T_Y\big|_Z$, getting from (4.3.1) that

$$\kappa_X = i_A^*\big(i_{2*}^A(p^*\mathrm{Td}(\mathbf{t}_X^0)\cdot[N])\big)\cdot\mathrm{Td}^{-1}(\mathbf{t}_X^0).$$

Since $p^* = i_2^* q^*$ (with i_2^* the pullback on operational Chow rings), the projection formula gives

$$\kappa_X = i_A^*\big(q^*\mathrm{Td}(\mathbf{t}_X^0)\cdot i_{2*}^A([N])\big)\cdot\mathrm{Td}^{-1}(\mathbf{t}_X^0).$$

But the right-hand side of the last equality is precisely $i_A^*([N])$, as $q\circ i = \mathrm{id}_{\pi_0(X)}$.

Proof of Lemma 4.3.2 Let $J \subset \mathcal{O}_Y$ be the ideal of Z and

$$\widetilde{Y} = \mathrm{Spec} \bigoplus_{n=0}^{\infty} J^n\cdot t^n \xrightarrow{\varrho} \mathbb{A}^1 = \mathrm{Spec}\,\mathbb{C}[t]$$

be the deformation to the normal cone. The morphism ϱ is flat, with $\varrho^{-1}(0) = N$ and $\varrho^{-1}(t) \simeq Y$, $t\neq 0$. Let $\varepsilon_t\colon \varrho^{-1}(t)\hookrightarrow\widetilde{Y}$ be the embedding and

$$\varepsilon_t^!\colon A_\bullet(\widetilde{Y}) \to A_{\bullet-1}(\varrho^{-1}(t))$$

be the specialization map of [7, 10.1]. By [7, Example 18.3.8]

$$\tau_{\varrho^{-1}(t)}(\mathcal{O}_{\varrho^{-1}(t)}) = \varepsilon_t^!\tau_{\widetilde{Y}}(\mathcal{O}_{\widetilde{Y}}),\ t\in\mathbb{A}^1.$$

Moreover, $\tau_{\widetilde{Y}}(\mathcal{O}_{\widetilde{Y}})$ is uniquely defined by its specializations for $t\neq 0$ [7, 11.1]. In other words, if $y\in A_\bullet(\widetilde{Y})$ is such that $\varepsilon_t^!(y) = \tau_Y(\mathcal{O}_Y)$, $t\neq 0$, then necessarily $y = \tau_{\widetilde{Y}}(\mathcal{O}_{\widetilde{Y}})$ and hence $\tau_N(\mathcal{O}_N) = \varepsilon_0^!(y)$.

We have a projection $\sigma\colon \widetilde{Y}\to Y$ induced by the embedding $\mathcal{O}_Y = J^0\cdot t^0 \subset \bigoplus J^n\cdot t^n$. The map σ is the identity on each $\varrho^{-1}(t) = Y$, $t\neq 0$ and is equal to ip on $\varrho^{-1}(0) = N$. Let now $y = \big(\sigma^*\mathrm{Td}(T_Y)\big)[\widetilde{Y}]\in A_\bullet(\widetilde{Y})$. Here we view $\mathrm{Td}(T_Y)$ as an element of $A^\bullet(Y)$, so $\sigma^*\mathrm{Td}(T_Y) \in A^\bullet(\widetilde{Y}) = A^\bullet(\widetilde{Y} \to \widetilde{Y})$ and y is the value of $\sigma^*\mathrm{Td}(T_Y)$ on $[\widetilde{Y}] \in A_\bullet(\widetilde{Y})$. Then, clearly, y satisfies the above condition on $\varepsilon_t^!(y)$, $t \neq 0$, so

$$\tau_N(\mathcal{O}_N) = \varepsilon_0^!(y) = \varepsilon_0^!\big(\sigma^*\mathrm{Td}(T_Y)\big)[\widetilde{Y}] = p^* i^* \mathrm{Td}(T_Y)[N]$$

as claimed. □

4.4 A Riemann–Roch theorem for dg–manifolds

Let X be a $[0,1]$–manifold. A dg–vector bundle $E^\bullet$ on X will be called finitely generated, if the complex $\bar{E}^\bullet$ of vector bundles on $\pi_0(X)$ (see Section 1.1) is finite. In this case $\underline{H}^j(E^\bullet) = 0$ except for finitely many j and so we have the class $[\underline{H}^\bullet(E^\bullet)] \in K_\circ(\pi_0(X))$.

4.4.1 Theorem

$$\tau_{\pi_0(X)}[\underline{H}^\bullet(E^\bullet)] = \mathrm{ch}(\bar{E}^\bullet) \cdot \mathrm{Td}(\mathbf{t}_X^\bullet) \cdot [X]^{\mathrm{vir}}.$$

Here the first two factors on the right are considered as endomorphisms of $A_\bullet\big(\pi_0(X)\big) \otimes \mathbb{Q}$ and applied successively to $[X]^{\mathrm{vir}}$.

This is a consequence of (4.1.3), of Theorem 4.2.3, and the following fact.

4.4.2 Theorem *We have the equality in $K_\circ(\pi_0(X))$:*

$$[\underline{H}^\bullet(E^\bullet)] = [\bar{E}^\bullet] \cdot [\underline{H}^\bullet(\mathcal{O}_X^\bullet)]$$

(product of an element of K° with an element of $K_\circ$).

Proof We use the equivariant set-up and the notation from the proof of Theorem 3.3.1. Since the $\mathbb{G}_m$–equivariant pushforward $i_* = i_{2*} i_{3*}$ is injective, it is enough to show that

(4.4.3) $$i_{3*}[\underline{H}^\bullet(E^\bullet)] = i_{3*}\big([\bar{E}^\bullet] \cdot [\underline{H}^\bullet(\mathcal{O}_X^\bullet)]\big).$$

This would follow if we proved the following equality in $K_\circ^{\mathbb{G}_m}(N)$:

(4.4.4) $$i_{3*}\big[\underline{H}^\bullet(E^\bullet)] = [p^*\bar{E}^\bullet \otimes \Lambda^\bullet(p^*K^*)\big].$$

The proof of (4.4.4) proceeds similarly to the case $E^\bullet = \mathcal{O}_X^\bullet$; see (3.3.5)–(3.3.6). To be precise, $\Lambda^\bullet(p^*K^*)$ has the Koszul differential, so the RHS of (4.4.4) is equal to

$$\big[\underline{H}^\bullet\big(p^*\bar{E}^\bullet \otimes \Lambda^\bullet(p^*K^*)\big)\big]$$

which, in view of the quasi-isomorphism $K \to \mathbf{t}^{\geq 1}[1]$ gives

(4.4.5) $$[p^*\bar{E}^\bullet \otimes \Lambda^\bullet(p^*K^*)\big] = \big[\underline{H}^\bullet(p^*(\bar{E}^\bullet \otimes S(\omega^{\leq -1})))\big].$$

By Proposition 1.3.2(2),

$$p^*(E^\bullet|_{\pi_0(X)}) \simeq \widetilde{\mathrm{gr}}_J E^\bullet. \tag{4.4.6}$$

On the other hand, by Proposition 1.3.3,

$$\mathrm{gr}_{p^*\mathcal{D}} p^*(E^\bullet|_{\pi_0(X)}) \simeq p^*\bar{E}^\bullet \otimes p^* S(\omega^{\leq -1}). \tag{4.4.7}$$

The spectral sequence of the filtered complex $(p^*(E^\bullet|_{\pi_0(X)}), p^*\mathcal{D})$ (together with finite generation of $E^\bullet$) implies then that $[\underline{H}^\bullet p^*(E^\bullet|_{\pi_0(X)})]$ makes sense and

$$[\underline{H}^\bullet p^*(E^\bullet|_{\pi_0(X)})] = [\underline{H}^\bullet(p^*\bar{E}^\bullet \otimes p^* S(\omega^{\leq -1}))]. \tag{4.4.8}$$

Next, (4.4.7) and the spectral sequence of the filtered complex $(E^\bullet, J)$ implies

$$[\underline{H}^\bullet(p^* E^\bullet|_{\pi_0(X)})] = i_{3*}[\underline{H}^\bullet(E^\bullet)]. \tag{4.4.9}$$

Combining (4.4.5), (4.4.8) and (4.4.9) proves the equality (4.4.4) and therefore Theorems 4.4.2 and 4.4.1. □

4.4.10 Corollary *For two finitely generated dg–bundles $E^\bullet, F^\bullet$ on X we have the equality in $K_\circ(\pi_0(X))$:*

$$\left[\underline{H}^\bullet\left(E^\bullet \otimes_{\mathcal{O}_X^\bullet} F^\bullet\right)\right] = [\bar{E}^\bullet]\cdot[\underline{H}^\bullet(F^\bullet)].$$

4.5 Consequences for the Euler characteristic

Let us assume, in the situation of Section 4.4 that $\pi_0(X)$ is projective. Then the Euler characteristic

$$\chi\left(\pi_0(X),\ \underline{H}^\bullet(E^\bullet)\right) = \sum (-1)^i \chi\left(\pi_0(X),\ \underline{H}^i(E^\bullet)\right)$$

is defined. Theorem 4.4.1 allows us to establish a simple formula for this Euler characteristic.

Since $\pi_0(X)$ is projective, we have the degree map

$$\deg\colon A_0(\pi_0(X)) \to \mathbb{Z}.$$

For any $\varphi \in A^\bullet(\pi_0(X)) = A^\bullet\left(\pi_0(X) \to \pi_0(X)\right)$ we define

$$\int_{[X]^{\mathrm{vir}}} \varphi = \deg\left((\varphi\cdot[X]^{\mathrm{vir}})_0\right).$$

Here the subscript 0 means the degree 0 component of $\varphi\cdot[X]^{\mathrm{vir}} \in A_\bullet(\pi_0(X))$.

4.5.1 Theorem

$$\chi\big(\pi_0(X),\ \underline{H}^\bullet(E^\bullet)\big) = \int_{[X]^{\mathrm{vir}}} \mathrm{ch}(\bar{E}^\bullet)\cdot \mathrm{Td}(\mathbf{t}_X^\bullet).$$

Proof This is a direct consequence of Theorem 4.4.1 and the fact that τ commutes with direct image (for the map $\pi_0(X) \to \mathrm{pt}$). □

4.6 Chern numbers and the cobordism class of a [0, 1]–manifold

Let X be a $[0,1]$–manifold of virtual dimension d. Let $P(d)$ be the set of partitions of d into ordered summands, ie, of sequences $I = (i_1, \dots, i_p)$ with $i_\nu \in \mathbb{Z}_+$ and $\sum i_\nu = d$. For each $I \in P(d)$ we define the I–th Chern number of X to be

(4.6.1) $$c_I(X) = \int_{[X]^{\mathrm{vir}}} c_{i_1}(\mathbf{t}_X^\bullet)\cdots c_{i_p}(\mathbf{t}_X^\bullet) \quad \in \quad \mathbb{Z}.$$

Let ΩU^d be the cobordism group of compact almost complex manifolds of real dimension $2d$; see Ravenel [13]. For each such manifold M the tangent bundle T_M is a complex vector bundle so it has Chern classes $c_i(T_M) \in H^{2i}(M,\mathbb{Z})$, and for each $I \in P(d)$ we have the Chern number

(4.6.2) $$c_I(M) = \int_{[M]} c_{i_1}(T_M)\cdots c_{i_p}(T_M) \quad \in \quad \mathbb{Z}.$$

Here $[M]$ is the usual fundamental class of M. The following is well known; see Ravenel [13]:

4.6.3 Proposition (a) *The Chern numbers are cobordism invariant.*

(b) *If two almost complex manifolds have the same Chern numbers, then they are cobordant.*

Our next result shows that a $[0,1]$–manifold over $\mathbb{C}$ can be seen as a "virtual" smooth complex manifold. This agrees with the intuition that working with dg–manifolds is a replacement of deforming to transverse intersection, a technique that typically leads outside of algebraic geometry.

4.6.4 Theorem *Let $k = \mathbb{C}$ and X be a $[0,1]$–manifold over $\mathbb{C}$ of virtual dimension d. Then there exists a (unique, up to cobordism) almost complex manifold M of real dimension $2d$ such that $c_I(M) = c_I(X)$ for all $I \in P(d)$.*

Proof We first recall the concept of Schur functors [12]. Let $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots)$ be a weakly decreasing sequence of nonnegative integers terminating in zeroes. Let also Vect_k be the category of finite-dimensional k–vector spaces. Then we have the Schur functor $\Sigma^\alpha\colon \mathrm{Vect}_k \to \mathrm{Vect}_k$. If $V = k^d$, then $\Sigma^\alpha(k^d)$ is "the" space of the irreducible representation of the algebraic group GL_d/k with highest weight α. The functor Σ^α can be applied to vector bundles (and projective modules over any commutative k–algebra). In particular, if $k = \mathbb{C}$ and M is an almost complex manifold, then we have the complex vector bundle $\Sigma^\alpha(T_M)$ on M. In this case the number

$$\phi_\alpha(M) = \int_{[M]} \mathrm{ch}(\Sigma^\alpha(T_M)) \cdot \mathrm{Td}(T_M)$$

is expressible as a universal $\mathbb{Q}$–linear combination of the Chern numbers of M:

(4.6.5)
$$\phi_\alpha(M) = \sum_I q_\alpha^I c_I(M), \quad q_\alpha^I \in \mathbb{Q}. \qquad \square$$

The following is a reformulation of the Hattori–Stong theorem; see Ravenel [13] and Stong [14]:

4.6.6 Theorem *Let $(\lambda_I)_{I\in P(d)}$ be a system of integers labelled by $P(d)$. Then the following are equivalent:*

(1) *There exists an almost complex manifold M (unique up to cobordism) such that $c_I(M) = \lambda_I$ for all $I \in P(d)$.*

(2) *For any α as above the number $\sum_I q_\alpha^I \lambda_I$ is an integer.*

We now prove that the condition (ii) holds for $\lambda_I = c_I(X)$ where X is a [0, 1]–manifold of virtual dimension d. Indeed, the Schur functors apply equally well to dg–bundles on X. See, eg, Akin, Buchsbaum and Weyman [1] for Schur functors of complexes. If $E^\bullet$ is a finitely generated bundle, then so is $\Sigma^\alpha(E^\bullet)$. Further, Schur functors commute with restrictions of bundles, so in particular,

$$\overline{\Sigma^\alpha(E^\bullet)} = \Sigma^\alpha(\bar{E}^\bullet).$$

Now, applying Theorem 4.5.1, we see that

$$\sum_I q_\alpha^I c_I(X) = \int_{[X]^{\mathrm{vir}}} \mathrm{ch}(\Sigma^\alpha \mathbf{t}_X^\bullet) \cdot \mathrm{Td}(\mathbf{t}_X^\bullet) = \chi(X, \Sigma^\alpha T_X^\bullet) \in \mathbb{Z},$$

whence the statement.

4.6.7 Example Let X be a smooth projective variety of dimension n over $\mathbb{C}$, let E be a vector bundle of rank r on X, and let $s \in \Gamma(X, E)$ be a global section. We denote by Z the zero locus of s (assumed to be nonempty), and by i its inclusion in X. The Koszul complex $(\Lambda^\bullet E, d = d_s)$ is a sheaf of dg–algebras on X, hence we get a $[0,1]$–manifold $X_{E,s}$ of virtual dimension $n - r$, with $\pi_0(X_{E,s}) = Z$ and $\mathbf{t}^\bullet = [T_X\big|_Z \to E\big|_Z]$. The homological virtual class satisfies $i_*[X_{E,s}]^{\mathrm{vir}} = c_r(E)$. The cobordism class in this case can be realized explicitly as the zero locus M in X of any other section s' of E which is *regular*, and $X_{E,s}$ deforms algebraically to $M \sim_{\mathrm{qis}} X_{E,s'}$.

4.6.8 Remark In general, it may be impossible to find an algebraic deformation as in 4.6.7 above of a $[0,1]$–manifold to a smooth variety of the expected dimension. Our cobordism class rectifies this defect, at least as far as numerical invariants are concerned. For example, it follows from Theorem 4.6.6 that any "genus" of complex manifolds which is expressible by universal formulas in Chern numbers, can be extended to $[0,1]$–manifolds and will satisfy the same properties in this enlarged setting.

5 Localization

5.1 Background

Let $G = (\mathbb{G}_m)^n$ be an n–dimensional algebraic torus over k. For a G–scheme Z we denote by $K_G(Z)$ the Grothendieck group of G–equivariant coherent sheaves on Z and by $K_G^\circ(Z)$ the Grothendieck ring of G–vector bundles on Z. We denote by $\mathrm{Rep}(G) = K_G(\mathrm{pt})$ the representation ring of G (which is a Laurent polynomial ring) and by $\mathrm{FRep}(G)$ its field of fractions. The following is well-known; see Thomason [15, Lemma 3.2].

5.1.1 Lemma *If the G–action on Z is trivial, and Z is quasiprojective, then, for every G–bundle E satisfying $E^G = 0$, the element $[\Lambda^\bullet(E)]$ is invertible in the localization $K_G^\circ(Z) \otimes_{\mathrm{Rep(G)}} \mathrm{FRep}(G)$.*

Let Y be a smooth quasiprojective G–variety and $Z \subset Y$ an invariant closed subscheme. We will need a version of the Bott localization formula for Z.

Denote $\epsilon\colon Z^G \to Z$, $\widetilde{\epsilon}\colon Y^G \to Y$ the embeddings of the fixed point loci, so we have the Cartesian square of closed embeddings:

(5.1.2)
$$\begin{tikzcd} Z \arrow[r, "j"] & Y \\ Z^G \arrow[u, "\epsilon"] \arrow[r, "\widetilde{j}"] & Y^G \arrow[u, "\widetilde{\epsilon}"'] \end{tikzcd}$$

Note that $\widetilde{\epsilon}$ is a regular embedding (and Y^G is smooth). Let $\mathcal{N}$ be the normal bundle of Y^G in Y and $\mathcal{N}^*$ its dual bundle. Let

$$\textbf{(5.1.3)} \qquad \epsilon^! \colon K_G(Z) \to K_G(Z^G)$$

be the K–theoretic Gysin map defined by putting, for each coherent G–sheaf $\mathcal{F}$ on Z:

$$\textbf{(5.1.4)} \qquad \epsilon^!([\mathcal{F}]) = \sum_l (-1)^l \big[\widetilde{j}^* \underline{\mathrm{Tor}}_l^{\mathcal{O}_Y}(j_*\mathcal{F}, \mathcal{O}_{Y^G})\big].$$

Here the Tor–sheaves are supported on Z^G. This is a K–theoretic analog of the refined Gysin map of Fulton [7]. Like that map, $\epsilon^!$ depends not only on the morphism ϵ, but on the entire diagram (5.1.2).

5.1.5 Theorem *For any $\xi \in K_G(Z)$ we have the equality*

$$\xi = \epsilon_*\left(\frac{\epsilon^!(\xi)}{\big[\Lambda^\bullet(\mathcal{N}^*|_{Z^G})\big]}\right)$$

in the group $K_G(Z) \otimes_{\mathrm{Rep}(G)} \mathrm{FRep}(G)$.

Proof By the result of Thomason [15, Theorem 2.1],

$$\textbf{(5.1.6)} \qquad \epsilon_* \colon K_G(Z^G) \otimes_{\mathrm{Rep}(G)} \mathrm{FRep}(G) \to K_G(Z) \otimes_{\mathrm{Rep}(G)} \mathrm{FRep}(G)$$

is an isomorphism, so $\xi = \epsilon_*(\eta)$ for some η in the left-hand side of (5.1.6). On the other hand, for any coherent G–sheaf $\mathcal{L}$ on Z^G we have

$$\begin{aligned} \epsilon^!\epsilon_*[\mathcal{L}] = \big[\underline{\mathrm{Tor}}_\bullet^{\mathcal{O}_Y}(\widetilde{j}_*\mathcal{L}, \mathcal{O}_{Y^G})\big] &= \big[\widetilde{j}_*\mathcal{L} \otimes_{\mathcal{O}_{Y^G}} \underline{\mathrm{Tor}}_\bullet^{\mathcal{O}_Y}(\mathcal{O}_{Y^G}, \mathcal{O}_{Y^G})\big] \\ &= [\mathcal{L}] \cdot \big[\Lambda^\bullet(\mathcal{N}^*|_{Z^G})\big]. \end{aligned}$$

Therefore
$$\epsilon^!\xi = \eta \cdot \big[\Lambda^\bullet(\mathcal{N}^*|_{Z^G})\big].$$

This means that the fraction in the RHS of the equality claimed in Theorem 5.1.5, is equal to η, and the equality is true since $\xi = \epsilon_*(\eta)$. □

5.2 The setup

Let X be a $[0,1]$–manifold with G–action. Then we have the fixed point (dg–) submanifold $X^G \subset X$, with

$$(X^G)^0 = (X^0)^G, \quad \pi_0(X^G) = \pi_0(X)^G,$$
$$\mathcal{O}_{X^G}^\bullet = \big(\mathcal{O}_X^\bullet|_{(X^0)^G}\big)_G \quad \text{(the coinvariants).}$$

Let $i\colon X^G \hookrightarrow X$ be the embedding and $\nu^\bullet = i^* T_X^\bullet / T_{X^G}^\bullet$ be the dg–normal bundle of X^G. It has the induced G–action. As in Section 1.1 we denote by $\overline{\nu}^\bullet$ the restriction of $\nu^\bullet$ to $\pi_0(X)^G$ in the sense of dg–manifolds. Thus we have a split exact sequence of complexes of vector bundles

$$0 \to \mathbf{t}^\bullet_{(X)^G} \to \mathbf{t}^\bullet_X\big|_{\pi_0(X)^G} \to \overline{\nu}^\bullet \to 0, \quad \mathbf{t}^\bullet_{X^G} = \left(\mathbf{t}^\bullet_X\big|_{\pi_0(X)^G}\right)^G. \tag{5.2.1}$$

This shows the following:

5.2.2 Proposition *X^G is again a $[0,1]$–manifold, and $\overline{\nu}^\bullet$ is fiberwise exact outside of degrees 0, 1.*

Therefore

$$\overline{\nu}^{\bullet\prime} = \left\{\overline{\nu}^0 \to \mathrm{Ker}\{\overline{\nu}^1 \xrightarrow{d} \overline{\nu}^2\}\right\} \tag{5.2.3}$$

is a 2–term G–complex of bundles on $\pi_0(X)^G$ quasi-isomorphic to $\overline{\nu}^\bullet$. This is precisely the "moving part" of the obstruction theory $\mathbf{t}^\bullet_X$ in the sense of [8].

5.3 K–theoretic localization for $[0,1]$–manifolds

In the setup of Section 5.2 let $E^\bullet$ be a finitely generated G–equivariant dg–vector bundle on X. We denote by

$$i^* E^\bullet = (i^0)^{-1} E^\bullet \otimes_{(i^0)^{-1}\mathcal{O}^\bullet_X} \mathcal{O}^\bullet_{X^G}$$

the restriction of $E^\bullet$ to X^G in the sense of dg–manifolds. We have the class $[\underline{H}^\bullet(E^\bullet)] \in K_G(\pi_0(X))$. In particular, for $E^\bullet = \mathcal{O}^\bullet_X$ we get the G–equivariant version of the K–theoretic virtual class

$$[X]^{\mathrm{vir},G}_K = [\underline{H}^\bullet(\mathcal{O}^\bullet_X)] \in K_G(\pi_0(X)),$$

and, furthermore,

$$[\underline{H}^\bullet(i^*\mathcal{O}^\bullet_X)] = [X^G]^{\mathrm{vir}}_K.$$

5.3.1 Theorem *In $K_G(\pi_0(X)) \otimes_{\mathrm{Rep}(G)} \mathrm{FRep}(G)$ we have the equality*

$$[\underline{H}^\bullet(E^\bullet)] = \pi_0(i)_* \left(\frac{\left[\underline{H}^\bullet(i^*E^\bullet)\right]}{\left[\Lambda^\bullet(\overline{\nu}^{\bullet\prime *})\right]} \right),$$

where $[\Lambda^\bullet(\overline{\nu}^{\bullet\prime *})]$ is defined as $[\Lambda^\bullet(\overline{\nu}^{0\prime *})]/[\Lambda^\bullet(\overline{\nu}^{1\prime *})]$; see (5.2.3).

Proof We apply Theorem 5.1.5 to $Y = X^0$, $Z = \pi_0(X)$, so $\epsilon = \pi_0(i)$, $\widetilde{\epsilon} = i^0$, and we keep the notation j, $\widetilde{j}$ for the other two morphisms. We take $\xi = [\underline{H}^\bullet(E^\bullet)]$. Then $j_*\xi = [E^\bullet] \in K_G(X^0)$. Because $E^\bullet$ is, in particular, a complex of vector bundles on X^0, taking Tor's of $\underline{H}^\bullet(E^\bullet)$ with $\mathcal{O}_{X^{0G}}$, as in (5.1.4), gives the same element of K–theory as just tensoring $E^\bullet$ with $\mathcal{O}_{X^{0G}}$, ie, forming the restriction $E^\bullet\big|_{X^{0G}}$. In other words,

$$\pi_0(i)^! \, [\underline{H}^\bullet(E^\bullet)] = \left[\underline{H}^\bullet\left(E^\bullet\big|_{X^{0G}}\right)\right]. \tag{5.3.2}$$

Note further that $\mathcal{N}$, being the normal bundle of X^{0G} in X^0, is the same as ν^0, so $\mathcal{N}\big|_{\pi_0(X^G)} = \overline{\nu}^0$. So Theorem 5.1.5 gives

$$[\underline{H}^\bullet(E^\bullet)] = \pi_0(i)_*\left(\frac{\left[\underline{H}^\bullet\left(E^\bullet\big|_{X^{0G}}\right)\right]}{\left[\Lambda^\bullet\left(\overline{\nu}^{0*}\right)\right]}\right). \tag{5.3.3}$$

To prove Theorem 5.3.1 it then suffices to prove the following equality in $K_G(\pi_0(X^G))$:

$$[\Lambda^\bullet(\overline{\nu}^{1\prime *})] \cdot \left[\underline{H}^\bullet\left(E^\bullet\big|_{X^{0G}}\right)\right] = [\underline{H}^\bullet(i^*E^\bullet)]. \tag{5.3.4}$$

Let $I^\bullet \subset \mathcal{O}^\bullet_X$ be the dg–ideal of X^G, so $I^0 \subset \mathcal{O}_{X^0}$ is the ideal of X^{0G}. Then we have

$$E^\bullet\big|_{X^{0G}} = E^\bullet/I^0E^\bullet, \quad i^*E^\bullet = E^\bullet/I^\bullet E^\bullet = \left(E^\bullet\big|_{X^{0G}}\right)/I^{\le -1}\cdot\left(E^\bullet\big|_{X^{0G}}\right). \tag{5.3.5}$$

Further, the usual interpretation of the conormal bundle holds in the dg–situation as well: $I^\bullet/(I^\bullet)^2 = \nu^*$. Therefore $I^{\le -1}/(I^{\le -1})^2 \cdot I^0 = (\nu^*)^{\le -1}$, and we deduce for the $I^{\le -1}$–adic filtration:

$$\begin{aligned}(I^{\le -1})^d \cdot \left(E^\bullet\big|_{X^{0G}}\right)\Big/(I^{\le -1})^{d+1}\cdot\left(E^\bullet\big|_{X^{0G}}\right) \\ = i^*(E^\bullet) \otimes_{\mathcal{O}^\bullet_{X^G}} S^d((\nu^*)^{\le -1}).\end{aligned} \tag{5.3.6}$$

Notice that Corollary 4.4.10 is applicable to equivariant K-groups as well. Applying it to the dg–variety X^G, we get

$$\begin{aligned}\left[\underline{H}^\bullet\left(i^*E^\bullet \otimes_{\mathcal{O}^\bullet_{X^G}} S^d((\nu^*)^{\le -1})\right)\right] &= [S^d((\overline{\nu}^*)^{\le -1})] \cdot [\underline{H}^\bullet(i^*E^\bullet)] \\ &= [S^d(\overline{\nu}^{1\prime *})] \cdot [\underline{H}^\bullet(i^*E)],\end{aligned} \tag{5.3.7}$$

where the last equality follows from the quasi-isomorphism of $(\overline{\nu}^*)^{\le -1}$ with $\overline{\nu}^{1\prime *}$.

Now, at the formal level, if we replace $E^\bullet\big|_{X^{0G}}$ by the (infinite) sum of the quotients of the $I^{\le -1}$–adic filtration, given by (5.3.6), we get the sum of the classes of the symmetric powers of $\overline{\nu}^{1\prime *}$ which is (formally) inverse to the class of the exterior algebra in (5.3.4). This can be made rigorous by performing the deformation to the normal cone to X^G in X, ie, by considering the $I^\bullet$–adic filtration in $\mathcal{O}_X^\bullet$ and its associated graded sheaf of algebras $\mathrm{gr}_I\, \mathcal{O}_X^\bullet$. Its spectrum is $\mathcal{N}_{X^G/X}$, the (dg)-normal bundle to X^G in X considered as a dg–manifold. Let us denote it $\widehat{X}$. Note that its underlying scheme $\widehat{X}^0$ is $\mathcal{N}_{X^{0G}/X^0}$, the normal bundle to X^{0G} in X^0. At the same time $\widehat{X}^G = X^G$. Let $\widehat{i}\colon X^G \to \widehat{X}$ be the embedding. Taking the I–adic filtration in $E^\bullet$, we have that $\mathrm{gr}_I\, E^\bullet$ is a module over $\mathrm{gr}_I\, \mathcal{O}_X^\bullet$ and thus gives a dg–vector bundle $\widehat{E}^\bullet$ on $\widehat{X}$. As in (3.3.4)–(3.3.5), the argument with a spectral sequence of coherent sheaves on $\mathcal{N}_{X^{0G}/X^0}$, converging for Noetherian reasons, gives that

$$\textbf{(5.3.8)}\qquad \big[\underline{H}^\bullet(E^\bullet|_{X^{0G}})\big] = \big[\underline{H}^\bullet(\widehat{E}^\bullet|_{X^{0G}})\big], \quad [\underline{H}^\bullet(i^*E)] = [\underline{H}^\bullet(\widehat{i}^*\widehat{E})].$$

So we can and will assume in proving (5.3.4) that $X = \widehat{X}$ coincides with the normal bundle to the fixed point locus. In this case, the $I^\bullet$–adic filtration comes from a grading, so

$$E^\bullet|_{X^{0G}} = \bigoplus_{d=0}^{\infty} (i^*E) \otimes_{\mathcal{O}^\bullet_{X^G}} S^d((\nu^*)^{\le -1}),$$

and the left-hand side of (5.3.4) becomes, by Corollary 4.4.10,

$$\textbf{(5.3.9)}\qquad \Big[\underline{H}^\bullet\Big(\Lambda^\bullet((\nu^*)^{\le -1}) \otimes_{\mathcal{O}^\bullet_{X^G}} S^\bullet((\nu^*)^{\le -1}) \otimes_{\mathcal{O}^\bullet_{X^G}} i^*E\Big)\Big].$$

Let d be the differential in the triple tensor product of complexes in (5.3.9). We can add to d another summand δ, the Koszul differential on $\Lambda^\bullet \otimes S^\bullet$ tensored with the identity on the third factor, and we can arrange the tensor product into a double complex. The cohomology with respect to δ is then i^*E, so $\underline{H}^\bullet_d(\underline{H}^\bullet_\delta) = \underline{H}^\bullet(i^*E)$, and a spectral sequence argument shows that its class in $K_G(\pi_0(X^G))$ is the same as the class of $\underline{H}^\bullet_{d+\delta}$. On the other hand, the class of $\underline{H}^\bullet_{d+\delta}$ is equal to that of $\underline{H}^\bullet_d$, as we see from the other spectral sequence corresponding to the double complex. This proves the equality (5.3.4) and Theorem 5.3.1. □

References

[1] **K Akin**, **D A Buchsbaum**, **J Weyman**, *Schur functors and Schur complexes*, Adv. in Math. 44 (1982) 207–278 MR658729

[2] **K Behrend**, *Differential graded schemes I: Perfect resolving algebras* `arXiv:math.AG/0212225`

[3] **K Behrend**, **B Fantechi**, *The intrinsic normal cone*, Invent. Math. 128 (1997) 45–88 MR1437495

[4] **H Cartan**, **S Eilenberg**, *Homological algebra*, Princeton Univ, Press (1956) MR0077480

[5] **I Ciocan-Fontanine**, **M Kapranov**, *Derived Quot schemes*, Ann. Sci. École Norm. Sup. (4) 34 (2001) 403–440 MR1839580

[6] **I Ciocan-Fontanine**, **M M Kapranov**, *Derived Hilbert schemes*, J. Amer. Math. Soc. 15 (2002) 787–815 MR1915819

[7] **W Fulton**, *Intersection theory*, Ergebnisse series (3) 2, Springer, Berlin (1984) MR732620

[8] **T Graber**, **R Pandharipande**, *Localization of virtual classes*, Invent. Math. 135 (1999) 487–518 MR1666787

[9] **M Kontsevich**, *Enumeration of rational curves via torus actions*, from: "The moduli space of curves (Texel Island, 1994)", (R Dijkgraaf, C Faber, G van der Geer, editors), Progr. Math. 129, Birkhäuser, Boston (1995) 335–368 MR1363062

[10] **Y-P Lee**, *Quantum K–theory. I. Foundations*, Duke Math. J. 121 (2004) 389–424 MR2040281

[11] **J Li**, **G Tian**, *Virtual moduli cycles and Gromov–Witten invariants of algebraic varieties*, J. Amer. Math. Soc. 11 (1998) 119–174 MR1467172

[12] **I G Macdonald**, *Symmetric functions and Hall polynomials*, second edition, Oxford Math. Monogr., Oxford Sci. Publ., The Clarendon Press, Oxford Univ. Press, New York (1995) MR1354144 With contributions by A Zelevinsky

[13] **D C Ravenel**, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Math. 121, Academic Press, Orlando, FL (1986) MR860042

[14] **R E Stong**, *Relations among characteristic numbers. I*, Topology 4 (1965) 267–281 MR0192515

[15] **R W Thomason**, *Une formule de Lefschetz en K–théorie équivariante algébrique*, Duke Math. J. 68 (1992) 447–462 MR1194949

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