Orbifold Gromov–Witten theory of the symmetric product of $\mathcal{A}_r$

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Let $\mathcal{A}_r$ be the minimal resolution of the type A_r surface singularity. We study the equivariant orbifold Gromov–Witten theory of the n–fold symmetric product stack $[\mathrm{Sym}^n(\mathcal{A}_r)]$ of $\mathcal{A}_r$. We calculate the divisor operators, which turn out to determine the entire theory under a nondegeneracy hypothesis. This, together with the results of Maulik and Oblomkov, shows that the Crepant Resolution Conjecture for $\mathrm{Sym}^n(\mathcal{A}_r)$ is valid. More strikingly, we complete a tetrahedron of equivalences relating the Gromov–Witten theories of $[\mathrm{Sym}^n(\mathcal{A}_r)]/\mathrm{Hilb}^n(\mathcal{A}_r)$ and the relative Gromov–Witten/Donaldson–Thomas theories of $\mathcal{A}_r \times \mathbb{P}^1$.

14N35; 14H10

0 Introduction

0.1 Results

Let $\mathcal{A}_r$ be the minimal resolution of the type A_r surface singularity. The symmetric group $\mathfrak{S}_n$ on n letters acts on the n–fold Cartesian product $\mathcal{A}_r^n$ by permuting coordinates. Thus, we obtain a quotient scheme $\mathrm{Sym}^n(\mathcal{A}_r) := \mathcal{A}_r^n/\mathfrak{S}_n$, the n–fold symmetric product of $\mathcal{A}_r$, and a quotient stack $[\mathrm{Sym}^n(\mathcal{A}_r)]$, the n–fold symmetric product stack of $\mathcal{A}_r$. The stack $[\mathrm{Sym}^n(\mathcal{A}_r)]$ is a smooth orbifold, whose coarse moduli space is the symmetric product $\mathrm{Sym}^n(\mathcal{A}_r)$.

In this article, we compare the equivariant orbifold Gromov–Witten theory of the symmetric products of $\mathcal{A}_r$ with the equivariant Gromov–Witten theory of the crepant resolutions in the spirit of Bryan and Graber's Crepant Resolution Conjecture [4].

Let $\mathbb{T} = \mathbb{C}^\times \times \mathbb{C}^\times$ be a two-dimensional torus. The (localized) $\mathbb{T}$–equivariant cohomology of a point is generated by t_1 and t_2. Our main objects are the 3–point functions

$$\langle\langle \alpha_1, \alpha_2, \alpha_3 \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} \in \mathbb{Q}(t_1, t_2)[u, s_1, \dots, s_r],$$

which encode 3–point extended Gromov–Witten invariants of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ (see (2-6)). Note that the equivariant orbifold quantum cohomology is traditionally defined by the above functions with the quantum parameter u set to be 0. Thus, 3–point extended orbifold invariants provide more enumerative information than usual 3–point orbifold invariants.

The above 3–point functions add a multiplicative structure to the equivariant Chen–Ruan cohomology $H^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)];\mathbb{Q})$. The multiplication so obtained is called the small (extended) orbifold quantum product.

The quotient space $\mathrm{Sym}^n(\mathcal{A}_r)$ admits a unique crepant resolution of singularities, namely the Hilbert scheme $\mathrm{Hilb}^n(\mathcal{A}_r)$ of n points in $\mathcal{A}_r$. The $\mathbb{T}$–equivariant quantum cohomology of $\mathrm{Hilb}^n(\mathcal{A}_r)$ has been explored by Maulik and Oblomkov [15], so we need only deal with the quantum ring of the orbifold $[\mathrm{Sym}^n(\mathcal{A}_r)]$. We fully cover 2–point extended Gromov–Witten invariants of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ and find that the calculation of these invariants is tantamount to the question of counting certain branched covers of rational curves. Our discovery can be summarized in the following statement.

Theorem 0.1 *Given any positive integers r and n, two-point extended equivariant Gromov–Witten invariants of the symmetric product stack $[\mathrm{Sym}^n(\mathcal{A}_r)]$ in nonzero degrees are expressible in terms of equivariant orbifold Poincaré pairings and one-part double Hurwitz numbers.*

One-part double Hurwitz numbers, as shown by Goulden, Jackson and Vakil [9], admit explicit closed formulas (Proposition 3.13), and therefore Theorem 0.1 provides a complete solution to the divisor operators, ie, the operators of quantum multiplication by divisor classes. These operators correspond naturally to the divisor operators on the Hilbert scheme $\mathrm{Hilb}^n(\mathcal{A}_r)$:

Theorem 0.2 *After making the change of variables $q=-e^{iu}$, where $i^2=-1$, and extending scalars to an appropriate field F, there is a linear isomorphism of equivariant quantum cohomologies*

$$L\colon H^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)];F)\to H^*_{\mathbb{T}}(\mathrm{Hilb}^n(\mathcal{A}_r);F)$$

which preserves gradings, Poincaré pairings and respects small quantum product by divisors. In other words, for any Chen–Ruan cohomology classes α_1,α_2 and divisor D, we have the following identity for 3–point functions:

$$\langle\!\langle \alpha_1, D, \alpha_2\rangle\!\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}=\langle L(\alpha_1), L(D), L(\alpha_2)\rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)}.$$

Here $\langle -,-,-\rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)}$ are the 3–point functions of $\mathrm{Hilb}^n(\mathcal{A}_r)$ in variables t_1,t_2, $q,s_1,\dots,s_r$ (see (4-2)).

In addition to the relation to the Hilbert schemes, the orbifold theory is related to the relative Gromov–Witten theory of threefolds.

Theorem 0.3 *Given cohomology-weighted partitions $\lambda_1(\vec{\eta}_1)$, $\lambda_2(\vec{\eta}_2)$ of n and $\alpha = 1(1)^n$, (2) or D_k, $k = 1, \dots, r$* (see Section 1.2.2 and (3-1) for these classes), *we have*

$$\langle\langle \lambda_1(\vec{\eta}_1), \alpha, \lambda_2(\vec{\eta}_2) \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} = \mathrm{GW}(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta}_1), \alpha, \lambda_2(\vec{\eta}_2)},$$

where the right hand side is a shifted partition function (see Section 4.1).

0.2 Tetrahedron of equivalences

The above theorems form a triangle of equivalences. We can include the Donaldson–Thomas theory to make up a tetrahedron. In fact, Theorems 0.2 and 0.3, in conjunction with the results of Maulik and Oblomkov [13; 14; 15], establish the following equivalences for divisor operators.

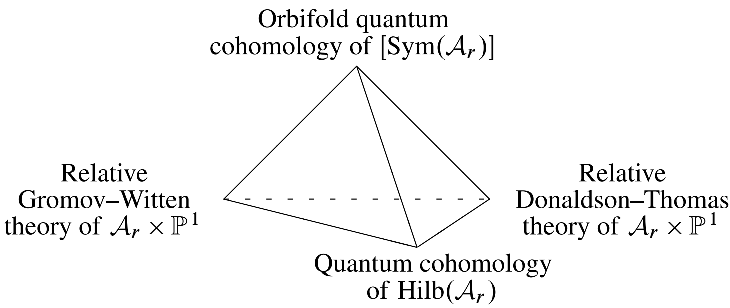

Figure 1. A tetrahedron of equivalences

Before the study of the Gromov–Witten theory of $[\mathrm{Sym}^n(\mathcal{A}_r)]$, the case of the affine plane $\mathbb{C}^2$ was the only known example for the above tetrahedron to hold for all operators; see Bryan and Graber [4], Bryan and Pandharipande [5] and Okounkov and Pandharipande [17; 18]. If the generation conjecture (Conjecture 5.1) of Maulik and Oblomkov is assumed, these four theories will be equivalent in our case of $\mathcal{A}_r$ as well. The base triangle of equivalences is the work of Maulik and Oblomkov. And the triangle facing the rightmost corner is worked out in this paper:

Theorem 0.4 *Let L be as in Theorem 0.2 and $\lambda_1(\vec{\eta}_1)$, $\lambda_2(\vec{\eta}_2)$, $\lambda_3(\vec{\eta}_3)$ any cohomology-weighted partitions of n. Assuming the generation conjecture, the identities*

$$\begin{aligned} \langle\langle \lambda_1(\vec{\eta}_1), \lambda_2(\vec{\eta}_2), \lambda_3(\vec{\eta}_3) \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} &= \langle L(\lambda_1(\vec{\eta}_1)), L(\lambda_2(\vec{\eta}_2)), L(\lambda_3(\vec{\eta}_3)) \rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)} \\ &= \mathrm{GW}(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta}_1), \lambda_2(\vec{\eta}_2), \lambda_3(\vec{\eta}_3)} \end{aligned}$$

hold under the substitution $q = -e^{iu}$.

We can make a more general statement on the Gromov–Witten theories of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ and $\mathrm{Hilb}^n(\mathcal{A}_r)$ due to the WDVV equations.

Theorem 0.5 *Let $q = -e^{iu}$. Assuming the generation conjecture, the map L in Theorem 0.2 equates the extended multipoint functions of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ to the multipoint functions of $\mathrm{Hilb}^n(\mathcal{A}_r)$. Moreover, these functions are rational functions in t_1, t_2, $q, s_1, \dots, s_r$. (Multipoint functions are those with at least three insertions.)*

This theorem provides a new evidence for the Crepant Resolution Conjecture. We will see that Theorems 0.4 and 0.5 are valid in the case of $n = 2, r = 1$ even without presuming the generation conjecture (see Section 5.1).

0.3 Outline of the paper

The aim of Section 1 is to give a brief introduction to the resolved surface $\mathcal{A}_r$ and Chen–Ruan's orbifold cohomology for a symmetric product. In Section 2, we review some background on orbifold Gromov–Witten theory and define extended Gromov–Witten invariants as well as their connected counterparts.

Section 3 is the main part of this paper. We provide explicit formulas for any 2–point extended invariants in nonzero degrees (Theorem 0.1). In Section 4, we show Theorems 0.2 and 0.3, which establish the tetrahedron of equivalences for divisor operators. Section 5, due to the first author, proves Theorem 0.4 and discusses multipoint functions of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ as well as the full version of the Crepant Resolution Conjecture (Theorem 0.5).

0.4 Notation and convention

The following notation will be used without further comment. Some other notation will be introduced along the way.

(1) To avoid doubling indices, we identify

$$A^i(X) = H^{2i}(X;\mathbb{Q}), \quad A_i(X) = H_{2i}(X;\mathbb{Q}) \quad \text{and} \quad A_i(X;\mathbb{Z}) = H_{2i}(X;\mathbb{Z}),$$

just to name a few, for any complex variety X to appear in this article (note that we drop $\mathbb{Q}$ but not $\mathbb{Z}$). They will be referred to as cohomology or homology groups rather than Chow groups.

(2) For any curve C on a complex variety X, the curve class $[C]$ is simply denoted by C.

(3) An orbifold $\mathcal{X}$ is a smooth Deligne–Mumford stack of finite type over $\mathbb{C}$. Denote by $c\colon \mathcal{X} \to X$ the canonical map to the coarse moduli space.

(4) For every positive integer s, μ_s is the cyclic subgroup of $\mathbb{C}^\times$ of order s. For any finite group G, $\mathcal{B}G$ is the classifying stack of G, ie, $[\operatorname{Spec}\mathbb{C}/G]$.

(5) (a) $\mathbb{T} = (\mathbb{C}^\times)^2$ is always a two-dimensional torus, and t_1, t_2 are the generators of the $\mathbb{T}$–equivariant cohomology $A^*_{\mathbb{T}}(\text{point})$ of a point, that is, $A^*_{\mathbb{T}}(\text{point}) = \mathbb{Q}[t_1, t_2]$.
(b) $V_{\mathfrak{m}} = V \otimes_{\mathbb{Q}[t_1,t_2]} \mathbb{Q}(t_1, t_2)$ for each $\mathbb{Q}[t_1, t_2]$–module V.

(6) Given any object $\mathcal{O}$, $\mathcal{O}^n$ means that $\mathcal{O}$ repeats itself n times.

(7) For $i = 1, 2$, ϵ_i is a function on the set of nonnegative integers such that

$$\epsilon_i(m) = \begin{cases} 0 & \text{if } m < i, \\ 1 & \text{if } m \geq i. \end{cases}$$

(8) Let σ be a partition of a nonnegative integer.
(a) $\ell(\sigma)$ is the length of σ. Unless otherwise stated, σ is presumed to be written as

$$\sigma = (\sigma_1, \dots, \sigma_{\ell(\sigma)}).$$

To emphasize, if σ_k is another partition, it is simply $(\sigma_{k1}, \dots, \sigma_{k\ell(\sigma_k)})$.
(b) Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_{\ell(\sigma)})$ be an $\ell(\sigma)$–tuple of cohomology classes associated to σ so that we may form a cohomology-weighted partition $\sigma(\vec{\alpha}) := \sigma_1(\alpha_1) \cdots \sigma_{\ell(\sigma)}(\alpha_{\ell(\sigma)})$. The group $\operatorname{Aut}(\sigma(\vec{\alpha}))$ is defined to be the group of permutations on $\{1, 2, \dots, \ell(\sigma)\}$ fixing

$$\big((\sigma_1, \alpha_1), \dots, (\sigma_{\ell(\sigma)}, \alpha_{\ell(\sigma)})\big).$$

Let $\operatorname{Aut}(\sigma)$ be the group $\operatorname{Aut}(\sigma(\vec{\alpha}))$ when all entries of $\vec{\alpha}$ are identical. Its order is simply $\prod_{i=1}^{n} m_i!$ if $\sigma = (1^{m_1}, \dots, n^{m_n})$.
(c) $|\sigma| = n$ if $\sigma_1 + \cdots + \sigma_{\ell(\sigma)} = n$, and $o(\sigma) = \operatorname{lcm}(|\sigma_1|, \dots, |\sigma_{\ell(\sigma)}|)$ is the order of any permutation of cycle type σ.
(d) $(2) := (2, 1^{n-2})$ and $1 := (1^n)$ are partitions of length $n-1$ and length n respectively.

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1 Preliminaries

1.1 Resolutions of cyclic quotient surface singularities

We fix a positive integer r once and for all. Let the cyclic group μ_{r+1} act on $\mathbb{C}^2$ by the diagonal matrices

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix},$$

where $\zeta \in \mu_{r+1}$. The quotient $\mathbb{C}^2/\mu_{r+1}$ is a surface singularity. We denote by

$$\pi\colon \mathcal{A}_r \to \mathbb{C}^2/\mu_{r+1}$$

its minimal resolution. It is well-known that π can be obtained via a sequence of $\lfloor \frac{r+1}{2} \rfloor$ blow-ups at the unique singularity. The exceptional locus $\mathrm{Ex}(\pi)$ of π is a chain of (-2)–curves,

$$\bigcup_{i=1}^{r} E_i,$$

with E_{i-1} and E_i intersect transversally. The intersection numbers of the exceptional curves are given by

$$E_i \cdot E_j = \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the intersection matrix is negative definite (as expected from the general theory of complex surfaces). Additionally, $E_1, \dots, E_r$ give a basis for $A_1(\mathcal{A}_r; \mathbb{Z})$. We also have two noncompact curves E_0 and E_{r+1} attached to E_1 and E_r respectively. The curve E_0 (resp. E_{r+1}) can be arranged to map to the μ_{r+1}–orbit of the x–axis (resp. the y–axis).

The natural action of $\mathbb{T}$ on $\mathbb{C}^2$ comes with tangent weights t_1 and t_2 at the origin. It commutes with the μ_{r+1}–action, so we have an induced $\mathbb{T}$–action on the quotient $\mathbb{C}^2/\mu_{r+1}$ and thus on the resolved surface $\mathcal{A}_r$. We fix these actions of $\mathbb{T}$ throughout the article.

The $\mathbb{T}$–invariant compact curves on $\mathcal{A}_r$ are $E_1, \dots, E_r$. The $\mathbb{T}$–fixed points are the nodes of the chain $\bigcup_{i=0}^{r+1} E_i$ of curves. Precisely, they are

$$x_1, \dots, x_{r+1},$$

where $\{x_i\} = E_{i-1} \cap E_i$. Let us assume that L_i and R_i are respectively the weights of the $\mathbb{T}$–action on the tangent spaces to E_{i-1} and E_i at x_i. We have $L_1 = (r+1)t_1$,

$R_{r+1} = (r+1)t_2$ and the equalities

$$L_i + R_i = t_1 + t_2, \quad R_i = -L_{i+1},$$

for each $i = 1, \dots r$.

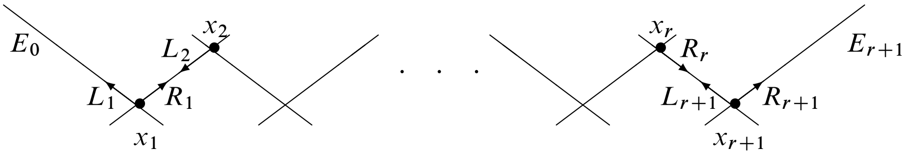

Figure 2. The middle chain (without E_0 and E_{r+1}) is the exceptional locus $\mathrm{Ex}(\pi)$. The labeled vectors stand for the tangent weights at the fixed points.

The above information will be sufficient for our calculation of Gromov–Witten invariants. Certainly, one can also compute explicitly to obtain

$$(L_i, R_i) = \big((r-i+2)t_1 + (1-i)t_2, (-r+i-1)t_1 + i\,t_2\big).$$

From now on, we let

$$\{\omega_1, \dots, \omega_r\}$$

be the dual basis of $\{E_1, \dots, E_r\}$ with respect to the Poincaré pairing.

1.2 Chen–Ruan cohomology

Given any smooth complex variety X, the symmetric group $\mathfrak{S}_n$ acts on X^n by $g(z)_i = z_{g(i)}$ for all $g \in \mathfrak{S}_n$, $z \in X^n$. The n–fold symmetric product $\mathrm{Sym}^n(X)$ is defined to be $X^n/\mathfrak{S}_n$, and the n–fold symmetric product stack $[\mathrm{Sym}^n(X)]$ is defined to be the quotient stack $[X^n/\mathfrak{S}_n]$. The space $\mathrm{Sym}^n(X)$ is in general singular and is the coarse moduli space of the (smooth) orbifold $[\mathrm{Sym}^n(X)]$.

1.2.1 Stack of cyclotomic gerbes The stack of cyclotomic gerbes to $[\mathrm{Sym}^n(X)]$ (consult Abramovich, Graber and Vistoli [2; 3]), denoted by $\overline{I}[\mathrm{Sym}^n(X)]$, is defined to be

$$\coprod_{s\in\mathbb{N}} \mathrm{HomRep}(\mathcal{B}\mu_s, [\mathrm{Sym}^n(X)])/\mathcal{B}\mu_s,$$

where $\mathrm{HomRep}(\mathcal{B}\mu_s, [\mathrm{Sym}^n(X)])$ is the stack of representable morphisms from the classifying stack $\mathcal{B}\mu_s$ to $[\mathrm{Sym}^n(X)]$. There is another natural stack associated to $[\mathrm{Sym}^n(X)]$. It is the inertia stack $I[\mathrm{Sym}^n(X)] := \coprod_{s\in\mathbb{N}} \mathrm{HomRep}(\mathcal{B}\mu_s, [\mathrm{Sym}^n(X)])$. In fact, the stack of cyclotomic gerbes is obtained by rigidifying the inertia stack, ie, removing the action of the cyclic groups μ_s's (consult [2; 3] and Abramovich, Corti and Vistoli [1]).

The stack $\bar{I}[\mathrm{Sym}^n(X)]$ is isomorphic to a disjoint union of orbifolds

$$\coprod_{[g]\in C} [X^n_g/\overline{C(g)}],$$

where C is the set of the conjugacy classes of $\mathfrak{S}_n$, $C(g)$ is the centralizer of g, $\overline{C(g)}$ is the quotient group $C(g)/\langle g\rangle$, and X^n_g is the g–fixed locus of X^n. On the other hand, $I[\mathrm{Sym}^n(X)]$ is just $\coprod_{[g]\in C}[X^n_g/C(g)]$.

The Chen–Ruan cohomology [7]

$$A^*_{\mathrm{orb}}([\mathrm{Sym}^n(X)])$$

is the cohomology $A^*(\bar{I}[\mathrm{Sym}^n(X)])$ of the stack of cyclotomic gerbes in X. Thus, it is simply

$$\bigoplus_{[g]\in C} A^*(X^n_g/C(g)) = \bigoplus_{[g]\in C} A^*(X^n_g)^{C(g)}.$$

Remark 1.1 As both $\bar{I}[\mathrm{Sym}^n(X)]$ and $I[\mathrm{Sym}^n(X)]$ have the same coarse moduli space, the Chen–Ruan cohomology is identical to the cohomology of the inertia stack. We focus on the stack of cyclotomic gerbes because it is the space where the evaluation maps land (see (2-1)).

As there is a one-to-one correspondence between the conjugacy classes of $\mathfrak{S}_n$ and the partitions of n, the connected components of $\bar{I}[\mathrm{Sym}^n(X)]$ can be labeled with the partitions of n. If $[g]$ is the conjugacy class corresponds to the partition λ, we may write

$$X(\lambda) = X^n_g/C(g),$$
$$\overline{X(\lambda)} = X^n_g/\overline{C(g)}.$$

The component $[X^n/\mathfrak{S}_n]$ is called the untwisted sector while all other components of the stack $\bar{I}[\mathrm{Sym}^n(X)]$ are called twisted sectors.

Additionally, for $\alpha \in A^i(X(\lambda))$, the orbifold (Chow) degree of α is defined to be $i+\mathrm{age}(\lambda)$, where $\mathrm{age}(\lambda) = n-\ell(\lambda)$ is the age of the sector $[\overline{X(\lambda)}]$.

1.2.2 Bases Assume that X admits a $\mathbb{T}$–action. We can see easily that there are induced $\mathbb{T}$–actions on the above spaces. So we may put the cohomologies into an equivariant context by considering $\mathbb{T}$–equivariant cohomologies.

Let us sketch a basis, constructed in Cheong [8], for the equivariant Chen–Ruan cohomology of the stack $[\mathrm{Sym}^n(X)]$. Indeed, any cohomology-weighted partition $\lambda(\vec{\eta})$

with η_i's cohomology classes on X defines a class on the sector $X(\lambda)$, which we denote by $\lambda(\vec{\eta})$ as well: Pick a representative permutation $g \in \mathfrak{S}_n$ of cycle type λ. It has a cycle decomposition $g = g_1 \dots g_{\ell(\lambda)}$ with g_i being a λ_i–cycle. We let

$$\lambda(\vec{\eta}) = \left(|\mathrm{Aut}(\lambda(\vec{\eta}))| \prod_{i=1}^{\ell(\lambda)} \lambda_i \right)^{-1} \sum_{h \in C(g)} \bigotimes_{i=1}^{\ell(\lambda)} h^{-1} g_i h(\eta_i) \in A^*_{\mathbb{T}}(X^n_g)^{C(g)} = A^*_{\mathbb{T}}(X(\lambda)).$$

Here the class $h^{-1}g_i h(\eta_i)$ is the pullback of η_i by the isomorphism $X^{\lambda_i}_{h^{-1}g_i h} \cong X$, and the term $(|\mathrm{Aut}(\lambda(\vec{\eta}))| \prod_{i=1}^{\ell(\lambda)} \lambda_i)^{-1}$ is a normalization factor to ensure that no repetition occurs. It is easy to see that the above expression is independent of the decomposition of g.

Let $\mathfrak{B}$ be a basis for $A^*_{\mathbb{T}}(X)$. The classes $\lambda(\vec{\eta})$'s, running over all partitions λ of n and all $\eta_i \in \mathfrak{B}$, serve as a basis for the Chen–Ruan cohomology $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])$. For classes $\lambda(\vec{\eta}) \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])$ and $\rho(\vec{\xi}) \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^m(X)])$, keep in mind that the class

$$\lambda_1(\eta_1) \cdots \lambda_{\ell(\lambda)}(\eta_{\ell(\lambda)}) \rho_1(\xi_1) \cdots \rho_{\ell(\rho)}(\xi_{\ell(\rho)}) \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^{n+m}(X)])$$

is denoted by

$$\lambda(\vec{\eta})\rho(\vec{\xi}).$$

We use the shorthand

$$(2)$$

for the divisor class $2(1)1(1)^{n-2}$. Also, we define the age of $\lambda(\vec{\eta})$, denoted by

$$\mathrm{age}(\lambda(\vec{\eta})),$$

to be the age of the sector $[\overline{X(\lambda)}]$, ie, $n - \ell(\lambda)$.

Fixed-point basis We can work with $\lambda(\vec{\eta})$'s with η_k's in the localized cohomology $A^*_{\mathbb{T}}(X)_{\mathfrak{m}}$ to give a basis for $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])_{\mathfrak{m}}$.

Assume that X has exactly p $\mathbb{T}$–fixed points $z_1, \dots, z_p$. For partitions $\sigma_1, \dots, \sigma_p$, we denote the class

$$\sigma_{11}([z_1]) \cdots \sigma_{1\ell(\sigma_1)}([z_1]) \cdots \sigma_{p1}([z_p]) \cdots \sigma_{p\ell(\sigma_p)}([z_p])$$

by

$$\widetilde{\sigma} := (\sigma_1, \dots, \sigma_p).$$

The classes $\widetilde{\sigma}$'s form a basis for $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])_{\mathfrak{m}}$. Note also that each $\widetilde{\sigma}$ corresponds to a $\mathbb{T}$–fixed point, which we denote by

$$[\widetilde{\sigma}],$$

in the sector indexed by the partition $(\sigma_{11},\dots,\sigma_{1\ell(\sigma_1)},\dots,\sigma_{p1},\dots,\sigma_{p\ell(\sigma_p)})$. So we refer to $\widetilde{\sigma}$'s as $\mathbb{T}$–fixed point classes.

Moreover, given $\widetilde{\delta}\in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])_{\mathfrak{m}}$ and $\widetilde{\sigma}\in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^m(X)])_{\mathfrak{m}}$ $(m\le n)$, we say that

$$\widetilde{\delta}\supset\widetilde{\sigma}$$

if σ_k is a subpartition of δ_k, $\forall k=1,\dots,p$; in this case, we let

$$\widetilde{\delta}-\widetilde{\sigma}=(\delta_1-\sigma_1,\dots,\delta_p-\sigma_p)\in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^{n-m}(X)])_{\mathfrak{m}}.$$

(for instance, the difference $(1,1,2,2,3)-(1,2,3)$ of two partitions is the partition $(1,2)$.)

Tangent weights Given any fixed-point class $\widetilde{\sigma}$, let

$$t(\widetilde{\sigma})=e_{\mathbb{T}}(T_{[\widetilde{\sigma}]}\overline{I}[\mathrm{Sym}^m(X)]),$$

ie, the $\mathbb{T}$–equivariant Euler class of the tangent space to $\overline{I}[\mathrm{Sym}^m(X)]$ at the fixed point $[\widetilde{\sigma}]$. A simple analysis shows that $t(\widetilde{\sigma})=\prod_{k=1}^p e_{\mathbb{T}}(T_{z_k}X)^{\ell(\sigma_k)}$. Thus, for each $\widetilde{\delta}\supset\widetilde{\sigma}$,

$$t(\widetilde{\delta})=t(\widetilde{\sigma})t(\widetilde{\delta}-\widetilde{\sigma}). \tag{1-1}$$

1.2.3 Coefficients with respect to fixed-point basis We denote the $\mathbb{T}$–equivariant orbifold pairings on the Chen–Ruan cohomology $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^m(\mathcal{A}_r)])$ by

$$\langle\cdot\,|\,\cdot\rangle.$$

For $\theta(\vec{\xi})\in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^m(X)])_{\mathfrak{m}}$, we let

$$\alpha_{\theta(\vec{\xi})}(\widetilde{\sigma})=\frac{\langle\theta(\vec{\xi})\,|\,\widetilde{\sigma}\rangle}{\langle\widetilde{\sigma}\,|\,\widetilde{\sigma}\rangle} \tag{1-2}$$

be the components of $\theta(\vec{\xi})$ relative to fixed-point classes $\widetilde{\sigma}$'s. We intend to present an algorithm to calculate $\alpha_{\lambda(\vec{\eta})}(\widetilde{\delta})$.

First of all, we have two properties by direct verification:

(1) Suppose the classes $\lambda(\vec{\eta})$ and $\rho(\vec{\varepsilon})\in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])_{\mathfrak{m}}$ have explicit forms

$$\prod_{i=1}^n\prod_{j=1}^{m_i} i(\eta_{ij})\quad\text{and}\quad\prod_{i=1}^n\prod_{j=1}^{\ell_i} i(\varepsilon_{ij}),$$

respectively, we have

$$\langle \lambda(\vec{\eta}) \mid \rho(\vec{\varepsilon})\rangle = \begin{cases} 0 & \text{if } m_i \neq \ell_i \text{ for some } i, \\ \displaystyle\prod_{i=1}^{n} \Big\langle \prod_{j=1}^{m_i} i(\eta_{ij}) \Big| \prod_{j=1}^{m_i} i(\varepsilon_{ij}) \Big\rangle & \text{if } m_i = \ell_i \text{ for each } i. \end{cases}$$

(2) Given $\eta_1, \dots, \eta_n \in A^*_{\mathbb{T}}(X)_{\mathfrak{m}}$, $\mathbb{T}$–fixed points $y_1, \dots, y_n$ of X. For $m \leq n$, the coefficient $\alpha_{i(\eta_1)\cdots i(\eta_n)}(i([y_1])\cdots i([y_n]))$ equals

$$\sum_{\xi_1,\dots,\xi_n} \alpha_{i(\xi_1)\cdots i(\xi_m)}(i([y_1])\cdots i([y_m]))\alpha_{i(\xi_{m+1})\cdots i(\xi_n)}(i([y_{m+1}])\cdots i([y_n])),$$

where the sum is over all possible $i(\xi_1)\cdots i(\xi_m)$ and $i(\xi_{m+1})\cdots i(\xi_n)$ such that

$$i(\xi_1)\cdots i(\xi_n) = i(\eta_1)\cdots i(\eta_n).$$

We may combine (1) with (2) to get a general statement.

Proposition 1.2 *Given $\lambda(\vec{\eta}), \widetilde{\delta} \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])_{\mathfrak{m}}$ and $\widetilde{\sigma} \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^m(X)])_{\mathfrak{m}}$ with $\widetilde{\delta} \supset \widetilde{\sigma}$,*

$$\alpha_{\lambda(\vec{\eta})}(\widetilde{\delta}) = \sum_P \alpha_{\theta(\vec{\xi})}(\widetilde{\sigma})\alpha_{\mu(\vec{\gamma})}(\widetilde{\delta} - \widetilde{\sigma}), \tag{1-3}$$

where the index P under the summation symbol means that the sum is taken over all possible $\theta(\vec{\xi}) \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^m(X)])_{\mathfrak{m}}$ and $\mu(\vec{\gamma}) \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^{n-m}(X)])_{\mathfrak{m}}$ satisfying the equality $\lambda(\vec{\eta}) = \theta(\vec{\xi})\mu(\vec{\gamma})$. □

In the proposition, $\widetilde{\delta}$ is separated into two parts $\widetilde{\sigma}$ and $\widetilde{\delta} - \widetilde{\sigma}$. In general, we can break it as many parts as possible. The form (1-3) is, however, convenient for our use.

2 Extended Gromov–Witten theory of orbifolds

To make our exposition as self-contained as possible, we review some relevant background on orbifold Gromov–Witten theory. We take the algebro-geometric approach in the sense of Abramovich, Graber and Vistoli's works [2; 3]. The reader may also want to consult the original work [6] of Chen and Ruan in symplectic category.

In what follows, we utilize the isomorphism

$$A_1(\mathrm{Sym}^n(X); \mathbb{Z}) \cong A_1(X^n; \mathbb{Z})^{\mathfrak{S}_n} \cong A_1(X; \mathbb{Z}).$$

In other words, we may view $E_1, \dots, E_r$ as a basis for $A_1(\mathrm{Sym}^n(\mathcal{A}_r); \mathbb{Z})$.

2.1 The space of twisted stable maps

For any curve class $\beta \in A_1(X;\mathbb{Z})$, the moduli space

$$\overline{M}_{0,k}([\mathrm{Sym}^n(X)],\beta)$$

parametrizes genus zero, k–pointed, twisted stable map (or orbifold stable map in [6])

$$f\colon (\mathcal{C},\mathcal{P}_1,\dots,\mathcal{P}_k)\to[\mathrm{Sym}^n(X)]$$

with the following conditions:

- $(\mathcal{C},\mathcal{P}_1,\dots,\mathcal{P}_k)$ is an twisted nodal k–pointed curve. The marking $\mathcal{P}_i$ is an étale gerbe banded by μ_{r_i}, where r_i is the order of the stabilizer of the twisted point. Moreover, over a node, $\mathcal{C}$ has a chart isomorphic to $\mathrm{Spec}\,\mathbb{C}[u,v]/(uv)/\mu_s$ where μ_s acts on $\mathrm{Spec}\,\mathbb{C}[u,v]$ by $\xi\cdot(u,v)=(\xi u,\xi^{-1}v)$, and the canonical map $c\colon\mathcal{C}\to C$ is given by $x=u^s$, $y=v^s$ in this chart.
- f is a representable morphism and induces a genus zero, k–pointed, degree β stable map $f_c\colon (C,c(\mathcal{P}_1),\dots,c(\mathcal{P}_k))\to\mathrm{Sym}^n(X)$ by passing to coarse moduli spaces. Note that the canonical map $c\colon\mathcal{C}\to C$ is an isomorphism away from the nodes and marked gerbes and that whenever we say that f is of degree β, we actually mean f_c is.

There are evaluation maps on the moduli space $\overline{M}_{0,k}([\mathrm{Sym}^n(X)],\beta)$, which take values in the stack of cyclotomic gerbes in X. At the level of $\mathrm{Spec}(\mathbb{C})$–points, the i–th evaluation map

$$\mathrm{ev}_i\colon \overline{M}_{0,k}([\mathrm{Sym}^n(X)],\beta)\to\bar{I}[\mathrm{Sym}^n(X)] \tag{2-1}$$

is defined by $[f\colon(\mathcal{C},\mathcal{P}_1,\dots,\mathcal{P}_k)\to[\mathrm{Sym}^n(X)]]\longmapsto[f|_{\mathcal{P}_i}\colon\mathcal{P}_i\to[\mathrm{Sym}^n(X)]]$.

The moduli space $\overline{M}_{0,k}([\mathrm{Sym}^n(X)],\beta)$ can be decomposed into open and closed substacks:

$$\overline{M}_{0,k}([\mathrm{Sym}^n(X)],\beta)=\coprod_{\sigma_1,\dots,\sigma_k}\mathrm{ev}_1^{-1}([\overline{X(\sigma_1)}])\cap\cdots\cap\mathrm{ev}_k^{-1}([\overline{X(\sigma_k)}])$$

where the union is taken over all partitions $\sigma_1,\dots,\sigma_k$ of n. Let

$$\overline{M}([\mathrm{Sym}^n(X)],\sigma_1,\dots,\sigma_k;\beta)=\mathrm{ev}_1^{-1}([\overline{X(\sigma_1)}])\cap\cdots\cap\mathrm{ev}_k^{-1}([\overline{X(\sigma_k)}]).$$

Note that its virtual dimension is given by

$$-K_{[\mathrm{Sym}^n(X)]}\cdot\beta+n\cdot\dim(X)+k-3-\sum_{i=1}^{k}\mathrm{age}(\sigma_i).$$

The twisted map f representing an element of $\overline{M}([\mathrm{Sym}^n(X)], \sigma_1, \dots, \sigma_k; \beta)$ amounts to the commutative diagram

$$\begin{tikzcd} P_{\mathcal{C}} \arrow[r, "f'"] \arrow[d] & X^n \arrow[d, "\pi"] \\ \mathcal{C} \arrow[r, "f"] \arrow[d, "c"'] & {[\mathrm{Sym}^n(X)]} \arrow[d, "c"] \\ C \arrow[r, "f_c"] & \mathrm{Sym}^n(X). \end{tikzcd} \tag{2-2}$$

Here π is the natural map, $P_{\mathcal{C}} = \mathcal{C} \times_{[\mathrm{Sym}^n(X)]} X^n$ is a scheme by representability of f, and f' is $\mathfrak{S}_n$–equivariant. Away from the marked points and nodes, $P_{\mathcal{C}}$ is a principal $\mathfrak{S}_n$–bundle of C. It is branched over the markings with ramification types $\sigma_1, \dots, \sigma_k$.

Additionally, there is such a diagram

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & X \\ (C, c(\mathcal{P}_1), \dots, c(\mathcal{P}_k)) & \end{tikzcd} \tag{2-3}$$

associated to f that $p\colon \widetilde{C} \to C$ is an admissible cover branched over $c(\mathcal{P}_1), \dots, c(\mathcal{P}_k)$ with monodromy given by $\sigma_1, \dots, \sigma_k$, and $\widetilde{f}\colon \widetilde{C} \to X$ is a degree β morphism such that if $\Sigma \subset C$ is a rational curve possessing less than 3 special points, then there is a component of $p^{-1}(\Sigma)$ which is not $\widetilde{f}$–contracted. In fact, (2-3) is induced by the diagram (2-2) by taking f' mod $\mathfrak{S}_{n-1}$ and composing with the n–th projection.

The diagram (2-3) will be particularly helpful later in the descriptions of $\mathbb{T}$–fixed loci for the space of twisted stable maps to $[\mathrm{Sym}^n(\mathcal{A}_r)]$. The reader should look closely at the above notation. We will use the diagrams (2-2) and (2-3) without further comment.

2.2 The space of connected coverings

Denote by

$$\overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)], \beta)$$

the locus in $\overline{M}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)], \beta)$ which parametrizes connected covers (ie, each cover $\widetilde{C}$ associated to $[f\colon \mathcal{C} \to [\mathrm{Sym}^n(\mathcal{A}_r)]] \in \overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)], \beta)$ is connected). Note that the space $\overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)], \beta)$ is generally not connected, however.

Just like the situation in Section 2.1, $\overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)],\beta)$ also admits an evaluation map to the stack $\overline{I}[\mathrm{Sym}^n(\mathcal{A}_r)]$ of cyclotomic gerbes in $\mathcal{A}_r$. Moreover, we define the space

$$\overline{M}^{\circ}([\mathrm{Sym}^n(\mathcal{A}_r)],\sigma_1,\dots,\sigma_k;\beta)$$

to be the intersection

$$\overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)],\beta)\cap \mathrm{ev}_1^{-1}([\overline{\mathcal{A}_r(\sigma_1)}])\cap\cdots\cap \mathrm{ev}_k^{-1}([\overline{\mathcal{A}_r(\sigma_k)}]).$$

These spaces of connected coverings will help us define the connected version of orbifold Gromov–Witten invariants, which will play a special role in our determination of the usual (ie, not necessarily connected) orbifold invariants.

2.3 Gromov–Witten invariants

For any Chen–Ruan cohomology classes $\alpha_i\in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)])$ $(i=1,\dots,k)$, the k–point equivariant Gromov–Witten invariant is defined by

$$\langle\alpha_1,\dots,\alpha_k\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}_{\beta}:=\int_{[\overline{M}_{0,k}([\mathrm{Sym}^n(X)],\beta)]^{\mathrm{vir}}_{\mathbb{T}}}\mathrm{ev}_1^*(\alpha_1)\cdots\mathrm{ev}_k^*(\alpha_k),\tag{2-4}$$

where the symbol $[\ ]^{\mathrm{vir}}_{\mathbb{T}}$ indicates the $\mathbb{T}$–equivariant virtual fundamental class. However, it is convenient to express the integral in (2-4) as a sum of integrals against the virtual fundamental classes of the components $\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)],\sigma_1,\dots,\sigma_k;\beta)$'s.

Note that the moduli space over which the integral takes is not necessarily compact. But (2-4) is well-defined if the integral is written as a sum of residue integrals over $\mathbb{T}$–fixed components via the virtual localization formula (see Graber and Pandharipande [10]). Alternatively, the definition (2-4) is valid when some insertions have compact supports, eg, $\mathbb{T}$–fixed point classes. So by extending scalars, we may treat (2-4) as a $\mathbb{Q}(t_1,t_2)$–combination of invariants with at least one compactly supported insertion. In general, the invariant takes values in $\mathbb{Q}(t_1,t_2)$.

Extended version Let us identify $A_0([\overline{\mathcal{A}_r((2))}];\mathbb{Z})$ with $\mathbb{Z}$. We may define k–point extended Gromov–Witten invariant $\langle\alpha_1,\dots,\alpha_k\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}_{(a,\beta)}$ in twisted degree $(a,\beta)\in\mathbb{Z}\oplus A_1(\mathcal{A}_r;\mathbb{Z})$. We set the invariant to be zero in case $a<0$. If $a\geq 0$, we include additional a unordered markings in the twisted stable map of degree β above such that these markings go to the age one sector under the corresponding evaluation maps. To make this precise, we present a formula

$$\langle\alpha_1,\dots,\alpha_k\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}_{(a,\beta)}=\frac{1}{a!}\langle\alpha_1,\dots,\alpha_k,(2)^a\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}_{\beta}.\tag{2-5}$$

In the expression, the last a insertions are all (2). For later convenience of explanation, we refer to the markings associated to $\alpha_1,\dots,\alpha_k$ as distinguished marked points and to the other a markings as simple marked points. Also the markings corresponding to the twisted sectors are called twisted and are otherwise called untwisted.

The expression (2-5) is almost identical to the nonextended version except for the appearance of the factor $1/a!$ due to the fact that we do not order simple markings. Additionally, we say that $\langle \alpha_1, \dots, \alpha_k \rangle_{(a,\beta)}^{[\mathrm{Sym}^n(\mathcal{A}_r)]}$ is in nonzero (resp. zero) degree if it is a Gromov–Witten invariant (up to a multiple) in nonzero (resp. zero) degree and that $\langle \alpha_1, \dots, \alpha_k \rangle_{(a,\beta)}^{[\mathrm{Sym}^n(\mathcal{A}_r)]}$ is multipoint if $k \geq 3$.

Like ordinary Gromov–Witten theory, if $\beta \neq 0$ or $k \geq 3$, we have a forgetful morphism

$$ft_{k+1} \colon \overline{M}([\mathrm{Sym}^n(X)], \sigma_1, \dots, \sigma_k, 1; \beta) \to \overline{M}([\mathrm{Sym}^n(X)], \sigma_1, \dots, \sigma_k; \beta)$$

defined by forgetting the last untwisted marked points. The (untwisted) divisor equation holds as well in the orbifold case. Unfortunately, we are not allowed to forget twisted markings in general.

Connected version We define k–point connected Gromov–Witten invariant as the contribution of the space $\overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)], \beta)$ to the extended Gromov–Witten invariant, namely,

$$\langle \alpha_1, \dots, \alpha_k \rangle_{\beta}^{[\mathrm{Sym}^n(\mathcal{A}_r)],\mathrm{conn}} = \int_{[\overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)], \beta)]^{\mathrm{vir}}_{\mathbb{T}}} \mathrm{ev}_1^*(\alpha_1) \cdots \mathrm{ev}_k^*(\alpha_k).$$

Note that $\overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)], \beta)$ is compact whenever $\beta \neq 0$, in which case the corresponding connected invariant is an element of $\mathbb{Q}[t_1, t_2]$.

Similarly, the connected invariant has an extended version. We define k–point extended connected invariant by

$$\langle \alpha_1, \dots, \alpha_k \rangle_{(a,\beta)}^{[\mathrm{Sym}^n(\mathcal{A}_r)],\mathrm{conn}} = \frac{1}{a!} \langle \alpha_1, \dots, \alpha_k, (2)^a \rangle_{\beta}^{[\mathrm{Sym}^n(\mathcal{A}_r)],\mathrm{conn}}.$$

Orbifold quantum product For any classes $\alpha_1, \dots, \alpha_k \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)])$, we define the extended k–point function of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ by

$$\langle\langle \alpha_1, \dots, \alpha_k \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} = \sum_{a=0}^{\infty} \sum_{\beta \in A_1(\mathcal{A}_r;\mathbb{Z})} \langle \alpha_1, \dots, \alpha_k \rangle_{(a,\beta)}^{[\mathrm{Sym}^n(\mathcal{A}_r)]} u^a s_1^{\beta\cdot\omega_1} \cdots s_r^{\beta\cdot\omega_r} \tag{2-6}$$

and denote by

$$\langle \alpha_1, \dots, \alpha_k \rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}$$

the usual k–point function $\langle\langle \alpha_1, \dots, \alpha_k \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}|_{u=0}$.

Now let $\{\gamma\}$ be a basis for the Chen–Ruan cohomology $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)])$ and let $\{\gamma^\vee\}$ be its dual basis. Define the small (extended) orbifold quantum product on $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)])$ in this way:

$$\alpha_1 *_{\mathrm{orb}} \alpha_2 = \sum_\gamma \langle\langle \alpha_1, \alpha_2, \gamma\rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} \gamma^\vee.$$

Equivalently, $\alpha_1 *_{\mathrm{orb}} \alpha_2$ is defined to be the unique element satisfying

$$\langle \alpha_1 *_{\mathrm{orb}} \alpha_2 \mid \alpha\rangle = \langle\langle \alpha_1, \alpha_2, \alpha\rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} \quad \forall \alpha.$$

The associativity of the product follows from the WDVV equation, and $1(1)^n$ is the multiplicative identity because of the fundamental class axiom. By extending scalars, we work with

$$QA^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)]),$$

which is defined as the vector space

$$A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)]) \otimes_{\mathbb{Q}[t_1,t_2]} \mathbb{Q}(t_1,t_2)((u,s_1,\dots,s_r))$$

endowed with quantum multiplication $*_{\mathrm{orb}}$.

3 Divisor operators

For any divisor classes D, we want to study the operators

$$D *_{\mathrm{orb}} -$$

on the (small) quantum cohomology of the orbifold $[\mathrm{Sym}^n(\mathcal{A}_r)]$. We call them divisor operators. Let

$$D_k = 1(1)^{n-1}1(\omega_k), \quad k = 1,\dots,r. \tag{3-1}$$

These classes, along with (2), form a basis for divisors on $[\mathrm{Sym}^n(\mathcal{A}_r)]$. Thus, the divisor operators are determined by

$$(2) *_{\mathrm{orb}} -, \quad D_1 *_{\mathrm{orb}} -, \quad \dots, \quad D_r *_{\mathrm{orb}} -,$$

which are governed by 2–point extended invariants to be calculated in this section.

Fix a nonnegative integer a throughout the rest of this section. We shorten our notation by declaring

$$\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \sigma_1,\dots,\sigma_k;(a,\beta)) = \overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \sigma_1,\dots,\sigma_k,(2)^a;\beta).$$

Also, we use

$$\vec{g} = (g_1,\dots,g_{r+1})$$

to denote an $(r+1)$–tuple, whose entries are either all partitions or all nonnegative integers. In the case of integers, define

$$|\vec{g}| = \ell,$$

if the entries of $\vec{g}$ add up to ℓ. Moreover, given a partition σ_0 and a multipartition $\vec{\sigma}$, we put

$$\hat{\sigma} := (\sigma_0, \vec{\sigma}) = (\sigma_0, \dots, \sigma_{r+1}),$$

which we also realize as a partition of the integer $\sum_{k=0}^{r+1} |\sigma_k|$.

3.1 Valuations

Let Λ_1 and Λ_2 be partitions of n. For each $\mathbb{T}$–fixed connected component F of the moduli space $\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \Lambda_1, \Lambda_2, (a, \beta))$, the virtual normal bundle to F is denoted by

$$N_F^{\mathrm{vir}}.$$

Let $[f\colon \mathcal{C} \to [\mathrm{Sym}^n(\mathcal{A}_r)] \in F$ and $\coprod_v \mathcal{C}_v$ the union of one-dimensional, contracted, connected components of $\mathcal{C}$. We have a natural morphism

$$\phi_F\colon F \to F^c := \prod_v \overline{M}_{0,\mathrm{val}(v)} \tag{3-2}$$

defined by $\phi_F([f]) = ([c(\mathcal{C}_v)])_v$. That is, all noncontracted components, zero-dimensional contracted components, stack structures at special points, and the map f are forgotten. Here $\mathrm{val}(v)$ denotes the number of special points on $\mathcal{C}_v$.

Given nonnegative integers i, j, s with $1 \le i \le j \le r$ and $s \le a$. We consider effective curve classes

$$\mathcal{E}_{ij} = E_i + \cdots + E_j.$$

(Note that $\mathcal{E}_{ii} = E_i$.)

Later in this section, we will introduce what we call the $\mathbb{T}$–fixed components of types I and II (Section 3.1.1) in $\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \Lambda_1, \Lambda_2, (a, \beta))$ for $\beta = d\mathcal{E}_{ij}$. Before proceeding, we state a lemma by the first author on (t_1+t_2)–valuations, which will greatly simplify our virtual localization calculation. The proof will be given in Section 3.1.2.

Lemma 3.1 *Given any $\mathbb{T}$–fixed component F of $\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \Lambda_1, \Lambda_2, (a, \beta))$. If $\beta = d\mathcal{E}_{ij}$ for some d, i, j, and F is of type I, then the inverse Euler class $1/e_{\mathbb{T}}(N_F^{\mathrm{vir}})$ has valuation 1 with respect to $(t_1 + t_2)$. Otherwise, $1/e_{\mathbb{T}}(N_F^{\mathrm{vir}})$ has valuation at least 2.*

3.1.1 Fixed components of types I and II For β not a multiple of $\mathcal{E}_{ij}$ for any i, j, we do not need a detailed description of the $\mathbb{T}$–fixed connected components of the moduli space $\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \Lambda_1, \Lambda_2, (a, \beta))$, and we will be able to show the vanishing of the corresponding two-point extended invariants by applying Lemma 3.1 (see Section 3.2).

In this section, we focus on the classes $\beta = d\mathcal{E}_{ij}$ for positive integers d, i, j. We divide the $\mathbb{T}$–fixed components of the moduli space

$$\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \Lambda_1, \Lambda_2, (a, d\mathcal{E}_{ij}))$$

into two types: type I and type II.

In the following, we describe the components of types I and II (for $\beta = d\mathcal{E}_{ij}$). Our description is based on the fixed points to which the first two (distinguished) markings are sent and on the configurations of the admissible covers associated to the twisted stable maps (see the diagram (2-3) and the discussion there).

Given a nonnegative integer $s \leq a$. For each $b_0^L \in \{0, \dots, s\}$ and $u_0^L \in \{0, \dots, a-s\}$, put $b_0^R = s - b_0^L$ and $u_0^R = a - s - u_0^L$. We let

$$\{\overline{M}_0^{b_0^L, \sigma_0, u_0^L}(1)\} \quad (\text{resp. } \{\overline{M}_0^{b_0^L, \sigma_0, u_0^L}(2)\}) \tag{3-3}$$

be the set consisting of all $\mathbb{T}$–fixed connected components of the moduli space

$$\overline{M}^\circ([\mathrm{Sym}^{|\lambda_0|}(\mathcal{A}_r)], \lambda_0, \rho_0, (2)^s, 1^{a-s}; d\mathcal{E}_{ij})$$

(see Section 2.2) such that each element $[f_0\colon \mathcal{C} \to [\mathrm{Sym}^{|\lambda_0|}(\mathcal{A}_r)]] \in \overline{M}_0^{b_0^L, \sigma_0, u_0^L}(1)$ (resp. $\overline{M}_0^{b_0^L, \sigma_0, u_0^L}(2)$) has the following properties:

(i) f_0 has its source curve decomposed as

$$\mathcal{C} = \mathcal{C}_{L0} \cup \mathcal{D}_0 \cup \mathcal{C}_{R0}.$$

Here $\mathcal{C}_{k0}$'s are disjoint f_0–contracted components, $\mathcal{D}_0$ is a chain of noncontracted components with $f_{0*}([\mathcal{D}_0]) = d\mathcal{E}_{ij}$, and $\mathcal{C}_{k0} \cap \mathcal{D}_0 = \{\mathcal{P}_k\}$ is a twisted point for $k = L, R$.

Let D_0, C, P_k be coarse moduli spaces of $\mathcal{D}_0, \mathcal{C}, \mathcal{P}_k$ respectively ($k = L, R$) and $\widetilde{C}_0$ the admissible cover associated to $\mathcal{C}$.

(ii) $\widetilde{C}_0 := \widetilde{C}_{L0} \cup \widetilde{D}_0 \cup \widetilde{C}_{R0}$ is connected with admissible covers $\widetilde{D}_0 \to D_0$ and $\widetilde{C}_{k0} \to C_{k0}$ ($k = L, R$). Moreover,
- each irreducible component of the cover $\widetilde{D}_0 \to D_0$ is totally branched over two points (either nodes or markings) and branched nowhere else.

- the covering $\widetilde{C}_{L0} \to C_{L0}$ is branched with monodromy
$$\lambda_0,\ (2)^{b_0^L},\ 1^{u_0^L},\ \sigma_0 \text{ (resp. } \lambda_0,\ \rho_0,\ (2)^{b_0^L},\ 1^{u_0^L},\ \sigma_0)$$
around markings and P_L.
- the covering $\widetilde{C}_{R0} \to C_{R0}$ is branched with monodromy
$$\rho_0,\ (2)^{b_0^R},\ 1^{u_0^R},\ \sigma_0 \quad \text{(resp. } (2)^{b_0^R},\ 1^{u_0^R},\ \sigma_0)$$
around markings and P_R.

(iii) In the cover $\widetilde{D}_0$, there exists a unique chain ε formed by rational curves not contracted by $\widetilde{f}_0$. Additionally,
- ε possesses $j-i+1$ irreducible components which are mapped to $E_i,\dots,E_j$ with the same degree d under the map $\widetilde{f}_0$.
- the contracted components attached to the two ends of ε collapse to x_i and x_{j+1} respectively. □

Now we turn our attention to $\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \Lambda_1, \Lambda_2, (a, d\mathcal{E}_{ij}))$. We fix $\vec{b}^L$ and $\vec{b}^R$, tuples of nonnegative integers, with $|\vec{b}^L| = u_0^L$ and $|\vec{b}^R| = u_0^R$. We define
$$\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b_0^L,u_0^L}(\vec{\lambda}, \vec{b}^L \mid \vec{b}^R, \vec{\rho})[i,j,s] = \{\overline{M}^{b_0^L,\sigma_0,u_0^L}(1)\}$$
where each $\overline{M}^{b_0^L,\sigma_0,u_0^L}(1)$ is a union of $\mathbb{T}$–fixed connected components of the space $\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \Lambda_1, \Lambda_2, (a, d\mathcal{E}_{ij}))$ (so $\Lambda_1 = \widehat{\lambda}$ and $\Lambda_2 = \widehat{\rho}$ as partitions) such that any element $[f\colon \mathcal{C} \to [\mathrm{Sym}^n(\mathcal{A}_r)]] \in \overline{M}^{b_0^L,\sigma_0,u_0^L}(1)$ satisfies the following properties:

(a) The domain curve $\mathcal{C}$ of f decomposes into three pieces
$$\mathcal{C} = \mathcal{C}_L \cup \mathcal{D} \cup \mathcal{C}_R,$$
where $\mathcal{C}_k$'s are disjoint f–contracted components; $\mathcal{D}$ is a chain of noncontracted components, which maps to $[\mathrm{Sym}^n(\mathcal{A}_r)]$ with degree $d\mathcal{E}_{ij}$; and the intersection $\mathcal{C}_k \cap \mathcal{D} := \{\mathcal{Q}_k\}$ is a twisted point for $k = L, R$.

As in (2-3), there is an associated morphism $\widetilde{f}\colon \widetilde{C} \to \mathcal{A}_r$. Let D, C, C_k, Q_k be coarse moduli spaces of $\mathcal{D}, \mathcal{C}, \mathcal{C}_k, \mathcal{Q}_k$ respectively ($k = L, R$).

(b) $\mathcal{C}_L$ carries $b_0^L + u_0^L + 1$ marked points, and $\mathcal{C}_R$ carries the other $b_0^R + u_0^R + 1 = a - b_0^L - u_0^L + 1$ marked points.

(c) The covering $\widetilde{C} \to C$ has components
$$\widetilde{C}_k := \widetilde{C}_{Lk} \cup \widetilde{D}_k \cup \widetilde{C}_{Rk}, \quad k = 0, \dots, r+1.$$

For $k \neq 0$, $\widetilde{C}_k$, if nonempty, is contracted to x_k in $\mathcal{A}_r$. (Note that $\widetilde{C}_k$ is possibly empty or disconnected for $k \neq 0$, and we include empty sets just for the simplicity of notation.)

(d) For $k = 0, \dots, r+1$,

- the covering $\coprod_{k=0}^{r+1} \widetilde{C}_{Lk} \to C_L$ (resp. $\coprod_{k=0}^{r+1} \widetilde{C}_{Rk} \to C_R$) is ramified with monodromy
$$\widehat{\lambda},\ (2)^{b_0^L + u_0^L},\ \widehat{\sigma} \quad (\text{resp. } \widehat{\rho},\ (2)^{b_0^R + u_0^R},\ \widehat{\sigma})$$
around markings and Q_L (resp. Q_R);
- each irreducible component of the cover $\widetilde{D}_k \to D$ is totally branched over two points and branched nowhere else;
- each $\widetilde{C}_{Lk} \to C_L$ (resp. $\widetilde{C}_{Rk} \to C_R$) is a covering ramified with monodromy
$$\lambda_k,\ (2)^{b_k^L},\ 1^{b_0^L + u_0^L - b_k^L},\ \sigma_k \quad (\text{resp. } \rho_k,\ (2)^{b_k^R},\ 1^{b_0^R + u_0^R - b_k^R},\ \sigma_k)$$
around markings and Q_L (resp. Q_R).

(e) The diagram of maps

$$\begin{array}{ccc} \widetilde{C}_0 & \xrightarrow{\tilde{f}|_{\widetilde{C}_0}} & \mathcal{A}_r \\ \downarrow & & \\ C & & \end{array} \tag{3-4}$$

corresponds to $[f_0] \in \overline{M}_0^{b_0^L, \sigma_0, u_0^L}(1)$ above. □

Note that $\mathcal{F}^{\vec{\sigma}}_{\lambda_0, \sigma_0, \rho_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{b}^L \mid \vec{b}^R, \vec{\rho})[i, j, s]$ does not exist for certain parameters. If it does, it is indexed by $\overline{M}_0^{b_0^L, \sigma_0, u_0^L}(1)$'s.

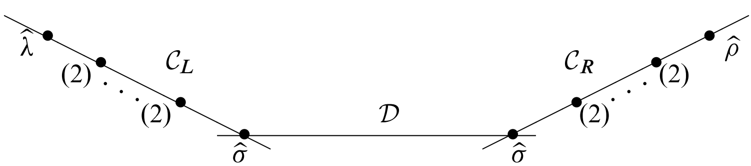

Figure 3. This is the configuration of a typical domain curve $\mathcal{C}$ for $\overline{M}^{b_0^L, \sigma_0, u_0^L}(1)$. Each straight line represents a chain of curves. All markings and $\mathcal{Q}_k$'s are labeled with their monodromy, and there are $b_0^k + u_0^k$ copies of (2) on $\mathcal{C}_k$, $k = L, R$. In case $b_0^k + u_0^k = 0$, $\mathcal{C}_k$ is simply a twisted point. Details on the covering $\widetilde{\mathcal{C}}$ associated to $\mathcal{C}$ are included in the above properties.

Define

$$\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho},\vec{b}^L\mid\vec{b}^R)[i,j,s]=\{\overline{M}^{b_0^L,\sigma_0,u_0^L}(2)\}$$

in an analogous manner. The differences occur in properties (b), (d) and (e). Precisely, (b) the curve $\mathcal{C}_L$ carries $b_0^L+u_0^L+2$ marked points while the curve $\mathcal{C}_R$ carries the other $b_0^R+u_0^R$ marked points; (d) the covering $\coprod_{k=0}^{r+1}\widetilde{C}_{Lk}\to C_L$ (resp. $\coprod_{k=0}^{r+1}\widetilde{C}_{Rk}\to C_R$) is ramified with monodromy

$$\widehat{\lambda},\ \widehat{\rho},\ (2)^{b_0^L+u_0^L},\ \widehat{\sigma}\quad(\text{resp. }(2)^{b_0^R+u_0^R},\ \widehat{\sigma})$$

around markings and Q_L (resp. Q_R), and the monodromy associated to the cover $\widetilde{C}_{Lk}\to C_L$ (resp. $\widetilde{C}_{Rk}\to C_R$) is now

$$\lambda_k,\ \rho_k,\ (2)^{b_k^L},\ 1^{b_0^L+u_0^L-b_k^L},\ \sigma_k\quad(\text{resp. }(2)^{b_k^R},\ 1^{b_0^R+u_0^R-b_k^R},\ \sigma_k);$$

(e) the diagram (3-4) corresponds to $[f_0]\in\overline{M}_0^{b_0^L,\sigma_0,u_0^L}(2)$.

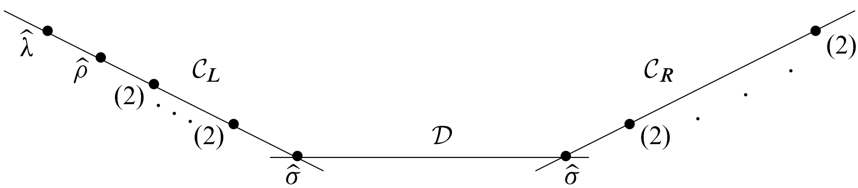

Figure 4. This is the configuration of a typical domain curve $\mathcal{C}$ for $\overline{M}^{b_0^L,\sigma_0,u_0^L}(2)$. There are $b_0^k+u_0^k$ copies of (2) on $\mathcal{C}_k$, $k=L,R$. $\mathcal{C}_L$ is always a twisted curve. $\mathcal{C}_R$ is of dimension $\epsilon_2(b_0^R+u_0^R)$; in particular, it is a twisted point when $b_0^R+u_0^R\le 1$.

Let

$$\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho};\vec{b}^L,\vec{b}^R)[i,j,s]$$

be the union

$$\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b_0^L,u_0^L}(\vec{\lambda},\vec{b}^L\mid\vec{b}^R,\vec{\rho})[i,j,s]\cup\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho},\vec{b}^L\mid\vec{b}^R)[i,j,s].$$

The components of its elements are said to be of type I.

The $\mathbb{T}$–fixed connected component of $\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)],\Lambda_1,\Lambda_2,(a,d\mathcal{E}_{ij}))$ which is not a component of any element in $\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho};\vec{b}^L,\vec{b}^R)[i,j,s]$ is said to be of type II.

We will suppress the indices $b_0^L,\sigma_0,u_0^L,(k)$ $(k=L,R)$ from $\overline{M}^{b_0^L,\sigma_0,u_0^L}(k)$ and $\overline{M}_0^{b_0^L,\sigma_0,u_0^L}(k)$ and simply write $\overline{M}$ and $\overline{M}_0$. For each

$$\overline{M}\in\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho};\vec{b}^L,\vec{b}^R)[i,j,s],$$

we let

$$\overline{M}_{\mathbb{T}}$$

be the collection of all $\mathbb{T}$–fixed components of $\overline{M}$.

3.1.2 Proof of Lemma 3.1 Let F be any $\mathbb{T}$–fixed connected component of

$$\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \Lambda_1, \Lambda_2; (a, \beta)).$$

Let $f\colon \mathcal{C} \to [\mathrm{Sym}^n(\mathcal{A}_r)]$ represent a point of F. As discussed earlier, there are a morphism $\widetilde{f}\colon \widetilde{C} \to \mathcal{A}_r$ and an ordinary stable map $f_c\colon C \to \mathrm{Sym}^n(\mathcal{A}_r)$ associated to f. We set $\tau = -(r+1)^2 t_1^2$. To establish the assertion, we need to analyze the contribution of following situations to the inverse Euler class $1/e_{\mathbb{T}}(N_F^{\mathrm{vir}})$.

(1) *Infinitesimal deformations and obstructions of f with $\mathcal{C}$ held fixed:*

(a) Any contracted component contributes zero (t_1+t_2)–valuation: Let $\mathcal{C}' \subset \mathcal{C}$ be a contracted component and pick any connected component Z of the cover associated to $\mathcal{C}'$. We see that Z contributes

$$\frac{e_{\mathbb{T}}(H^1(Z, \widetilde{f}^* T\mathcal{A}_r))}{e_{\mathbb{T}}(H^0(Z, \widetilde{f}^* T\mathcal{A}_r))} \tag{3-5}$$

and is collapsed by $\widetilde{f}$ to x_k for some k. So the numerator is, by Mumford's relation, congruent modulo t_1+t_2 to

$$\Lambda^\vee(L_k)\Lambda^\vee(R_k) \equiv \tau^g,$$

where $g = \mathrm{rank}(H^0(Z, \omega_Z))$ and $\Lambda^\vee(t) = \sum_{i=0}^g c_i(H^0(Z, \omega_Z)^\vee) t^{g-i}$. The denominator of (3-5) is $e_{\mathbb{T}}(T_{x_k}\mathcal{A}_r)$. Thus, the contribution of Z is simply

$$\tau^{g-1} \mod (t_1+t_2).$$

In other words, the contribution of $\mathcal{C}'$, being the product of the contributions of such Z's, is not divisible by t_1+t_2.

(b) The nodes joining contracted curves to noncontracted curves have zero (t_1+t_2)–valuation because each of them gives some positive power of τ modulo (t_1+t_2).

(c) Noncontracted curves: Suppose $\mathcal{D}$ is a noncontracted component with $\widetilde{D}$ its associated (possibly disconnected) covering. Its contribution is

$$\frac{e_{\mathbb{T}}(H^1(\mathcal{D}, f^*T[\mathrm{Sym}^n(\mathcal{A}_r)]))^{\mathrm{mov}}}{e_{\mathbb{T}}(H^0(\mathcal{D}, f^*T[\mathrm{Sym}^n(\mathcal{A}_r)]))^{\mathrm{mov}}} = \frac{e_{\mathbb{T}}(H^1(\widetilde{D}, \widetilde{f}^*T\mathcal{A}_r))^{\mathrm{mov}}}{e_{\mathbb{T}}(H^0(\widetilde{D}, \widetilde{f}^*T\mathcal{A}_r))^{\mathrm{mov}}}.$$

Here $(\ )^{\mathrm{mov}}$ stands for the moving part. It is clear from (a) that each $\widetilde{f}$–contracted component of $\widetilde{D}$ has zero (t_1+t_2)–valuation. However, any irreducible component Σ of $\widetilde{D}$ that is not $\widetilde{f}$–contracted contributes

$$\frac{t_1+t_2}{\tau} \quad \mathrm{mod}\ (t_1+t_2)^2. \tag{3-6}$$

This can be seen as follows. Assume that $\widetilde{f}$ maps Σ to $E := \widetilde{f}(\Sigma)$ with degree $\ell > 0$. Let $S_1 = \{0, \ldots, 2\ell-2\} - \{\ell-1\}$ and $S_2 = \{0, \ldots, 2\ell\} - \{\ell\}$.

The moving part of $e_{\mathbb{T}}(H^1(\Sigma, \widetilde{f}^* T\mathcal{A}_r)))$ arises from

$$H^1(\Sigma, \widetilde{f}^* N_{E/\mathcal{A}_r}) = H^0(\Sigma, \omega_\Sigma \otimes \widetilde{f}^* N^\vee_{E/\mathcal{A}_r})^\vee.$$

The curve E having self-intersection -2 implies $N_{E/\mathcal{A}_r} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$, and so the invertible sheaf $\omega_\Sigma \otimes \widetilde{f}^* N^\vee_{E/\mathcal{A}_r}$ has degree $2\ell-2$. Hence, the moving part is

$$(t_1+t_2) \prod_{k\in S_1} \frac{k(\frac{\ell-1}{\ell}(r+1)t_1) + (2\ell-2-k)(\frac{1-\ell}{\ell}(r+1)t_1)}{2\ell-2} \quad \mathrm{mod}\ (t_1+t_2)^2$$

(which is simply (t_1+t_2) for $\ell = 1$). We further simplify it to get

$$(t_1+t_2)\tau^{\ell-1} \prod_{k=1}^{\ell-1} \left(\frac{\ell-k}{\ell}\right)^2 \quad \mathrm{mod}\ (t_1+t_2)^2. \tag{3-7}$$

On the other hand, $e_{\mathbb{T}}(H^0(\Sigma, \widetilde{f}^* T\mathcal{A}_r))^{\mathrm{mov}}$ equals $e_{\mathbb{T}}(H^0(\Sigma, \widetilde{f}^* TE))^{\mathrm{mov}}$, that is congruent modulo (t_1+t_2) to

$$\prod_{k\in S_2} \frac{k(-(r+1)t_1) + (2\ell-k)((r+1)t_1)}{2\ell} \equiv \tau^\ell \prod_{k=1}^{\ell-1} \left(\frac{\ell-k}{\ell}\right)^2. \tag{3-8}$$

Dividing (3-7) by (3-8) gives (3-6).

(2) *Infinitesimal automorphisms of $\mathcal{C}$:*

We need only investigate the nonspecial points p, which lie on noncontracted curves Σ and are mapped to fixed points. In fact, each p gives the weight of the tangent space to Σ at p, which has zero (t_1+t_2)–valuation.

(3) *Infinitesimal deformations of $\mathcal{C}$:*

Given any node $\mathcal{P}$ joining two curves $\mathcal{V}_1$ and $\mathcal{V}_2$. Let P, V_1, V_2 be coarse moduli spaces of $\mathcal{P}, \mathcal{V}_1, \mathcal{V}_2$ respectively and $\mathrm{Stab}(\mathcal{P})$ the stabilizer of $\mathcal{P}$. In each of the following, we study the contribution arising from smoothing the node $\mathcal{P}$.

(a) $\mathcal{V}_1$ and $\mathcal{V}_2$ are noncontracted: We may assume that the restriction of f_c to V_k is a d_k–sheeted covering

$$f_c|_{V_k}\colon V_k \to \Sigma_k := f_c(V_k) \cong \mathbb{P}^1$$

for some $d_k > 0$, $k = 1, 2$. The node-smoothing contribution is

$$|\mathrm{Stab}(\mathcal{P})| \left(\frac{w_1}{d_1} + \frac{w_2}{d_2} \right)^{-1}, \tag{3-9}$$

where w_k is the tangent weight of the rational curve Σ_k at the fixed point $f_c(P)$. Thus, (3-9) is proportional to $(t_1+t_2)^{-1}$ only if $d_1 = d_2$ and $w_1 + w_2$ is a multiple of $t_1 + t_2$.

(b) $\mathcal{V}_1$ is noncontracted but $\mathcal{V}_2$ is contracted: Let w be the tangent weight of V_1 at the node P and $\mathcal{L}$ the tautological line bundle formed by the cotangent space $T_P^* V_2$. Denote by ψ the first Chern class of $\mathcal{L}$. The node smoothing contributes

$$\frac{|\mathrm{Stab}(\mathcal{P})|}{w - \psi}. \tag{3-10}$$

So, neither (t_1+t_2) nor $(t_1+t_2)^{-1}$ is generated in this case.

Thus, only the situations described in (1)(c) and (3)(a) may produce any power of (t_1+t_2). We conclude that F gives positive $(t_1{+}t_2)$–valuation because the number of noncontracted curves is more than the number of nodes joining them.

Let $\beta = d\mathcal{E}_{ij}$. Suppose F is of type I, in which case we have a unique chain of noncontracted rational components for the cover associated to $\mathcal{C}$. The discussion in (3)(a) shows that each node in the chain gives $(t_1{+}t_2)$–valuation -1. In total, the node smoothing contributes $i - j$ in valuation. On the other hand, the chain has $j - i + 1$ irreducible components. By our calculation in (1)(c), $1/e_{\mathbb{T}}(N_F^{\mathrm{vir}})$ has valuation 1, which establishes the first assertion.

Assume that F is of type II. If the associated cover has at least two disjoint chains of noncontracted rational curves, a $(t_1{+}t_2)$–valuation at least 2 is obtained because each chain gives valuation at least 1. Otherwise, the cover has a unique chain but property (e) (and hence (iii)) in Section 3.1.1 is not fulfilled for each i, j, s. In this case, we have the same consequence by the discussion in (3)(a) and the calculation in (1)(c).

If β is not a multiple of $\mathcal{E}_{ij}$ for any i, j, and F is any component, then the discussion in the preceding paragraph still works. This completes the proof of the second assertion. □

3.2 Reduction

From now on, fix cohomology-weighted partitions $\mu_1(\vec{\eta}_1)$ and $\mu_2(\vec{\eta}_2)$ of n such that each entry of the $\ell(\mu_i)$–tuple $\vec{\eta}_i$ is 1 or a divisor on the surface $\mathcal{A}_r$. We concentrate on the 2–point extended invariant

$$\langle \mu_1(\vec{\eta}_1), \mu_2(\vec{\eta}_2)\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}_{(a,\beta)} \tag{3-11}$$

in twisted degree (a,β), $\beta \neq 0$. We will leave out the superscript $[\mathrm{Sym}^n(\mathcal{A}_r)]$ when there is no likelihood of confusion.

Let us write

$$\mu_i(\vec{\eta}_i) = \kappa_i(\vec{\eta}_{i1})\theta_i(\vec{\eta}_{i2})$$

where all entries of $\vec{\eta}_{i1}$'s are 1 and all entries of $\vec{\eta}_{i2}$'s are divisors, $i = 1, 2$. We may assume that

$$\ell(\kappa_1) \leq \ell(\kappa_2).$$

Use the identity

$$1 = \sum_{k=1}^{r+1} \frac{1}{L_k R_k}[x_k],$$

we see readily that (3-11) is a $\mathbb{Q}(t_1,t_2)$–linear combination of the invariants of the form

$$\langle \kappa_{11}([x_{m_1}]) \cdots \kappa_{1\ell(\kappa_1)}([x_{m_{\ell(\kappa_1)}}])\theta_1(\vec{\eta}_{12}), \mu_2(\vec{\eta}_2)\rangle_{(a,\beta)}. \tag{3-12}$$

Additionally, (3-12) is an element of $\mathbb{Q}[t_1,t_2]$ as the first insertion has compact support. Also, the sum of the degrees of the insertions is at most 1 larger than the virtual dimension. Precisely, the difference is

$$\ell(\kappa_1) - \ell(\kappa_2) + 1.$$

Thus, the invariant (3-12) is a linear polynomial if $\ell(\kappa_1) = \ell(\kappa_2)$; otherwise, it is a rational number.

Assume that β is not a multiple of $\mathcal{E}_{ij}$ for any i, j. By Lemma 3.1, the invariant (3-12) is zero by divisibility of $(t_1+t_2)^2$ (each of the two insertions is a linear combination of fixed-point classes with coefficients being 0 or having nonnegative (t_1+t_2)–valuation; for details, consult the discussion preceding Lemma 3.6). It follows that (3-11) is zero as well. So we can now set our mind on the invariant

$$\langle \mu_1(\vec{\eta}_1), \mu_2(\vec{\eta}_2)\rangle_{(a,d\mathcal{E}_{ij})}, \quad d, i, j > 0.$$

By virtual localization, (3-12) can be expressed as a sum of residue integrals over $\mathbb{T}$–fixed components. By Lemma 3.1, the invariant (3-12) is $\alpha(t_1+t_2)$ for some rational number α, and it suffices to evaluate (3-12) over all $\mathbb{T}$–fixed components of the elements in the union $\coprod^{i,j} \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho};\vec{b}^L,\vec{b}^R)[i,j,s]$, where $\coprod^{i,j}$ means that only i, j are fixed and other parameters can vary.

For any nonnegative integer s, let $I(s)$ be the total contribution of the components of the elements lying in the union $\coprod^{i,j,s} \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho};\vec{b}^L,\vec{b}^R)[i,j,s]$ (all but i, j, s vary) to the invariant (3-12) with $\beta = d\mathcal{E}_{ij}$.

The following lemma, due to the first author, is crucial for obtaining our description of 2–point extended invariants of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ in Section 3.4.

Lemma 3.2 *For any $s < a$,*

$$I(s) \equiv 0 \mod (t_1+t_2)^2.$$

3.3 Proof of Lemma 3.2

Before proving Lemma 3.2, let us explain our strategy briefly.

For any $\mathbb{T}$–fixed component F that can possibly make a contribution to $I(s)$ ($s < a$), we relate it to F^c via the morphism ϕ_F (see (3-2)). In this process, Hurwitz numbers and $\deg(\phi_F)$ emerge. Working modulo $(t_1+t_2)^2$, we use these ingredients and an expression of the inverse equivariant Euler class $1/e_{\mathbb{T}}(N_F^{\mathrm{vir}})$ to write $I(s)$ in terms of some specific connected invariants. We will see later that each of these connected invariants has an identity class insertion and is equal to 0, and so $I(s)$ will vanish modulo $(t_1+t_2)^2$.

Sections 3.3.1, 3.3.2, and 3.3.3 serve as preparation. The proof of Lemma 3.2 is given in Section 3.3.4.

3.3.1 Counting branched covers In order to demonstrate Lemma 3.2, we count certain coverings of (chains of) rational curves. Let us now review some related notions and fix notation.

For partitions $\lambda_1, \dots, \lambda_s$ of n, the Hurwitz number

$$H(\lambda_1, \dots, \lambda_s)$$

is the weighted number of possibly disconnected covers $\pi\colon X \to (\mathbb{P}^1, p_1, \dots, p_s)$ such that π are branched over $p_1, \dots, p_s$ with ramification profiles $\lambda_1, \dots, \lambda_s$ and unbranched away from $p_1, \dots, p_s$. (Each cover is counted with weight 1 over the size of its automorphism group.)

The Hurwitz number $H(\lambda_1,\dots,\lambda_s)$ is essentially a combinatorial object. It can be described combinatorially by

$$\frac{1}{n!}|\mathcal{H}(\lambda_1,\dots,\lambda_s)|.$$

Here $\mathcal{H}(\lambda_1,\dots,\lambda_s)$ is the set consisting of $(g_1,\dots,g_s)\in\prod_{i=1}^s\mathfrak{S}_n$ satisfying

(i) for each $i=1,\dots,s$, g_i has cycle type λ_i;

(ii) $g_1\cdots g_s=1$.

Let us introduce some other Hurwitz-type numbers. Let

$$\mathcal{H}_\sigma(\lambda_1,\dots,\lambda_s\mid\rho_1,\dots,\rho_t)$$

be the subset of $\mathcal{H}(\lambda_1,\dots,\lambda_s,\rho_1,\dots,\rho_t)$ such that each element $(g_1,\dots,g_s,h_1,\dots,h_t)$ has an additional property that $g_1\cdots g_s$ has cycle type σ (and so $h_1\cdots h_t$ has the same cycle type as well). Put

$$H_\sigma(\lambda_1,\dots,\lambda_s\mid\rho_1,\dots,\rho_t)=\frac{|\mathcal{H}_\sigma(\lambda_1,\dots,\lambda_s\mid\rho_1,\dots,\rho_t)|}{n!}$$

(in case σ is a vacuous partition, we set $H_\sigma(\lambda_1,\dots,\lambda_s\mid\rho_1,\dots,\rho_t)=1$).

We readily find the following relations.

Lemma 3.3 *The number $H_\sigma(\lambda_1,\dots,\lambda_s\mid\rho_1,\dots,\rho_t)$ is exactly the product*

$$|C(\sigma)|\,H(\lambda_1,\dots,\lambda_s,\sigma)\,H(\sigma,\rho_1,\dots,\rho_t).$$

Moreover, we have the equality

$$H(\lambda_1,\dots,\lambda_s,\rho_1,\dots,\rho_t)=\sum_{|\sigma|=n}H_\sigma(\lambda_1,\dots,\lambda_s\mid\rho_1,\dots,\rho_t).$$

3.3.2 Degrees Let

$$\overline{M}\in\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b_0^L,u_0^L}(\vec{\lambda},\vec{b}^L\mid\vec{b}^R,\vec{\rho})[i,j,s]$$
$$\text{(resp. }\overline{M}\in\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho},\vec{b}^L\mid\vec{b}^R)[i,j,s]).$$

As mentioned earlier, there are natural morphisms

$$\phi_F\colon F\to\overline{M}_{0,b_0^L+u_0^L+2}\times\overline{M}_{0,b_0^R+u_0^R+2}\quad\text{(resp. }\overline{M}_{0,b_0^L+u_0^L+3}\times\overline{M}_{0,b_0^R+u_0^R+1})$$

for $F\in\overline{M}_{\mathbb{T}}$ and

$$\phi_{\overline{M}_0}\colon\overline{M}_0\to\overline{M}_{0,b_0^L+u_0^L+2}\times\overline{M}_{0,b_0^R+u_0^R+2}\quad\text{(resp. }\overline{M}_{0,b_0^L+u_0^L+3}\times\overline{M}_{0,b_0^R+u_0^R+1}).$$

Obviously, $F^c = \overline{M}_0^c$ (see (3-2)).

We intend to show Lemma 3.2 by localization, which will be reduced to integrals over F^c's. So it is necessary to understand the degree

$$\deg(\phi_F)$$

of the morphism ϕ_F.

For $F \in \overline{M}_{\mathbb{T}}$ with $\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b_0^L,u_0^L}(\vec{\lambda}, \vec{b}^L \mid \vec{b}^R, \vec{\rho})[i,j,s]$, we let $[f\colon \mathcal{C}_L \cup \mathcal{D} \cup \mathcal{C}_R \to [\mathrm{Sym}^n(\mathcal{A}_r)]] \in F$ (in the notation of Section 3.1.1) be a typical element. The degree of ϕ_F is the product $m_1 \cdot m_2$. Here

- $m_1 = c_0 (o(\hat{\sigma})^{-1} \prod_{k=1}^{r+1} |C(\sigma_k)|)^{\varepsilon(F)}$ is a factor arising from the nodes, which are glued over the stack of cyclotomic gerbes. Here c_0 is an overall factor coming from nodes of the cover $\widetilde{C}_0 \to C$ (we do not have to give a careful description here as c_0 will be cancelled by an identical term in $\deg(\phi_{\overline{M}_0})$), and $\varepsilon(F)$ is the number $\epsilon_1(b_0^L + u_0^L) + \epsilon_1(b_0^R + u_0^R) + j - i$ (the terms $\epsilon_1(b_0^L + u_0^L)$ and $\epsilon_1(b_0^R + u_0^R)$ record the dimensions of C_L and C_R respectively).
- m_2 is given by

$$d^{j-i+1} m_0 \cdot \prod_{k=1}^{r+1} H(\lambda_k, (2)^{b_k^L}, 1^{b_0^L + u_0^L - b_k^L}, \sigma_k) H(\sigma_k, \sigma_k)^{j-i+1} H(\sigma_k, (2)^{b_k^R}, 1^{b_0^R + u_0^R - b_k^R}, \rho_k),$$

where d^{j-i+1} is an automorphism factor that takes care of the restriction $f|_{\mathcal{D}}$ forgotten by ϕ_F, m_0 is the contribution of $\widetilde{C}_0$, and the other terms account for the overall contribution of $\coprod_{k=1}^{r+1} \widetilde{C}_k$.

Also, the degree of $\phi_{\overline{M}_0}$ can be calculated in a similar fashion. That is,

$$\deg(\phi_{\overline{M}_0}) = c_0 \left(\frac{1}{o(\sigma_0)} \right)^{\varepsilon(F)} d^{j-i+1} m_0.$$

By Lemma 3.3, we may write $\deg(\phi_F)$ as

$$\deg(\phi_{\overline{M}_0}) \left(\frac{o(\sigma_0)}{o(\hat{\sigma})} \right)^{\varepsilon(F)} \cdot \prod_{k=1}^{r+1} H_{\sigma_k}(\lambda_k, (2)^{b_k^L}, 1^{b_0^L + u_0^L - b_k^L} \mid (2)^{b_k^R}, 1^{b_0^R + u_0^R - b_k^R}, \rho_k). \tag{3-13}$$

Similarly, for $F \in \overline{M}_{\mathbb{T}}$ with $\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho},\vec{b}^L \mid \vec{b}^R)[i,j,s]$, $\deg(\phi_F)$ is given by

$$\deg(\phi_{\overline{M}_0})\left(\frac{o(\sigma_0)}{o(\widehat{\sigma})}\right)^{\varepsilon(F)} \cdot \prod_{k=1}^{r+1} H_{\sigma_k}(\lambda_k,\rho_k,(2)^{b_k^L},1^{b_0^L+u_0^L-b_k^L} \mid (2)^{b_k^R},1^{b_0^R+u_0^R-b_k^R}), \tag{3-14}$$

where $\varepsilon(F)$ is now set to be $1+\epsilon_2(b_0^R+u_0^R)+j-i$.

Remark 3.4 The term $(o(\sigma_0)/o(\widehat{\sigma}))^{\varepsilon(F)}$ will cancel with a similar term in $1/e_{\mathbb{T}}(N_F^{\mathrm{vir}})$ (see Lemma 3.5 below). Moreover, forgetting the indices involving the partition 1 does not change the value of the Hurwitz-type numbers. We did not do this in the above formulas so as to keep track of the ramification profiles corresponding to the simple marked points.

3.3.3 Virtual normal bundles Let us determine $1/e_{\mathbb{T}}(N_F^{\mathrm{vir}})$ modulo $(t_1+t_2)^2$ for each component F of type I.

Lemma 3.5 *Given any $\mathbb{T}$–fixed connected component $F \in \overline{M}_{\mathbb{T}}$ where $\overline{M}$ is in $\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho};\vec{b}^L,\vec{b}^R)[i,j,s]$, we have the congruence equation*

$$\frac{1}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})} \equiv \left(\frac{o(\widehat{\sigma})}{o(\sigma_0)}\right)^{\varepsilon(F)} \frac{\tau^{\frac{1}{2}(a-s-\ell(\vec{\lambda})-\ell(\vec{\rho}))}}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})} \mod (t_1+t_2)^2.$$

Here $\tau = -(r+1)^2 t_1^2$, and $\varepsilon(F)$'s are as in (3-13), (3-14) respectively.

Proof We just investigate the case where $\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b_0^L,u_0^L}(\vec{\lambda},\vec{b}^L \mid \vec{b}^R,\vec{\rho})$ and $F \in \overline{M}_{\mathbb{T}}$, the other case being similar.

Let $p = \sum_{k=0}^{r+1} b_k^L$ and $q = \sum_{k=0}^{r+1} b_k^R$, and so $p+q=a$. Assume that $p,q>0$. Pick any point $[f] \in F$. Again, we follow the notation of Section 3.1.1. The contribution from the contracted component $\mathcal{C}_L$ is

$$\frac{e_{\mathbb{T}}(H^1(\mathcal{C}_L, f^*[\mathrm{Sym}^n(\mathcal{A}_r)]))}{e_{\mathbb{T}}(H^0(\mathcal{C}_L, f^*[\mathrm{Sym}^n(\mathcal{A}_r)]))} \equiv \tau^{\sum_k (g_k-1)} \mod (t_1+t_2).$$

Here g_k's are the genera of connected components of the covering associated to $\mathcal{C}_L$. We find, by Riemann–Hurwitz formula, that $\sum_k (g_k-1) = \frac{1}{2}(p-\ell(\widehat{\lambda})-\ell(\widehat{\sigma}))$. Hence $\mathcal{C}_L$ contributes

$$\tau^{\frac{1}{2}(p-\ell(\widehat{\lambda})-\ell(\widehat{\sigma}))} \mod (t_1+t_2).$$

Similarly, $\mathcal{C}_R$ contributes

$$\tau^{\frac{1}{2}(q-\ell(\widehat{\rho})-\ell(\widehat{\sigma}))} \mod (t_1+t_2).$$

And the contribution from nodes joining contracted components to $\mathcal{D}$ is

$$\tau^{2\ell(\widehat{\sigma})} \mod (t_1+t_2).$$

These three contributions, taken together, yield

$$\tau^{\frac{1}{2}(a-\ell(\widehat{\lambda})-\ell(\widehat{\rho})+2\ell(\widehat{\sigma}))} \mod (t_1+t_2).$$

One can check that the same formula holds when $p=0$ or $q=0$.

As for the cover $\widetilde{C}_{L0}\cup\widetilde{D}_0\cup\widetilde{C}_{R0}$, by a similar argument, the combined contribution of $\widetilde{C}_{L0},\widetilde{C}_{R0}$ and nodes joining $\widetilde{C}_{L0},\widetilde{C}_{R0}$ to $\widetilde{D}_0$ is given by

$$\tau^{\frac{1}{2}(s-\ell(\lambda_0)-\ell(\rho_0)+2\ell(\sigma_0))} \mod (t_1+t_2).$$

Further, the covers $\widetilde{D}_1,\dots,\widetilde{D}_{r+1}$ (including the nodes inside) contribute

$$\frac{1}{\tau^{\ell(\vec{\sigma})}} \mod (t_1+t_2).$$

We now study the infinitesimal deformations of $\mathcal{C}$. Let $k=L,R$. When $\mathcal{C}_k$ is a curve, smoothing the node $\mathcal{P}_k$ joining $\mathcal{C}_k$ to $\mathcal{D}$ contributes

$$\frac{o(\widehat{\sigma})}{w_k-\psi_k},$$

where w_k is the $\mathbb{T}$–weight of the tangent space to $c(\mathcal{D})$ at the point $c(\mathcal{P}_k)$, and ψ_k is the class associated to $T^*_{c(\mathcal{P}_k)}\mathcal{C}_k$ (see (3-10)). By property (e) in Section 3.1.1, $\widetilde{f}\colon \widetilde{C}_0\to\mathcal{A}_r$ corresponds to the point $[f_0\colon \mathcal{C}_{L0}\cup\mathcal{D}_0\cup\mathcal{C}_{R0}\to[\mathrm{Sym}^{|\lambda_0|}(\mathcal{A}_r)]]\in\overline{M}_0$, so

$$\frac{o(\sigma_0)}{w_k-\psi_k}$$

is the factor smoothing nodes joining $\mathcal{C}_{k0}$ and $\mathcal{D}_0$ and is $o(\sigma_0)/o(\widehat{\sigma})$ times the preceding factor. Similarly, the overall contributions of node smoothing inside $\mathcal{D}$ and node smoothing inside $\mathcal{D}_0$ differ by a factor $(o(\widehat{\sigma})/o(\sigma_0))^{j-i}$. Hence, deformations of $\mathcal{C}$ contribute the product of $(o(\widehat{\sigma})/o(\sigma_0))^{\varepsilon(F)}$ with the contribution of the deformations of $\mathcal{C}_{L0}\cup\mathcal{D}_0\cup\mathcal{C}_{R0}$. The term

$$\left(\frac{o(\widehat{\sigma})}{o(\sigma_0)}\right)^{\varepsilon(F)}\frac{1}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})}$$

is the combined contribution of the deformations of $\mathcal{C}$ and the unique noncontracted connected component $\widetilde{C}_0$ of the associated cover $\widetilde{C}$.

Putting all these together, we get

$$\frac{1}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})} \equiv \left(\frac{o(\widehat{\sigma})}{o(\sigma_0)}\right)^{\varepsilon(F)} \frac{1}{e_{\mathbb{T}}(N_{\overline{M}_0}^{\mathrm{vir}})} \cdot \frac{\tau^{\frac{1}{2}(a-\ell(\widehat{\lambda})-\ell(\widehat{\rho})+2\ell(\widehat{\sigma}))}}{\tau^{\frac{1}{2}(s-\ell(\lambda_0)-\ell(\rho_0)+2\ell(\sigma_0))}} \cdot \frac{1}{\tau^{\ell(\vec{\sigma})}}$$
$$\equiv \left(\frac{o(\widehat{\sigma})}{o(\sigma_0)}\right)^{\varepsilon(F)} \frac{\tau^{\frac{1}{2}(a-s-\ell(\vec{\lambda})-\ell(\vec{\rho}))}}{e_{\mathbb{T}}(N_{\overline{M}_0}^{\mathrm{vir}})} \mod (t_1+t_2)^2,$$

as desired. □

3.3.4 Vanishing and relation to connected invariants Now we are ready to prove Lemma 3.2, ie, for any $s < a$,

$$I(s) \equiv 0 \mod (t_1+t_2)^2.$$

Given $\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho};\vec{b}^L,\vec{b}^R)[i,j,s]$ and $F \in \overline{M}_{\mathbb{T}}$, we let

$$\iota_F\colon F \to \overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)],\Lambda_1,\Lambda_2,(a,d\mathcal{E}_{ij}))$$

be the natural inclusion (as partitions, $\Lambda_1 = \widehat{\lambda}$, and $\Lambda_2 = \widehat{\rho}$). We fix a nonnegative integer $s < a$ and positive integers i, j, d with $i \le j$ from here on. We also fix $\mathbb{T}$–fixed point classes $\vec{A}$, $\vec{B}$ and define

$$\mathcal{I} = \sum_{\overline{M}} \sum_{F\in \overline{M}_{\mathbb{T}}} \int_F \frac{\iota_F^*(\mathrm{ev}_1^*(\vec{A})\mathrm{ev}_2^*(\vec{B}))}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})},$$

where $\overline{M}$ is taken over all possible elements in

$$\coprod_{\sigma_0,b_0^L,u_0^L,\vec{\sigma},\vec{b}^L,\vec{b}^R} \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho};\vec{b}^L,\vec{b}^R)[i,j,s]. \tag{3-15}$$

We would like to deduce the lemma by replacing the two insertions of the invariant (3-12) ($\beta = d\mathcal{E}_{ij}$) with $\mathbb{T}$–fixed point classes. By Proposition 1.2, the coefficient

$$\frac{\langle \kappa_{11}([x_{m_1}]) \cdots \kappa_{1\ell(\kappa_1)}([x_{m_{\ell(\kappa_1)}}])\theta_1(\vec{\eta}_{12}) \mid \vec{A}\rangle}{\langle \vec{A} \mid \vec{A}\rangle} \cdot \frac{\langle \mu_2(\vec{\eta}_2) \mid \vec{B}\rangle}{\langle \vec{B} \mid \vec{B}\rangle}$$

is either zero or has nonnegative valuation with respect to t_1+t_2, and so Lemma 3.2 follows from the following lemma.

Lemma 3.6 $\mathcal{I} \equiv 0 \mod (t_1+t_2)^2$.

Proof of Lemma 3.6 The lemma is clear if the condition

$$\lambda_k \subset A_k, \quad \rho_k \subset B_k, \quad \forall\, k = 1, \dots, r+1, \tag{3-16}$$

does not hold, in which case $\mathcal{I}$ is identically zero. Now we assume (3-16), and the idea of the proof in this case is to relate $\mathcal{I}$ to certain connected invariants. We put

$$\overline{\lambda}_k = A_k - \lambda_k, \quad \overline{\rho}_k = B_k - \rho_k.$$

That is, we may write

$$\vec{A} = ((\lambda_1, \overline{\lambda}_1), \dots, (\lambda_{r+1}, \overline{\lambda}_{r+1})), \quad \vec{B} = ((\rho_1, \overline{\rho}_1), \dots, (\rho_{r+1}, \overline{\rho}_{r+1})).$$

Let

$$\overline{A} = (\overline{\lambda}_1, \dots, \overline{\lambda}_{r+1}) \quad \text{and} \quad \overline{B} = (\overline{\rho}_1, \dots, \overline{\rho}_{r+1})$$

be $\mathbb{T}$–fixed point classes.

For simplicity, we drop the index $[i, j, s]$ from (3-15). First, it is good to have some observations on hand.

Lemma 3.7 *For any partition σ_0 and $(r+1)$–tuples $\vec{b}^L, \vec{b}^R$, $\vec{\sigma}$,*

$$J_1(\sigma_0; b_0^L, u_0^L) := \sum_{\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0, \sigma_0, \rho_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{b}^L \,|\, \vec{b}^R, \vec{\rho})} \deg(\phi_{\overline{M}_0}) \int_{\overline{M}_0^c} \frac{\iota^*_{\overline{M}_0}(\mathrm{ev}_1^*(\overline{A})\mathrm{ev}_2^*(\overline{B}))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})}$$

is

$$\sum_{\overline{M} \in \mathcal{F}^{\vec{\theta}}_{\lambda_0, \sigma_0, \rho_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{c}^L \,|\, \vec{c}^R, \vec{\rho})} \deg(\phi_{\overline{M}_0}) \int_{\overline{M}_0^c} \frac{\iota^*_{\overline{M}_0}(\mathrm{ev}_1^*(\overline{A})\mathrm{ev}_2^*(\overline{B}))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})},$$

and

$$J_2(\sigma_0; b_0^L, u_0^L) := \sum_{\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0, \rho_0, \sigma_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{\rho}, \vec{b}^L \,|\, \vec{b}^R)} \deg(\phi_{\overline{M}_0}) \int_{\overline{M}_0^c} \frac{\iota^*_{\overline{M}_0}(\mathrm{ev}_1^*(\overline{A})\mathrm{ev}_2^*(\overline{B}))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})}$$

is

$$\sum_{\overline{M} \in \mathcal{F}^{\vec{\theta}}_{\lambda_0, \rho_0, \sigma_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{\rho}, \vec{c}^L \,|\, \vec{c}^R)} \deg(\phi_{\overline{M}_0}) \int_{\overline{M}_0^c} \frac{\iota^*_{\overline{M}_0}(\mathrm{ev}_1^*(\overline{A})\mathrm{ev}_2^*(\overline{B}))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})},$$

for any $\vec{c}^L$, $\vec{c}^R$ and $\vec{\theta}$ satisfying $|\theta_k| = |\sigma_k|$ for each $k = 1, \dots, r+1$. Here the collections of unions of $\mathbb{T}$–fixed components under the summation symbols are all nonempty.

Proof The first identity follows as

$$\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b_0^L,u_0^L}(\vec{\lambda},\vec{b}^L \mid \vec{b}^R,\vec{\rho}) \quad \text{and} \quad \mathcal{F}^{\vec{\theta}}_{\lambda_0,\sigma_0,\rho_0;b_0^L,u_0^L}(\vec{\lambda},\vec{c}^L \mid \vec{c}^R,\vec{\rho})$$

have the same number of elements and the same configuration for the unique noncontracted connected component of the associated cover (see the description in Section 3.1.1). The second identity holds for similar reasons. □

We apply Lemma 3.1 to the connected invariant

$$\langle \bar{A}, \bar{B}, (2)^s, 1^{a-s}\rangle^{\mathrm{conn}}_{d\mathcal{E}_{ij}} \tag{3-17}$$

and find that (3-17) is given by

$$\sum_{\sigma_0,b_0^L,u_0^L} (J_1(\sigma_0;b_0^L,u_0^L)+J_2(\sigma_0;b_0^L,u_0^L)) \quad \mathrm{mod}\ (t_1+t_2)^2.$$

As $a-s>0$, (3-17) is zero. We then have

$$\sum_{\sigma_0,b_0^L,u_0^L} (J_1(\sigma_0;b_0^L,u_0^L)+J_2(\sigma_0;b_0^L,u_0^L)) \equiv 0 \quad \mathrm{mod}\ (t_1+t_2)^2. \tag{3-18}$$

Here is an elementary but helpful combinatorial fact.

Lemma 3.8 *Given nonnegative integers k, p and $p_1,\dots,p_k$ with $p_1+\cdots+p_k=p$. For any nonnegative integer $m\le p$,*

$$\binom{p}{p_1,\dots,p_k} = \sum_{m_1,\dots,m_k} \binom{m}{m_1,\dots,m_k}\binom{p-m}{p_1-m_1,\dots,p_k-m_k}.$$

Note that $\binom{\ell}{\ell_1,\dots,\ell_k}:=0$ if ℓ is smaller than some of ℓ_i's or if some entries are negative integers. □

We continue the proof of Lemma 3.6. We set again $\tau=-(r+1)^2t_1^2$. Let

$$\theta = \frac{1}{2}(a-s+\ell(\vec{\lambda})+\ell(\vec{\rho})).$$

For any $(r+1)$–tuple $\vec{q}$ with $|\vec{q}|=a-s$, let

$$Q(\vec{q}) = \{(\vec{b}^L,\vec{b}^R) \mid b_k^L+b_k^R=q_k,\ \forall k=1,\dots,r+1\}.$$

Fix σ_0, b_0^L, u_0^L, we consider two cases:

(1) The total contribution of $\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b^L_0,u^L_0}(\vec{\lambda},\vec{b}^L\mid\vec{b}^R,\vec{\rho})$'s to $\mathcal{I}$ with the constraint $(\vec{b}^L,\vec{b}^R)\in Q(\vec{q})$ is congruent modulo $(t_1+t_2)^2$ to

$$\sum_{(\vec{b}^L,\vec{b}^R)\in Q(\vec{q})}\sum_{\vec{\sigma}}\sum_{\overline{M}}\sum_{F\in\overline{M}_{\mathbb{T}}}\int_F\frac{\iota_F^*(\mathrm{ev}_1^*(A)\mathrm{ev}_2^*(B))}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})} \tag{3-19}$$

where $\overline{M}\in\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b^L_0,u^L_0}(\vec{\lambda},\vec{b}^L\mid\vec{b}^R,\vec{\rho})$ runs through all elements. Equation (1-1) implies that for each $F\in\overline{M}_{\mathbb{T}}$,

$$\iota_F^*(\mathrm{ev}_1^*(A)\cdot\mathrm{ev}_2^*(B))\equiv\tau^{\ell(\vec{\lambda})+\ell(\vec{\rho})}\,\iota_{\overline{M}_0}^*(\mathrm{ev}_1^*(\overline{A})\cdot\mathrm{ev}_2^*(\overline{B}))\mod (t_1+t_2).$$

Applying the pushforward ϕ_{F*} and Lemma 3.5, (3-19) is given by

$$\tau^\theta\sum_{(\vec{b}^L,\vec{b}^R)\in Q(\vec{q})}\sum_{\vec{\sigma}}\sum_{\overline{M}\in\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b^L_0,u^L_0}(\vec{\lambda},\vec{b}^L\mid\vec{b}^R,\vec{\rho})}\sum_{F\in\overline{M}_{\mathbb{T}}}\deg(\phi_F)$$
$$\cdot\left(\frac{o(\hat{\sigma})}{o(\sigma_0)}\right)^{\varepsilon(F)}\int_{\overline{M}_0^c}\frac{\iota_{\overline{M}_0}^*(\mathrm{ev}_1^*(\overline{A})\mathrm{ev}_2^*(\overline{B}))}{e_{\mathbb{T}}(N_{\overline{M}_0}^{\mathrm{vir}})}\mod (t_1+t_2)^2.$$

By (3-13), (3-19) is congruent modulo $(t_1+t_2)^2$ to

$$\tau^\theta\sum_{(\vec{b}^L,\vec{b}^R)\in Q(\vec{q})}\binom{u_0^L}{b_1^L,\dots b_{r+1}^L}\binom{u_0^R}{b_1^R,\dots,b_{r+1}^R}$$
$$\cdot\sum_{\vec{\sigma}}\prod_{k=1}^{r+1}H_{\sigma_k}(\lambda_k,(2)^{b_k^L},1^{b_0^L+u_0^L-b_k^L}\mid(2)^{b_k^R},1^{b_0^R+u_0^R-b_k^R},\rho_k)$$
$$\cdot\sum_{\overline{M}\in\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b^L_0,u^L_0}(\vec{\lambda},\vec{b}^L\mid\vec{b}^R,\vec{\rho})}\deg(\phi_{\overline{M}_0})\int_{\overline{M}_0^c}\frac{\iota_{\overline{M}_0}^*(\mathrm{ev}_1^*(\overline{A})\mathrm{ev}_2^*(\overline{B}))}{e_{\mathbb{T}}(N_{\overline{M}_0}^{\mathrm{vir}})},$$

where the product

$$\binom{u_0^L}{b_1^L,\dots b_{r+1}^L}\binom{u_0^R}{b_1^R,\dots,b_{r+1}^R}$$

is the number of choices to distribute simple ramification points lying above simple markings. By Lemmas 3.3, 3.7 and 3.8, the above expression can be simplified to

$$\binom{a-s}{q_1,\dots,q_{r+1}}\tau^\theta\prod_{k=1}^{r+1}H(\lambda_k,(2)^{q_k},1^{a-q_k},\rho_k)J_1(\sigma_0;b_0^L,u_0^L).$$

(2) By a similar argument, the total contribution of $\mathcal{F}^{\vec{\sigma}}_{\vec{\lambda}_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho},\vec{b}^L \mid \vec{b}^R)$'s to $\mathcal{I}$ with the constraint $(\vec{b}^L,\vec{b}^R)\in Q(\vec{q})$ is congruent modulo $(t_1+t_2)^2$ to

$$\binom{a-s}{q_1,\dots,q_{r+1}}\tau^\theta\prod_{k=1}^{r+1}H(\lambda_k,(2)^{q_k},1^{a-q_k},\rho_k)J_2(\sigma_0;b_0^L,u_0^L).$$

As a consequence, $\mathcal{I}$ is given by

$$H\cdot\sum_{\sigma_0,b_0^L,u_0^L}(J_1(\sigma_0;b_0^L,u_0^L)+J_2(\sigma_0;b_0^L,u_0^L))\mod(t_1+t_2)^2,$$

where $$H=\sum_{|\vec{q}|=a-s}\binom{a-s}{q_1,\dots,q_{r+1}}\tau^\theta\prod_{k=1}^{r+1}H(\lambda_k,(2)^{q_k},1^{a-q_k},\rho_k)\neq 0.$$

By (3-18),

$$\mathcal{I}\equiv 0\mod(t_1+t_2)^2.$$

This shows Lemma 3.6 and ends the proof of Lemma 3.2. □

3.4 Combinatorial descriptions of two-point extended invariants

By Lemma 3.2, we can deduce the following formula on 2–point extended invariants in nonzero degrees.

Theorem 3.9 *Given cohomology-weighted partitions $\mu_1(\vec{\eta}_1)$ and $\mu_2(\vec{\eta}_2)$ of n such that each entry of the $\ell(\mu_i)$–tuple $\vec{\eta}_i$ is 1 or a divisor class on $\mathcal{A}_r$ for $i=1,2$. For any curve class $\beta\neq 0$, the invariant*

$$\langle\mu_1(\vec{\eta}_1),\mu_2(\vec{\eta}_2)\rangle_{(a,\beta)}\tag{3-20}$$

is given by the sum

$$\sum\langle\theta(\vec{\xi}_1)\mid\theta(\vec{\xi}_2)\rangle\langle\nu_1(\vec{\gamma}_1),\nu_2(\vec{\gamma}_2)\rangle^{\mathrm{conn}}_{(a,\beta)}.\tag{3-21}$$

Here the sum is taken over all possible cohomology-weighted partitions $\theta(\vec{\xi}_1)$, $\theta(\vec{\xi}_2)$, $\nu_1(\vec{\gamma}_1)$, $\nu_2(\vec{\gamma}_2)$ satisfying $\mu_1(\vec{\eta}_1)=\theta(\vec{\xi}_1)\nu_1(\vec{\gamma}_1)$ and $\mu_2(\vec{\eta}_2)=\theta(\vec{\xi}_2)\nu_2(\vec{\gamma}_2)$. (In particular, ν_1,ν_2 are subpartitions of μ_1,μ_2 respectively and $\mu_1-\nu_1=\theta=\mu_2-\nu_2$).

Proof of Theorem 3.9 The statement is clear if β is not a multiple of $\mathcal{E}_{ij}$ for each i, j because both (3-20) and (3-21) vanish (see Theorem 3.14 below). Now fix $i, j, d > 0$ and let $\beta = d\mathcal{E}_{ij}$.

We learn by Lemma 3.2 that $I(a)$ is the only possible contribution to (3-12). In other words, only

$$\mathcal{F}_{\sigma_0,b}(\lambda_0,\rho_0;\vec{\sigma}) := \mathcal{F}_1 \cup \mathcal{F}_2,$$

ranging over all possible $\lambda_0, \rho_0, \sigma_0, b, \vec{\sigma}$, can possibly make a contribution. Here

$$\mathcal{F}_1 = \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b,0}(\vec{\sigma},(0,\dots,0)\mid(0,\dots,0),\vec{\sigma})[i,j,a], \tag{3-22}$$

$$\mathcal{F}_2 = \mathcal{F}^{(1^n)}_{\lambda_0,\rho_0,\sigma_0;b,0}(\vec{\sigma},\vec{\sigma},(0,\dots,0)\mid(0,\dots,0))[i,j,a]. \tag{3-23}$$

(With notation of Section 3.1.1, any admissible cover $\widetilde{C}$, associated to the components of the elements in $\mathcal{F}_{\sigma_0,b}(\lambda_0,\rho_0;\vec{\sigma})$, has all those simple ramification points that are branched over simple markings in the connected component $\widetilde{C}_0$, and each $\widetilde{C}_k$ $(k \neq 0)$ is either empty or a chain of rational curves.)

Thus, in order to evaluate the invariant (3-20), it is enough to perform localization calculations over $\mathbb{T}$–fixed components of the elements in $\mathcal{F}_{\sigma_0,b}(\lambda_0,\rho_0;\vec{\sigma})$'s because (3-20) is a linear combination of invariants of the form (3-12).

We have a lemma on the inverse Euler classes of virtual normal bundles.

Lemma 3.10 *Given $F \in \overline{M}_{\mathbb{T}}$ with $\overline{M} \in \mathcal{F}_1 \cup \mathcal{F}_2$, we have*

$$\frac{1}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})} = \left(\frac{o(\hat{\sigma})}{o(\sigma_0)}\right)^{\varepsilon_k(F)} \frac{1}{t(\tilde{\sigma})\, e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})}$$

for $\overline{M} \in \mathcal{F}_k$, $k = 1, 2$. Here $\varepsilon_1(F) = \epsilon_1(b) + \epsilon_1(a-b) + j - i$ and $\varepsilon_2(F) = 1 + \epsilon_2(a-b) + j - i$.

Proof All contracted connected components of the associated cover are necessarily of genus 0. The proof of Lemma 3.5 can be carried through. □

We let

$$\mathcal{I}(\nu_1,\nu_2) \quad \text{and} \quad \mathcal{I}(\nu_1,\nu_2;\vec{\sigma})$$

be the contributions to (3-20) of $\coprod_{\sigma_0,b,\vec{\sigma}} \mathcal{F}_{\sigma_0,b}(\nu_1,\nu_2;\vec{\sigma})$ and $\coprod_{\sigma_0,b} \mathcal{F}_{\sigma_0,b}(\nu_1,\nu_2;\vec{\sigma})$ respectively.

Now we compute $\mathcal{I}(\nu_1,\nu_2;\vec{\sigma})$. In order for the contribution not to vanish, the partitions ν_1 and ν_2 must be subpartitions of μ_1 and μ_2 respectively.

We assume $\nu_1 \subset \mu_1$, $\nu_2 \subset \mu_2$. The configurations (3-22) and (3-23) force $\mu_1 - \nu_1 = \mu_2 - \mu_2$. We set $\theta = \mu_1 - \nu_1$.

Recall $\alpha_{\lambda(\vec{\eta})}(\widetilde{\delta})$ is the component of $\lambda(\vec{\eta})$ relative to the fixed-point class $\widetilde{\delta}$ (see (1-2)).

Lemma 3.11 *Given $\overline{M} \in \coprod_{b,\sigma_0} \mathcal{F}_{\sigma_0}^{\vec{\sigma}}(\nu_1, \nu_2, \vec{\sigma})$. For $k = 1, 2$ and each $F \in \overline{M}_{\mathbb{T}}$,*

$$\iota_F^* \mathrm{ev}_k^*(\mu_k(\vec{\eta}_k)) = t(\widetilde{\sigma}) \sum_{P_k} \alpha_{\theta(\vec{\xi}_k)}(\widetilde{\sigma})\, \iota_{\overline{M}_0}^* \mathrm{ev}_k^*(\nu_k(\vec{\gamma}_k)). \tag{3-24}$$

Here P_k means that we take the sum over all possible $\theta(\vec{\xi}_k)$ and $\nu_k(\vec{\gamma}_k)$ satisfying $\mu_k(\vec{\eta}_k) = \theta(\vec{\xi}_k)\nu_k(\vec{\gamma}_k)$.

Proof The left side of (3-24) is $\sum_{\widetilde{\delta} \supset \widetilde{\sigma}} \alpha_{\mu_k(\vec{\eta}_k)}(\widetilde{\delta})\, t(\widetilde{\delta})$. By Proposition 1.2, it equals

$$\sum_{\widetilde{\delta} \supset \widetilde{\sigma}} \sum_{P_k} \alpha_{\theta(\vec{\xi}_k)}(\widetilde{\sigma}) \alpha_{\nu_1(\vec{\gamma}_k)}(\widetilde{\delta} - \widetilde{\sigma})\, t(\widetilde{\delta}) = t(\widetilde{\sigma}) \sum_{P_k} \alpha_{\theta(\vec{\xi}_k)}(\widetilde{\sigma}) \sum_{\widetilde{\epsilon}} \alpha_{\nu_1(\vec{\gamma}_k)}(\widetilde{\epsilon})\, t(\widetilde{\epsilon}),$$

which gives the right side of (3-24). □

It follows from Lemma 3.11 that for each $F \in \overline{M}_{\mathbb{T}}$, $\iota_F^*(\mathrm{ev}_1^*(\mu_1(\vec{\eta}_1)) \cdot \mathrm{ev}_2^*(\mu_2(\vec{\eta}_2)))$ coincides with

$$t(\widetilde{\sigma})^2 \sum_{Q} \alpha_{\theta(\vec{\xi}_1)}(\widetilde{\sigma}) \alpha_{\theta(\vec{\xi}_2)}(\widetilde{\sigma}) \iota_{\overline{M}_0}^* (\mathrm{ev}_1^*(\nu_1(\vec{\gamma}_1)) \cdot \mathrm{ev}_2^*(\nu_2(\vec{\gamma}_2))). \tag{3-25}$$

In the formula, the index Q means that the sum is over all possible $\theta(\vec{\xi}_1)$, $\theta(\vec{\xi}_2)$, $\nu_1(\vec{\gamma}_1))$ and $\nu_2(\vec{\gamma}_2))$ satisfying $\mu_1(\vec{\eta}_1) = \theta(\vec{\xi}_1)\nu_1(\vec{\gamma}_1)$ and $\mu_2(\vec{\eta}_2) = \theta(\vec{\xi}_2)\nu_2(\vec{\gamma}_2)$. Applying (3-25) and Lemma 3.10, the contribution $\mathcal{I}(\nu_1, \nu_2; \vec{\sigma})$ is

$$\frac{t(\widetilde{\sigma})}{a!} \sum_{Q} \alpha_{\theta(\vec{\xi}_1)}(\widetilde{\sigma}) \alpha_{\theta(\vec{\xi}_2)}(\widetilde{\sigma}) \sum_{\sigma_0, b, \overline{M}_0} \mathbb{H}(\widetilde{\sigma}) \int_{\overline{M}_0} \frac{\iota_{\overline{M}_0}^* (\mathrm{ev}_1^*(\nu_1(\vec{\gamma}_1)) \cdot \mathrm{ev}_2^*(\nu_2(\vec{\gamma}_2))}{e_{\mathbb{T}}(N_{\overline{M}_0}^{\mathrm{vir}})},$$

where $\mathbb{H}(\widetilde{\sigma}) = \prod_{k=1}^{r+1} H(\sigma_k, \sigma_k)$ is a product of Hurwitz numbers. Thus, $\mathcal{I}(\nu_1, \nu_2; \vec{\sigma})$ is simplified to

$$\mathbb{H}(\widetilde{\sigma}) t(\widetilde{\sigma}) \sum_{Q} \alpha_{\theta(\vec{\xi}_1)}(\widetilde{\sigma}) \alpha_{\theta(\vec{\xi}_2)}(\widetilde{\sigma}) \langle \nu_1(\vec{\gamma}_1), \nu_2(\vec{\gamma}_2) \rangle_{(a, d\mathcal{E}_{ij})}^{\mathrm{conn}}.$$

Adding up all possible $\mathcal{I}(\nu_1, \nu_2; \vec{\sigma})$'s, we obtain

$$\mathcal{I}(\nu_1, \nu_2) = \sum_{Q} \sum_{\widetilde{\sigma}} \mathbb{H}(\widetilde{\sigma}) t(\widetilde{\sigma}) \alpha_{\theta(\vec{\xi}_1)}(\widetilde{\sigma}) \alpha_{\theta(\vec{\xi}_2)}(\widetilde{\sigma}) \langle \nu_1(\vec{\gamma}_1), \nu_2(\vec{\gamma}_2) \rangle_{(a, d\mathcal{E}_{ij})}^{\mathrm{conn}}.$$

Moreover,

$$\langle \theta(\vec{\xi}_1) \mid \theta(\vec{\xi}_2)\rangle = \sum_{\tilde{\sigma}} \alpha_{\theta(\vec{\xi}_1)}(\tilde{\sigma})\alpha_{\theta(\vec{\xi}_2)}(\tilde{\sigma})\langle \tilde{\sigma} \mid \tilde{\sigma}\rangle = \sum_{\tilde{\sigma}} \alpha_{\theta(\vec{\xi}_1)}(\tilde{\sigma})\alpha_{\theta(\vec{\xi}_2)}(\tilde{\sigma})\mathbb{H}(\tilde{\sigma})t(\tilde{\sigma}).$$

This implies that

$$\mathcal{I}(\nu_1,\nu_2) = \sum_{Q} \langle \theta(\vec{\xi}_1) \mid \theta(\vec{\xi}_2)\rangle \langle \nu_1(\vec{\gamma}_1), \nu_2(\vec{\gamma}_2)\rangle^{\mathrm{conn}}_{(a,d\mathcal{E}_{ij})}.$$

Consequently, by taking into account of all $\mathcal{I}(\nu_1,\nu_2)$'s, we deduce that (3-20) equals

$$\sum \langle \theta(\vec{\xi}_1) \mid \theta(\vec{\xi}_2)\rangle \langle \nu_1(\vec{\gamma}_1), \nu_2(\vec{\gamma}_2)\rangle^{\mathrm{conn}}_{(a,d\mathcal{E}_{ij})},$$

where the sum is taken over all possible choices stated in the theorem. This finishes the proof. □

Remark 3.12 In [13], the statement for the relative theory of $\mathcal{A}_r \times \mathbb{P}^1$ which corresponds to Theorem 3.9 is obtained by writing each involved (disconnected) relative invariant as a product of connected invariants. In [15], a similar statement for $\mathrm{Hilb}^n(\mathcal{A}_r)$ is proved by reducing to a certain product of moduli spaces involving punctual Hilbert schemes. It would be great if a similar phenomenon occurred in the theory of $\mathrm{Sym}^n(\mathcal{A}_r)$ since this would simplify the proofs of Lemma 3.2 and Theorem 3.9. Unfortunately, although the $\mathbb{T}$–fixed components can be arranged according to the configurations of the covers associated to the source curves, they are seemingly not related to the product of moduli spaces parametrizing the components of the associated covers in general. Thus, it seems that we can not directly apply the ideas from [13; 15] to prove our results above.

Now, it remains to determine the two-point extended connected invariants explicitly. For partitions μ,ν of n, we denote the Hurwitz number $H(\mu,\nu,(2)^b)$ (see Section 3.3.1) by

$$H^g_{\mu,\nu}$$

where $g = \frac{1}{2}(b+2-\ell(\mu)-\ell(\nu))$ is determined by the Riemann–Hurwitz formula. In general, it is not easy to obtain a closed formula for $H^g_{\mu,\nu}$. However, when $\nu=(n)$, we have the following result due to Goulden, Jackson and Vakil.

Proposition 3.13 [9] *Given any partition μ of n. The so-called one-part double Hurwitz number $H^g_{\mu,(n)}$ is the coefficient of t^{2g} in the power series expansion of*

$$\frac{(2g+\ell(\mu)-1)!\,n^{2g+\ell(\mu)-2}}{|\mathrm{Aut}(\mu)|}\frac{t/2}{\sinh(t/2)}\prod_{i=1}^{\ell(\mu)}\frac{\sinh(\mu_i t/2)}{\mu_i t/2}.$$

Given nonnegative integers a_1, a_2, let g_{a_1}, g_{a_2}, $g(a)$ be integers satisfying

$$a_k = 2g_{a_k} - 1 + \ell(\mu_k) \quad (k = 1, 2),$$
$$g(a) = \frac{1}{2}(a - \ell(\mu_1) - \ell(\mu_2) + 2).$$

For $k = 1, 2$, we put

$$[\mu_k(\vec{\gamma}_k)] = \frac{|\mathrm{Aut}(\mu_k)|}{|\mathrm{Aut}(\mu_k(\vec{\gamma}_k))|}.$$

Our connected invariants can be expressed in terms of one-part double Hurwitz numbers.

Theorem 3.14 *Assume that γ_{1k}, $\gamma_{2\ell}$'s are $E_1, \dots, E_r$ or 1 and β is a nonzero effective curve class. If $\beta = d\mathcal{E}_{ij}$ for some d, i, j and all γ_{1k}, $\gamma_{2\ell}$'s are either E_i or E_j, the invariant*

$$\langle \mu_1(\vec{\gamma}_1), \mu_2(\vec{\gamma}_2)\rangle^{\mathrm{conn}}_{(a,\beta)} \tag{3-26}$$

is given by

$$\frac{(t_1+t_2)(-1)^{g(a)}(-1-\delta_{1,r})^{\ell(\mu_1)+\ell(\mu_2)} d^{a-1}[\mu_1(\vec{\gamma}_1)][\mu_2(\vec{\gamma}_2)]}{n^{a-2}} \sum_{a_1+a_2=a} \frac{H^{g_{a_1}}_{\mu_1,(n)} H^{g_{a_2}}_{\mu_2,(n)}}{a_1!\,a_2!},$$

where $\delta_{1,r}$ is the Kronecker delta (which is 1 if $r = 1$ and 0 otherwise). Otherwise, (3-26) vanishes. Thus, by Proposition 3.13, the invariant (3-26) admits an explicit closed formula.

Proof Let $r > 1$. According to Lemma 3.1 and the discussion preceding Lemma 3.6, (3-26) is divisible by $t_1 + t_2$.

As mentioned earlier, (3-26) is a polynomial in t_1 and t_2. So if at least one of γ_{1k}, $\gamma_{2\ell}$'s is 1, the invariant must be zero because of (t_1+t_2)–divisibility and the fact that the sum of the degrees of the insertions is at most $\ell(\mu_1) + \ell(\mu_2) - 1$, which is the virtual dimension.

Assume that all γ_{1k}, $\gamma_{2\ell}$'s are $E_1, \dots, E_r$, in which case (3-26) is proportional to $(t_1 + t_2)$. By Lemma 3.1 again, the invariant is zero if β is not a multiple of $\mathcal{E}_{ij}$ for all i, j.

Now we assume further that $\beta = d\mathcal{E}_{ij}$ for some d, i, j. We may evaluate (3-26) modulo $(t_1 + t_2)^2$, so any $\mathbb{T}$–fixed component that contributes a factor $(t_1 + t_2)^k$ for some $k \geq 2$ may be ruled out. That is, it is enough to investigate those $\mathbb{T}$–fixed components defined in (3-3) with $s = a$, which we denote by F's. However, in order for the contributions of these components to (3-26) not to vanish, the ramification points lying above the distinguished markings must map to x_i or x_{j+1}. As a result, (3-26) vanishes if one of γ_{1k}, $\gamma_{2\ell}$'s is E_k for some $k \neq i, j$. This completes the proof of the second assertion.

Now we show the first assertion. Let

$$P = -\frac{1}{L_i}[x_i], \quad Q = -\frac{1}{R_{j+1}}[x_{j+1}].$$

It remains to evaluate (3-26) with γ_{1k}, $\gamma_{2\ell}$'s equal to E_i or E_j. We check that $E_i \sim P$ and $E_i \sim Q$ ("$\sim$" means that the difference between the left side and the right side can be written in terms of classes $[x_{i+1}], \dots, [x_j]$ and 1 as long as we are working modulo (t_1+t_2)); similarly, $E_j \sim P$ and $E_j \sim Q$.

By the vanishing claims just verified, we can replace all γ_{1k}'s with P and all $\gamma_{2\ell}$'s with Q. The invariant (3-26) is not exactly the resulting invariant

$$\langle \mu_{11}(P)\cdots\mu_{1\ell(\mu_1)}(P), \mu_{21}(Q)\cdots\mu_{2\ell(\mu_2)}(Q)\rangle^{\mathrm{conn}}_{(a,d\mathcal{E}_{ij})}.$$

Instead, it is congruent modulo $(t_1+t_2)^2$ to

$$\text{(3-27)} \quad J := [\mu_1(\vec{\gamma}_1)][\mu_2(\vec{\gamma}_2)]\langle \mu_{11}(P)\cdots\mu_{1\ell(\mu_1)}(P), \\ \mu_{21}(Q)\cdots\mu_{2\ell(\mu_2)}(Q)\rangle^{\mathrm{conn}}_{(a,d\mathcal{E}_{ij})}.$$

With (3-27) in mind, we can thus execute localization calculations over those F's with one more constraint on the source curve $\mathcal{C}_0$: $\mathcal{C}_{L0}$ carries the marking corresponding to μ_1, and $\mathcal{C}_{R0}$ carries the marking corresponding to μ_2 because the ramification points associated to μ_1 (resp. μ_2) are mapped to x_i (resp. x_{j+1}). This means that in (3-3), $\lambda_0 = \mu_1$, $\rho_0 = \mu_2$, and $\sigma_0 = (n)$.

To summarize, in order to evaluate (3-27), we only have to consider such $\mathbb{T}$–fixed components, denoted by F_{a_1,a_2}, where the source curve $\mathcal{C}$ decomposes into three pieces $\mathcal{C}_{a_1} \cup \Sigma \cup \mathcal{C}_{a_2}$: $\mathcal{C}_{a_k}$ is a contracted component carrying a_k simple markings, and its unique distinguished marking corresponds to μ_k; the intersection $\mathcal{C}_{a_1} \cap \mathcal{C}_{a_2}$ is empty; the cover $\widetilde{C}_{a_k}$ associated to $\mathcal{C}_{a_k}$ is of genus g_{a_k}; and Σ is a chain of noncontracted components, which connects $\mathcal{C}_{a_1}$ and $\mathcal{C}_{a_2}$, and the two twisted points of intersection have stack structures given by the monodromy (n). Note that $\mathcal{C}_{a_k}$'s are twisted points whenever they contain less than three special points and are otherwise twisted curves.

In this way, we reduce our calculation to the integral over

$$\overline{M}(\mathcal{B}\mathfrak{S}_n, \mu_1, (n); a_1) \times \overline{M}(\mathcal{B}\mathfrak{S}_n, \mu_2, (n); a_2),$$

followed by division by the product of the automorphism factor d^{j-i+1} and the distribution factor $a_1!\,a_2!$ of simple marked points.

Let $\epsilon_1\colon \overline{M}(\mathcal{B}\mathfrak{S}_n,\mu_1,(n);a_1)\to\overline{M}_{0,a_1+2}$ be the natural morphism mapping $\mathcal{C}_{a_1}$ to its coarse moduli space C_{a_1} (the node $\mathcal{C}_{a_1}\cap\Sigma$ is mapped to the marking Q_1) and $\mathcal{L}_1$ the tautological line bundle formed by the cotangent space $T^*_{Q_1}C_{a_1}$. Let $\psi_1=c_1(\mathcal{L}_1)$. We define $\epsilon_2\colon \overline{M}(\mathcal{B}\mathfrak{S}_n,\mu_2,(n);a_2)\to\overline{M}_{0,a_1+2}$ and ψ_2 in a similar way.

To proceed, we summarize the contributions of virtual normal bundles (see Lemma 3.1). Set $\theta=(r+1)t_1$.

- Contracted components: For $k=1,2$, $\mathcal{C}_{a_k}$ contributes
$$(-1)^{g_{a_k}-1}\theta^{2g_{a_k}-2}\mod (t_1+t_2).$$
- A chain of noncontracted components: The contribution of each node smoothing is just $((t_1+t_2)/d)^{-1}$. All other node contributions are L_kR_k, $k=i,\dots,j+1$, each of which equals $-\theta^2 \mod t_1+t_2$. Furthermore, all noncontracted curves contribute $((t_1+t_2)/-\theta^2)^{j-i+1}\mod (t_1+t_2)^2$. Hence the total contribution equals
$$-\theta^2 d^{j-i}(t_1+t_2)\mod (t_1+t_2)^2.$$
- Smoothing nodes joining a contracted curve to a noncontracted curve: The contributions are given by
$$\frac{1}{(1/n)(nR_i/d-\epsilon_1^*\psi_1)},\quad \frac{1}{(1/n)(nL_{j+1}/d-\epsilon_2^*\psi_2)}.$$

The contribution of the component F_{a_1,a_2} to J, denoted by I_{a_1,a_2}, is congruent modulo $(t_1+t_2)^2$ to

$$-\theta^2 d^{j-i}(t_1+t_2)\frac{[\mu_1(\vec{\gamma}_1)][\mu_2(\vec{\gamma}_2)]}{d^{j-i+1}a_1!\,a_2!}\,\theta^{\ell(\mu_1)}(-\theta)^{\ell(\mu_2)}\cdot\frac{(-1)^{a_1}}{\theta^4}\\ \times\prod_{k=1}^{2}(-1)^{g_{a_k}}\theta^{2g_{a_k}}\frac{1}{n^{a_k-1}}\left(\frac{d}{\theta}\right)^{a_k}\int_{\overline{M}(\mathcal{B}\mathfrak{S}_n,\mu_k,(n);a_k)}\epsilon_k^*\psi_k^{a_k-1}.$$

Note that each factor in the second line is replaced with 1 in case $a_k=0$. Simplifying the expression yields

$$\frac{(t_1+t_2)(-1)^{g(a)+\ell(\mu_1)+\ell(\mu_2)}d^{a-1}[\mu_1(\vec{\gamma}_1)][\mu_2(\vec{\gamma}_2)]}{n^{a-2}a_1!\,a_2!}\prod_{k=1}^{2}\int_{\overline{M}(\mathcal{B}\mathfrak{S}_n,\mu_k,(n);a_k)}\epsilon_k^*\psi_k^{a_k-1}.$$

For $a_k > 0$,

$$\int_{\overline{M}(\mathcal{B}\mathfrak{S}_n,\mu_k,(n);a_k)} \epsilon_k^*\psi_k^{a_k-1} = \deg(\epsilon_k)\int_{\overline{M}_{0,a_k+2}} \psi_k^{a_k-1} = H^{g_{a_k}}_{\mu_k,(n)}.$$

We conclude that I_{a_1,a_2} is congruent to

$$\frac{(t_1+t_2)(-1)^{g(a)+\ell(\mu_1)+\ell(\mu_2)}d^{a-1}[\mu_1(\vec{\gamma}_1)][\mu_2(\vec{\gamma}_2)]}{n^{a-2}}\cdot\frac{H^{g_{a_1}}_{\mu_1,(n)}H^{g_{a_2}}_{\mu_2,(n)}}{a_1!\,a_2!} \mod (t_1+t_2)^2.$$

The theorem then follows by summing I_{a_1,a_2} over all possible a_1,a_2 with $a_1+a_2=a$ and using the fact that (3-26) is a multiple of t_1+t_2.

The case $r=1$ is similar, and so we omit the proof. □

By applying the intersection matrix with respect to the curve classes $E_1,\dots,E_r$, we arrive at the following statement.

Corollary 3.15 *Let γ_{1k}, $\gamma_{2\ell}$'s be 1 or divisors on $\mathcal{A}_r$ and β a nonzero effective curve class. If $\beta = d\mathcal{E}_{ij}$ for some d,i,j, the connected invariant $\langle\mu_1(\vec{\gamma}_1),\mu_2(\vec{\gamma}_2)\rangle^{\mathrm{conn}}_{(a,\beta)}$ is given by*

$$(t_1+t_2)[\mu_1(\vec{\gamma}_1)][\mu_2(\vec{\gamma}_2)]\prod_{k=1}^{\ell(\mu_1)}(\mathcal{E}_{ij}\cdot\gamma_{1k})\prod_{k=1}^{\ell(\mu_2)}(\mathcal{E}_{ij}\cdot\delta_k)$$
$$\cdot\frac{(-1)^{g(a)}d^{a-1}}{n^{a-2}}\sum_{a_1+a_2=a}\frac{H^{g_{a_1}}_{\mu_1,(n)}H^{g_{a_2}}_{\mu_2,(n)}}{a_1!\,a_2!}.$$

Otherwise, it is zero. □

Theorem 3.9 and Theorem 3.14 provide an effective method to compute 2–point extended invariants of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ in nonzero degrees. With the equations in the following proposition, this also determines the divisor operators as a consequence of 3–point extended invariants in degree zero being determined by the Gromov–Witten theory of $[\mathrm{Sym}^n(\mathbb{C}^2)]$; see Cheong [8].

Proposition 3.16 *Given any classes $\alpha_1,\dots,\alpha_k\in A^*_{\mathbb{T},\mathrm{orb}}[\mathrm{Sym}^n(\mathcal{A}_r)]$. We have*

$$\langle\langle\alpha_1,\dots,\alpha_k,(2)\rangle\rangle = \frac{d}{du}\langle\langle\alpha_1,\dots,\alpha_k\rangle\rangle, \tag{3-28}$$

and for each $\ell=1,\dots,r$,

$$\langle\langle\alpha_1,\dots,\alpha_k,D_\ell\rangle\rangle = \langle\langle\alpha_1,\dots,\alpha_k,D_\ell\rangle\rangle|_{s_1,\dots,s_r=0} + s_\ell\frac{d}{ds_\ell}\langle\langle\alpha_1,\dots,\alpha_k\rangle\rangle. \tag{3-29}$$

Proof By definition,

$$\langle \alpha_1, \dots, \alpha_k, (2) \rangle_{(a,\beta)} = (a+1) \langle \alpha_1, \dots, \alpha_k \rangle_{(a+1,\beta)},$$

and by the untwisted divisor equation ($\beta \neq 0$ or $k \geq 3$),

$$\langle \alpha_1, \dots, \alpha_k, D_\ell \rangle_{(a,\beta)} = (\omega_\ell \cdot \beta) \langle \alpha_1, \dots, \alpha_k \rangle_{(a,\beta)}.$$

These relations yield (3-28) and (3-29). (Note, however, that (3-29) is read as

$$\langle\!\langle \alpha_1, \dots, \alpha_k, D_\ell \rangle\!\rangle = s_\ell \frac{d}{ds_\ell} \langle\!\langle \alpha_1, \dots, \alpha_k \rangle\!\rangle$$

for $k \geq 3$.) □

The sine function $\sin(u)$ is a rational function of e^{iu}, where $i^2 = -1$. It is straightforward to verify that extended 3–point functions involving (2) or D_ℓ are rational functions in $t_1, t_2, e^{iu}, s_1, \dots, s_r$ by the above equations.

4 Comparison to other theories

4.1 Relative Gromov–Witten theory of threefolds

Given $\mathbb{P}^1$ with k distinct marked points $p_1, \dots, p_k$, and partitions $\lambda_1, \dots, \lambda_k$ of a positive integer n. Following Maulik [13], we let

$$\overline{M}_g^\bullet(\mathcal{A}_r \times \mathbb{P}^1, (\beta, n); \lambda_1, \dots, \lambda_k)$$

be the moduli space parametrizing relative stable maps to $\mathcal{A}_r \times \mathbb{P}^1$ with the following data:

- The domains are nodal curves of genus g and are allowed to be disconnected.
- The relative stable maps have degree $(\beta, n) \in A_1(\mathcal{A}_r \times \mathbb{P}^1; \mathbb{Z})$ and have nonzero degrees on any connected components.
- The maps are ramified over the divisor $\mathcal{A}_r \times p_i$ with ramification type λ_i. The ramification points are taken to be marked and ordered.

Given any cohomology-weighed partition $\lambda_i(\vec{\eta}_i)$, $i = 1, \dots, k$, we have an evaluation map

$$\mathrm{ev}_{ij} \colon \overline{M}_g^\bullet(\mathcal{A}_r \times \mathbb{P}^1, (\beta, n); \lambda_1, \dots, \lambda_k) \to \mathcal{A}_r$$

corresponding to the ramification point of type λ_{ij} over the divisor $\mathcal{A}_r \times p_i$. The genus g relative invariant $\langle \lambda_1(\vec{\eta}_1), \dots, \lambda_k(\vec{\eta}_k)\rangle_{g,\beta}^{\mathcal{A}_r\times\mathbb{P}^1}$ is defined by

$$\frac{1}{\prod_{i=1}^k |\mathrm{Aut}(\lambda_i(\vec{\eta}_i))|} \int_{[\bar{M}_g^\bullet(\mathcal{A}_r\times\mathbb{P}^1,(\beta,n);\lambda_1,\dots,\lambda_k)]^{\mathrm{vir}}_{\mathbb{T}}} \prod_{i=1}^{k}\prod_{j=1}^{l(\lambda_i)} \mathrm{ev}_{ij}^*(\eta_{ij}).$$

We are interested in the shifted partition function

$$\mathrm{GW}(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta}_1),\dots,\lambda_k(\vec{\eta}_k)}$$

defined as

$$u^{2n-\sum_{i=1}^k \mathrm{age}(\lambda_i)} \sum_{g,\beta} \langle \lambda_1(\vec{\eta}_1), \dots, \lambda_k(\vec{\eta}_k)\rangle_{g,\beta}^{\mathcal{A}_r\times\mathbb{P}^1} u^{2g-2} s_1^{\beta\cdot\omega_1}\cdots s_r^{\beta\cdot\omega_r}.$$

Our results recover certain relative Gromov–Witten invariants by the following equalities.

Proposition 4.1 *For $\alpha = 1(1)^n$, (2) or D_k, $k = 1, \dots, r$,*

$$\langle\!\langle \lambda_1(\vec{\eta}_1), \alpha, \lambda_2(\vec{\eta}_2)\rangle\!\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} = \mathrm{GW}(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta}_1),\alpha,\lambda_2(\vec{\eta}_2)}. \tag{4-1}$$

Proof When specialized to $s_1 = \dots = s_r = 0$, the equality (4-1) has been justified in Cheong [8]. In particular, (4-1) is valid for $\alpha = 1(1)^n$ without the constraint.

For $\alpha = (2)$ or D_k, the coefficients of $u^i s_1^{j_1} \dots s_r^{j_r}$, where $j_1 + \dots + j_r > 0$, match up on both sides of (4-1) by a direct comparison of [13, Proposition 4.4] with our results in Section 3.4. Hence, (4-1) follows as well in this case. □

4.2 Quantum cohomology of Hilbert schemes of points

4.2.1 Nakajima basis We review the Nakajima basis for the equivariant cohomology $A_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r)$ of the Hilbert scheme $\mathrm{Hilb}^n(\mathcal{A}_r)$ of n points in $\mathcal{A}_r$.

Let λ be a partition of n and $\vec{\eta} = (\eta_1, \dots, \eta_{\ell(\lambda)})$ an associated $\ell(\lambda)$–tuple with entries in $A_{\mathbb{T}}^*(\mathcal{A}_r)$. Let $|0\rangle = 1 \in A_{\mathbb{T}}^0(\mathrm{Hilb}^0(\mathcal{A}_r))$, we define

$$\mathfrak{a}_\lambda(\vec{\eta}) = \frac{1}{|\mathrm{Aut}(\lambda(\vec{\eta}))|} \prod_{i=1}^{\ell(\lambda)} \frac{1}{\lambda_i} \mathfrak{p}_{-\lambda_i}(\eta_i)|0\rangle,$$

where $\mathfrak{p}_{-\lambda_i}(\eta_i)\colon A_{\mathbb{T}}^*(\mathrm{Hilb}^k(\mathcal{A}_r)) \to A_{\mathbb{T}}^{*+\lambda_i-1+\deg(\eta_i)/2}(\mathrm{Hilb}^{k+\lambda_i}(\mathcal{A}_r))$ are Heisenberg creation operators; see Grojnowski [11], Li, Qin and Wang [12] and Nakajima [16].

Choose a basis $\mathfrak{B}$ for $A_{\mathbb{T}}^*(\mathcal{A}_r)$. The classes $\mathfrak{a}_\lambda(\vec{\eta})$'s, running through all partitions λ of n and all $\eta_i \in \mathfrak{B}$, give a basis for $A_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r))$. They are called the Nakajima basis associated to $\mathfrak{B}$.

4.2.2 Quantum cup product Let $\rho_*^{\mathrm{HC}}\colon A_1(\mathrm{Hilb}^n(\mathcal{A}_r);\mathbb{Z}) \to A_1(\mathrm{Sym}^n(\mathcal{A}_r);\mathbb{Z})$ be the homomorphism induced by the Hilbert–Chow morphism $\rho^{\mathrm{HC}}\colon \mathrm{Hilb}^n(\mathcal{A}_r) \to \mathrm{Sym}^n(\mathcal{A}_r)$. There are isomorphisms

$$A_1(\mathrm{Hilb}^n(\mathcal{A}_r);\mathbb{Z}) \cong \mathrm{Ker}(\rho_*^{\mathrm{HC}}) \oplus A_1(\mathrm{Sym}^n(\mathcal{A}_r);\mathbb{Z}) \cong \mathrm{Ker}(\rho_*^{\mathrm{HC}}) \oplus A_1(\mathcal{A}_r;\mathbb{Z}).$$

Let ℓ be the class dual to the divisor $-\mathfrak{a}_1(1)^{n-2}a_2(1)$ on $\mathrm{Hilb}^n(\mathcal{A}_r)$. It is an effective rational curve class generating the kernel $\mathrm{Ker}(\rho_*^{\mathrm{HC}})$. For any classes $\alpha_1,\dots,\alpha_k$ on $\mathrm{Hilb}^n(\mathcal{A}_r)$, we consider the k–point function

$$\langle \alpha_1,\dots,\alpha_k\rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)} = \sum_{d=0}^{\infty} \sum_{\beta\in A_1(\mathcal{A}_r;\mathbb{Z})} \langle \alpha_1,\dots,\alpha_k\rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)}_{(d\ell,\beta)} q^d s_1^{\beta\cdot\omega_1}\cdots s_r^{\beta\cdot\omega_r}. \tag{4-2}$$

Now given any basis $\{\delta\}$ for $A_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r))$ and $\{\delta^\vee\}$ its dual basis. Define the small quantum cup product $*_{\mathrm{q}}$ on $A_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r))$ by the 3–point functions as follows:

$$\alpha_1 *_{\mathrm{q}} \alpha_2 = \sum_{\delta} \langle \alpha_1,\alpha_2,\delta\rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)}\delta^\vee.$$

Like the orbifold case, we define

$$QA_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r))$$

as the vector space $A_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r)) \otimes_{\mathbb{Q}[t_1,t_2]} \mathbb{Q}(t_1,t_2)((q,s_1,\dots,s_r))$ with the multiplication $*_{\mathrm{q}}$.

4.2.3 SYM/HILB correspondence In Section 3, we provide a combinatorial description of any divisor operator on the quantum ring $A_{\mathbb{T},\mathrm{orb}}^*([\mathrm{Sym}^n(\mathcal{A}_r)])$. In [15], on the other hand, any divisor operator on $A_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r))$ is expressed in terms of the action of affine Lie algebra $\widehat{\mathfrak{gl}}(r+1)$ on the basic representations. These two expressions are actually equivalent via the correspondence L given in the work [8] of the first author.

Let us make the substitution $q=-e^{iu}$ where i is a square root of -1, and put

$$F = \mathbb{Q}(i,t_1,t_2)((u,s_1,\dots,s_r)) \quad\text{and}\quad K = \mathbb{Q}(t_1,t_2)((u,s_1,\dots,s_r)).$$

We recall the map L. It is defined by

$$L(\lambda(\vec{\eta})) = (-i)^{\mathrm{age}(\lambda)}\mathfrak{a}_\lambda(\vec{\eta}).$$

Obviously, L is a one-to-one correspondence and extends to a F–linear isomorphism

$$L\colon QA_{\mathbb{T},\mathrm{orb}}^*([\mathrm{Sym}^n(\mathcal{A}_r)]) \otimes_K F \to QA_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r)) \otimes_K F.$$

(The Chow degree of $\mathfrak{a}_\lambda(\vec{\eta})$ is clearly $n-\ell(\lambda)+\sum_{k=1}^{\ell(\lambda)}\deg(\eta_k)$, the orbifold degree of $\lambda(\vec{\eta})$.)

Denote by $\langle\cdot|\cdot\rangle$ as well the equivariant Poincaré pairing on $\mathrm{Hilb}^n(\mathcal{A}_r)$. We know from [8] that L preserves (orbifold) Poincaré pairings, ie,

$$\langle\lambda(\vec{\eta})\,|\,\rho(\vec{\xi})\rangle=\langle L(\lambda(\vec{\eta}))\,|\,L(\rho(\vec{\xi}))\rangle$$

for all partitions λ,ρ of n and cohomology classes η_i,ξ_j's on $\mathcal{A}_r$. Further, we have the following SYM/HILB correspondence.

Proposition 4.2 *The F–linear isomorphism L respects quantum multiplication by divisors:*

$$L(D *_{\mathrm{orb}} \alpha)=L(D)*_{\mathrm{q}} L(\alpha) \tag{4-3}$$

for any class α and divisor D.

Proof For cohomology-weighted partitions $\lambda_1(\vec{\eta}_1)$, $\lambda_2(\vec{\eta}_2)$ and $\alpha=(2)$ or D_k,

$$\begin{aligned}\langle\!\langle\lambda_1(\vec{\eta}_1),\alpha,\lambda_2(\vec{\eta}_2)\rangle\!\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}&=\mathrm{GW}(\mathcal{A}_r\times\mathbb{P}^1)_{\lambda_1(\vec{\eta}_1),\alpha,\lambda_2(\vec{\eta}_2)}\\&=\langle L(\lambda_1(\vec{\eta}_1)),L(\alpha),L(\lambda_2(\vec{\eta}_2))\rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)}.\end{aligned}$$

The first equality is Proposition 4.1 while the second equality is [15, Proposition 6.6].

As L preserves Poincaré pairings, it follows from the above equalities that

$$\langle L(\lambda_1(\vec{\eta}_1)*_{\mathrm{orb}}\alpha)\,|\,L(\lambda_2(\vec{\eta}_2))\rangle=\langle L(\lambda_1(\vec{\eta}_1))*_{\mathrm{q}}L(\alpha)\,|\,L(\lambda_2(\vec{\eta}_2))\rangle.$$

This implies that L respects quantum multiplication by (2) and D_k's. The equality (4-3) now follows due to the fact that (2) and D_k's give a basis for divisor classes. □

5 The Crepant Resolution Conjecture

Let us study a simple example before discussing the full version of Bryan–Graber Crepant Resolution Conjecture.

5.1 An example

We would like to give an explicit expression for the divisor operator $D_1 *_{\mathrm{orb}} -$ on the quantum ring $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^2(\mathcal{A}_1)])$. Let us substitute $q=-e^{iu}$ so that

$$\sin(\gamma u)=\frac{1}{2i}\left((-q)^\gamma-\frac{1}{(-q)^\gamma}\right).$$

Consider the following basis

$$\mathfrak{B} := \{1(E_1)1(E_1),\ 2(E_1),\ 1(1)1(E_1),\ 2(1),\ 1(1)1(1)\},$$

whose elements are ordered according to their orbifold degrees. The matrix representation of the operator $D_1 *_{\mathrm{orb}} -$ with respect to $\mathfrak{B}$ is given by

$$\begin{pmatrix} 2\theta(1-\frac{1}{1+sq}-\frac{1}{1+s/q}) & i\theta(\frac{1}{1+sq}-\frac{1}{1+s/q}) & -1 & 0 & 0 \\ -2i\theta(\frac{1}{1+sq}-\frac{1}{1+s/q}) & \theta(2-\frac{1}{1+sq}-\frac{1}{1+s/q}-\frac{2}{1-s}) & 0 & -1 & 0 \\ 2t_1t_2 & 0 & \frac{-\theta(1+s)}{1-s} & 0 & -\frac{1}{2} \\ 0 & 4t_1t_2 & 0 & 0 & 0 \\ 0 & 0 & 4t_1t_2 & 0 & 0 \end{pmatrix}$$

where $\theta = t_1+t_2$ and $s = s_1$. This is also the matrix representation of the operator

$$L(D_1) *_{\mathrm{q}} -$$

with respect to the ordered basis $L(\mathfrak{B})$; see Maulik and Oblomkov [15].

It is straightforward to check that $D_1 *_{\mathrm{orb}} -$ has distinct eigenvalues. In particular, we have a basis $\{v_1, \dots, v_5\}$ of eigenvectors. By quantum multiplication by D_1 and the identity 1, we find

$$v_i *_{\mathrm{orb}} v_i = \begin{cases} a_i v_i & \text{for some } a_i \neq 0, \\ 0 & \text{for all } i \neq j. \end{cases}$$

So by replacing v_i with v_i/a_i, we may assume that $\{v_1, \dots, v_5\}$ is an idempotent basis; in which case,

$$1 = \sum_{i=1}^{5} v_i. \tag{5-1}$$

Moreover, the Vandermonde matrix associated to the eigenvalues of $D_1 *_{\mathrm{orb}} -$ is invertible. In other words, by (5-1), the set

$$\{1, D_1, D_1^2, D_1^3, D_1^4\}$$

is a basis for the quantum cohomology $QA^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^2(\mathcal{A}_1)])$. Similarly, $L(D_1)$ generates the quantum ring $QA^*_{\mathbb{T}}(\mathrm{Hilb}^2(\mathcal{A}_1)) \otimes_K F$. We conclude that

$$L\colon QA^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^2(\mathcal{A}_1)]) \otimes_K F \to QA^*_{\mathbb{T}}(\mathrm{Hilb}^2(\mathcal{A}_1)) \otimes_K F$$

is indeed an F–algebra isomorphism. □

This simple example raises the question: Do divisor classes generate the whole quantum ring? In response to this, one may wish to examine the eigenvalues of divisor operators for bigger n. This, however, seems a difficult task to perform directly.

If one of the operators $(2) *_{\mathrm{orb}} -$, $D_k *_{\mathrm{orb}} -$'s turns out to have distinct eigenvalues, the ring structure will be determined, and L will be an F–algebra isomorphism. The hypothesis has yet to be entirely verified and may seem a little too good to be true. It is reasonable to expect something weaker (maybe certain combinations of these operators work).

5.2 Generation Conjecture

The following statement is referred to as generation conjecture. The reader is urged to consult [15] for a partial evidence of the conjecture.

Conjecture 5.1 [15] *Let L be as in Section 4.2.3. The commuting family of the operators*

$$L((2)) *_{\mathrm{q}} -, \quad L(D_1) *_{\mathrm{q}} -, \quad \dots, \quad L(D_r) *_{\mathrm{q}} -$$

on the quantum cohomology of $\mathrm{Hilb}^n(\mathcal{A}_r)$ *is nonderogatory. That is, its joint eigenspaces are one-dimensional.*

Let us briefly explain some consequences of the nonderogatory conjecture on our quantum cohomology rings. Set $R = \mathbb{Q}(i, t_1, t_2, q, s_1, \dots, s_r)$ and $q = -e^{iu}$. Since the quantum ring $A^*_{\mathbb{T}}(\mathrm{Hilb}^n(\mathcal{A}_r)) \otimes_{\mathbb{Q}[t_1,t_2]} R$ is semisimple, it admits a basis, say $\{v_1, \dots, v_m\}$, of idempotent eigenvectors summing to the identity 1. The basis elements are also the simultaneous eigenvectors for $L((2)) *_{\mathrm{q}} -, L(D_1) *_{\mathrm{q}} -, \dots, L(D_r) *_{\mathrm{q}} -$.

Suppose $e_{0k}, e_{1k}, \dots, e_{rk}$ are respectively the eigenvalues of the operators $L((2)) *_{\mathrm{q}} -$, $L(D_1) *_{\mathrm{q}} -, \dots, L(D_r) *_{\mathrm{q}} -$ corresponding to the eigenvector v_k. The nonderogatory property ensures that we can find numbers $a_0, a_1, \dots, a_r$ such that

$$\sum_{j=0}^{r} a_j e_{j1}, \dots, \sum_{j=0}^{r} a_j e_{jm}$$

is a sequence of distinct elements. Thus, the Vandermonde argument given earlier shows that the element $a_0 \cdot L((2)) + \sum_{j=1}^{r} a_j \cdot L(D_j)$ generates $A^*_{\mathbb{T}}(\mathrm{Hilb}^n(\mathcal{A}_r)) \otimes_{\mathbb{Q}[t_1,t_2]} R$. This implies that $a_0 \cdot (2) + \sum_{j=1}^{r} a_j \cdot D_j$ generates the quantum cohomology of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ over R as well. We thus obtain the following "corollary"[1].

"Corollary" 5.2 *The divisor classes* (2) *and* $D_1, \dots, D_r$ *generate the quantum cohomology ring* $QA^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)])$*, and any extended three-point function is a rational function in* $t_1, t_2, e^{iu}, s_1, \dots, s_r$*. Under the substitution* $q = -e^{iu}$*, the map*

[1] Whenever we use double quotation marks (" "), we emphasize that the statements or words inside come with the hypothesis of the generation conjecture.

$$L\colon QA^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)])\otimes_K F \to QA^*_{\mathbb{T}}(\mathrm{Hilb}^n(\mathcal{A}_r))\otimes_K F$$

gives an isomorphism of F–algebras. □

On the other hand, we can match the orbifold Gromov–Witten theory with the relative Gromov–Witten theory.

"Corollary" 5.3 *The equality*

$$\langle\!\langle \lambda_1(\vec{\eta}_1), \lambda_2(\vec{\eta}_2), \lambda_3(\vec{\eta}_3)\rangle\!\rangle = \mathrm{GW}(\mathcal{A}_r\times\mathbb{P}^1)_{\lambda_1(\vec{\eta}_1),\lambda_2(\vec{\eta}_2),\lambda_3(\vec{\eta}_3)}$$

holds for any cohomology-weighted partitions $\lambda_1(\vec{\eta}_1)$, $\lambda_2(\vec{\eta}_2)$, $\lambda_3(\vec{\eta}_3)$ of n. □

5.3 Multipoint functions

Once the generation conjecture holds, all extended 3–point functions are known by "Corollary" 5.2. In this situation, we are actually able to generalize "Corollary" 5.2 to cover multipoint invariants. This can be done by proceeding in an analogous manner to Okounkov and Pandharipande's determination of multipoint invariants of $\mathrm{Hilb}^n(\mathbb{C}^2)$ [18].

Let $\mathcal{B}$ be a basis for the Chen–Ruan cohomology $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)])$. We recall the WDVV equation from [3], but we write it in terms of extended functions to better suit our needs. For the time being, we drop the superscript $[\mathrm{Sym}^n(\mathcal{A}_r)]$.

Proposition 5.4 [3] *Given Chen–Ruan cohomology classes $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ and $\beta_1,\dots,\beta_k$. Let S be the set $\{1,\dots,k\}$, we have*

$$\sum_{S_1\coprod S_2=S}\sum_{\gamma\in\mathcal{B}}\langle\!\langle \alpha_1,\alpha_2,\beta_{S_1},\gamma\rangle\!\rangle\langle\!\langle \gamma^\vee,\beta_{S_2},\alpha_3,\alpha_4\rangle\!\rangle$$
$$= \sum_{S_1\coprod S_2=S}\sum_{\gamma\in\mathcal{B}}\langle\!\langle \alpha_1,\alpha_3,\beta_{S_1},\gamma\rangle\!\rangle\langle\!\langle \gamma^\vee,\beta_{S_2},\alpha_2,\alpha_4\rangle\!\rangle.$$

Here, for instance, $\langle\!\langle \alpha_1,\alpha_2,\beta_{S_1},\gamma\rangle\!\rangle := \langle\!\langle \alpha_1,\alpha_2,\beta_{i_1},\dots,\beta_{i_\ell},\gamma\rangle\!\rangle$ if $S_1=\{i_1,\dots,i_\ell\}$.

"Proposition" 5.5 *All extended multipoint functions of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ can be determined from extended three-point functions and all are rational functions in t_1,t_2, $e^{iu},s_1,\dots,s_r$.*

Proof We may see this by induction. Suppose that any extended m–point function with $m\le k$ is known and is a rational function in t_1,t_2, $e^{iu},s_1,\dots,s_r$. To determine extended $(k+1)$–point functions, it suffices to study

$$N := \langle\!\langle \alpha_0,\alpha_1,\dots,\alpha_k\rangle\!\rangle$$

for $\alpha_0 = (2)^{\ell} *_{\mathrm{orb}} D_1^{m_1} *_{\mathrm{orb}} \cdots *_{\mathrm{orb}} D_r^{m_r}$, where ℓ, $m_1, \dots, m_r$ are nonnegative integers. We may assume that $\ell + m_1 + \cdots + m_r \geq 2$ in light of Proposition 3.16 and the fundamental class axiom. Let us write $\alpha_0 = D *_{\mathrm{orb}} \delta$ for some $D = (2)$ or D_j. Clearly,

$$N = \sum_{\gamma \in \mathcal{B}} \langle\!\langle D, \delta, \gamma \rangle\!\rangle \langle\!\langle \gamma^{\vee}, \alpha_1, \dots, \alpha_k \rangle\!\rangle.$$

Let $S = \{1, \dots, k-2\}$. By the WDVV equation,

$$\begin{aligned}\sum_{\gamma \in \mathcal{B}} \langle\!\langle D, \delta, \gamma \rangle\!\rangle \langle\!\langle \gamma^{\vee}, \alpha_S, \alpha_{k-1}, \alpha_k \rangle\!\rangle + \sum_{\gamma \in \mathcal{B}} \langle\!\langle D, \delta, \alpha_S, \gamma \rangle\!\rangle \langle\!\langle \gamma^{\vee}, \alpha_{k-1}, \alpha_k \rangle\!\rangle \\ = \sum_{\gamma \in \mathcal{B}} \langle\!\langle D, \alpha_{k-1}, \gamma \rangle\!\rangle \langle\!\langle \gamma^{\vee}, \alpha_S, \delta, \alpha_k \rangle\!\rangle + \sum_{\gamma \in \mathcal{B}} \langle\!\langle D, \alpha_{k-1}, \alpha_S, \gamma \rangle\!\rangle \langle\!\langle \gamma^{\vee}, \delta, \alpha_k \rangle\!\rangle \\ + \text{ (terms with extended } m\text{–point functions, } 3 \leq m \leq k).\end{aligned}$$

This says that N is determined by lower-point functions and extended $(k+1)$–point functions with a δ–insertion. By replacing $D *_{\mathrm{orb}} \delta$ with δ if necessary and continuing the above procedure, we conclude that N can be calculated from lower-point functions and is a rational function in t_1, t_2, $e^{iu}, s_1, \dots, s_r$. By induction, our claim is thus justified. □

"Theorem" 5.6 (The Crepant Resolution Conjecture) *Let $q = -e^{iu}$ and $k \geq 3$. For any Chen–Ruan cohomology classes $\alpha_1, \dots, \alpha_k$ on $[\mathrm{Sym}^n(\mathcal{A}_r)]$, we have*

$$\langle\!\langle \alpha_1, \dots, \alpha_k \rangle\!\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} = \langle L(\alpha_1), \dots, L(\alpha_k) \rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)}.$$

In particular, $\langle \alpha_1, \dots, \alpha_k \rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} = \langle L(\alpha_1), \dots, L(\alpha_k) \rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)}|_{q=-1}$.

Proof We suppress the indices $[\mathrm{Sym}^n(\mathcal{A}_r)]$ and $\mathrm{Hilb}^n(\mathcal{A}_r)$. The proof of "Proposition" 5.5 works as well for multipoint functions on $\mathrm{Hilb}^n(\mathcal{A}_r)$. What makes things nice is that we get exactly the same set of WDVV equations on both $[\mathrm{Sym}^n(\mathcal{A}_r)]$ and $\mathrm{Hilb}^n(\mathcal{A}_r)$ sides via L provided that we have the equalities

$$\langle\!\langle \alpha_1, \alpha_2, \alpha_3, D \rangle\!\rangle = \langle L(\alpha_1), L(\alpha_2), L(\alpha_3), L(D) \rangle$$

for $D = (2)$ and D_j ($j = 1, \dots, r$). But these are clear by (both twisted and untwisted) divisor equations and "Corollary" 5.2. Thus by a recursive argument, we conclude that L preserves (extended) multipoint functions, and the first claim follows. The second claim is now clear. □

5.4 Closing remarks

All "results" discussed above are honestly true for the case $n=2$ and $r=1$ since the divisor operator $D_1 *_{\mathrm{orb}} -$ has distinct eigenvalues and determines the orbifold quantum product.

Also, in the definition of the map L, we may choose $-i$ instead of i, in which setting the correct change of variables is $q=-e^{-iu}$. Indeed, the transformation $q \longmapsto 1/q$ takes

$$\langle\!\langle \lambda_1(\vec{\eta}_1),\ldots,\lambda_k(\vec{\eta}_k)\rangle\!\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} \text{ to } (-1)^{\sum_{j=1}^k \mathrm{age}(\lambda_j)}\langle\!\langle \lambda_1(\vec{\eta}_1),\ldots,\lambda_k(\vec{\eta}_k)\rangle\!\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}.$$

To illustrate this, just look at the matrix in Section 5.1. There we observe that terms involving q and $1/q$ agree up to a sign.

The calculation of $[\mathrm{Sym}^n(\mathcal{A}_r)]$–invariants in Section 3 gives an indication that these invariants might be closer, geometrically and combinatorially, to the relative invariants of $\mathcal{A}_r \times \mathbb{P}^1$ than the invariants of $\mathrm{Hilb}^n(\mathcal{A}_r)$. In reality, it is the form the relative invariants take that motivates our calculation. We do know that $\mathrm{GW}(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta}_1),\ldots,\lambda_k(\vec{\eta}_k)}$ can be "reduced" to the 3–point case by the degeneration formula; consult [13]. It is, however, unclear if the WDVV equation "behaves" in a similar way to the degeneration formula. At the moment, we expect that the equality

$$\langle\!\langle \lambda_1(\vec{\eta}_1),\ldots,\lambda_k(\vec{\eta}_k)\rangle\!\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} = \mathrm{GW}(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta}_1),\ldots,\lambda_k(\vec{\eta}_k)}$$

should be true. Particularly, the usual k–point function $\langle \lambda_1(\vec{\eta}_1),\ldots,\lambda_k(\vec{\eta}_k)\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}$ should be the coefficient of

$$u^{\sum_{i=1}^k \mathrm{age}(\lambda_i)-2n} \quad \text{in } \mathrm{Z}'(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta}_1),\ldots,\lambda_k(\vec{\eta}_k)}.$$

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