

Boundary distance, lens maps and entropy of geodesic flows of Finsler metrics

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We show that a small perturbation of the boundary distance function of a simple Finsler metric on the n -disc is also the boundary distance function of some Finsler metric. (Simple metrics form an open class containing all flat metrics.) The lens map is a map that sends the exit vector to the entry vector as a geodesic crosses the disc. We show that a small perturbation of a lens map of a simple Finsler metric is in its turn the lens map of some Finsler metric. We use this result to construct a smooth perturbation of the metric on the standard 4-dimensional sphere to produce positive metric entropy of the geodesic flow. Furthermore, this flow exhibits local generation of metric entropy; that is, positive entropy is generated in arbitrarily small tubes around one trajectory.

[53C60](#), [37A35](#), [37J40](#)

1 Introduction

In this paper we prove three theorems. At first glance, one of them may seem unrelated to the others, however it heavily depends on them. Rigorous formulations can be found later in this section; here we give a very informal description to provide the reader with intuition.

We deal with the following situation. We have a disc $D = D^n$. This disc is equipped with a Finsler metric. For a geometer, Finsler metrics generalize Riemannian ones. As in the Riemannian case they allow us to measure lengths of tangent vectors. However the assumption that this length comes from a quadratic form on the tangent space at every point is dropped. For people from dynamical systems or classical mechanics, Finsler metrics are quadratically homogeneous Lagrangians. Thus a Finsler metric determines two structures: a distance function on $D \times D$ and a Lagrangian flow on TD .

We work with a class of “nice” Finsler metrics called *simple*. The precise definition can be found later in this section. To get basic intuition, the reader may think of small, smooth perturbations of a flat metric on a Euclidean ball.

Given a simple Finsler metric, we have a number of related objects, which we now describe.

First of all, we have distances between boundary points of the disc. This is a function on $\partial D \times \partial D$ called the *boundary distance function* associated to the metric inside.

The first question that arises is: if we perturb the boundary distance function, can we find a metric inside which realizes it?

For Riemannian metrics, the answer is *no*. There is a rather obscure obstruction relying on the Besicovitch inequality (see Besicovitch [1]). Indeed, consider the standard Euclidean unit square. Slightly decrease the boundary distances between points close to opposite vertices. The distances between the pairs of opposite sides remains 1, therefore in the Riemannian case the area must be at least 1 by the Besicovitch inequality. On the other hand, the main result of Ivanov [4] implies that a slight decrease of the boundary distance function of a flat metric results in a decrease of area. This shows that this boundary distance function cannot be realized by a Riemannian metric.

So far, the authors do not know if a small perturbation of the boundary distance function of a Riemannian metric in a neighborhood of a single pair of points in $\partial D \times \partial D$ can be realized by a Riemannian metric. Here we ignore all distances between pairs of points outside this neighborhood. This seems to be a very intriguing problem.

We are particularly interested in this problem since our dynamical application requires a perturbation localized near one geodesic. We do not know how to make it Riemannian, nor how to do that in low dimensions.

We can however prove that a sufficiently small perturbation of the boundary distance function of a simple Finsler metric is the boundary distance function of some Finsler metric. Moreover this Finsler metric is a small perturbation of the original one. See [Theorem 1](#) for the precise formulation.

The theorem contains two parts. First we work with Finsler metrics without assuming reversibility. The second part, which requires additional effort, provides a construction respecting reversibility. Reversibility of a Finsler metric means that its Finsler norms are symmetric. Geometers often prefer to assume this symmetry for it gives rise to a usual metric space with a symmetric distance function. In dynamics, there are many situations when one wants to look at non-reversible Finsler metrics. They arise in physics (eg magnetic fields) and correspond to non-reversible Lagrangian flows. Non-reversible Finsler metrics determine distance functions that satisfy all axioms of distance other than the symmetry one.

Next we proceed to the lens map (sometimes also called Poincaré map or scattering relation). At this point we suggest that the reader think about our disc equipped with a thin collar and some smooth extension of the metric to that collar.

Our Finsler metric determines a Lagrangian flow whose Lagrangian is given by the square of the Finsler norm. Thus we can look at trajectories. They enter our disc. There are entry and exit points for each trajectory of the flow. Of course, the flow is defined on the (unit) tangent bundle, hence the entry and exit points are tangent vectors.

We prefer now to turn to the Hamiltonian language, in which case we think about the entry and exit co-vectors, identified with tangent vectors by the Legendre transform. Thus we get a scattering relation which sends the entry co-vector to the exit co-vector. It is referred to as the *dual lens map*. This map is symplectic.

There is a very clear relationship between the dual lens map and the boundary distance function of a simple metric. As will be discussed below, they uniquely determine each other. Namely, the derivative of the boundary distance function with respect to its first and second arguments at $p, q \in \partial D$ is essentially the same thing as the entry and exit co-vectors corresponding to the geodesic segment $[pq]$. This follows from the formula for the derivative of the distance function, see equations (2-1) and (2-2).

Our second result, [Theorem 2](#), shows that if one perturbs the dual lens map of a simple Finsler metric in the class of symplectic maps, we also get the dual lens map of a simple Finsler metric.

We hope that these results are of some interest on their own. We also use them to give partial answers to some classic problems in dynamical systems and entropy theory. Here the word “entropy” by default means the measure-theoretic or “metric” entropy.

In our final result, [Theorem 3](#), we show that the Riemannian metric of the standard 4-dimensional sphere can be perturbed (in C^∞) in such a way that the geodesic flow on the unit tangent bundle is of positive entropy. This perturbation is in the class of Finsler metrics, or, equivalently, in the class of Lagrangian systems.

The perturbation takes place in a small tube around one trajectory of the non-perturbed periodic flow. This allows us to give an example in which positive metric entropy is generated in arbitrarily small neighborhoods of one periodic trajectory; see [Corollary 5.2](#). There is no fixed terminology however many authors call this property *entropy non-expansiveness*. This property has been of interest to a number of mathematicians including Bowen, Knieper, Newhouse, and Burago; see eg Bowen [\[2\]](#), Burago [\[3\]](#), Newhouse [\[6\]](#). In the literature this notion was mostly discussed for topological entropy. Obviously local generation of metric entropy implies that for topological entropy.

Now we proceed with precise definitions and formulations.

Boundary distance

Recall that a Finsler metric on a smooth manifold M is a function $\varphi: TM \rightarrow \mathbb{R}_+$ which is smooth outside the zero section and such that its restriction to each fiber $T_x M$,

$x \in M$, is a Minkowski norm. The latter means that the function $\varphi|_{T_x M}$ is positively homogeneous of degree 1, positive outside zero, and its square is quadratically convex.

The word “norm” here is used in a slightly more general sense than in functional analysis, namely we do not assume that it is symmetric. Finsler metrics whose norms are symmetric (ie $\varphi(-v) = \varphi(v)$ for all $v \in M$) are said to be *reversible*. Whenever we assume reversibility, we mention this explicitly.

The value $\varphi(v)$ is interpreted as the length of a tangent vector $v \in TM$. Then one defines the length of a smooth path and a distance function $d_\varphi: M \times M \rightarrow \mathbb{R}$. As with norms, we consider non-symmetric distance functions. By definition, the distance $d_\varphi(x, y)$ between points $x, y \in M$ is the infimum of φ -lengths of paths starting at x and ending at y . Note that d_φ satisfies the triangle inequality:

$$d_\varphi(x, y) + d_\varphi(y, z) \geq d_\varphi(x, z).$$

The distance function d_φ is symmetric if and only if φ is reversible.

The *boundary distance function* bd_φ of φ is the restriction of d_φ to $\partial M \times \partial M$. That is, $bd_\varphi(x, y)$ is the length of a φ -shortest path in M connecting the boundary points x and y .

Geodesics in a Finsler manifold are smooth curves which are locally shortest paths. From a dynamical viewpoint, geodesics are trajectories of a Lagrangian system whose Lagrangian is $\frac{1}{2}\varphi^2$. We only consider geodesics parametrized by arc length.

Let $D = D^n$ denote the n -dimensional disc. A Finsler metric φ on D is called *simple* if the following conditions hold:

- (1) Every pair of points in D is connected by a unique geodesic.
- (2) Geodesics depend smoothly on their endpoints.
- (3) The boundary is strictly convex; that is, geodesics never touch it at their interior points.

These conditions imply that all geodesics are minimizing and have no conjugate points.

Our main result, [Theorem 1](#), asserts that every C^∞ -small perturbation of the boundary distance function bd_φ can be realized as the boundary distance function of a C^∞ -small perturbation of φ .

Note that the distance function is not smooth at the diagonal of $D \times D$. To avoid silly technicalities, we work in the following set-up. We only consider perturbations of bd_φ which are identical in a neighborhood of the diagonal Δ of $\partial D \times \partial D$, and perturbations of φ which are identical near the boundary.

Theorem 1 Let φ be a simple Finsler metric on $D = D^n$, $f = bd_\varphi$, and U_0 a neighborhood of the diagonal $\Delta \subset \partial D \times \partial D$. Then for every function $\tilde{f}: \partial D \times \partial D \rightarrow \mathbb{R}$ such that $\tilde{f}|_{U_0} = f|_{U_0}$ and \tilde{f} is sufficiently C^∞ -close to f on $\partial D \times \partial D \setminus U_0$, there exists a simple Finsler metric $\tilde{\varphi}$ such that $bd_{\tilde{\varphi}} = \tilde{f}$ and $\tilde{\varphi}$ coincides with φ in a neighborhood of ∂D .

The choice of $\tilde{\varphi}$ can be made in such a way that $\tilde{\varphi}$ converges to φ whenever \tilde{f} converges to f (in C^∞).

In addition, if φ is reversible and \tilde{f} is symmetric, then $\tilde{\varphi}$ can be chosen reversible.

Lens map

Let φ be a Finsler metric on D . We denote by UTD the unit sphere bundle of φ . By U_{in} we denote the set of unit vectors tangent to D at the boundary ∂D and such that they point inwards. Analogously U_{out} is the set of unit tangent vectors at the boundary pointing outwards.

For a vector $v \in U_{in}$, we can look at the geodesic with the initial vector v . As it hits the boundary again, we get its velocity vector $\beta(v) \in U_{out}$. This defines a map $\beta: U_{in} \rightarrow U_{out}$. This map is the *lens map* of φ (sometimes also called *Poincaré map* or *scattering relation*). If φ is reversible, then the lens map β is reversible in the following sense: $-\beta(-\beta(v)) = v$ for every $v \in U_{in}$.

Now we proceed to our next theorem, [Theorem 2](#). It answers the question: which perturbations of β are realized as lens maps of Finsler metrics? The answer is that the perturbed map should be symplectic. To give a reasonably precise formulation, however, we need more definitions and notation.

First, to employ the symplectic structure, we need, as usual, to pass to the cotangent bundle T^*D . Let $\varphi^*: T^*D \rightarrow \mathbb{R}$ be the fiber-wise dual norm to φ . That is, for $x \in D$ and $\alpha \in T_x^*D$, one defines

$$\varphi^*(\alpha) = \sup\{\alpha(v) \mid v \in UT_x M\}.$$

We denote by UT^*D the unit sphere bundle of φ^* .

Let $\mathcal{L}: TD \rightarrow T^*D$ be the Legendre transform of the Lagrangian $\frac{1}{2}\varphi^2$. Due to homogeneity of φ , \mathcal{L} is norm-preserving. In particular it induces a diffeomorphism between UTD and UT^*D . For a tangent vector $v \in UT_x D$, its Legendre transform $\mathcal{L}(v)$ is the unique co-vector $\alpha \in UT_x^*D$ such that $\alpha(v) = 1$.

Then we introduce the dual lens map. Namely, consider the subsets $U_{in}^* = \mathcal{L}(U_{in})$ and $U_{out}^* = \mathcal{L}(U_{out})$ of UT^*D . The *dual lens map* of φ is the map $\sigma: U_{in}^* \rightarrow U_{out}^*$ defined by $\sigma = \mathcal{L} \circ \beta \circ \mathcal{L}^{-1}$, where β is the lens map of φ .

Note that U_{in}^* and U_{out}^* are $(2n - 2)$ -dimensional submanifolds of T^*D . Consider the restriction of the canonical symplectic 2-form of T^*D to U_{in}^* and U_{out}^* . This restriction is non-degenerate and hence symplectic. Indeed, recall that the geodesic flow of φ is Hamiltonian when considered on T^*D . The non-degeneracy claimed above follows from the fact that U_{in}^* and U_{out}^* lie in the energy level of this Hamiltonian flow and are transverse to this flow. For the same reason, the dual lens map σ is symplectic.

If φ is reversible then its dual lens map is symmetric in the following sense:

$$-\sigma(-\sigma(\alpha)) = \alpha \quad \text{for all } \alpha \in U_{in}^*.$$

Note that reversibility of φ implies that $-\alpha \in U_{out}^*$ if $\alpha \in U_{in}^*$, and vice versa.

Theorem 2 *Assume that $n \geq 3$. Let φ be a simple metric on $D = D^n$ and σ its dual lens map. Let W be the complement of a compact set in U_{in}^* .*

Then every sufficiently small symplectic perturbation $\tilde{\sigma}$ of σ such that $\tilde{\sigma}|_W = \sigma|_W$ is realized by the dual lens map of a simple metric $\tilde{\varphi}$ which coincides with φ in some neighborhood of ∂D .

The choice of $\tilde{\varphi}$ can be made in such a way that $\tilde{\varphi}$ converges to φ whenever $\tilde{\sigma}$ converges to σ (in C^∞).

In addition, if φ is a reversible Finsler metric and $\tilde{\sigma}$ is symmetric in the sense that $-\tilde{\sigma}(-\tilde{\sigma}(\alpha)) = \alpha$ for all $\alpha \in U_{in}^$, then $\tilde{\varphi}$ can be chosen reversible as well.*

Remark The domain U_{in}^* and range U_{out}^* of the dual lens map depend on the metric φ . This is not an issue in [Theorem 2](#) because $\tilde{\varphi}$ is required to coincide with φ in a neighborhood of ∂D , and the sets U_{in}^* and U_{out}^* are determined by the restriction of φ to $TD|_{\partial D}$.

Remark The assumption that $n \geq 3$ in [Theorem 2](#) is essential. In dimension 2 there is a single obstruction, namely the map $\tilde{\sigma}$ should satisfy a certain integral identity; see [Remark 4.4](#).

Positive entropy

The previous two results are applied to prove the following theorem.

Theorem 3 *The standard metric of S^4 can be perturbed in the class of reversible Finsler metrics so that the resulting metric has positive metric entropy of its geodesic flow. The perturbation can be chosen to be arbitrarily small in C^∞ .*

Note that the analogue of this result for topological entropy is well-known; see Paterlain [\[7\]](#) and Knieper and Weiss [\[5\]](#).

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2 Preliminaries

The following preliminaries are copied with minor modifications from [\[4\]](#).

Let $M = (M^n, \varphi)$ be a Finsler manifold, possibly with boundary. In this paper we only consider the case when M is either the disc $D = D^n$ with a simple metric φ , or an open subset of such a disc.

We say that a continuous function $f: M \rightarrow \mathbb{R}$ is *distance-like* (with respect to φ) if f is smooth on $M \setminus \partial M$ and $\varphi^*(d_x f) = 1$ for all $x \in M$. Note that every distance-like function f is forward 1-Lipschitz, that is,

$$f(y) - f(x) \leq d_\varphi(x, y)$$

for all $x, y \in M$.

If $M = D$ and φ is a simple metric, then for every $p \in \partial D$ the functions $d_\varphi(p, \cdot)$ and $-d_\varphi(\cdot, p)$ are distance-like. Note that $d_\varphi(\cdot, p)$ is *not* distance-like in general (unless the metric is reversible).

The Finslerian *gradient* of a distance-like function $f: M \rightarrow \mathbb{R}$ at $x \in M$, denoted by $\text{grad}_\varphi f(x)$, is the unique tangent vector $v \in U_x TM$ such that $d_x f(v) = 1$. Equivalently, $\text{grad}_\varphi f(x) = \mathcal{L}^{-1}(d_x f)$, where \mathcal{L} is the Legendre transform determined by φ . Beware that, unlike in the Riemannian case, grad_φ is a non-linear operator.

We introduce the following notation for a simple metric φ on D . For distinct points $x, y \in D$, the unique arc-length geodesic segment starting at x and ending at y is denoted by $[xy]$. By v_{xy} and w_{xy} we denote the velocity vectors of this geodesic segment at x and y , respectively. Thus $v_{xy} \in UT_x D$ and $w_{xy} \in UT_y D$.

Observe that the gradient $\text{grad}_\varphi d_\varphi(x, \cdot)$ at y equals w_{xy} . Indeed, the directional derivative of $d_\varphi(x, \cdot)$ along w_{xy} equals 1. Thus the derivative of the function $d_\varphi(x, \cdot)$ at y is given by

$$(2-1) \quad d_y d_\varphi(x, \cdot) = \mathcal{L}(w_{xy}).$$

A similar argument for the distance-like function $-d_\varphi(\cdot, y)$ shows that

$$(2-2) \quad d_x d_\varphi(\cdot, y) = -\mathcal{L}(v_{xy}).$$

One can see that these formulas are special cases of the first variation formula for Finslerian length.

Definition 2.1 (see [4]) Let $M = (M^n, \varphi)$ be a Finsler manifold and S a smooth manifold diffeomorphic to S^{n-1} . We say that a continuous function $F: S \times M \rightarrow \mathbb{R}$ is an *enveloping function* for φ if F is smooth outside $S \times \partial M$ and the following two conditions are satisfied:

- (1) For every $p \in S$, the function $F_p := F(p, \cdot)$ is distance-like.
- (2) For every $x \in M \setminus \partial M$, the map $p \mapsto d_x F_p$ is a diffeomorphism from S to $UT_x^* D$.

Note that an enveloping function uniquely determines the metric. Indeed, at every point $x \in M \setminus \partial M$ the unit co-tangent sphere $UT_x^* M$ can be recovered from F as the image of S under the map $p \mapsto d_x F_p$. This unit sphere determines the dual norm $\varphi^*|_{T_x^* M}$ and hence the original norm $\varphi|_{T_x M}$.

We need the notion of enveloping function for the following situation. Let $M = D$, $S = \partial D$ and assume that φ is simple. Then the function $F: S \times D \rightarrow \mathbb{R}$ given by $F(p, x) = d_\varphi(p, x)$ is an enveloping function for φ .

The following lemma characterizes enveloping functions of this type.

Lemma 2.2 Let $F: \partial D \times D \rightarrow \mathbb{R}$ be an enveloping function for φ such that the following conditions hold:

- (1) $F(p, p) = 0$ for every $p \in \partial D$.
- (2) For every distinct $p, q \in \partial D$, the function $F_p = F(p, \cdot)$ is smooth at q and $\text{grad}_\varphi F_p(q)$ points out of D .

Then $F(p, x) = d_\varphi(p, x)$ for all $p \in \partial D$ and $x \in D$.

Proof Let $p \in \partial D$ and $x \in D \setminus \partial D$. Consider the backward gradient curve of F_p through x ; namely, let $\gamma: (-t_0, 0] \rightarrow D$ be the maximal backward solution of the ODE $\dot{\gamma}(t) = \text{grad}_\varphi F_p(\gamma(t))$ with the initial condition $\gamma(0) = x$.

Since F_p is distance-like, we have $\varphi(\dot{\gamma}(t)) = 1$ and $\frac{d}{dt} F_p(\gamma(t)) = 1$ for all t . Hence γ is parametrized by arc length and $F_p(\gamma(t)) = F_p(x) + t$ for all $t \in (-t_0, 0]$. Thus for every $t \leq 0$ we have

$$F_p(x) - F_p(\gamma(t)) = -t = \text{length}_\varphi(\gamma|_{[-t, 0]}) \geq d_\varphi(\gamma(t), x)$$

for all $t \in (-t_0, 0]$. On the hand, we have $F_p(x) - F_p(\gamma(t)) \leq d_\varphi(\gamma(t), x)$ since F_p is forward 1-Lipschitz. Therefore

$$(2-3) \quad F_p(x) - F_p(\gamma(t)) = d_\varphi(\gamma(t), x)$$

for all $t \in (-t_0, 0]$.

Since F_p is bounded, (2-3) implies that γ is not trapped in D and therefore eventually hits the boundary. Due to condition (2), the only point where it may hit the boundary is p . Thus $\gamma(-t_0) = p$ and then (2-3) for $t = -t_0$ implies that

$$F_p(x) = F_p(x) - F_p(p) = d_\varphi(p, x).$$

The lemma follows. □

The key point of the proof of [Theorem 1](#) is that a small perturbation of an enveloping function is again an enveloping function for another Finsler metric. The precise statement that we need is the following lemma (borrowed from [\[4\]](#) with minor modifications).

Lemma 2.3 *Let φ be a simple metric on D and $F: S \times D \rightarrow \mathbb{R}$ an enveloping function for φ . Let $U \subset D$ be an open set separated from ∂D . Then every function $\tilde{F}: S \times U \rightarrow \mathbb{R}$ which is sufficiently C^∞ -close to $F|_{S \times U}$ is an enveloping function of some Finsler metric $\tilde{\varphi}$ on U .*

Furthermore, $\tilde{\varphi}$ tends to $\varphi|_U$ in C^∞ as \tilde{F} goes to F .

Proof Note that the derivatives of F are bounded on $S \times U$ since U is contained in a compact subset of $D \setminus \partial D$. For every $x \in U$, the map $p \mapsto d_x F_p$ from S to $T_x^* D$ parametrizes $UT_x^* D$ which is a quadratically convex surface in $T_x^* D$. If \tilde{F} is close to F then the map $p \mapsto d_x \tilde{F}_p$ is close to $p \mapsto d_x F_p$ and hence it also parametrizes a quadratically convex surface close to the original one. This surface is the unit sphere of some Minkowski norm on $T_x^* D$. The dual norm on $T_x D$ is the desired metric $\tilde{\varphi}$ at x . The entire construction is continuous with respect to the relevant C^∞ topologies. □

3 Proof of [Theorem 1](#)

Let $n \geq 2$, $D = D^n$ and $S = \partial D$. Let φ , f and \tilde{f} be as in [Theorem 1](#). Recall that f is the boundary distance function of φ , and \tilde{f} is a perturbation of f supported outside a neighborhood U_0 of the diagonal $\Delta \subset S \times S$.

For $r > 0$ and $p \in D$, let $B_r(p)$ denote the forward d_φ -ball of radius r centered at p , namely $B_r(p) = \{x \in D : d_\varphi(p, x) < r\}$. By $U_r(S)$ we denote the forward r -neighborhood of the boundary, that is, $U_r(S) = \bigcup_{p \in S} B_r(p)$.

Since \tilde{f} coincides with $f = bd_\varphi$ in a neighborhood U_0 of the diagonal, there exists an $\varepsilon > 0$ such that $\tilde{f}(x, y) = f(x, y) = d_\varphi(x, y)$ for all $x, y \in S$ with $d_\varphi(x, y) < 5\varepsilon$. Fix this ε for the rest of this section.

Define a function $F: S \times D \rightarrow \mathbb{R}$ by $F(p, x) = d_\varphi(p, x)$. Then F is an enveloping function for φ . In order prove [Theorem 1](#), we construct an enveloping function $\tilde{F}: S \times D \rightarrow \mathbb{R}$ for the desired metric $\tilde{\varphi}$. We make \tilde{F} out of F and a function G which takes care of a neighborhood of the boundary (see [Lemma 3.2](#) below).

Fix $p \in S$ and define a function $H_p: D \rightarrow \mathbb{R}$ by

$$(3-1) \quad H_p(x) = \max_{y \in S} \{ \tilde{f}(p, y) - d_\varphi(x, y) \}.$$

Note that if $\tilde{f} = f$, then $H_p(x) = d_\varphi(p, x)$ and the maximum in (3-1) is attained at the (unique) point $y \in S$ such that $x \in [py]$.

Though H_p is defined on the whole disc D , we are going to restrict it to the set $U_\delta(S) \setminus B_\varepsilon(p)$, where δ is a sufficiently small positive constant (depending on φ and ε).

For every $q \in S \setminus \{p\}$ we define a tangent vector $\tilde{w}_{pq} \in UT_q D$ pointing out of D , as follows. Consider the derivative $d_q \tilde{f}(p, \cdot) \in T_q^* S$. If $\tilde{f} = f$, the φ -norm of this derivative is strictly less than 1. This follows from (2-1) and the fact that the geodesic $[pq]$ is transverse to the boundary. Hence the φ^* -norm of this derivative is less than 1 whenever \tilde{f} is sufficiently close to f (in C^∞ , on $S \times S \setminus U_0$). Hence there exists a unique vector $w \in UT_q D$ pointing out of D and such that $\mathcal{L}(w)|_{T_q S} = d_q \tilde{f}(p, \cdot)$. We take this vector w for \tilde{w}_{pq} . Note that \tilde{w}_{pq} depends smoothly on p, q and \tilde{f} .

In the case when $\tilde{f} = f$ we have $\tilde{w}_{pq} = w_{pq}$ by (2-1), where w_{pq} is the velocity vector of the geodesic $[pq]$ at q (see the notation introduced in [Section 2](#)). Therefore \tilde{w}_{pq} tends to w_{pq} (in C^∞ , as a function of p and q), and $\tilde{w}_{pq} = w_{pq}$ if $d_\varphi(p, q) < 5\varepsilon$.

For every distinct $p, q \in S$ and $t \geq 0$ define $\gamma_{pq}(-t) = \exp_q(-t\tilde{w}_{pq})$, where \exp_q is the exponential map of the metric φ at q . That is, γ_{pq} is a geodesic which hits the boundary at q with velocity \tilde{w}_{pq} . This geodesic is parametrized so that $q = \gamma_{pq}(0)$. The domain of γ_{pq} is an interval of the form $[-T, 0]$, where T depends smoothly on p, q and \tilde{f} . Hence $T \geq \min\{d_\varphi(p, q), \varepsilon\}$, provided that \tilde{f} is sufficiently close to f .

In the case when $\tilde{f} = f$ we have $\gamma_{pq} = [pq]$. Since \tilde{w}_{pq} is a C^∞ -small perturbation of w_{pq} , it follows that the geodesics $\{\gamma_{pq}\}_{q \in S \setminus \{p\}}$ can intersect only within the ball $B_\varepsilon(p)$ and cover the set $D \setminus B_\varepsilon(p)$.

Lemma 3.1 *Let $p, q \in S$ and let $t \geq 0$ be such that $\gamma(-t) \in D \setminus B_\varepsilon(p)$. Then*

$$(3-2) \quad H_p(\gamma_{pq}(-t)) = \tilde{f}(p, q) - t,$$

provided that \tilde{f} is sufficiently close to f .

Proof Let $x = \gamma_{pq}(-t)$. Then $d_\varphi(x, q) = t$. Let $y \in S$ be a point of maximum in (3-1). Since the derivative at a point of maximum is zero, we have

$$d_y(\tilde{f}(p, \cdot) - d_\varphi(x, \cdot))|_{T_y S} = 0.$$

Hence

$$(3-3) \quad d_y \tilde{f}(p, \cdot) = d_y d_\varphi(x, \cdot)|_{T_y S} = \mathcal{L}(w_{xy})|_{T_y S},$$

where the second equality follows from (2-1). This implies that $w_{xy} = \tilde{w}_{py}$, therefore x belongs to the geodesic γ_{py} . This implies that $y = q$, because otherwise the geodesics γ_{pq} and γ_{py} would intersect outside $B_\varepsilon(p)$.

Thus q is a point of maximum in (3-1). Now (3-1) takes the form

$$H_p(x) = \tilde{f}(p, q) - d_\varphi(x, q) = \tilde{f}(p, q) - t,$$

and the lemma follows. □

Since for a fixed p the geodesics γ_{pq} cover $D \setminus B_\varepsilon(p)$, the identity (3-2) uniquely determines the function H_p in $D \setminus B_\varepsilon(p)$. Moreover, the map $(q, t) \mapsto \gamma_{pq}(-t)$ is a diffeomorphism from an appropriate subset of $S \times \mathbb{R}_+$ to $D \setminus B_\varepsilon(p)$. Hence (3-2) implies that H_p is smooth on $D \setminus B_\varepsilon(p)$.

Now let us show that $H_p|_{D \setminus B_\varepsilon(p)}$ is a distance-like function. First observe that H_p is forward 1-Lipschitz. Indeed, H_p is the supremum of functions of the form $x \mapsto \tilde{f}(p, y) - d_\varphi(x, y)$, each of which is forward 1-Lipschitz by the triangle inequality for d_φ . Next, (3-2) implies that H_p grows at unit rate along each geodesic γ_{pq} within $D \setminus B_\varepsilon(p)$. Hence $H_p|_{D \setminus B_\varepsilon(p)}$ is distance-like and

$$(3-4) \quad \text{grad}_\varphi H_p(\gamma_{pq}(-t)) = \dot{\gamma}_{pq}(-t)$$

whenever $\gamma_{pq}(-t) \in D \setminus B_\varepsilon(p)$.

Lemma 3.2 *There exist a neighborhood $V \subset D$ of S and a function $G: S \times V \rightarrow \mathbb{R}$ such that*

- (1) G is an enveloping function for $\varphi|_V$.
- (2) G coincides with \tilde{f} on $S \times S$.

- (3) G coincides with F in a neighborhood of Δ in $S \times V$. (Recall that Δ is the diagonal of $S \times S$.)
- (4) For every pair of distinct points $p, q \in S$, the gradient of the function $G_p = G(p, \cdot)$ at q points out of D .

Furthermore, G is constructed out of φ and \tilde{f} in such a way that G converges to $F|_{S \times V}$ (in C^∞) as \tilde{f} goes to f .

Proof Since the boundary is strictly convex with respect to φ , there exists a $\delta = \delta(\varphi, \varepsilon) > 0$ such that the following holds: if $p, q \in S$, $x \in [pq] \cap U_\delta(S)$ and $\varepsilon \leq d_\varphi(p, x) \leq 2\varepsilon$, then $d_\varphi(p, q) < 3\varepsilon$. Let $V = U_\delta(S)$.

For each $p \in S$ define a function $G_p: V \rightarrow \mathbb{R}$ by

$$(3-5) \quad G_p(x) = \begin{cases} H_p(x) & \text{if } d_\varphi(p, x) > \varepsilon, \\ d_\varphi(p, x) & \text{if } d_\varphi(p, x) < 2\varepsilon. \end{cases}$$

The domains in the two cases in (3-5) overlap, yet the two formulas yield the same value. Indeed, let $x \in V$ be such that $\varepsilon < d_\varphi(p, x) < 2\varepsilon$ and let $q \in S$ be such that $x \in [pq]$. Then $d_\varphi(p, q) < 3\varepsilon$ by the choice of δ . Then $\tilde{w}_{pq} = w_{pq}$ and hence $\gamma_{pq} = [pq] \ni x$. By Lemma 3.1 it follows that

$$H_p(x) = \tilde{f}(p, q) - d_\varphi(x, q) = d_\varphi(p, q) - d_\varphi(x, q) = d_\varphi(p, x).$$

Thus the two formulas in (3-5) agree on the overlap. Hence (3-5) defines a continuous function $G_p: V \rightarrow \mathbb{R}$.

Define $G: S \times V \rightarrow \mathbb{R}$ by $G(p, x) = G_p(x)$. Clearly G is smooth outside Δ and G tends to $F|_{S \times V}$ as \tilde{f} goes to f . We are going to show that G satisfies the assertions of the lemma. The third assertion is trivial by construction.

To prove the second assertion, consider $p, q \in S$. If $d_\varphi(p, q) < 2\varepsilon$ then $G(p, q) = d_\varphi(p, q) = \tilde{f}(p, q)$ by (3-5) and the choice of ε . In the other case, namely if $d_\varphi(p, q) \geq 2\varepsilon$, we have $G(p, q) = H_p(q) = \tilde{f}(p, q)$ by setting $t = 0$ in Lemma 3.1.

The fourth assertion follows from the fact that $\text{grad}_\varphi G_p(q) = \tilde{w}_{pq}$ for all distinct $p, q \in S$. In the case $d_\varphi(p, q) > \varepsilon$ this fact follows from (3-4), and in the case $d_\varphi(p, q) \leq \varepsilon$ we have $\text{grad}_\varphi G_p = w_{pq} = \tilde{w}_{pq}$.

It remains to verify the first assertion of the lemma. First observe that G_p is distance-like because both H_p and $d_\varphi(p, \cdot)$ are distance-like on V . Let $x \in V \setminus S$. Since G_p is distance-like, we have a smooth map $p \mapsto d_x G_p$ from S to $UT_x^* D$. It remains to prove that this map is a diffeomorphism. Recall that G is a C^∞ -small perturbation of

$F|_{S \times V}$ and the map $p \mapsto d_x F_p$ is a diffeomorphism from S to $UT_x^* D$. Furthermore F and G coincide within the set $\Delta_{2\varepsilon} = \{(p, x) : |px| < 2\varepsilon\}$. Hence $d_x G_p = d_x F_p$ for $(p, x) \in \Delta_{2\varepsilon}$. Outside $\Delta_{2\varepsilon}$ the derivatives of F are uniformly bounded. Hence a sufficiently small perturbation supported outside $\Delta_{2\varepsilon}$ yields a diffeomorphism $p \mapsto d_x G_p$ from S to $UT_x^* D$. \square

Proof of Theorem 1 Let $G: S \times V \rightarrow \mathbb{R}$ be a function constructed in Lemma 3.2. We glue the desired function \tilde{F} out of F and G using a partition of unity. Let $h: D \rightarrow [0, 1]$ be a smooth function such that $h = 1$ outside V and $h = 0$ in a neighborhood V_0 of S . Define $\tilde{F}: S \times D \rightarrow \mathbb{R}$ by

$$\tilde{F}(p, x) = h(x)F(p, x) + (1 - h(x))G(p, x).$$

Then \tilde{F} is an enveloping function for a Finsler metric $\tilde{\varphi}$ which coincides with φ within V_0 . Indeed, $\tilde{F} = G$ in $V_0 \subset V$ and hence Lemma 3.2 implies that \tilde{F} is an enveloping function for φ within V_0 . Outside V_0 , \tilde{F} converges to F (in C^∞) as $\tilde{f} \rightarrow f$. Hence by Lemma 2.3 it is an enveloping function of some Finsler metric $\tilde{\varphi}$ which is close to φ .

Since \tilde{F} is an extension of \tilde{f} , Lemma 2.2 implies that \tilde{f} is the boundary distance function of $\tilde{\varphi}$. This finishes the proof of Theorem 1 in the non-reversible case.

In order to make $\tilde{\varphi}$ reversible (provided that φ is reversible and \tilde{f} is symmetric) we need some preparation.

Notation 3.3 For a set $U \subset S \times S$ denote by $\Gamma(U)$ the set of velocity vectors of all geodesics $[pq]$ such that $(p, q) \in U$.

Remark 3.4 The construction of $\tilde{\varphi}$ also guarantees the following property which we need below. If $U \subset S \times S$ is an open set such that $\tilde{f}|_U = f|_U$, then $\tilde{\varphi}$ coincides with φ on $\Gamma(U)$.

Indeed, let $(p, q) \in U$ and $x \in [pq]$. The argument in Step 2 of the proof of Lemma 3.2 shows that every local maximum in (3-1) is global. Hence $H_p(x) = F(p, x)$. Going through the subsequent parts of the construction one sees that $\tilde{F}(p, x) = F(p, x)$. Since U is open, this argument also shows that $\tilde{F}(p', x) = F(p', x)$ for all p' from a neighborhood of p in S . Therefore $\tilde{\varphi}^*$ coincides with φ^* in a neighborhood of the co-vector $d_x F_p$. Let v be the velocity vector of $[pq]$ at x . Then $d_x F_p$ is the Legendre transform of v . Hence $\tilde{\varphi}(v) = \varphi(v)$ as claimed.

Now we are in position to prove the last assertion of Theorem 1. Let R denote the involution of $S \times S$ permuting its arguments, ie $R(p, q) = (q, p)$. Recall that \tilde{f} and

f coincide in a neighborhood U_0 of the diagonal $\Delta \subset S \times S$. Fix an open covering $\{U_i\}_{i=1}^N$ of $S \times S \setminus U_0$ such that for every i the sets U_i and $R(U_i)$ have disjoint closures.

Connect f to \tilde{f} by a sequence of functions $f = f_0, f_1, \dots, f_N = \tilde{f}$ where every $f_i: S \times S \rightarrow \mathbb{R}$ is a symmetric function obtained from f_{i-1} by a C^∞ -small perturbation supported in $U_i \cup R(U_i)$. By induction, we construct a sequence of reversible metrics φ_i on D such that f_i is the boundary distance function of φ_i for each i . The induction step goes as follows. First apply **Theorem 1** with φ_{i-1} in place of φ and f_i in place of \tilde{f} . This yields a non-reversible metric $\tilde{\varphi}_i$ whose boundary distance function is f_i . By **Remark 3.4**, $\tilde{\varphi}_i$ coincides with φ_{i-1} outside $\Gamma(U_i \cup R(U_i))$, where $\Gamma(\cdot)$ is defined as in **Notation 3.3** however with respect to φ_{i-1} rather than φ . Since $\Gamma(U_i)$ and $\Gamma(R(U_i))$ have disjoint closures, we can change the metric within $\Gamma(R(U_i))$ so that it becomes reversible. Namely, we define the desired metric φ_i by

$$\varphi_i(v) = \begin{cases} \tilde{\varphi}_i(-v) & \text{if } v \in \Gamma(R(U_i)), \\ \tilde{\varphi}_i(v) & \text{otherwise,} \end{cases}$$

for every $v \in UTD$. This metric has the same boundary distance function. Indeed, if $(p, q) \notin R(U_i)$ then the geodesic $[pq]$ of the metric $\tilde{\varphi}_i$ remains a geodesic as we replace $\tilde{\varphi}_i$ by φ_i . Hence $d_{\varphi_i}(p, q) = d_{\tilde{\varphi}_i}(p, q) = f_i(p, q)$ if $(p, q) \notin R(U_i)$. This identity holds for $(p, q) \in R(U_i)$ because in this case $(q, p) \notin R(U_i)$ and the distance is symmetric.

It remains to verify that the set of functions \tilde{f} reachable by sequences f_0, \dots, f_N such that the construction works, contains a neighborhood of f in our space of boundary distance functions. (Recall that this is the space of functions on $S \times S$ coinciding with f in U_0 , equipped with the C^∞ topology.) The covering $\{U_i\}$ and hence the number of steps in the construction is determined by U_0 . The construction which produces $\tilde{\varphi}$ out of φ and \tilde{f} is explicit and hence continuous in its arguments φ and \tilde{f} (regarded as elements of their respective functional spaces). Furthermore it is defined for an open set of pairs (φ, \tilde{f}) . The decomposition and symmetrization procedures above also have similar continuity properties. Therefore if \tilde{f} is chosen from a suitable neighborhood of bd_φ , then each pair (φ_{i-1}, f_i) in the inductive construction belongs to the domain where **Theorem 1** applies. □

4 Proof of Theorem 2

As in the previous sections, let φ be a simple metric on $D = D^n$ and $S = \partial D$. Recall that in **Theorem 2** we assume that $n \geq 3$. The metric φ (or its restriction to a neighborhood of S) determines the sets $U_{in}^*, U_{out}^* \subset T^*D$ as in **Section 1**.

For the proof of the theorem we need a zoo of notation. The good news is that it has clear geometric meaning.

We think of trajectories of a Hamiltonian flow on the unit co-tangent bundle UT^*D . They enter UT^*D and leave it. Thus we have an entry co-vector in U_{in}^* and an exit co-vector in U_{out}^* for each trajectory. Of course each of them can be viewed as a pair consisting of the base point in ∂D and the co-vector component.

A diffeomorphism $\sigma: U_{in}^* \rightarrow U_{out}^*$ tells us the exit co-vector from the entry one. In the following notation and constructions we use only certain properties of σ and do not assume that it arose from a flow. We apply these constructions to both σ and $\tilde{\sigma}$.

First we need a map P_σ which takes an entry co-vector and produces two points $p, q \in \partial D$ which are the entry and exit base points of the “trajectory” of σ determined by that co-vector. Analogously, Q_σ takes an exit co-vector and produces the entry and exit base points.

Formally, we define maps $P_\sigma: U_{in}^* \rightarrow S \times S$ and $Q_\sigma: U_{out}^* \rightarrow S \times S$ by

$$\begin{aligned} P_\sigma(\alpha) &= (\pi(\alpha), \pi(\sigma(\alpha))), & \alpha \in U_{in}^*, \\ Q_\sigma(\alpha) &= (\pi(\sigma^{-1}(\alpha)), \pi(\alpha)), & \alpha \in U_{out}^*, \end{aligned}$$

where $\pi: T^*D \rightarrow D$ is the bundle projection. Note that $P_\sigma = Q_\sigma \circ \sigma$.

Next we need the inverse maps P_σ^{-1} and Q_σ^{-1} . The map P_σ^{-1} takes two distinct points from the boundary ∂D as its input and tells us the entry co-vector. Similarly, Q_σ^{-1} takes two distinct boundary points and reports the exit co-vector. In order to ensure that P_σ^{-1} and Q_σ^{-1} exist we assume that σ is nice in the following sense:

Definition 4.1 We say that a map $\sigma: U_{in}^* \rightarrow U_{out}^*$ is *nice* if the following conditions are satisfied:

- (1) σ is a symplectic diffeomorphism between U_{in}^* and U_{out}^* .
- (2) P_σ is a diffeomorphism between U_{in}^* and $S \times S \setminus \Delta$.
- (3) Q_σ is a diffeomorphism between U_{out}^* and $S \times S \setminus \Delta$.

Here, as in the previous sections, Δ denotes the diagonal of $S \times S$.

In fact, the third condition in [Definition 4.1](#) follows from the second one and the identity $P_\sigma = Q_\sigma \circ \sigma$. We include them both in the definition for clarity of exposition. The assumption that σ is symplectic is crucial. Nonetheless it is used only in one place, namely in the proof of [Lemma 4.3](#).

If σ is the dual lens map of a simple metric, then σ is nice. Furthermore, if $\tilde{\sigma}$ is obtained from a nice map σ by a sufficiently small compactly supported perturbation (as in [Theorem 2](#)), then $\tilde{\sigma}$ is nice as well.

The last bit of notation is the 1-form λ_σ on $S \times S \setminus \Delta$ defined as follows. For $p, q \in S$, $p \neq q$, $\xi \in T_p S$, $\eta \in T_q S$, define

$$(4-1) \quad \lambda_\sigma(\xi, \eta) = -P_\sigma^{-1}(p, q)(\xi) + Q_\sigma^{-1}(p, q)(\eta).$$

Here (ξ, η) is regarded as an element of $T_{(p,q)}(S \times S \setminus \Delta)$ through the standard identification $T(S \times S) \simeq TS \times TS$. The maps P_σ^{-1} and Q_σ^{-1} in (4-1) are correctly defined since σ is nice. Moreover, $P_\sigma^{-1}(p, q) \in (U_{in}^*)_p$ and $Q_\sigma^{-1}(p, q) \in (U_{out}^*)_q$, where $(U_{in}^*)_p$ and $(U_{out}^*)_q$ are the fibers of U_{in}^* and U_{out}^* over p and q respectively. Therefore the terms $P_\sigma^{-1}(p, q)(\xi)$ and $Q_\sigma^{-1}(p, q)(\eta)$ in (4-1) make sense.

The following lemma explains where the definition of λ_σ comes from. The lemma tells us that if σ arises from a simple metric, then λ_σ is the derivative of the distance function, as follows:

Lemma 4.2 *Let $f = bd_\varphi$ and let $\sigma: U_{in}^* \rightarrow U_{out}^*$ be a nice map. Then σ is the dual lens map of φ if and only if $df = \lambda_\sigma$ on $S \times S \setminus \Delta$.*

Proof We begin with the “only if” part. Let σ be the dual lens map of φ . Let $p, q \in S$, $p \neq q$, $\alpha = \mathcal{L}(v_{pq})$ and $\beta = \mathcal{L}(w_{pq})$, where v_{pq} and w_{pq} are the velocity vectors of the geodesic $[pq]$ at p and q ; see the notation in [Section 2](#). Then $\beta = \sigma(\alpha)$ and $P_\sigma(\alpha) = Q_\sigma(\beta) = (p, q)$. By (2-2) and (2-1) we have $d_p f(\cdot, q) = -\alpha$ and $d_q f(p, \cdot) = \beta$. Hence for every $\xi \in T_p S$ and $\eta \in T_q S$,

$$df(\xi, \eta) = -\alpha(\xi) + \beta(\eta) = -P_\sigma^{-1}(p, q)(\xi) + Q_\sigma^{-1}(p, q)(\eta) = \lambda_\sigma(\xi, \eta).$$

This proves the “only if” part of the lemma.

To prove the “if” part, let $\hat{\sigma}$ be the dual lens map of φ . By the “only if” part we have $df = \lambda_{\hat{\sigma}}$ on $S \times S \setminus \Delta$. Thus it suffices to prove that $\sigma = \hat{\sigma}$ provided that $\lambda_\sigma = \lambda_{\hat{\sigma}}$.

Let $\alpha \in U_{in}^*$, $p = \pi(\alpha) \in S$, $\beta = \hat{\sigma}(\alpha) \in U_{out}^*$ and $q = \pi(\beta) \in S \setminus \{p\}$. Then $P_{\hat{\sigma}}(\alpha) = Q_{\hat{\sigma}}(\beta) = (p, q)$. Since $\lambda_{\hat{\sigma}} = \lambda_\sigma$, substituting $\eta = 0$ into (4-1) yields that

$$P_\sigma^{-1}(p, q)(\xi) = P_{\hat{\sigma}}^{-1}(p, q)(\xi) = \alpha(\xi)$$

for every $\xi \in T_p S$. That is, $P_\sigma^{-1}(p, q)|_{T_p S} = \alpha|_{T_p S}$. Since $P_\sigma^{-1}(p, q) \in (U_{in}^*)_p$ and a co-vector from $(U_{in}^*)_p$ is uniquely determined by its restriction to $T_p S$, it follows that $P_\sigma^{-1}(p, q) = \alpha$. Hence $\sigma(\alpha) \in (U_{out}^*)_q$.

Similarly, substituting $\xi = 0$ into (4-1) yields that $Q_\sigma^{-1}(p, q)|_{T_q S} = \beta|_{T_q S}$, and therefore $Q_\sigma^{-1}(p, q) = \beta$. Hence $\sigma^{-1}(\beta) \in (U_{in}^*)_p$. Thus

$$(4-2) \quad \pi(\sigma^{-1}(\beta)) = p = \pi(\alpha) = \pi(\sigma^{-1}(\sigma(\alpha))).$$

Since σ is nice, Q_σ is a diffeomorphism from U_{out}^* to $S \times S \setminus \Delta$. In particular, the restriction $Q_\sigma|_{(U_{out}^*)_q}$ is a diffeomorphism from $(U_{out}^*)_q$ to $(S \setminus \{q\}) \times \{q\}$. Hence the first coordinate map of this restriction, namely $\pi \circ \sigma^{-1}|_{(U_{out}^*)_q}$, is injective. Since $\beta \in (U_{out}^*)_q$ and $\sigma(\alpha) \in (U_{out}^*)_q$, this injectivity and (4-2) imply that $\sigma(\alpha) = \beta$. Since α is an arbitrary element of U_{in}^* and $\beta = \hat{\sigma}(\alpha)$, it follows that $\sigma = \hat{\sigma}$. This finishes the proof of the “if” part of the lemma. \square

Lemma 4.3 λ_σ is a closed 1–form for every nice map $\sigma: U_{in}^* \rightarrow U_{out}^*$.

Proof Let ν denote the canonical 1–form on T^*D . Recall that the canonical 1–form is defined as follows: for $\alpha \in T^*D$ and $\tau \in T_\alpha T^*D$,

$$\nu(\tau) = \alpha(d\pi(\tau)),$$

where $\pi: T^*D \rightarrow D$ is the bundle projection. The two terms in (4-1) are actually pull-backs of ν by P_σ^{-1} and Q_σ^{-1} , namely

$$\lambda_\sigma = -(P_\sigma^{-1})^*\nu + (Q_\sigma^{-1})^*\nu.$$

Hence

$$d\lambda_\sigma = -(P_\sigma^{-1})^*(d\nu) + (Q_\sigma^{-1})^*(d\nu).$$

Let ω be the canonical symplectic form on T^*D . Recall that $\omega = d\nu$. Therefore

$$\begin{aligned} d\lambda_\sigma &= -(P_\sigma^{-1})^*\omega + (Q_\sigma^{-1})^*\omega \\ &= -(P_\sigma^{-1})^*\omega + (\sigma \circ P_\sigma^{-1})^*\omega \\ &= -(P_\sigma^{-1})^*\omega + (P_\sigma^{-1})^*(\sigma^*\omega) = 0. \end{aligned}$$

Here we use the identity $P_\sigma = Q_\sigma \circ \sigma$ and the fact that $\sigma^*\omega = \omega$ because σ is symplectic. Thus $d\lambda_\sigma = 0$ and the lemma follows. \square

Proof of Theorem 2 Now let σ be the dual lens map of φ and let $\tilde{\sigma}$ be a perturbation of σ as in the statement of Theorem 2. Recall that $\tilde{\sigma}$ is nice provided that the perturbation is sufficiently small. Then by Lemma 4.3 the 1–form $\lambda_{\tilde{\sigma}}$ is closed. Since $n \geq 3$, the space $S \times S \setminus \Delta$ is simply connected and therefore $\lambda_{\tilde{\sigma}}$ is exact. Hence there exists a smooth function $\tilde{f}: S \times S \setminus \Delta \rightarrow \mathbb{R}$ such that $d\tilde{f} = \lambda_{\tilde{\sigma}}$.

Since $\tilde{\sigma}$ coincides with σ outside a compact set, $\lambda_{\tilde{\sigma}}$ and λ_{σ} also coincide outside a compact subset of $S \times S \setminus \Delta$. That is, there exists a tubular neighborhood U of Δ in $S \times S$ such that $\lambda_{\tilde{\sigma}}|_{U \setminus \Delta} = \lambda_{\sigma}|_{U \setminus \Delta}$. This fact and [Lemma 4.2](#) imply that

$$(4-3) \quad d\tilde{f}|_{U \setminus \Delta} = df|_{U \setminus \Delta},$$

where f is the boundary distance function of φ . Since $n \geq 3$, the set $U \setminus \Delta$ is connected. This and (4-3) imply that $\tilde{f} - f$ is constant on $U \setminus \Delta$. Since \tilde{f} is defined up to an additive constant, we may assume that $\tilde{f} = f$ on $U \setminus \Delta$. Now we can extend \tilde{f} to the whole $S \times S$ by setting it equal to zero on Δ . Then \tilde{f} is a small perturbation of f in the sense of [Theorem 1](#). Hence by [Theorem 1](#) there exists a simple Finsler metric $\tilde{\varphi}$ whose boundary distance function is \tilde{f} . Moreover, $\tilde{\varphi}$ coincides with φ in a neighborhood of the boundary and is close to φ in C^∞ . Now [Lemma 4.2](#) applied to $\tilde{\varphi}$ in place of φ implies that $\tilde{\sigma}$ is the dual lens map of σ .

It remains to prove the last assertion of [Theorem 2](#), namely that $\tilde{\varphi}$ can be chosen reversible if φ is reversible and $\tilde{\sigma}$ satisfies the symmetry condition: $-\tilde{\sigma}(-\tilde{\sigma}(\alpha)) = \alpha$ for all $\alpha \in U_{in}^*$. By the last assertion of [Theorem 1](#), it suffices to verify that \tilde{f} is symmetric; that is, $\tilde{f}(p, q) = \tilde{f}(q, p)$ for all $p, q \in S$. First observe that the symmetry condition on $\tilde{\sigma}$ implies that

$$(4-4) \quad P_{\tilde{\sigma}}^{-1}(q, p) = -Q_{\tilde{\sigma}}^{-1}(p, q)$$

for all $p, q \in S, p \neq q$. Indeed, if $\alpha = P_{\tilde{\sigma}}^{-1}(p, q)$ and $\beta = Q_{\tilde{\sigma}}^{-1}(p, q)$, then $\tilde{\sigma}(\alpha) = \beta$, hence $\tilde{\sigma}(-\beta) = -\alpha$ by the symmetry condition. Therefore $P_{\tilde{\sigma}}^{-1}(q, p) = -\beta$, and (4-4) follows.

Now consider the function $\tilde{g}: S \times S \rightarrow \mathbb{R}$ defined by $\tilde{g}(x, y) = \tilde{f}(y, x)$. Let $p, q \in S, p \neq q, \xi \in T_p S$ and $\eta \in T_q S$. Then

$$\begin{aligned} d_{(p,q)}\tilde{g}(\xi, \eta) &= d_{(q,p)}\tilde{f}(\eta, \xi) = -P_{\tilde{\sigma}}^{-1}(q, p)(\eta) + Q_{\tilde{\sigma}}^{-1}(q, p)(\xi) \\ &= Q_{\tilde{\sigma}}^{-1}(p, q)(\eta) - P_{\tilde{\sigma}}^{-1}(p, q)(\xi) = d_{(p,q)}\tilde{f}(\xi, \eta). \end{aligned}$$

Here we used the identity $d\tilde{f} = \lambda_{\tilde{\sigma}}$, the definition of $\lambda_{\tilde{\sigma}}$ and (4-4). Thus $d\tilde{g} = d\tilde{f}$, hence $\tilde{g} = \tilde{f}$; that is, \tilde{f} is symmetric. This finishes the proof of [Theorem 2](#). \square

Remark 4.4 The above argument does not work in dimension $n = 2$ for two reasons. The first issue is that $S \times S \setminus \Delta$ is not simply connected, so a priori the closed 1-form $\lambda_{\tilde{\sigma}}$ may not be exact. However it is easy to see that this 1-form is exact even in dimension 2. Indeed, the fundamental group of $S \times S \setminus \Delta$ is cyclic and its generator can be realized by a loop arbitrarily close to Δ . Since $\lambda_{\tilde{\sigma}}$ and λ_{σ} coincide in a

neighborhood of Δ , their integrals over this loop are equal. Since $\lambda_\sigma = df$ is exact, these integrals are zero. Hence $\lambda_{\tilde{\sigma}}$ is exact as well.

The second issue with $n = 2$ is that $U \setminus \Delta$ is not connected, where U is a tubular neighborhood of Δ in $S \times S$. In fact, $U \setminus \Delta$ consists of two components. These components are represented by points (p, q_1) and (p, q_2) in $S \times S \setminus \Delta$, where $p \in S$ is an arbitrary point and $q_1, q_2 \in S$ are close to p and lie on different sides of it. It may happen that the function \tilde{f} (an antiderivative of $\lambda_{\tilde{\sigma}}$) approaches different values as we tend to the diagonal Δ from different sides. In this case \tilde{f} cannot be extended to the whole $S \times S$ as a continuous function. To rule out this situation we need to impose an additional condition on $\tilde{\sigma}$, namely

$$(4-5) \quad \int_{\{p\} \times (S \setminus \{p\})} \lambda_{\tilde{\sigma}} = 0$$

for some (and then all) $p \in S$. This condition is a necessary and sufficient one and it is easy to give examples of when it is not satisfied.

Taking into account the definition of $\lambda_{\tilde{\sigma}}$, the condition (4-5) can be interpreted as follows. For a fixed $p \in S$, the image $\tilde{\sigma}(U_{in}^*|_p)$ is a graph of a differential 1-form on $S \setminus \{p\}$, and (4-5) is equivalent to the requirement that the (improper) integral of this 1-form over $S \setminus \{p\}$ be zero.

5 Proof of Theorem 3

In this section we deduce Theorem 3 from Theorem 2. Consider the unit disc $D^6 \subset \mathbb{R}^6$ equipped with the standard symplectic structure. We need the following lemma.

Lemma 5.1 *There exists a symplectomorphism $\theta: D^6 \rightarrow D^6$ which is arbitrarily close to the identity in C^∞ , coincides with the identity map near the boundary, and has positive metric entropy.*

Proof Let M be a surface of genus ≥ 2 equipped with a Riemannian metric of negative curvature. Let $I_1 = (-1, 1)$, $I_2 = (-2, 2)$ and $N = M \times I_2$. Let B^*M denote the bundle of unit balls in T^*M .

First we construct a compactly supported Hamiltonian $H: T^*N \rightarrow \mathbb{R}$ whose Hamiltonian flow has positive metric entropy. Let $\psi: \mathbb{R} \rightarrow \mathbb{R}_+$ be a smooth cut-off function which equals 1 on $[-1, 1]$ and vanishes outside $(-\frac{3}{2}, \frac{3}{2})$. For a co-vector $(x, y, \xi, \eta) \in T^*N$, where $x \in M$, $y \in I_2$, $\xi \in T_x^*M$, $\eta \in T_y^*I_2$, we define

$$H(x, y, \xi, \eta) = \psi(y) \cdot \psi(|\xi|) \cdot \psi(|\eta|) \cdot \frac{|\xi|^2}{2},$$

where $|\xi|$ is the norm of ξ determined by the Riemannian metric of M .

Observe that H is compactly supported. Consider the Hamiltonian flow of H restricted to the set

$$B^*M \times B^*I_1 \subset T^*M \times T^*I_2 = T^*N.$$

On this set we have $H(x, y, \xi, \eta) = |\xi|^2/2$. Recall that the standard Hamiltonian of our Riemannian metric is given by $H_{Riem}(x, \xi) = |\xi|^2/2$. Hence the Hamiltonian flow of H restricted to $B^*M \times B^*I_1$ is the product of the geodesic flow of our Riemannian metric (restricted to B^*M) and the trivial flow on B^*I_1 . Hence the metric entropy of this flow is positive.

Now let $U \subset \mathbb{R}^3$ be an open set diffeomorphic to N (in other words, U is a thickened handlebody). A diffeomorphism between N and U induces a symplectomorphism between T^*N and T^*U . This symplectomorphism sends the Hamiltonian flow of H to a compactly supported Hamiltonian flow on $T^*U \subset T^*\mathbb{R}^3 \simeq \mathbb{R}^6$. The support of this flow fits into D^6 upon suitable rescaling.

Thus there exists a Hamiltonian flow $\{\Phi^t\}$ on D^6 which vanishes near the boundary and has positive metric entropy. Finally, in order to construct the desired symplectomorphism θ , we take a map Φ^t for a sufficiently small $t > 0$. \square

Proof of Theorem 3 Consider the sphere S^4 with its standard Riemannian metric, and let $D \subset S^4$ be a small geodesic ball. Then D is diffeomorphic to D^4 , and the metric of S^4 restricts to a simple Riemannian metric on D . Using the notation introduced before [Theorem 2](#), consider the dual lens map $\sigma: U_{in}^* \rightarrow U_{out}^*$.

Fix a co-vector $\alpha \in U_{in}^*$ and let $\omega = \beta^*(\alpha)$. Let $V \Subset U_{in}^*$ be a neighborhood of α in U_{in}^* . We choose V to be symplectomorphic to a rescaled copy of the interior of the standard disc D^6 . By [Lemma 5.1](#) there exists a compactly supported symplectomorphism $\theta: V \rightarrow V$ with positive metric entropy, and arbitrarily close to the identity map in the C^∞ topology. We extend θ to a map from the entire U_{in}^* to itself so that it is the identity on $U_{in}^* \setminus V$. We use the same letter θ for this extension.

Define a perturbed dual lens map $\tilde{\sigma}: U_{in}^* \rightarrow U_{out}^*$ by $\tilde{\sigma} = \sigma \circ \theta$. If θ is sufficiently close to the identity, then by [Theorem 2](#) the map $\tilde{\sigma}$ can be realized as the dual lens map of a Finsler metric $\tilde{\varphi}$. Furthermore, $\tilde{\varphi}$ can be chosen close to the original Riemannian metric and coinciding with it in a neighborhood of the boundary of D . We paste the metric $\tilde{\varphi}$ on D with the original metric on $S^4 \setminus D$ and obtain a smooth Finsler metric on S^4 . This metric is the desired perturbation with positive entropy.

Indeed, since the original geodesic flow is periodic, the return map of the perturbed geodesic flow equals θ on V (and remains the identity elsewhere). Thus the perturbed

flow contains an invariant open set where the flow is a suspension of θ . Hence the perturbed geodesic flow has positive metric entropy. \square

The disc D in the proof of [Theorem 3](#) can be chosen arbitrarily small. Thus the perturbation of the geodesic flow is confined to an arbitrarily small neighborhood of one (periodic) orbit of the original flow. This allows us to construct an example with local generation of metric entropy:

Corollary 5.2 *There exists a smooth perturbation of the standard metric of S^4 (in the class of reversible Finsler metrics) such that the following holds. There is a periodic trajectory γ such that for every $\varepsilon > 0$ the ε -neighborhood of γ contains an open invariant set in which the metric entropy is positive. In particular, the flow is entropy non-expansive in the sense of [\[2\]](#).*

Proof Begin with a sequence $\{\gamma_i\}$ of disjoint geodesics in S^4 converging to a geodesic γ . For each i choose a neighborhood U_i of γ_i in such a way that these neighborhoods are disjoint. Construct a small perturbation of the metric within each U_i as in [Theorem 3](#). These perturbations should be so small that their derivatives go to zero as $i \rightarrow \infty$. Then the union of these perturbations provides the desired example. \square

Note that both metric and topological entropy in the ε -neighborhoods in [Corollary 5.2](#) must vanish as $\varepsilon \rightarrow 0$. This follows from the fact that ε -local entropy of any C^∞ flow tends to zero as $\varepsilon \rightarrow 0$; see [\[6\]](#).

References

- [1] **A S Besicovitch**, *On two problems of Loewner*, J. London Math. Soc. 27 (1952) 141–144 [MR0047126](#)
- [2] **R Bowen**, *Entropy-expansive maps*, Trans. Amer. Math. Soc. 164 (1972) 323–331 [MR0285689](#)
- [3] **D Y Burago**, *A new approach to the calculation of the entropy of the geodesic flow and of related dynamical systems*, Dokl. Akad. Nauk SSSR 299 (1988) 1041–1044 [MR947227](#) In Russian; translated in Soviet Math. Dokl. 37 (1988) 501–504
- [4] **S Ivanov**, *Local monotonicity of Riemannian and Finsler volume with respect to boundary distances*, Geom. Dedicata 164 (2013) 83–96 [MR3054617](#)
- [5] **G Knieper, H Weiss**, *A surface with positive curvature and positive topological entropy*, J. Differential Geom. 39 (1994) 229–249 [MR1267890](#)
- [6] **S E Newhouse**, *Continuity properties of entropy*, Ann. of Math. 129 (1989) 215–235 [MR986792](#)

- [7] **G P Paternain**, *Real analytic convex surfaces with positive topological entropy and rigid body dynamics*, Manuscripta Math. 78 (1993) 397–402 [MR1208649](#)

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