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THETA PRODUCTS AND ETA QUOTIENTS OF LEVEL 24 AND WEIGHT 2

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Abstract: We find bases for the spaces $M_2\Big(\Gamma_0(24), \Big(\frac{d}{\cdot}\Big)\Big)$ $(d = 1, 8, 12, 24)$ of modular forms. We determine the Fourier coefficients of all 35 theta products $\varphi[a_1, a_2, a_3, a_4](z)$ in these spaces. We then deduce formulas for the number of representations of a positive integer n by diagonal quaternary quadratic forms with coefficients 1, 2, 3 or 6 in a uniform manner, of which 14 are Ramanujan's universal quaternary quadratic forms. We also find all the eta quotients in the Eisenstein spaces $E_2\Big(\Gamma_0(24), \Big(\frac{d}{\cdot}\Big)\Big)$ $(d = 1, 8, 12, 24)$ and give their Fourier coefficients.

Keywords: Dedekind eta function, eta quotients, theta products, Eisenstein series, modular forms, cusp forms, Fourier coefficients, Fourier series.

1. Introduction and notation

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ denote the sets of positive integers, non-negative integers, integers, rational numbers and complex numbers, respectively. Let $N \in \mathbb{N}$. Let $\Gamma_0(N)$ be the modular subgroup defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\Big|\, a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ c \equiv 0 \pmod{N} \right\}.$$

We define a Dirichlet character χ_t for each $t \in \{-24, -8, -4, -3, 1, 8, 12, 24\}$ by

$$\chi_t(m) = \left(\frac{t}{m}\right) \text{ for } m \in \mathbb{Z}. \tag{1.1}$$

Note that χ_1 is the trivial character. Let χ and ψ be Dirichlet characters. For $n \in \mathbb{N}$ we define the generalized sums of divisors functions $\sigma_{(\chi,\psi)}(n)$ by

$$\sigma_{(\chi,\psi)}(n) = \sum_{1 \leqslant m \mid n} \chi(m)\psi(n/m)m. \tag{1.2}$$

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If $n \notin \mathbb{N}$ we set $\sigma_{(\chi,\psi)}(n) = 0$. If $\chi = \psi = \chi_1$, then $\sigma_{(\chi,\psi)}(n)$ coincides with the sum of divisors function

$$\sigma(n) = \sum_{1 \leqslant m \mid n} m.$$

Let $k \in \mathbb{Z}$. We write $M_k(\Gamma_0(N), \chi)$ to denote the space of modular forms of weight k with multiplier system χ for $\Gamma_0(N)$, and $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$ to denote the subspaces of Eisenstein forms and cusp forms of $M_k(\Gamma_0(N), \chi)$, respectively. We also write $M_k(\Gamma_0(N))$, $E_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$ for $M_k(\Gamma_0(N), \chi_1)$, $E_k(\Gamma_0(N), \chi_1)$ and $S_k(\Gamma_0(N), \chi_1)$, respectively. It is known (see [14, p. 83]) that

$$M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi). \tag{1.3}$$

The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}). \tag{1.4}$$

Throughout the remainder of this paper we set $q = q(z) = e^{2\pi i z}$ with $z \in \mathbb{H}$. Let $N \in \mathbb{N}$ and $r_\delta \in \mathbb{Z}$ for each positive divisor δ of N. We define an eta quotient of level N by the product formula

$$f(z) = \prod_{1 \leqslant \delta \mid N} \eta^{r_\delta}(\delta z). \tag{1.5}$$

We define an Eisenstein series $E_{t_1,t_2}(q)$ by

$$E_{t_1,t_2}(z) = C_{t_1,t_2} + \sum_{n=1}^{\infty} \sigma_{(\chi_{t_1},\chi_{t_2})}(n) q^n \tag{1.6}$$

for each

$$\begin{aligned}(t_1, t_2) =&(-8,-3), (-3,-8), (1,24), (24,1), (1,1), (1,8), (8,1),\\ &(1,12), (12,1), (-3,-4), (-4,-3),\end{aligned}$$

where

$$C_{8,1} = -\frac{1}{2}, \qquad C_{12,1} = -1, \qquad C_{24,1} = -3, \qquad C_{1,1} = -\frac{1}{24},$$
$$C_{-8,-3} = C_{-3,-8} = C_{1,24} = C_{1,8} = C_{1,12} = C_{-3,-4} = C_{-4,-3} = 0.$$

For $(t_1, t_2) = (1, 1)$ we write

$$L(q) = E_{1,1}(z) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) q^n.$$

For $1 < d \mid 24$ we set

$$L_d(q) = L(q) - dL(q^d). \tag{1.7}$$

Ramanujan's theta function $\varphi(z)$ is defined by

$$\varphi(z) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

It is well known [5, p. 11] that $\varphi(z)$ can be expressed as

$$\varphi(z) = \frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)}. \tag{1.8}$$

Let $a_1,\ a_2,\ a_3,\ a_4 \in \mathbb{N}$ and $n \in \mathbb{N}_0$. We define

$$N(a_1, a_2, a_3, a_4; n) = \mathrm{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \,|\, n = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2\}.$$

For notational convenience we set

$$\varphi[a_1, a_2, a_3, a_4](z) = \varphi(a_1z)\varphi(a_2z)\varphi(a_3z)\varphi(a_4z).$$

We have

$$\varphi[a_1, a_2, a_3, a_4](z) = \sum_{n=0}^{\infty} N(a_1, a_2, a_3, a_4; n)q^n, \tag{1.9}$$

which is independent of the order of a_1, a_2, a_3, a_4. We have 35 theta products $\varphi[a_1, a_2, a_3, a_4](z)$ of level 24 and weight 2. We group them according to the modular spaces to which they belong, namely

$$\begin{cases} \varphi[1,1,1,1](z), & \varphi[1,1,2,2](z), & \varphi[1,1,3,3](z), & \varphi[1,1,6,6](z), \\ \varphi[1,2,3,6](z), & \varphi[2,2,2,2](z), & \varphi[2,2,3,3](z), & \varphi[2,2,6,6](z), \\ \varphi[3,3,3,3](z), & \varphi[3,3,6,6](z), & \varphi[6,6,6,6](z); \end{cases} \tag{1.10}$$

$$\begin{cases} \varphi[1,1,1,2](z), & \varphi[1,1,3,6](z), & \varphi[1,2,2,2](z), & \varphi[1,2,3,3](z), \\ \varphi[1,2,6,6](z), & \varphi[2,2,3,6](z), & \varphi[3,3,3,6](z), & \varphi[3,6,6,6](z); \end{cases} \tag{1.11}$$

$$\begin{cases} \varphi[1,1,1,3](z), & \varphi[1,1,2,6](z), & \varphi[1,2,2,3](z), & \varphi[1,3,3,3](z), \\ \varphi[1,3,6,6](z), & \varphi[2,2,2,6](z), & \varphi[2,3,3,6](z), & \varphi[2,6,6,6](z); \end{cases} \tag{1.12}$$

$$\begin{cases} \varphi[1,1,1,6](z), & \varphi[1,1,2,3](z), & \varphi[1,2,2,6](z), & \varphi[1,3,3,6](z), \\ \varphi[1,6,6,6](z), & \varphi[2,2,2,3](z), & \varphi[2,3,3,3](z), & \varphi[2,3,6,6](z) \end{cases} \tag{1.13}$$

are in $M_2(\Gamma_0(24), \chi_1)$, $M_2(\Gamma_0(24), \chi_8)$, $M_2(\Gamma_0(24), \chi_{12})$ and $M_2(\Gamma_0(24), \chi_{24})$, respectively.

In this paper we give bases for the spaces $M_2\Big(\Gamma_0(24), \Big(\frac{d}{\cdot}\Big)\Big)$ $(d = 1, 8, 12, 24)$ of modular forms. We determine the Fourier cofficients of all 35 theta products in (1.10)–(1.13). We then deduce explicit formulas for $N(a_1, a_2, a_3, a_4; n)$, where $a_1, a_2, a_3, a_4 \in \{1, 2, 3, 6\}$, in a uniform manner, of which 14 are Ramanujan's universal quaternary quadratic forms given in [13]. We also find all the eta quotients in the Eisenstein spaces $E_2\Big(\Gamma_0(24), \Big(\frac{d}{\cdot}\Big)\Big)$ $(d = 1, 8, 12, 24)$ and give their Fourier coefficients.

2. Bases for $M_2(\Gamma_0(24), \chi_i)$ for $i \in \{1, 8, 12, 24\}$

We deduce from [14, Sec. 6.1, p. 93] that

$$\dim S_2(\Gamma_0(24)) = 1, \qquad \dim E_2(\Gamma_0(24)) = 7. \tag{2.1}$$

We also deduce from [14, Sec. 6.3, p. 98] that

$$\dim S_2(\Gamma_0(24), \chi_8) = 2, \quad \dim S_2(\Gamma_0(24), \chi_{12}) = 0, \quad \dim S_2(\Gamma_0(24), \chi_{24}) = 2, \tag{2.2}$$

and

$$\dim E_2(\Gamma_0(24), \chi_8) = 4, \quad \dim E_2(\Gamma_0(24), \chi_{12}) = 8, \quad \dim E_2(\Gamma_0(24), \chi_{24}) = 4. \tag{2.3}$$

We define the eta quotients $A(q)$, $B_1(q)$, $B_2(q)$, $C_1(q)$, $C_2(q)$ and the integers $a(n)$, $b_1(n)$, $b_2(n)$, $c_1(n)$, $c_2(n)$ by

$$A(q) = \eta(2z)\eta(4z)\eta(6z)\eta(12z) = \sum_{n=1}^{\infty} a(n)q^n, \tag{2.4}$$

$$B_1(q) = \frac{\eta(z)\eta^4(6z)\eta^2(8z)}{\eta(2z)\eta(3z)\eta(12z)} = \sum_{n=1}^{\infty} b_1(n)q^n, \tag{2.5}$$

$$B_2(q) = \frac{\eta^2(z)\eta(8z)\eta^4(12z)}{\eta(4z)\eta(6z)\eta(24z)} = \sum_{n=1}^{\infty} b_2(n)q^n, \tag{2.6}$$

$$C_1(q) = \frac{\eta(z)\eta(4z)\eta^4(6z)\eta^2(24z)}{\eta(2z)\eta(3z)\eta^2(12z)} = \sum_{n=1}^{\infty} c_1(n)q^n, \tag{2.7}$$

$$C_2(q) = \frac{\eta^2(z)\eta^4(4z)\eta(6z)\eta(24z)}{\eta^2(2z)\eta(8z)\eta(12z)} = \sum_{n=1}^{\infty} c_2(n)q^n. \tag{2.8}$$

We note that $A(q)$ is the only newform among all the eta quotients of weight 2 and level 24, see [10, Table I] or [11, Theorem 1]. We now give a basis for $M_2(\Gamma_0(24),\chi_i)$ for each $i \in \{1,8,12,24\}$.

Theorem 2.1. *Let* χ_t *be as in* (1.1) *for* $t = 1,8,12,24$. *Then*

$$\begin{aligned}
&\{L_d(q) \mid d = 2,3,4,6,8,12,24\} \cup \{A(q)\},\\
&\{E_{1,8}(z),\ E_{1,8}(3z),\ E_{8,1}(z),\ E_{8,1}(3z),\ B_1(q),\ B_2(q)\},\\
&\{E_{1,12}(z), E_{1,12}(2z), E_{12,1}(z), E_{12,1}(2z), E_{-4,-3}(z),\\
&\qquad\qquad E_{-4,-3}(2z), E_{-3,-4}(z), E_{-3,-4}(2z)\},\\
&\{E_{1,24}(z),\ E_{24,1}(z),\ E_{-3,-8}(z),\ E_{-8,-3}(z),\ C_1(q),\ C_2(q)\},
\end{aligned}$$

are bases for $M_2(\Gamma_0(24))$, $M_2(\Gamma_0(24),\chi_8)$, $M_2(\Gamma_0(24),\chi_{12})$ *and* $M_2(\Gamma_0(24),\chi_{24})$, *respectively.*

Proof. Appealing to [12, Theorem 1.64, p. 18] and [8, Corollary 2.3, p. 37] (see also [9, 7, 1]) one can show that $A(q) \in S_2(\Gamma_0(24))$; $B_1(q), B_2(q) \in S_2(\Gamma_0(24),\chi_8)$; $C_1(q), C_2(q) \in S_2(\Gamma_0(24),\chi_{24})$. The assertion (i) follows from (1.3), (2.1) and [14, Theorem 5.8, p. 88]. (ii) follows from (1.3), (2.2), (2.3) and [14, Theorem 5.9, p. 88] with $\epsilon = \chi_8$ and $\chi,\psi \in \{\chi_1,\chi_8\}$. (iii) follows from (1.3), (2.2) and (2.3) [14, Theorem 5.9, p. 88] with $\epsilon = \chi_{12}$ and $\chi,\psi \in \{\chi_1,\chi_{12},\chi_{-3},\chi_{-4}\}$. (iv) follows from (1.3), (2.2), (2.3) and [14, Theorem 5.9, p. 88] with $\epsilon = \chi_{24}$ and $\chi,\psi \in \{\chi_1,\chi_{24},\chi_{-3},\chi_{-8}\}$. ■

3. Theta products in $M_2(\Gamma_0(24),\chi_i)$ for $i \in \{1,8,12,24\}$

Theorems 3.1–3.4 below follow from (1.10)–(1.13) and Theorem 2.1.

Theorem 3.1. *Let* $\varphi[a_1,a_2,a_3,a_4](z) \in M_2(\Gamma_0(24))$ *be any of the theta products given in* (1.10), *and let* $L_d(q)$ *be as in* (1.7). *Then we have*

$$\begin{aligned}
\varphi[a_1,a_2,a_3,a_4](z) = {} & b_2L_2(q) + b_3L_3(q) + b_4L_4(q) + b_6L_6(q) + b_8(L_8(q)\\
& + b_{12}L_{12}(q) + b_{24}L_{24}(q) + xA(q),
\end{aligned}$$

where the coefficients b_2, b_3, b_4, b_6, b_8, b_{12}, b_{24} *and* x *are given at the right hand side of* Table 3.1.

Table 3.1. $\varphi[a_1,a_2,a_3,a_4](z) = b_2L_2(q)+b_3L_3(q)+b_4L_4(q)+b_6L_6(q)+b_8L_8(q)+b_{12}L_{12}(q)+b_{24}L_{24}(q)+xA(q)$.

a_1	a_2	a_3	a_4	b_2	b_3	b_4	b_6	b_8	b_{12}	b_{24}	x
1	1	1	1	0	0	8	0	0	0	0	0
1	1	2	2	2	0	-2	0	4	0	0	0
1	1	3	3	4	4	-4	-4	0	4	0	0
1	1	6	6	1	2	1	-1	-2	-1	2	2
1	2	3	6	$1/2$	-1	$-1/2$	$1/2$	1	$-1/2$	1	1
2	2	2	2	-4	0	0	0	4	0	0	0
2	2	3	3	1	2	1	-1	-2	-1	2	-2
2	2	6	6	-2	0	2	2	-2	-2	2	0
3	3	3	3	0	$-8/3$	0	0	0	$8/3$	0	0
3	3	6	6	0	$-4/3$	0	$2/3$	0	$-2/3$	$4/3$	0
6	6	6	6	0	0	0	$-4/3$	0	0	$4/3$	0

Proof. Let $\varphi[a_1,a_2,a_3,a_4](z)$ be any of the theta products listed in (1.10). By Theorem 2.1, $\varphi[a_1,a_2,a_3,a_4](z)$ must be a linear combination of $L_d(q)$ ($d = 2,3,4,6,8,12,24$) and $A(q)$, namely

$$\begin{aligned}\varphi[a_1,a_2,a_3,a_4](z) = {} & b_2L_2(q)+b_3L_3(q)+b_4L_4(q)+b_6L_6(q)+b_8L_8(q) \\ & + b_{12}L_{12}(q)+b_{24}L_{24}(q)+xA(q)\end{aligned} \tag{3.1}$$

for some scalars $b_2,b_3,b_4,b_6,b_8,b_{12},b_{24},x \in \mathbb{C}$. By [7, Theorem 3.13], the Sturm bound for the modular space $M_2(\Gamma_0(24))$ is 8. So, equating the coefficients of q^n for $0 \leqslant n \leqslant 8$ on both sides of (3.1), we find a system of linear equations. We solve this system and find the asserted coefficients. ∎

Similarly to Theorem 3.1, Theorems 3.2–3.4 follow from (1.11)–(1.13) and Theorem 2.1.

Theorem 3.2. *Let $\varphi[a_1,a_2,a_3,a_4](z) \in M_2(\Gamma_0(24),\chi_8)$ be any of the theta products given in* (1.11)*, where $\chi_8(n)$ is given by* (1.1)*. Let $E_{1,8}(z)$ and $E_{8,1}(z)$ be as in* (1.6)*. Then we have*

$$\begin{aligned}\varphi[a_1,a_2,a_3,a_4](z) = {} & b_1E_{1,8}(z)+b_2E_{1,8}(3z)+b_3E_{8,1}(z)+b_4E_{8,1}(3z) \\ & + x_1B_1(q)+x_2B_2(q),\end{aligned}$$

where the coefficients b_1, b_2, b_3, b_4, x_1 and x_2 are given at the right hand side of Table 3.2*.*

Table 3.2. $\varphi[a_1,a_2,a_3,a_4](z) = b_1E_{1,8}(z) + b_2E_{1,8}(3z) + b_3E_{8,1}(z) + b_4E_{8,1}(3z) + x_1B_1(q) + x_2B_2(q)$

a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	x_1	x_2
1	1	1	2	8	0	−2	0	0	0
1	1	3	6	16/5	−24/5	−4/5	−6/5	8/5	0
1	2	2	2	4	0	−2	0	0	0
1	2	3	3	8/5	48/5	2/5	−12/5	−8/5	8/5
1	2	6	6	4/5	24/5	2/5	−12/5	8/5	−4/5
2	2	3	6	8/5	−12/5	−4/5	−6/5	0	−4/5
3	3	3	6	0	8	0	−2	0	0
3	6	6	6	0	4	0	−2	0	0

Theorem 3.3. *Let $\varphi[a_1,a_2,a_3,a_4](z) \in M_2(\Gamma_0(24),\chi_{12})$ be any of the theta products given in* (1.12)*, where $\chi_{12}(n)$ is given by* (1.1)*. Let $E_{1,12}(z)$, $E_{12,1}(z)$, $E_{-3,-4}(z)$ and $E_{-4,-3}(z)$ be as in* (1.6)*. Then we have*

$$\varphi[a_1,a_2,a_3,a_4](z) = b_1E_{12,1}(z) + b_2E_{12,1}(2z) + b_3E_{1,12}(z) + b_4E_{1,12}(2z) + b_5E_{-4,-3}(z) + b_6E_{-4,-3}(2z) + b_7E_{-3,-4}(z) + b_8E_{-3,-4}(2z),$$

where the coefficients b_1, b_2, b_3, b_4, b_5, b_6, b_7 and b_8 are given at the right hand side of Table 3.3.

Table 3.3. $\varphi[a_1,a_2,a_3,a_4](z) = b_1E_{12,1}(z)+b_2E_{12,1}(2z)+b_3E_{1,12}(z)+b_4E_{1,12}(2z)+ b_5E_{-4,-3}(z) + b_6E_{-4,-3}(2z) + b_7E_{-3,-4}(z) + b_8E_{-3,-4}(2z)$.

a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
1	1	1	3	−1	0	6	0	3	0	−2	0
1	1	2	6	0	−1	3	0	0	3	1	0
1	2	2	3	0	−1	3	0	0	−3	−1	0
1	3	3	3	−1	0	2	0	−1	0	2	0
1	3	6	6	0	−1	1	0	0	1	1	0
2	2	2	6	0	−1	0	6	0	3	0	−2
2	3	3	6	0	−1	1	0	0	−1	−1	0
2	6	6	6	0	−1	0	2	0	−1	0	2

Theorem 3.4. *Let $\varphi[a_1,a_2,a_3,a_4](z) \in M_2(\Gamma_0(24),\chi_{24})$ be any of the theta products given in* (1.13)*, where $\chi_{24}(n)$ is given by* (1.1)*. Let $E_{1,24}(z)$, $E_{24,1}(z)$,*

$E_{-8,-3}(z)$ and $E_{-3,-8}(z)$ be as in (1.6). Then we have

$$\varphi[a_1,a_2,a_3,a_4](z) = b_1E_{24,1}(z) + b_2E_{1,24}(z) + b_3E_{-8,-3}(z) + b_4E_{-3,-8}(z) + x_1C_1(q) + x_2C_2(q),$$

where the coefficients b_1, b_2, b_3, b_4, x_1 and x_2 are given at the right hand side of Table 3.4.

Table 3.4. $\varphi[a_1,a_2,a_3,a_4](z) = b_1E_{24,1}(z) + b_2E_{1,24}(z) + b_3E_{-8,-3}(z) + b_4E_{-3,-8}(z) + x_1C_1(q) + x_2C_2(q)$.

a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	x_1	x_2
1	1	1	6	$-1/3$	4	1	$-4/3$	8	$8/3$
1	1	2	3	$-1/3$	4	-1	$4/3$	0	0
1	2	2	6	$-1/3$	2	1	$-2/3$	0	0
1	3	3	6	$-1/3$	$4/3$	$-1/3$	$4/3$	0	0
1	6	6	6	$-1/3$	$2/3$	$-1/3$	$2/3$	$8/3$	$4/3$
2	2	2	3	$-1/3$	2	-1	$2/3$	0	$-4/3$
2	3	3	3	$-1/3$	$4/3$	$1/3$	$-4/3$	$-8/3$	0
2	3	6	6	$-1/3$	$2/3$	$1/3$	$-2/3$	0	0

Remark 3.1. We obtain the following identities from Tables 3.1, 3.2 and 3.4.

$$\begin{aligned}
-\varphi[1,1,2,2](z) + 4\varphi[1,2,3,6](z) - 3\varphi[3,3,6,6](z) &= 4A(q),\\
-2\varphi[1,1,1,2](z) + 5\varphi[1,1,3,6](z) + 9\varphi[3,3,3,6](z) - 12\varphi[3,6,6,6](z) &= 8B_1(q),\\
2\varphi[1,2,2,2](z) - 5\varphi[2,2,3,6](z) - 6\varphi[3,3,3,6](z) + 9\varphi[3,6,6,6](z) &= 4B_2(q),\\
2\varphi[1,1,1,6](z) - 3\varphi[1,1,2,3](z) + 4\varphi[2,2,2,3](z) - 3\varphi[2,3,3,3](z) &= 24C_1(q),\\
2\varphi[1,1,2,3](z) - 2\varphi[1,2,2,6](z) - 3\varphi[2,2,2,3](z) + 3\varphi[2,3,6,6](z) &= 4C_2(q).
\end{aligned}$$

4. Representations by quaternary quadratic forms with coefficients 1, 2, 3 or 6

Let $a_1,a_2,a_3,a_4 \in \{1,2,3,6\}$. With the simplifying assumptions

$$\gcd(a_1,a_2,a_3,a_4) = 1 \qquad \text{and} \qquad a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4,$$

there are 26 diagonal quaternary quadratic forms $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$. Of these, 14 are Ramanujan's universal quaternary quadratic forms given in [13]. In Theorem 4.1 we give explicit formulas for $N(a_1,a_2,a_3,a_4;n)$ for these 14 universal quaternary quadratic forms. In Theorem 4.2 we give formulas for $N(a_1,a_2,a_3,a_4;n)$ for the remaining 12 quaternary quadratic forms. Both Theorems 4.1 and 4.2 follow from Theorems 3.1–3.4.

Theorem 4.1. *Let* $n \in \mathbb{N}$. *Then*

(i) $N(1,1,1,1;n) = 8\sigma(n) - 32\sigma(n/4)$,

(ii) $N(1,1,2,2;n) = 4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8)$,

(iii) $$N(1,1,3,3;n) = 4\sigma(n) - 8\sigma(n/2) - 12\sigma(n/3) + 16\sigma(n/4) + 24\sigma(n/6) - 48\sigma(n/12),$$

(iv) $$N(1,2,3,6;n) = \sigma(n) - \sigma(n/2) + 3\sigma(n/3) + 2\sigma(n/4) - 3\sigma(n/6) - 8\sigma(n/8) + 6\sigma(n/12) - 24\sigma(n/24) + a(n),$$

(v) $N(1,1,1,2;n) = 8\sigma_{(\chi_1,\chi_8)}(n) - 2\sigma_{(\chi_8,\chi_1)}(n)$,

(vi) $$N(1,1,3,6;n) = \frac{16}{5}\sigma_{(\chi_1,\chi_8)}(n) - \frac{24}{5}\sigma_{(\chi_1,\chi_8)}(n/3) - \frac{4}{5}\sigma_{(\chi_8,\chi_1)}(n) - \frac{6}{5}\sigma_{(\chi_8,\chi_1)}(n/3) + \frac{8}{5}b_1(n),$$

(vii) $N(1,2,2,2;n) = 4\sigma_{(\chi_1,\chi_8)}(n) - 2\sigma_{(\chi_8,\chi_1)}(n)$,

(viii) $$N(1,2,3,3;n) = \frac{8}{5}\sigma_{(\chi_1,\chi_8)}(n) + \frac{48}{5}\sigma_{(\chi_1,\chi_8)}(n/3) + \frac{2}{5}\sigma_{(\chi_8,\chi_1)}(n) - \frac{12}{5}\sigma_{(\chi_8,\chi_1)}(n/3) - \frac{8}{5}b_1(n) + \frac{8}{5}b_2(n),$$

(ix) $$N(1,1,1,3;n) = 6\sigma_{(\chi_1,\chi_{12})}(n) - \sigma_{(\chi_{12},\chi_1)}(n) - 2\sigma_{(\chi_{-3},\chi_{-4})}(n) + 3\sigma_{(\chi_{-4},\chi_{-3})}(n),$$

(x) $$N(1,1,2,6;n) = 3\sigma_{(\chi_1,\chi_{12})}(n) - \sigma_{(\chi_{12},\chi_1)}(n/2) + \sigma_{(\chi_{-3},\chi_{-4})}(n) + 3\sigma_{(\chi_{-4},\chi_{-3})}(n/2),$$

(xi) $$N(1,2,2,3;n) = 3\sigma_{(\chi_1,\chi_{12})}(n) - \sigma_{(\chi_{12},\chi_1)}(n/2) - \sigma_{(\chi_{-3},\chi_{-4})}(n) - 3\sigma_{(\chi_{-4},\chi_{-3})}(n/2),$$

(xii) $$N(1,1,1,6;n) = 4\sigma_{(\chi_1,\chi_{24})}(n) - \frac{1}{3}\sigma_{(\chi_{24},\chi_1)}(n) - \frac{4}{3}\sigma_{(\chi_{-3},\chi_{-8})}(n) + \sigma_{(\chi_{-8},\chi_{-3})}(n) + 8c_1(n) + \frac{8}{3}c_2(n),$$

(xiii) $$N(1,1,2,3;n) = 4\sigma_{(\chi_1,\chi_{24})}(n) - \frac{1}{3}\sigma_{(\chi_{24},\chi_1)}(n) + \frac{4}{3}\sigma_{(\chi_{-3},\chi_{-8})}(n) - \sigma_{(\chi_{-8},\chi_{-3})}(n),$$

(xiv) $$N(1,2,2,6;n) = 2\sigma_{(\chi_1,\chi_{24})}(n) - \frac{1}{3}\sigma_{(\chi_{24},\chi_1)}(n) - \frac{2}{3}\sigma_{(\chi_{-3},\chi_{-8})}(n) + \sigma_{(\chi_{-8},\chi_{-3})}(n).$$

Theorem 4.2. *Let $n \in \mathbb{N}$. Then*

(i) $$N(1,1,6,6;n) = 2\sigma(n) - 2\sigma(n/2) - 6\sigma(n/3) - 4\sigma(n/4) + 6\sigma(n/6) + 16\sigma(n/8) + 12\sigma(n/12) - 48\sigma(n/24) + 2a(n),$$

(ii) $$N(2,2,3,3;n) = 4\sigma(n) - 8\sigma(n/2) - 12\sigma(n/3) + 16\sigma(n/4) + 24\sigma(n/6) - 48\sigma(n/12),$$

(iii) $$N(1,2,6,6;n) = \frac{4}{5}\sigma_{(\chi_1,\chi_8)}(n) + \frac{24}{5}\sigma_{(\chi_1,\chi_8)}(n/3) + \frac{2}{5}\sigma_{(\chi_8,\chi_1)}(n) - \frac{12}{5}\sigma_{(\chi_8,\chi_1)}(n/3) + \frac{8}{5}b_1(n) - \frac{4}{5}b_2(n),$$

(iv) $$N(2,2,3,6;n) = \frac{8}{5}\sigma_{(\chi_1,\chi_8)}(n) - \frac{12}{5}\sigma_{(\chi_1,\chi_8)}(n/3) - \frac{4}{5}\sigma_{(\chi_8,\chi_1)}(n) - \frac{6}{5}\sigma_{(\chi_8,\chi_1)}(n/3) - \frac{4}{5}b_2(n),$$

(v) $$N(1,3,3,3;n) = 2\sigma_{(\chi_1,\chi_{12})}(n) - \sigma_{(\chi_{12},\chi_1)}(n) + 2\sigma_{(\chi_{-3},\chi_{-4})}(n) - \sigma_{(\chi_{-4},\chi_{-3})}(n),$$

(vi) $$N(1,3,6,6;n) = \sigma_{(\chi_1,\chi_{12})}(n) - \sigma_{(\chi_{12},\chi_1)}(n/2) + \sigma_{(\chi_{-3},\chi_{-4})}(n) + \sigma_{(\chi_{-4},\chi_{-3})}(n/2),$$

(vii) $$N(2,3,3,6;n) = \sigma_{(\chi_1,\chi_{12})}(n) - \sigma_{(\chi_{12},\chi_1)}(n/2) - \sigma_{(\chi_{-3},\chi_{-4})}(n) - \sigma_{(\chi_{-4},\chi_{-3})}(n/2),$$

(viii) $$N(1,3,3,6;n) = \frac{4}{3}\sigma_{(\chi_1,\chi_{24})}(n) - \frac{1}{3}\sigma_{(\chi_{24},\chi_1)}(n) + \frac{4}{3}\sigma_{(\chi_{-3},\chi_{-8})}(n) - \frac{1}{3}\sigma_{(\chi_{-8},\chi_{-3})}(n),$$

(ix) $$N(1,6,6,6;n) = \frac{2}{3}\sigma_{(\chi_1,\chi_{24})}(n) - \frac{1}{3}\sigma_{(\chi_{24},\chi_1)}(n) + \frac{2}{3}\sigma_{(\chi_{-3},\chi_{-8})}(n) - \frac{1}{3}\sigma_{(\chi_{-8},\chi_{-3})}(n) + \frac{8}{3}c_1(n) + \frac{4}{3}c_2(n),$$

(x) $$N(2,2,2,3;n) = 2\sigma_{(\chi_1,\chi_{24})}(n) - \frac{1}{3}\sigma_{(\chi_{24},\chi_1)}(n) + \frac{2}{3}\sigma_{(\chi_{-3},\chi_{-8})}(n) - \sigma_{(\chi_{-8},\chi_{-3})}(n) - \frac{4}{3}c_2(n),$$

(xi) $$N(2,3,3,3;n) = \frac{4}{3}\sigma_{(\chi_1,\chi_{24})}(n) - \frac{1}{3}\sigma_{(\chi_{24},\chi_1)}(n) - \frac{4}{3}\sigma_{(\chi_{-3},\chi_{-8})}(n) + \frac{1}{3}\sigma_{(\chi_{-8},\chi_{-3})}(n) - \frac{8}{3}c_1(n),$$

(xii) $$N(2,3,6,6;n) = \frac{1}{3}\big(2\sigma_{(\chi_1,\chi_{24})}(n) - \sigma_{(\chi_{24},\chi_1)}(n) - 2\sigma_{(\chi_{-3},\chi_{-8})}(n) + \sigma_{(\chi_{-8},\chi_{-3})}(n)\big).$$

Remark 4.1. The formula in Theorem 4.1(i) is the classical result of Jacobi [6, 16]. The formulas in Theorems 4.1(xiii)(xiv) and 4.2(viii)(xii) agree with the formulas given in [4, p. 1668]. Note that $A(n)$, $B(n)$, $C(n)$ and $D(n)$ in [4] are $\sigma_{(\chi_1,\chi_{24})}(n)$, $\sigma_{(\chi_{-8},\chi_{-3})}(n)$, $\sigma_{(\chi_{-3},\chi_{-8})}(n)$ and $\sigma_{(\chi_{24},\chi_1)}(n)$ in this paper, respectively. The formulas in Theorem 4.1(v)(vii) agree with those given in [15]. The formulas in Theorems 4.1(i)–(iv) and 4.2(i)(ii) agree with those given in [3].

5. Eta quotients in $E_2(\Gamma_0(24),\chi_i)$ for $i \in \{1,8,12,24\}$

We use arguments similar to the ones in [1, Theorem 3.1] to write a search algorithm in MAPLE to find all the eta quotients in $M_2(\Gamma_0(24),\chi_i)$ for $i \in \{1,8,12,24\}$. We found that there are exactly 819, 212, 800 and 212 eta quotients in $M_2(\Gamma_0(24),\chi_i)$ for $i \in \{1,8,12,24\}$, respectively. Of these, 282, 8, 800 and 8 eta quotients are in $E_2(\Gamma_0(24),\chi_i)$ for $i \in \{1,8,12,24\}$, respectively. We note that $M_2(\Gamma_0(24),\chi_{12}) = E_2(\Gamma_0(24),\chi_{12})$ as $\dim(S_2(\Gamma_0(24),\chi_{12})) = 0$.

Of the eta quotients in $E_2(\Gamma_0(24))$, 250 arise directly from those given in [17]. In Table 5.1 we list the remaining 32 eta quotients and their Fourier series expansions.

In Table 5.2 we list all 8 eta quotients in $E_2(\Gamma_0(24),\chi_{24})$. All 8 eta quotients in $E_2(\Gamma_0(24),\chi_8)$ arise directly from those in $E_2(\Gamma_0(8),\chi_8)$ given in [2], so we do not list them here.

All the eta quotients in $E_2(\Gamma_0(12),\chi_{12})$ are given in [1]. There are 395 eta quotients in $E_2(\Gamma_0(24),\chi_{12})$, which do not arise directly from those in $E_2(\Gamma_0(12),\chi_{12})$. We list these eta quotients and their Fourier series expansions in Table 5.3.

Theorems 5.1–5.3 follow from Theorem 2.1 directly.

Theorem 5.1. *Let* $f(z) = \prod_{1\leqslant\delta|24} \eta^{r_\delta}(\delta z) \in E_2(\Gamma_0(24))$ *be any of the eta quotients with the exponents* r_δ *given on the left hand side of* Table 5.1. *Then we have*

$$f(z) = b_2L_2(q) + b_3L_3(q) + b_4L_4(q) + b_6L_6(q) + b_8L_8(q) + b_{12}L_{12}(q) + b_{24}L_{24}(q),$$

where the coefficients b_j $(j \in \{2,3,4,6,8,12,24\})$ *are given at the right hand side of* Table 5.1.

Table 5.1. $f(z) = b_2L_2(q) + b_3L_3(q) + b_4L_4(q) + b_6L_6(q) + b_8L_8(q) + b_{12}L_{12}(q) + b_{24}L_{24}(q)$.

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_2	b_3	b_4	b_6	b_8	b_{12}	b_{24}
−1	4	−1	−5	−2	2	9	−2	−1/2	−1/3	5/4	1/3	−1/2	−5/12	1/6
−1	2	−1	1	0	−2	3	2	0	−1/3	−1/4	1/6	1/2	1/12	−1/6
−1	0	−1	3	2	2	1	−2	1/2	0	1/4	0	−1/2	−3/4	3/2
−1	−2	−1	9	4	−2	−5	2	1	0	−5/4	−3/2	1/2	15/4	−3/2
−2	5	2	−4	−5	1	6	1	0	−1/3	−1/2	5/6	1/4	−1/6	−1/12
−2	4	2	−1	−6	−1	9	−1	−1/2	−2/3	1/2	11/6	−1/2	−5/6	1/6
−2	1	2	4	−1	1	−2	1	1/2	0	−1/2	0	1/4	3/2	−3/4
−2	0	2	7	−2	−1	1	−1	1/2	0	1/2	3/2	−1/2	−3/2	3/2
−1	1	−1	−2	7	2	0	−2	1	1/3	−1	−1/3	0	−1/3	4/3
−1	−1	−1	4	9	−2	−6	2	1	1/3	−1	−5/3	0	11/3	−4/3
−2	2	2	−1	4	1	−3	1	1/2	1/3	0	−5/6	0	5/3	−2/3
−2	1	2	2	3	−1	0	−1	1	2/3	0	−1/3	0	−2/3	4/3
2	−1	−2	−2	1	1	4	1	0	−1/3	1/2	1/6	−1/4	−1/6	1/12
2	−2	−2	1	0	−1	7	−1	−1/2	2/3	1/2	−1/6	−1/2	−1/6	1/6
2	−5	−2	6	5	1	−4	1	5/2	0	−1/2	0	−1/4	−3/2	3/4
2	−6	−2	9	4	−1	−1	−1	−11/2	0	5/2	3/2	−1/2	−3/2	3/2
1	1	1	−4	−5	2	10	−2	−1/2	1/3	5/4	−2/3	−1/2	−1/12	1/6
1	−1	1	2	−3	−2	4	2	0	−1/3	1/4	5/6	−1/2	−5/12	1/6
1	−3	1	4	−1	2	2	−2	−5/2	0	5/4	0	−1/2	−3/4	3/2
1	−5	1	10	1	−2	−4	2	2	0	1/4	3/2	−1/2	−15/4	3/2
2	−4	−2	1	10	1	−5	1	5/2	1/3	−1	−1/6	0	−4/3	2/3
2	−5	−2	4	9	−1	−2	−1	−5	−2/3	2	5/3	0	−4/3	4/3
1	−2	1	−1	4	2	1	−2	−2	−1/3	0	2/3	0	−2/3	4/3
1	−4	1	5	6	−2	−5	2	2	1/3	0	2/3	0	−10/3	4/3
−1	9	−1	−6	−1	2	4	−2	5	3	−11	−3	4	3	0
−1	7	−1	0	1	−2	−2	2	1	3	1	−3	−4	3	0
−2	10	2	−5	−4	1	1	1	1/2	3	4	−15/2	−2	3	0
−2	9	2	−2	−5	−1	4	−1	5	6	−4	−15	4	6	0
2	4	−2	−3	2	1	−1	1	5/2	3	−5	−3/2	2	0	0
2	3	−2	0	1	−1	2	−1	−1	−6	−2	3	4	0	0
1	6	1	−5	−4	2	5	−2	2	−3	−10	6	4	0	0
1	4	1	1	−2	−2	−1	2	2	3	−2	−6	4	0	0

Theorem 5.2. *Let* $f(z) = \prod_{1\leqslant\delta|24} \eta^{r_\delta}(\delta z) \in E_2(\Gamma_0(24), \chi_{24})$ *be any of the eta quotients with the exponents* r_δ *given on the left hand side of* Table 5.2, *where* $\chi_{24}(n)$ *is given by* (1.1). *Then we have*

$$f(z) = b_1 E_{24,1}(z) + b_2 E_{1,24}(z) + b_3 E_{-8,-3}(z) + b_4 E_{-3,-8}(z),$$

where the coefficients b_1, b_2, b_3, b_4 *are given at the right hand side of* Table 5.2.

Table 5.2. $f(z) = b_1 E_{24,1}(z) + b_2 E_{1,24}(z) + b_3 E_{-8,-3}(z) + b_4 E_{-3,-8}(z)$.

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4
0	−2	−2	5	1	−2	8	−4	−1/3	2/3	1/3	−2/3
−1	−1	1	6	−2	−3	5	−1	0	2/3	1/3	0
−2	1	0	8	−2	−4	5	−2	−1/3	2	1	−2/3
−2	5	−4	−2	8	0	1	−2	−1/3	4/3	−1/3	4/3
−3	6	−1	−1	5	−1	−2	1	0	4/3	−1/3	0
−4	8	−2	1	5	−2	−2	0	−1/3	4	−1	4/3
1	−2	−1	5	−1	−1	6	−3	−1/3	0	0	−2/3
−1	5	−3	−2	6	1	−1	−1	−1/3	0	0	4/3

Theorem 5.3. *Let* $f(z) = \prod_{1\leqslant\delta|24} \eta^{r_\delta}(\delta z) \in E_2(\Gamma_0(24), \chi_{12})$ *be any of the eta quotients with the exponents* r_δ *given on the left hand side of* Table 5.3, *where* $\chi_{12}(n)$ *is given by* (1.1). *Then we have*

$$\begin{aligned} f(z) =& b_1 E_{12,1}(z) + b_2 E_{12,1}(2z) + b_3 E_{1,12}(z) + b_4 E_{1,12}(2z) + b_5 E_{-4,-3}(z) \\ &+ b_6 E_{-4,-3}(2z) + b_7 E_{-3,-4}(z) + b_8 E_{-3,-4}(2z), \end{aligned}$$

where the coefficients b_i $(1 \leqslant i \leqslant 8)$ *are given at the right hand side of* Table 5.3.

Table 5.3. $f(z) = b_1E_{12,1}(z) + b_2E_{12,1}(2z) + b_3E_{1,12}(z) + b_4E_{1,12}(2z) + b_5E_{-4,-3}(z) + b_6E_{-4,-3}(2z) + b_7E_{-3,-4}(z) + b_8E_{-3,-4}(2z)$

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
0	0	0	1	0	−2	−5	10	−1/16	1/16	1/48	−1/24	−1/48	−1/48	1/16	1/8
0	1	0	−2	−3	0	4	4	0	0	1/24	−1/12	1/12	1/12	−1/8	−1/4
0	2	0	−5	−6	2	13	−2	0	0	1/12	−1/6	−1/3	−1/3	1/4	1/2
0	3	0	−8	−9	4	22	−8	0	0	1/6	−1/3	4/3	4/3	−1/2	−1
0	0	0	−1	0	2	−3	6	−1/16	1/16	1/16	−1/8	1/16	1/16	−1/16	−1/8
0	1	0	−4	−3	4	6	0	0	0	1/8	−1/4	−1/4	−1/4	1/8	1/4
0	2	0	−7	−6	6	15	−6	0	0	1/4	−1/2	1	1	−1/4	−1/2
0	0	0	−3	0	6	−1	2	−1/16	1/16	3/16	−3/8	−3/16	−3/16	1/16	1/8
0	1	0	−6	−3	8	8	−4	0	0	3/8	−3/4	3/4	3/4	−1/8	−1/4
0	0	0	−5	0	10	1	−2	−1/16	1/16	9/16	−9/8	9/16	9/16	−1/16	−1/8
1	−2	−3	0	6	0	−2	4	0	0	−1/24	1/3	−1/12	−1/12	1/8	0
1	−1	−3	−3	3	2	7	−2	0	0	−1/12	2/3	1/3	1/3	−1/4	0
1	0	−3	−6	0	4	16	−8	0	−1	−1/6	4/3	−4/3	−4/3	1/2	0
1	−2	−3	−2	6	4	0	0	0	0	−1/8	1	1/4	1/4	−1/8	0
1	−1	−3	−5	3	6	9	−6	0	−1	−1/4	2	−1	−1	1/4	0
1	−2	−3	−4	6	8	2	−4	0	−1	−3/8	3	−3/4	−3/4	1/8	0
2	−5	−6	2	15	0	−8	4	−1/4	1/4	5/12	−4/3	1/12	1/12	−1/4	0
2	−4	−6	−1	12	2	1	−2	0	0	5/6	−8/3	−1/3	−1/3	1/2	0
2	−5	−6	0	15	4	−6	0	−1/4	1/4	5/4	−4	−1/4	−1/4	1/4	0
3	−7	−9	1	21	2	−5	−2	0	−1	−7/3	32/3	1/3	1/3	−1	0
1	−4	−3	6	8	−4	−8	8	−1/4	1/4	1/8	−1/3	0	1/3	1/8	0
1	−3	−3	3	5	−2	1	2	0	0	1/4	−2/3	0	−1/3	−1/4	0
1	−2	−3	0	2	0	10	−4	0	0	1/2	−4/3	0	1/3	1/2	0
1	−4	−3	4	8	0	−6	4	−1/4	1/4	3/8	−1	0	0	−1/8	0
1	−3	−3	1	5	2	3	−2	0	0	3/4	−2	0	0	1/4	0
1	−4	−3	2	8	4	−4	0	−1/4	1/4	9/8	−3	0	0	1/8	0
2	−6	−6	5	14	−2	−5	2	0	0	−1/2	8/3	0	1/3	1/2	0
2	−5	−6	2	11	0	4	−4	0	−1	−1	16/3	0	−1/3	−1	0
2	−6	−6	3	14	2	−3	−2	0	−1	−3/2	8	0	0	−1/2	0
3	−9	−9	7	23	−2	−11	2	−1	1	3	−32/3	0	−1/3	−1	0
0	−3	0	8	1	−4	−6	8	−1/4	1/4	1/8	−1/4	0	0	1/8	1/4
0	−2	0	5	−2	−2	3	2	0	0	1/4	−1/2	0	0	−1/4	−1/2
0	−1	0	2	−5	0	12	−4	0	0	1/2	−1	0	0	1/2	1
0	−3	0	6	1	0	−4	4	−1/4	1/4	3/8	−3/4	0	0	−1/8	−1/4
0	−2	0	3	−2	2	5	−2	0	0	3/4	−3/2	0	0	1/4	1/2
0	−3	0	4	1	4	−2	0	−1/4	1/4	9/8	−9/4	0	0	1/8	1/4
1	−5	−3	7	7	−2	−3	2	0	0	−1/4	2	0	0	1/4	0
1	−4	−3	4	4	0	6	−4	0	−1	−1/2	4	0	0	−1/2	0
1	−5	−3	5	7	2	−1	−2	0	−1	−3/4	6	0	0	−1/4	0
2	−8	−6	9	16	−2	−9	2	−1	1	5/2	−8	0	0	−1/2	0
1	−7	−3	13	9	−6	−9	6	−1	1	3/4	−2	0	1	1/4	0
1	−6	−3	10	6	−4	0	0	0	0	3/2	−4	0	−1	−1/2	0

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
1	−7	−3	11	9	−2	−7	2	−1	1	9/4	−6	0	0	−1/4	0
2	−9	−6	12	15	−4	−6	0	0	−1	−3	16	0	1	1	0
0	−6	0	15	2	−6	−7	6	−1	1	3/4	−3/2	0	0	1/4	1/2
0	−5	0	12	−1	−4	2	0	0	0	3/2	−3	0	0	−1/2	−1
0	−6	0	13	2	−2	−5	2	−1	1	9/4	−9/2	0	0	−1/4	−1/2
1	−8	−3	14	8	−4	−4	0	0	−1	−3/2	12	0	0	1/2	0
1	−10	−3	20	10	−8	−10	4	−4	4	9/2	−12	0	3	1/2	0
0	−9	0	22	3	−8	−8	4	−4	4	9/2	−9	0	0	1/2	1
0	−1	0	1	3	−1	−5	7	−1/8	1/8	1/12	−1/6	1/24	1/24	0	0
0	0	0	−2	0	1	4	1	0	0	1/6	−1/3	−1/6	−1/6	0	0
0	1	0	−5	−3	3	13	−5	0	0	1/3	−2/3	2/3	2/3	0	0
0	−1	0	−1	3	3	−3	3	−1/8	1/8	1/4	−1/2	−1/8	−1/8	0	0
0	0	0	−4	0	5	6	−3	0	0	1/2	−1	1/2	1/2	0	0
0	−1	0	−3	3	7	−1	−1	−1/8	1/8	3/4	−3/2	3/8	3/8	0	0
1	−3	−3	0	9	1	−2	1	0	0	−1/6	4/3	1/6	1/6	0	0
1	−2	−3	−3	6	3	7	−5	0	−1	−1/3	8/3	−2/3	−2/3	0	0
1	−3	−3	−2	9	5	0	−3	0	−1	−1/2	4	−1/2	−1/2	0	0
2	−6	−6	2	18	1	−8	1	−1/2	1/2	5/3	−16/3	−1/6	−1/6	0	0
1	−5	−3	6	11	−3	−8	5	−1/2	1/2	1/2	−4/3	0	1/3	0	0
1	−4	−3	3	8	−1	1	−1	0	0	1	−8/3	0	−1/3	0	0
1	−5	−3	4	11	1	−6	1	−1/2	1/2	3/2	−4	0	0	0	0
2	−7	−6	5	17	−1	−5	−1	0	−1	−2	32/3	0	1/3	0	0
0	−4	0	8	4	−3	−6	5	−1/2	1/2	1/2	−1	0	0	0	0
0	−3	0	5	1	−1	3	−1	0	0	1	−2	0	0	0	0
0	−4	0	6	4	1	−4	1	−1/2	1/2	3/2	−3	0	0	0	0
1	−6	−3	7	10	−1	−3	−1	0	−1	−1	8	0	0	0	0
1	−8	−3	13	12	−5	−9	3	−2	2	3	−8	0	1	0	0
0	−7	0	15	5	−5	−7	3	−2	2	3	−6	0	0	0	0
−1	1	3	−1	−3	0	1	4	0	0	1/24	1/6	1/12	1/12	−1/8	−1/2
−1	2	3	−4	−6	2	10	−2	0	0	1/12	1/3	−1/3	−1/3	1/4	1
−1	3	3	−7	−9	4	19	−8	0	−1	1/6	2/3	4/3	4/3	−1/2	−2
−1	1	3	−3	−3	4	3	0	0	0	1/8	1/2	−1/4	−1/4	1/8	1/2
−1	2	3	−6	−6	6	12	−6	0	−1	1/4	1	1	1	−1/4	−1
−1	1	3	−5	−3	8	5	−4	0	−1	3/8	3/2	3/4	3/4	−1/8	−1/2
0	−2	0	1	6	0	−5	4	−1/4	1/4	1/3	−2/3	−1/12	−1/12	0	0
0	−1	0	−2	3	2	4	−2	0	0	2/3	−4/3	1/3	1/3	0	0
0	−2	0	−1	6	4	−3	0	−1/4	1/4	1	−2	1/4	1/4	0	0
1	−4	−3	0	12	2	−2	−2	0	−1	−2/3	16/3	−1/3	−1/3	0	0
−1	−1	3	5	−1	−4	−5	8	−1/4	1/4	1/8	−1/6	0	−1/3	1/8	1/2
−1	0	3	2	−4	−2	4	2	0	0	1/4	−1/3	0	1/3	−1/4	−1
−1	1	3	−1	−7	0	13	−4	0	0	1/2	−2/3	0	−1/3	1/2	2
−1	−1	3	3	−1	0	−3	4	−1/4	1/4	3/8	−1/2	0	0	−1/8	−1/2
−1	0	3	0	−4	2	6	−2	0	0	3/4	−1	0	0	1/4	1
−1	−1	3	1	−1	4	−1	0	−1/4	1/4	9/8	−3/2	0	0	1/8	1/2
1	−6	−3	6	14	−2	−8	2	−1	1	2	−16/3	0	1/3	0	0

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
−1	−2	3	6	−2	−2	0	2	0	0	1/4	1	0	0	−1/4	−1
−1	−1	3	3	−5	0	9	−4	0	−1	1/2	2	0	0	1/2	2
−1	−2	3	4	−2	2	2	−2	0	−1	3/4	3	0	0	1/4	1
0	−5	0	8	7	−2	−6	2	−1	1	2	−4	0	0	0	0
−1	−4	3	12	0	−6	−6	6	−1	1	3/4	−1	0	−1	1/4	1
−1	−3	3	9	−3	−4	3	0	0	0	3/2	−2	0	1	−1/2	−2
−1	−4	3	10	0	−2	−4	2	−1	1	9/4	−3	0	0	−1/4	−1
−1	−5	3	13	−1	−4	−1	0	0	−1	3/2	6	0	0	−1/2	−2
−1	−7	3	19	1	−8	−7	4	−4	4	9/2	−6	0	−3	1/2	2
−1	0	3	−1	0	1	1	1	0	0	1/6	2/3	−1/6	−1/6	0	0
−1	1	3	−4	−3	3	10	−5	0	−1	1/3	4/3	2/3	2/3	0	0
−1	0	3	−3	0	5	3	−3	0	−1	1/2	2	1/2	1/2	0	0
0	−3	0	1	9	1	−5	1	−1/2	1/2	4/3	−8/3	1/6	1/6	0	0
−1	−2	3	5	2	−3	−5	5	−1/2	1/2	1/2	−2/3	0	−1/3	0	0
−1	−1	3	2	−1	−1	4	−1	0	0	1	−4/3	0	1/3	0	0
−1	−2	3	3	2	1	−3	1	−1/2	1/2	3/2	−2	0	0	0	0
−1	−3	3	6	1	−1	0	−1	0	−1	1	4	0	0	0	0
−1	−5	3	12	3	−5	−6	3	−2	2	3	−4	0	−1	0	0
−2	1	6	0	−3	0	−2	4	−1/4	1/4	5/12	−1/3	1/12	1/12	−1/4	−1
−2	2	6	−3	−6	2	7	−2	0	0	5/6	−2/3	−1/3	−1/3	1/2	2
−2	1	6	−2	−3	4	0	0	−1/4	1/4	5/4	−1	−1/4	−1/4	1/4	1
−1	−1	3	−1	3	2	1	−2	0	−1	2/3	8/3	1/3	1/3	0	0
−2	0	6	3	−4	−2	1	2	0	0	1/2	2/3	0	1/3	−1/2	−2
−2	1	6	0	−7	0	10	−4	0	−1	1	4/3	0	−1/3	1	4
−2	0	6	1	−4	2	3	−2	0	−1	3/2	2	0	0	1/2	2
−1	−3	3	5	5	−2	−5	2	−1	1	2	−8/3	0	−1/3	0	0
−2	−2	6	7	−2	−2	−3	2	−1	1	5/2	−2	0	0	−1/2	−2
−2	−3	6	10	−3	−4	0	0	0	−1	3	4	0	1	−1	−4
−2	0	6	0	0	1	−2	1	−1/2	1/2	5/3	−4/3	−1/6	−1/6	0	0
−2	−1	6	3	−1	−1	1	−1	0	−1	2	8/3	0	1/3	0	0
−3	2	9	−2	−6	2	4	−2	0	−1	7/3	4/3	−1/3	−1/3	1	4
−3	0	9	4	−4	−2	−2	2	−1	1	3	−4/3	0	1/3	−1	−4
−1	3	−1	−1	1	−2	−1	6	0	0	1/24	0	1/12	−1/4	−1/8	0
−1	5	−1	−7	−5	2	17	−6	0	0	1/6	0	4/3	1	−1/2	0
−1	3	−1	−3	1	2	1	2	0	0	1/8	0	−1/4	−1/4	1/8	0
−1	4	−1	−6	−2	4	10	−4	0	0	1/4	0	1	1	−1/4	0
−1	3	−1	−5	1	6	3	−2	0	0	3/8	0	3/4	3/4	−1/8	0
0	0	−4	1	10	−2	−7	6	−1/4	1/4	1/12	0	−1/12	1/4	1/4	0
0	2	−4	−5	4	2	11	−6	0	−1	1/3	0	−4/3	−1	1	0
0	0	−4	−1	10	2	−5	2	−1/4	1/4	1/4	0	1/4	1/4	−1/4	0
0	1	−4	−4	7	4	4	−4	0	−1	1/2	0	−1	−1	1/2	0
0	0	−4	−3	10	6	−3	−2	−1/4	−3/4	3/4	0	−3/4	−3/4	1/4	0
−1	1	−1	3	3	−2	−5	6	−1/4	1/4	1/8	0	0	0	1/8	0
−1	3	−1	−3	−3	2	13	−6	0	−1	1/2	0	0	0	1/2	0
−1	1	−1	1	3	2	−3	2	−1/4	1/4	3/8	0	0	0	−1/8	0

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
−1	2	−1	−2	0	4	6	−4	0	−1	3/4	0	0	0	1/4	0
−1	1	−1	−1	3	6	−1	−2	−1/4	−3/4	9/8	0	0	0	1/8	0
−1	0	−1	6	2	−4	−2	4	0	0	1/4	0	0	−1	−1/4	0
−1	1	−1	3	−1	−2	7	−2	0	0	1/2	0	0	1	1/2	0
0	−3	−4	8	11	−4	−8	4	−1	1	1/2	0	0	1	1/2	0
0	−2	−4	5	8	−2	1	−2	0	−1	1	0	0	−1	−1	0
−1	−2	−1	10	4	−4	−6	4	−1	1	3/4	0	0	0	1/4	0
−1	−1	−1	7	1	−2	3	−2	0	−1	3/2	0	0	0	−1/2	0
−1	−3	−1	13	3	−6	−3	2	0	0	3/2	0	0	−3	−1/2	0
0	−6	−4	15	12	−6	−9	2	−4	3	3	0	0	3	1	0
−1	−5	−1	17	5	−6	−7	2	−4	3	9/2	0	0	0	1/2	0
−2	4	2	0	−2	−3	−4	9	−1/8	1/8	1/24	0	−1/24	−3/8	1/8	1/2
−2	5	2	−3	−5	−1	5	3	0	0	1/12	0	1/6	1/2	−1/4	−1
−2	6	2	−6	−8	1	14	−3	0	0	1/6	0	−2/3	−1	1/2	2
−2	7	2	−9	−11	3	23	−9	0	−1	1/3	0	8/3	3	−1	−4
−2	4	2	−2	−2	1	−2	5	−1/8	1/8	1/8	0	1/8	1/8	−1/8	−1/2
−2	5	2	−5	−5	3	7	−1	0	0	1/4	0	−1/2	−1/2	1/4	1
−2	6	2	−8	−8	5	16	−7	0	−1	1/2	0	2	2	−1/2	−2
−2	4	2	−4	−2	5	0	1	−1/8	1/8	3/8	0	−3/8	−3/8	1/8	1/2
−2	5	2	−7	−5	7	9	−5	0	−1	3/4	0	3/2	3/2	−1/4	−1
−2	4	2	−6	−2	9	2	−3	−1/8	−7/8	9/8	0	9/8	9/8	−1/8	−1/2
−1	2	−1	−1	4	−1	−1	3	0	0	1/6	0	−1/6	−1/2	0	0
−1	3	−1	−4	1	1	8	−3	0	0	1/3	0	2/3	1	0	0
−1	2	−1	−3	4	3	1	−1	0	0	1/2	0	1/2	1/2	0	0
0	−1	−4	1	13	−1	−7	3	−1/2	1/2	1/3	0	1/6	1/2	0	0
0	0	−4	−2	10	1	2	−3	0	−1	2/3	0	−2/3	−1	0	0
0	−1	−4	−1	13	3	−5	−1	−1/2	−1/2	1	0	−1/2	−1/2	0	0
−2	3	2	1	−3	−1	1	3	0	0	1/4	0	0	0	−1/4	−1
−2	4	2	−2	−6	1	10	−3	0	0	1/2	0	0	0	1/2	2
−2	3	2	−1	−3	3	3	−1	0	0	3/4	0	0	0	1/4	1
−1	0	−1	3	6	−1	−5	3	−1/2	1/2	1/2	0	0	0	0	0
−1	1	−1	0	3	1	4	−3	0	−1	1	0	0	0	0	0
−1	0	−1	1	6	3	−3	−1	−1/2	−1/2	3/2	0	0	0	0	0
−2	1	2	7	−1	−5	−5	7	−1/2	1/2	1/4	0	0	−1	1/4	1
−2	2	2	4	−4	−3	4	1	0	0	1/2	0	0	1	−1/2	−2
−2	3	2	1	−7	−1	13	−5	0	−1	1	0	0	−1	1	4
−2	1	2	5	−1	−1	−3	3	−1/2	1/2	3/4	0	0	0	−1/4	−1
−2	2	2	2	−4	1	6	−3	0	−1	3/2	0	0	0	1/2	2
−2	1	2	3	−1	3	−1	−1	−1/2	−1/2	9/4	0	0	0	1/4	1
−1	−1	−1	6	5	−3	−2	1	0	0	1	0	0	−1	0	0
0	−4	−4	8	14	−3	−8	1	−2	1	2	0	0	1	0	0
−2	0	2	8	−2	−3	0	1	0	0	3/2	0	0	0	−1/2	−2
−1	−3	−1	10	7	−3	−6	1	−2	1	3	0	0	0	0	0
−2	−2	2	14	0	−7	−6	5	−2	2	3/2	0	0	−3	1/2	2
−2	−1	2	11	−3	−5	3	−1	0	−1	3	0	0	3	−1	−4
−2	−2	2	12	0	−3	−4	1	−2	1	9/2	0	0	0	−1/2	−2

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
−2	−5	2	21	1	−9	−7	3	−8	7	9	0	0	−9	1	4
−2	3	2	0	1	−2	−4	6	−1/4	1/4	1/6	0	1/12	−1/4	0	0
−2	5	2	−6	−5	2	14	−6	0	−1	2/3	0	4/3	1	0	0
−2	3	2	−2	1	2	−2	2	−1/4	1/4	1/2	0	−1/4	−1/4	0	0
−2	4	2	−5	−2	4	7	−4	0	−1	1	0	1	1	0	0
−2	3	2	−4	1	6	0	−2	−1/4	−3/4	3/2	0	3/4	3/4	0	0
−2	0	2	7	2	−4	−5	4	−1	1	1	0	0	−1	0	0
−2	1	2	4	−1	−2	4	−2	0	−1	2	0	0	1	0	0
−2	−3	2	14	3	−6	−6	2	−4	3	6	0	0	−3	0	0
−3	5	5	−2	−5	−1	2	3	0	0	1/3	0	1/6	1/2	−1/2	−2
−3	6	5	−5	−8	1	11	−3	0	0	2/3	0	−2/3	−1	1	4
−3	5	5	−4	−5	3	4	−1	0	0	1	0	−1/2	−1/2	1/2	2
−2	2	2	0	4	−1	−4	3	−1/2	1/2	2/3	0	−1/6	−1/2	0	0
−2	3	2	−3	1	1	5	−3	0	−1	4/3	0	2/3	1	0	0
−2	2	2	−2	4	3	−2	−1	−1/2	−1/2	2	0	1/2	1/2	0	0
−3	3	5	2	−3	−1	−2	3	−1/2	1/2	1	0	0	0	−1/2	−2
−3	4	5	−1	−6	1	7	−3	0	−1	2	0	0	0	1	4
−3	3	5	0	−3	3	0	−1	−1/2	−1/2	3	0	0	0	1/2	2
−3	2	5	5	−4	−3	1	1	0	0	2	0	0	1	−1	−4
−2	−1	2	7	5	−3	−5	1	−2	1	4	0	0	−1	0	0
−3	0	5	9	−2	−3	−3	1	−2	1	6	0	0	0	−1	−4
−4	5	8	−1	−5	−1	−1	3	−1/2	1/2	4/3	0	1/6	1/2	−1	−4
−4	6	8	−4	−8	1	8	−3	0	−1	8/3	0	−2/3	−1	2	8
−4	5	8	−3	−5	3	1	−1	−1/2	−1/2	4	0	−1/2	−1/2	1	4
−4	2	8	6	−4	−3	−2	1	−2	1	8	0	0	1	−2	−8
−3	8	1	−4	−4	0	2	4	0	0	1/8	0	1/4	1/4	−3/8	−1
−3	9	1	−7	−7	2	11	−2	0	0	1/4	0	−1	−1	3/4	2
−3	10	1	−10	−10	4	20	−8	0	−1	1/2	0	4	4	−3/2	−4
−3	8	1	−6	−4	4	4	0	0	0	3/8	0	−3/4	−3/4	3/8	1
−3	9	1	−9	−7	6	13	−6	0	−1	3/4	0	3	3	−3/4	−2
−3	8	1	−8	−4	8	6	−4	0	−1	9/8	0	9/4	9/4	−3/8	−1
−2	5	−2	−2	5	0	−4	4	−1/4	1/4	1/4	0	−1/4	−1/4	1/4	0
−2	6	−2	−5	2	2	5	−2	0	0	1/2	0	1	1	−1/2	0
−2	5	−2	−4	5	4	−2	0	−1/4	1/4	3/4	0	3/4	3/4	−1/4	0
−1	3	−5	−3	11	2	−1	−2	0	−1	1	0	−1	−1	1	0
−3	6	1	2	−2	−4	−4	8	−1/4	1/4	1/8	0	0	−1	1/8	1
−3	7	1	−1	−5	−2	5	2	0	0	1/4	0	0	1	−1/4	−2
−3	8	1	−4	−8	0	14	−4	0	0	1/2	0	0	−1	1/2	4
−3	6	1	0	−2	0	−2	4	−1/4	1/4	3/8	0	0	0	−1/8	−1
−3	7	1	−3	−5	2	7	−2	0	0	3/4	0	0	0	1/4	2
−3	6	1	−2	−2	4	0	0	−1/4	1/4	9/8	0	0	0	1/8	1
−2	4	−2	1	4	−2	−1	2	0	0	1/2	0	0	−1	−1/2	0
−2	5	−2	−2	1	0	8	−4	0	−1	1	0	0	1	1	0
−2	4	−2	−1	4	2	1	−2	0	−1	3/2	0	0	0	1/2	0
−1	1	−5	3	13	−2	−7	2	−1	1	1	0	0	1	1	0
−3	5	1	3	−3	−2	1	2	0	0	3/4	0	0	0	−3/4	−2

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
−3	6	1	0	−6	0	10	−4	0	−1	3/2	0	0	0	3/2	4
−3	5	1	1	−3	2	3	−2	0	−1	9/4	0	0	0	3/4	2
−2	2	−2	5	6	−2	−5	2	−1	1	3/2	0	0	0	1/2	0
−3	3	1	9	−1	−6	−5	6	−1	1	3/4	0	0	−3	1/4	2
−3	4	1	6	−4	−4	4	0	0	0	3/2	0	0	3	−1/2	−4
−3	3	1	7	−1	−2	−3	2	−1	1	9/4	0	0	0	−1/4	−2
−2	1	−2	8	5	−4	−2	0	0	−1	3	0	0	−3	−1	0
−3	2	1	10	−2	−4	0	0	0	−1	9/2	0	0	0	−3/2	−4
−3	0	1	16	0	−8	−6	4	−4	4	9/2	0	0	−9	1/2	4
−3	7	1	−4	−1	1	2	1	0	0	1/2	0	−1/2	−1/2	0	0
−3	8	1	−7	−4	3	11	−5	0	−1	1	0	2	2	0	0
−3	7	1	−6	−1	5	4	−3	0	−1	3/2	0	3/2	3/2	0	0
−2	4	−2	−2	8	1	−4	1	−1/2	1/2	1	0	1/2	1/2	0	0
−3	5	1	2	1	−3	−4	5	−1/2	1/2	1/2	0	0	−1	0	0
−3	6	1	−1	−2	−1	5	−1	0	0	1	0	0	1	0	0
−3	5	1	0	1	1	−2	1	−1/2	1/2	3/2	0	0	0	0	0
−2	3	−2	1	7	−1	−1	−1	0	−1	2	0	0	−1	0	0
−3	4	1	3	0	−1	1	−1	0	−1	3	0	0	0	0	0
−3	2	1	9	2	−5	−5	3	−2	2	3	0	0	−3	0	0
−4	8	4	−3	−4	0	−1	4	−1/4	1/4	1/2	0	1/4	1/4	−1/2	−2
−4	9	4	−6	−7	2	8	−2	0	0	1	0	−1	−1	1	4
−4	8	4	−5	−4	4	1	0	−1/4	1/4	3/2	0	−3/4	−3/4	1/2	2
−3	6	1	−4	2	2	2	−2	0	−1	2	0	1	1	0	0
−4	7	4	0	−5	−2	2	2	0	0	1	0	0	1	−1	−4
−4	8	4	−3	−8	0	11	−4	0	−1	2	0	0	−1	2	8
−4	7	4	−2	−5	2	4	−2	0	−1	3	0	0	0	1	4
−3	4	1	2	4	−2	−4	2	−1	1	2	0	0	−1	0	0
−4	5	4	4	−3	−2	−2	2	−1	1	3	0	0	0	−1	−4
−4	4	4	7	−4	−4	1	0	0	−1	6	0	0	3	−2	−8
−4	7	4	−3	−1	1	−1	1	−1/2	1/2	2	0	−1/2	−1/2	0	0
−4	6	4	0	−2	−1	2	−1	0	−1	4	0	0	1	0	0
−5	9	7	−5	−7	2	5	−2	0	−1	4	0	−1	−1	2	8
−5	7	7	1	−5	−2	−1	2	−1	1	4	0	0	1	−2	−8
−4	10	0	−3	0	−2	−3	6	−1/4	1/4	1/4	0	1/4	−3/4	−1/4	0
−4	12	0	−9	−6	2	15	−6	0	−1	1	0	4	3	−1	0
−4	10	0	−5	0	2	−1	2	−1/4	1/4	3/4	0	−3/4	−3/4	1/4	0
−4	11	0	−8	−3	4	8	−4	0	−1	3/2	0	3	3	−1/2	0
−4	10	0	−7	0	6	1	−2	−1/4	−3/4	9/4	0	9/4	9/4	−1/4	0
−4	7	0	4	1	−4	−4	4	−1	1	3/2	0	0	−3	−1/2	0
−4	8	0	1	−2	−2	5	−2	0	−1	3	0	0	3	1	0
−4	4	0	11	2	−6	−5	2	−4	3	9	0	0	−9	−1	0
−5	12	3	−5	−6	−1	3	3	0	0	1/2	0	1/2	3/2	−1	−4
−5	13	3	−8	−9	1	12	−3	0	0	1	0	−2	−3	2	8
−5	12	3	−7	−6	3	5	−1	0	0	3/2	0	−3/2	−3/2	1	4
−4	9	0	−3	3	−1	−3	3	−1/2	1/2	1	0	−1/2	−3/2	0	0
−4	10	0	−6	0	1	6	−3	0	−1	2	0	2	3	0	0
−4	9	0	−5	3	3	−1	−1	−1/2	−1/2	3	0	3/2	3/2	0	0

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
−5	10	3	−1	−4	−1	−1	3	−1/2	1/2	3/2	0	0	0	−1	−4
−5	11	3	−4	−7	1	8	−3	0	−1	3	0	0	0	2	8
−5	10	3	−3	−4	3	1	−1	−1/2	−1/2	9/2	0	0	0	1	4
−5	9	3	2	−5	−3	2	1	0	0	3	0	0	3	−2	−8
−4	6	0	4	4	−3	−4	1	−2	1	6	0	0	−3	0	0
−5	7	3	6	−3	−3	−2	1	−2	1	9	0	0	0	−2	−8
−6	12	6	−4	−6	−1	0	3	−1/2	1/2	2	0	1/2	3/2	−2	−8
−6	13	6	−7	−9	1	9	−3	0	−1	4	0	−2	−3	4	16
−6	12	6	−6	−6	3	2	−1	−1/2	−1/2	6	0	−3/2	−3/2	2	8
−6	9	6	3	−5	−3	−1	1	−2	1	12	0	0	3	−4	−16
−6	15	2	−6	−5	0	0	4	−1/4	1/4	3/4	0	3/4	3/4	−5/4	−4
−6	16	2	−9	−8	2	9	−2	0	0	3/2	0	−3	−3	5/2	8
−6	15	2	−8	−5	4	2	0	−1/4	1/4	9/4	0	−9/4	−9/4	5/4	4
−5	13	−1	−7	1	2	3	−2	0	−1	3	0	3	3	−1	0
−6	14	2	−3	−6	−2	3	2	0	0	3/2	0	0	3	−3/2	−8
−6	15	2	−6	−9	0	12	−4	0	−1	3	0	0	−3	3	16
−6	14	2	−5	−6	2	5	−2	0	−1	9/2	0	0	0	3/2	8
−5	11	−1	−1	3	−2	−3	2	−1	1	3	0	0	−3	−1	0
−6	12	2	1	−4	−2	−1	2	−1	1	9/2	0	0	0	−5/2	−8
−6	11	2	4	−5	−4	2	0	0	−1	9	0	0	9	−3	−16
−6	14	2	−6	−2	1	0	1	−1/2	1/2	3	0	−3/2	−3/2	0	0
−6	13	2	−3	−3	−1	3	−1	0	−1	6	0	0	3	0	0
−7	16	5	−8	−8	2	6	−2	0	−1	6	0	−3	−3	4	16
−7	14	5	−2	−6	−2	0	2	−1	1	6	0	0	3	−4	−16
−8	19	4	−7	−7	−1	1	3	−1/2	1/2	3	0	3/2	9/2	−4	−16
−8	20	4	−10	−10	1	10	−3	0	−1	6	0	−6	−9	8	32
−8	19	4	−9	−7	3	3	−1	−1/2	−1/2	9	0	−9/2	−9/2	4	16
−8	16	4	0	−6	−3	0	1	−2	1	18	0	0	9	−8	−32
−9	23	3	−11	−9	2	7	−2	0	−1	9	0	−9	−9	9	32
−9	21	3	−5	−7	−2	1	2	−1	1	9	0	0	9	−7	−32
2	−2	−2	2	4	−3	−6	9	1/8	−1/8	−1/24	1/6	1/24	−7/24	−1/8	0
2	−1	−2	−1	1	−1	3	3	0	0	−1/12	1/3	−1/6	1/6	1/4	0
2	0	−2	−4	−2	1	12	−3	0	0	−1/6	2/3	2/3	1/3	−1/2	0
2	1	−2	−7	−5	3	21	−9	0	−1	−1/3	4/3	−8/3	−7/3	1	0
2	−2	−2	0	4	1	−4	5	1/8	−1/8	−1/8	1/2	−1/8	−1/8	1/8	0
2	−1	−2	−3	1	3	5	−1	0	0	−1/4	1	1/2	1/2	−1/4	0
2	0	−2	−6	−2	5	14	−7	0	−1	−1/2	2	−2	−2	1/2	0
2	−2	−2	−2	4	5	−2	1	1/8	−1/8	−3/8	3/2	3/8	3/8	−1/8	0
2	−1	−2	−5	1	7	7	−5	0	−1	−3/4	3	−3/2	−3/2	1/4	0
2	−2	−2	−4	4	9	0	−3	1/8	−9/8	−9/8	9/2	−9/8	−9/8	1/8	0
3	−4	−5	1	10	−1	−3	3	0	0	1/3	−4/3	1/6	−1/6	−1/2	0
3	−3	−5	−2	7	1	6	−3	0	0	2/3	−8/3	−2/3	−1/3	1	0
3	−4	−5	−1	10	3	−1	−1	0	0	1	−4	−1/2	−1/2	1/2	0
4	−7	−8	3	19	−1	−9	3	1/2	−1/2	−4/3	16/3	−1/6	1/6	1	0
4	−6	−8	0	16	1	0	−3	0	−1	−8/3	32/3	2/3	1/3	−2	0
4	−7	−8	1	19	3	−7	−1	1/2	−3/2	−4	16	1/2	1/2	−1	0

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
2	−3	−2	3	3	−1	−1	3	0	0	1/4	−1	0	0	−1/4	0
2	−2	−2	0	0	1	8	−3	0	0	1/2	−2	0	0	1/2	0
2	−3	−2	1	3	3	1	−1	0	0	3/4	−3	0	0	1/4	0
3	−6	−5	5	12	−1	−7	3	1/2	−1/2	−1	4	0	0	1/2	0
3	−5	−5	2	9	1	2	−3	0	−1	−2	8	0	0	−1	0
3	−6	−5	3	12	3	−5	−1	1/2	−3/2	−3	12	0	0	−1/2	0
2	−5	−2	9	5	−5	−7	7	1/2	−1/2	−1/4	1	0	−1	−1/4	0
2	−4	−2	6	2	−3	2	1	0	0	−1/2	2	0	1	1/2	0
2	−3	−2	3	−1	−1	11	−5	0	−1	−1	4	0	−1	−1	0
2	−5	−2	7	5	−1	−5	3	1/2	−1/2	−3/4	3	0	0	1/4	0
2	−4	−2	4	2	1	4	−3	0	−1	−3/2	6	0	0	−1/2	0
2	−5	−2	5	5	3	−3	−1	1/2	−3/2	−9/4	9	0	0	−1/4	0
3	−7	−5	8	11	−3	−4	1	0	0	2	−8	0	−1	−1	0
4	−10	−8	10	20	−3	−10	1	2	−3	−8	32	0	1	2	0
2	−6	−2	10	4	−3	−2	1	0	0	3/2	−6	0	0	−1/2	0
3	−9	−5	12	13	−3	−8	1	2	−3	−6	24	0	0	1	0
2	−8	−2	16	6	−7	−8	5	2	−2	−3/2	6	0	−3	−1/2	0
2	−7	−2	13	3	−5	1	−1	0	−1	−3	12	0	3	1	0
2	−8	−2	14	6	−3	−6	1	2	−3	−9/2	18	0	0	1/2	0
2	−11	−2	23	7	−9	−9	3	8	−9	−9	36	0	−9	−1	0
1	0	1	0	−2	−2	0	6	0	0	1/24	−1/6	1/12	5/12	−1/8	−1/2
1	1	1	−3	−5	0	9	0	0	0	1/12	−1/3	−1/3	−2/3	1/4	1
1	2	1	−6	−8	2	18	−6	0	0	1/6	−2/3	4/3	5/3	−1/2	−2
1	0	1	−2	−2	2	2	2	0	0	1/8	−1/2	−1/4	−1/4	1/8	1/2
1	1	1	−5	−5	4	11	−4	0	0	1/4	−1	1	1	−1/4	−1
1	0	1	−4	−2	6	4	−2	0	0	3/8	−3/2	3/4	3/4	−1/8	−1/2
2	−3	−2	2	7	−2	−6	6	1/4	−1/4	−1/6	2/3	−1/12	−5/12	0	0
2	−2	−2	−1	4	0	3	0	0	0	−1/3	4/3	1/3	2/3	0	0
2	−1	−2	−4	1	2	12	−6	0	−1	−2/3	8/3	−4/3	−5/3	0	0
2	−3	−2	0	7	2	−4	2	1/4	−1/4	−1/2	2	1/4	1/4	0	0
2	−2	−2	−3	4	4	5	−4	0	−1	−1	4	−1	−1	0	0
2	−3	−2	−2	7	6	−2	−2	1/4	−5/4	−3/2	6	−3/4	−3/4	0	0
3	−5	−5	1	13	0	−3	0	0	0	4/3	−16/3	−1/3	−2/3	0	0
4	−8	−8	3	22	0	−9	0	1	−2	−16/3	64/3	1/3	2/3	0	0
1	−2	1	4	0	−2	−4	6	1/4	−1/4	−1/8	1/2	0	0	−1/8	−1/2
1	−1	1	1	−3	0	5	0	0	0	−1/4	1	0	0	1/4	1
1	0	1	−2	−6	2	14	−6	0	−1	−1/2	2	0	0	−1/2	−2
1	−2	1	2	0	2	−2	2	1/4	−1/4	−3/8	3/2	0	0	1/8	1/2
1	−1	1	−1	−3	4	7	−4	0	−1	−3/4	3	0	0	−1/4	−1
1	−2	1	0	0	6	0	−2	1/4	−5/4	−9/8	9/2	0	0	−1/8	−1/2
2	−4	−2	3	6	0	−1	0	0	0	1	−4	0	0	0	0
3	−7	−5	5	15	0	−7	0	1	−2	−4	16	0	0	0	0
1	−3	1	7	−1	−4	−1	4	0	0	1/4	−1	0	1	−1/4	−1
1	−2	1	4	−4	−2	8	−2	0	0	1/2	−2	0	−1	1/2	2
1	−3	1	5	−1	0	1	0	0	0	3/4	−3	0	0	1/4	1
2	−6	−2	9	8	−4	−7	4	1	−1	−1	4	0	−1	0	0

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
2	−5	−2	6	5	−2	2	−2	0	−1	−2	8	0	1	0	0
2	−6	−2	7	8	0	−5	0	1	−2	−3	12	0	0	0	0
1	−5	1	11	1	−4	−5	4	1	−1	−3/4	3	0	0	−1/4	−1
1	−4	1	8	−2	−2	4	−2	0	−1	−3/2	6	0	0	1/2	2
1	−5	1	9	1	0	−3	0	1	−2	−9/4	9	0	0	1/4	1
1	−6	1	14	0	−6	−2	2	0	0	3/2	−6	0	3	−1/2	−2
2	−9	−2	16	9	−6	−8	2	4	−5	−6	24	0	−3	0	0
1	−8	1	18	2	−6	−6	2	4	−5	−9/2	18	0	0	−1/2	−2
1	−1	1	0	1	−1	0	3	0	0	1/6	−2/3	−1/6	1/6	0	0
1	0	1	−3	−2	1	9	−3	0	0	1/3	−4/3	2/3	1/3	0	0
1	−1	1	−2	1	3	2	−1	0	0	1/2	−2	1/2	1/2	0	0
2	−4	−2	2	10	−1	−6	3	1/2	−1/2	−2/3	8/3	1/6	−1/6	0	0
2	−3	−2	−1	7	1	3	−3	0	−1	−4/3	16/3	−2/3	−1/3	0	0
2	−4	−2	0	10	3	−4	−1	1/2	−3/2	−2	8	−1/2	−1/2	0	0
1	−3	1	4	3	−1	−4	3	1/2	−1/2	−1/2	2	0	0	0	0
1	−2	1	1	0	1	5	−3	0	−1	−1	4	0	0	0	0
1	−3	1	2	3	3	−2	−1	1/2	−3/2	−3/2	6	0	0	0	0
1	−4	1	7	2	−3	−1	1	0	0	1	−4	0	1	0	0
2	−7	−2	9	11	−3	−7	1	2	−3	−4	16	0	−1	0	0
1	−6	1	11	4	−3	−5	1	2	−3	−3	12	0	0	0	0
0	0	4	1	−2	−2	−3	6	1/4	−1/4	−1/12	1/3	1/12	5/12	−1/4	−1
0	1	4	−2	−5	0	6	0	0	0	−1/6	2/3	−1/3	−2/3	1/2	2
0	2	4	−5	−8	2	15	−6	0	−1	−1/3	4/3	4/3	5/3	−1	−4
0	0	4	−1	−2	2	−1	2	1/4	−1/4	−1/4	1	−1/4	−1/4	1/4	1
0	1	4	−4	−5	4	8	−4	0	−1	−1/2	2	1	1	−1/2	−2
0	0	4	−3	−2	6	1	−2	1/4	−5/4	−3/4	3	3/4	3/4	−1/4	−1
1	−2	1	0	4	0	0	0	0	0	2/3	−8/3	1/3	2/3	0	0
2	−5	−2	2	13	0	−6	0	1	−2	−8/3	32/3	−1/3	−2/3	0	0
0	−1	4	2	−3	0	2	0	0	0	1/2	−2	0	0	1/2	2
1	−4	1	4	6	0	−4	0	1	−2	−2	8	0	0	0	0
0	−3	4	8	−1	−4	−4	4	1	−1	−1/2	2	0	1	−1/2	−2
0	−2	4	5	−4	−2	5	−2	0	−1	−1	4	0	−1	1	4
0	−3	4	6	−1	0	−2	0	1	−2	−3/2	6	0	0	1/2	2
0	−6	4	15	0	−6	−5	2	4	−5	−3	12	0	3	−1	−4
0	−1	4	1	1	−1	−3	3	1/2	−1/2	−1/3	4/3	−1/6	1/6	0	0
0	0	4	−2	−2	1	6	−3	0	−1	−2/3	8/3	2/3	1/3	0	0
0	−1	4	−1	1	3	−1	−1	1/2	−3/2	−1	4	1/2	1/2	0	0
0	−4	4	8	2	−3	−4	1	2	−3	−2	8	0	1	0	0
−1	1	7	−1	−5	0	3	0	0	0	1/3	−4/3	−1/3	−2/3	1	4
0	−2	4	1	4	0	−3	0	1	−2	−4/3	16/3	1/3	2/3	0	0
−1	−1	7	3	−3	0	−1	0	1	−2	−1	4	0	0	1	4
−2	1	10	0	−5	0	0	0	1	−2	−2/3	8/3	−1/3	−2/3	2	8
0	3	0	−1	−1	−1	−3	7	1/8	−1/8	0	0	1/8	1/8	−1/4	−1/2
0	4	0	−4	−4	1	6	1	0	0	0	0	−1/2	−1/2	1/2	1
0	5	0	−7	−7	3	15	−5	0	0	0	0	2	2	−1	−2
0	3	0	−3	−1	3	−1	3	1/8	−1/8	0	0	−3/8	−3/8	1/4	1/2

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
0	4	0	−6	−4	5	8	−3	0	0	0	0	3/2	3/2	−1/2	−1
0	3	0	−5	−1	7	1	−1	1/8	−1/8	0	0	9/8	9/8	−1/4	−1/2
1	1	−3	−2	5	1	0	1	0	0	0	0	1/2	1/2	−1/2	0
1	2	−3	−5	2	3	9	−5	0	−1	0	0	−2	−2	1	0
1	1	−3	−4	5	5	2	−3	0	−1	0	0	−3/2	−3/2	1/2	0
2	−2	−6	0	14	1	−6	1	1/2	−1/2	0	0	−1/2	−1/2	1	0
1	−1	−3	4	7	−3	−6	5	1/2	−1/2	0	0	0	−1	−1/2	0
1	0	−3	1	4	−1	3	−1	0	0	0	0	0	1	1	0
1	−1	−3	2	7	1	−4	1	1/2	−1/2	0	0	0	0	1/2	0
2	−3	−6	3	13	−1	−3	−1	0	−1	0	0	0	−1	−2	0
0	0	0	6	0	−3	−4	5	1/2	−1/2	0	0	0	0	−1/2	−1
0	1	0	3	−3	−1	5	−1	0	0	0	0	0	0	1	2
0	0	0	4	0	1	−2	1	1/2	−1/2	0	0	0	0	1/2	1
1	−2	−3	5	6	−1	−1	−1	0	−1	0	0	0	0	−1	0
1	−4	−3	11	8	−5	−7	3	2	−2	0	0	0	−3	−1	0
0	−3	0	13	1	−5	−5	3	2	−2	0	0	0	0	−1	−2
0	2	0	−1	2	0	−3	4	1/4	−1/4	0	0	−1/4	−1/4	0	0
0	3	0	−4	−1	2	6	−2	0	0	0	0	1	1	0	0
0	2	0	−3	2	4	−1	0	1/4	−1/4	0	0	3/4	3/4	0	0
1	0	−3	−2	8	2	0	−2	0	−1	0	0	−1	−1	0	0
1	−2	−3	4	10	−2	−6	2	1	−1	0	0	0	−1	0	0
0	−1	0	6	3	−2	−4	2	1	−1	0	0	0	0	0	0
−1	4	3	−3	−4	1	3	1	0	0	0	0	−1/2	−1/2	1/2	2
−1	5	3	−6	−7	3	12	−5	0	−1	0	0	2	2	−1	−4
−1	4	3	−5	−4	5	5	−3	0	−1	0	0	3/2	3/2	−1/2	−2
0	1	0	−1	5	1	−3	1	1/2	−1/2	0	0	1/2	1/2	0	0
−1	2	3	3	−2	−3	−3	5	1/2	−1/2	0	0	0	1	−1/2	−2
−1	3	3	0	−5	−1	6	−1	0	0	0	0	0	−1	1	4
−1	2	3	1	−2	1	−1	1	1/2	−1/2	0	0	0	0	1/2	2
−1	1	3	4	−3	−1	2	−1	0	−1	0	0	0	0	1	4
−1	−1	3	10	−1	−5	−4	3	2	−2	0	0	0	3	−1	−4
−1	3	3	−3	−1	2	3	−2	0	−1	0	0	1	1	0	0
−1	1	3	3	1	−2	−3	2	1	−1	0	0	0	1	0	0
−2	4	6	−2	−4	1	0	1	1/2	−1/2	0	0	−1/2	−1/2	1	4
−2	3	6	1	−5	−1	3	−1	0	−1	0	0	0	−1	2	8
−1	6	−1	−3	0	−1	1	3	0	0	0	0	−1/2	1/2	1/2	0
−1	7	−1	−6	−3	1	10	−3	0	0	0	0	2	1	−1	0
−1	6	−1	−5	0	3	3	−1	0	0	0	0	3/2	3/2	−1/2	0
0	3	−4	−1	9	−1	−5	3	1/2	−1/2	0	0	1/2	−1/2	−1	0
0	4	−4	−4	6	1	4	−3	0	−1	0	0	−2	−1	2	0
0	3	−4	−3	9	3	−3	−1	1/2	−3/2	0	0	−3/2	−3/2	1	0
−1	4	−1	1	2	−1	−3	3	1/2	−1/2	0	0	0	0	−1/2	0
−1	5	−1	−2	−1	1	6	−3	0	−1	0	0	0	0	1	0
−1	4	−1	−1	2	3	−1	−1	1/2	−3/2	0	0	0	0	1/2	0
−1	3	−1	4	1	−3	0	1	0	0	0	0	0	3	1	0
0	0	−4	6	10	−3	−6	1	2	−3	0	0	0	−3	−2	0
−1	1	−1	8	3	−3	−4	1	2	−3	0	0	0	0	−1	0

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
−2	7	2	−2	−3	−2	−2	6	1/4	−1/4	0	0	1/4	5/4	−1/2	−2
−2	8	2	−5	−6	0	7	0	0	0	0	0	−1	−2	1	4
−2	9	2	−8	−9	2	16	−6	0	−1	0	0	4	5	−2	−8
−2	7	2	−4	−3	2	0	2	1/4	−1/4	0	0	−3/4	−3/4	1/2	2
−2	8	2	−7	−6	4	9	−4	0	−1	0	0	3	3	−1	−4
−2	7	2	−6	−3	6	2	−2	1/4	−5/4	0	0	9/4	9/4	−1/2	−2
−1	5	−1	−3	3	0	1	0	0	0	0	0	1	2	0	0
0	2	−4	−1	12	0	−5	0	1	−2	0	0	−1	−2	0	0
−2	6	2	−1	−4	0	3	0	0	0	0	0	0	0	1	4
−1	3	−1	1	5	0	−3	0	1	−2	0	0	0	0	0	0
−2	4	2	5	−2	−4	−3	4	1	−1	0	0	0	3	−1	−4
−2	5	2	2	−5	−2	6	−2	0	−1	0	0	0	−3	2	8
−2	4	2	3	−2	0	−1	0	1	−2	0	0	0	0	1	4
−2	1	2	12	−1	−6	−4	2	4	−5	0	0	0	9	−2	−8
−2	6	2	−2	0	−1	−2	3	1/2	−1/2	0	0	−1/2	1/2	0	0
−2	7	2	−5	−3	1	7	−3	0	−1	0	0	2	1	0	0
−2	6	2	−4	0	3	0	−1	1/2	−3/2	0	0	3/2	3/2	0	0
−2	3	2	5	1	−3	−3	1	2	−3	0	0	0	3	0	0
−3	8	5	−4	−6	0	4	0	0	0	0	0	−1	−2	2	8
−2	5	2	−2	3	0	−2	0	1	−2	0	0	1	2	0	0
−3	6	5	0	−4	0	0	0	1	−2	0	0	0	0	2	8
−4	8	8	−3	−6	0	1	0	1	−2	0	0	−1	−2	4	16
−3	11	1	−6	−5	1	4	1	0	0	0	0	−3/2	−3/2	3/2	4
−3	12	1	−9	−8	3	13	−5	0	−1	0	0	6	6	−3	−8
−3	11	1	−8	−5	5	6	−3	0	−1	0	0	9/2	9/2	−3/2	−4
−2	8	−2	−4	4	1	−2	1	1/2	−1/2	0	0	3/2	3/2	−1	0
−3	9	1	0	−3	−3	−2	5	1/2	−1/2	0	0	0	3	−1/2	−4
−3	10	1	−3	−6	−1	7	−1	0	0	0	0	0	−3	1	8
−3	9	1	−2	−3	1	0	1	1/2	−1/2	0	0	0	0	1/2	4
−2	7	−2	−1	3	−1	1	−1	0	−1	0	0	0	3	2	0
−3	8	1	1	−4	−1	3	−1	0	−1	0	0	0	0	3	8
−3	6	1	7	−2	−5	−3	3	2	−2	0	0	0	9	−1	−8
−3	10	1	−6	−2	2	4	−2	0	−1	0	0	3	3	0	0
−3	8	1	0	0	−2	−2	2	1	−1	0	0	0	3	0	0
−4	11	4	−5	−5	1	1	1	1/2	−1/2	0	0	−3/2	−3/2	2	8
−4	10	4	−2	−6	−1	4	−1	0	−1	0	0	0	−3	4	16
−4	13	0	−5	−1	−1	−1	3	1/2	−1/2	0	0	−3/2	3/2	1	0
−4	14	0	−8	−4	1	8	−3	0	−1	0	0	6	3	−2	0
−4	13	0	−7	−1	3	1	−1	1/2	−3/2	0	0	9/2	9/2	−1	0
−4	10	0	2	0	−3	−2	1	2	−3	0	0	0	9	2	0
−5	15	3	−7	−7	0	5	0	0	0	0	0	−3	−6	4	16
−4	12	0	−5	2	0	−1	0	1	−2	0	0	3	6	0	0
−5	13	3	−3	−5	0	1	0	1	−2	0	0	0	0	4	16
−6	15	6	−6	−7	0	2	0	1	−2	0	0	−3	−6	8	32
−6	18	2	−8	−6	1	2	1	1/2	−1/2	0	0	−9/2	−9/2	5	16
−6	17	2	−5	−7	−1	5	−1	0	−1	0	0	0	−9	6	32
−8	22	4	−9	−8	0	3	0	1	−2	0	0	−9	−18	16	64

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
3	−1	−1	−1	−1	0	1	4	0	0	−1/8	1/2	−1/4	−1/4	3/8	1/2
3	0	−1	−4	−4	2	10	−2	0	0	−1/4	1	1	1	−3/4	−1
3	1	−1	−7	−7	4	19	−8	0	−1	−1/2	2	−4	−4	3/2	2
3	−1	−1	−3	−1	4	3	0	0	0	−3/8	3/2	3/4	3/4	−3/8	−1/2
3	0	−1	−6	−4	6	12	−6	0	−1	−3/4	3	−3	−3	3/4	1
3	−1	−1	−5	−1	8	5	−4	0	−1	−9/8	9/2	−9/4	−9/4	3/8	1/2
4	−4	−4	1	8	0	−5	4	−1/4	1/4	1/2	−2	1/4	1/4	−1/2	0
4	−3	−4	−2	5	2	4	−2	0	0	1	−4	−1	−1	1	0
4	−4	−4	−1	8	4	−3	0	−1/4	1/4	3/2	−6	−3/4	−3/4	1/2	0
5	−6	−7	0	14	2	−2	−2	0	−1	−4	16	1	1	−2	0
3	−3	−1	5	1	−4	−5	8	−1/4	1/4	1/8	−1/2	0	1	1/8	−1/2
3	−2	−1	2	−2	−2	4	2	0	0	1/4	−1	0	−1	−1/4	1
3	−1	−1	−1	−5	0	13	−4	0	0	1/2	−2	0	1	1/2	−2
3	−3	−1	3	1	0	−3	4	−1/4	1/4	3/8	−3/2	0	0	−1/8	1/2
3	−2	−1	0	−2	2	6	−2	0	0	3/4	−3	0	0	1/4	−1
3	−3	−1	1	1	4	−1	0	−1/4	1/4	9/8	−9/2	0	0	1/8	−1/2
4	−5	−4	4	7	−2	−2	2	0	0	−1	4	0	1	1	0
4	−4	−4	1	4	0	7	−4	0	−1	−2	8	0	−1	−2	0
4	−5	−4	2	7	2	0	−2	0	−1	−3	12	0	0	−1	0
5	−8	−7	6	16	−2	−8	2	−1	1	4	−16	0	−1	−2	0
3	−4	−1	6	0	−2	0	2	0	0	−3/4	3	0	0	3/4	1
3	−3	−1	3	−3	0	9	−4	0	−1	−3/2	6	0	0	−3/2	−2
3	−4	−1	4	0	2	2	−2	0	−1	−9/4	9	0	0	−3/4	−1
4	−7	−4	8	9	−2	−6	2	−1	1	3	−12	0	0	−1	0
3	−6	−1	12	2	−6	−6	6	−1	1	3/4	−3	0	3	1/4	−1
3	−5	−1	9	−1	−4	3	0	0	0	3/2	−6	0	−3	−1/2	2
3	−6	−1	10	2	−2	−4	2	−1	1	9/4	−9	0	0	−1/4	1
4	−8	−4	11	8	−4	−3	0	0	−1	−6	24	0	3	2	0
3	−7	−1	13	1	−4	−1	0	0	−1	−9/2	18	0	0	3/2	2
3	−9	−1	19	3	−8	−7	4	−4	4	9/2	−18	0	9	1/2	−2
3	−2	−1	−1	2	1	1	1	0	0	−1/2	2	1/2	1/2	0	0
3	−1	−1	−4	−1	3	10	−5	0	−1	−1	4	−2	−2	0	0
3	−2	−1	−3	2	5	3	−3	0	−1	−3/2	6	−3/2	−3/2	0	0
4	−5	−4	1	11	1	−5	1	−1/2	1/2	2	−8	−1/2	−1/2	0	0
3	−4	−1	5	4	−3	−5	5	−1/2	1/2	1/2	−2	0	1	0	0
3	−3	−1	2	1	−1	4	−1	0	0	1	−4	0	−1	0	0
3	−4	−1	3	4	1	−3	1	−1/2	1/2	3/2	−6	0	0	0	0
4	−6	−4	4	10	−1	−2	−1	0	−1	−4	16	0	1	0	0
3	−5	−1	6	3	−1	0	−1	0	−1	−3	12	0	0	0	0
3	−7	−1	12	5	−5	−6	3	−2	2	3	−12	0	3	0	0
2	−1	2	0	−1	0	−2	4	−1/4	1/4	1/4	−1	−1/4	−1/4	1/4	1
2	0	2	−3	−4	2	7	−2	0	0	1/2	−2	1	1	−1/2	−2
2	−1	2	−2	−1	4	0	0	−1/4	1/4	3/4	−3	3/4	3/4	−1/4	−1
3	−3	−1	−1	5	2	1	−2	0	−1	−2	8	−1	−1	0	0
2	−2	2	3	−2	−2	1	2	0	0	−1/2	2	0	−1	1/2	2
2	−1	2	0	−5	0	10	−4	0	−1	−1	4	0	1	−1	−4

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
2	−2	2	1	−2	2	3	−2	0	−1	−3/2	6	0	0	−1/2	−2
3	−5	−1	5	7	−2	−5	2	−1	1	2	−8	0	1	0	0
2	−4	2	7	0	−2	−3	2	−1	1	3/2	−6	0	0	1/2	2
2	−5	2	10	−1	−4	0	0	0	−1	−3	12	0	−3	1	4
2	−2	2	0	2	1	−2	1	−1/2	1/2	1	−4	1/2	1/2	0	0
2	−3	2	3	1	−1	1	−1	0	−1	−2	8	0	−1	0	0
1	0	5	−2	−4	2	4	−2	0	−1	−1	4	1	1	−1	−4
1	−2	5	4	−2	−2	−2	2	−1	1	1	−4	0	−1	1	4
2	1	−2	0	3	−2	−4	6	−1/4	1/4	0	0	−1/4	3/4	1/2	0
2	3	−2	−6	−3	2	14	−6	0	−1	0	0	−4	−3	2	0
2	1	−2	−2	3	2	−2	2	−1/4	1/4	0	0	3/4	3/4	−1/2	0
2	2	−2	−5	0	4	7	−4	0	−1	0	0	−3	−3	1	0
2	1	−2	−4	3	6	0	−2	−1/4	−3/4	0	0	−9/4	−9/4	1/2	0
2	−2	−2	7	4	−4	−5	4	−1	1	0	0	0	3	1	0
2	−1	−2	4	1	−2	4	−2	0	−1	0	0	0	−3	−2	0
2	−5	−2	14	5	−6	−6	2	−4	3	0	0	0	9	2	0
1	3	1	−2	−3	−1	2	3	0	0	0	0	−1/2	−3/2	1/2	2
1	4	1	−5	−6	1	11	−3	0	0	0	0	2	3	−1	−4
1	3	1	−4	−3	3	4	−1	0	0	0	0	3/2	3/2	−1/2	−2
2	0	−2	0	6	−1	−4	3	−1/2	1/2	0	0	1/2	3/2	0	0
2	1	−2	−3	3	1	5	−3	0	−1	0	0	−2	−3	0	0
2	0	−2	−2	6	3	−2	−1	−1/2	−1/2	0	0	−3/2	−3/2	0	0
1	1	1	2	−1	−1	−2	3	−1/2	1/2	0	0	0	0	1/2	2
1	2	1	−1	−4	1	7	−3	0	−1	0	0	0	0	−1	−4
1	1	1	0	−1	3	0	−1	−1/2	−1/2	0	0	0	0	−1/2	−2
1	0	1	5	−2	−3	1	1	0	0	0	0	0	−3	1	4
2	−3	−2	7	7	−3	−5	1	−2	1	0	0	0	3	0	0
1	−2	1	9	0	−3	−3	1	−2	1	0	0	0	0	1	4
0	3	4	−1	−3	−1	−1	3	−1/2	1/2	0	0	−1/2	−3/2	1	4
0	4	4	−4	−6	1	8	−3	0	−1	0	0	2	3	−2	−8
0	3	4	−3	−3	3	1	−1	−1/2	−1/2	0	0	3/2	3/2	−1	−4
0	0	4	6	−2	−3	−2	1	−2	1	0	0	0	−3	2	8
0	6	0	−3	−2	0	−1	4	−1/4	1/4	0	0	−3/4	−3/4	1	2
0	7	0	−6	−5	2	8	−2	0	0	0	0	3	3	−2	−4
0	6	0	−5	−2	4	1	0	−1/4	1/4	0	0	9/4	9/4	−1	−2
1	4	−3	−4	4	2	2	−2	0	−1	0	0	−3	−3	2	0
1	2	−3	2	6	−2	−4	2	−1	1	0	0	0	3	2	0
0	3	0	4	−1	−2	−2	2	−1	1	0	0	0	0	2	4
0	5	0	−3	1	1	−1	1	−1/2	1/2	0	0	3/2	3/2	0	0
−1	7	3	−5	−5	2	5	−2	0	−1	0	0	3	3	−2	−8
−1	5	3	1	−3	−2	−1	2	−1	1	0	0	0	−3	2	8
−2	10	2	−4	−4	−1	0	3	−1/2	1/2	0	0	−3/2	−9/2	2	8
−2	11	2	−7	−7	1	9	−3	0	−1	0	0	6	9	−4	−16
−2	10	2	−6	−4	3	2	−1	−1/2	−1/2	0	0	9/2	9/2	−2	−8
−2	7	2	3	−3	−3	−1	1	−2	1	0	0	0	−9	4	16
−3	14	1	−8	−6	2	6	−2	0	−1	0	0	9	9	−6	−16
−3	12	1	−2	−4	−2	0	2	−1	1	0	0	0	−9	2	16

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
5	−3	−3	0	3	−1	0	3	0	0	1/2	−2	1/2	−1/2	−1	0
5	−2	−3	−3	0	1	9	−3	0	0	1	−4	−2	−1	2	0
5	−3	−3	−2	3	3	2	−1	0	0	3/2	−6	−3/2	−3/2	1	0
6	−6	−6	2	12	−1	−6	3	1/2	−1/2	−2	8	−1/2	1/2	2	0
6	−5	−6	−1	9	1	3	−3	0	−1	−4	16	2	1	−4	0
6	−6	−6	0	12	3	−4	−1	1/2	−3/2	−6	24	3/2	3/2	−2	0
5	−5	−3	4	5	−1	−4	3	1/2	−1/2	−3/2	6	0	0	1	0
5	−4	−3	1	2	1	5	−3	0	−1	−3	12	0	0	−2	0
5	−5	−3	2	5	3	−2	−1	1/2	−3/2	−9/2	18	0	0	−1	0
5	−6	−3	7	4	−3	−1	1	0	0	3	−12	0	−3	−2	0
6	−9	−6	9	13	−3	−7	1	2	−3	−12	48	0	3	4	0
5	−8	−3	11	6	−3	−5	1	2	−3	−9	36	0	0	2	0
4	−2	0	1	0	−2	−3	6	1/4	−1/4	−1/4	1	−1/4	−5/4	1/4	1
4	−1	0	−2	−3	0	6	0	0	0	−1/2	2	1	2	−1/2	−2
4	0	0	−5	−6	2	15	−6	0	−1	−1	4	−4	−5	1	4
4	−2	0	−1	0	2	−1	2	1/4	−1/4	−3/4	3	3/4	3/4	−1/4	−1
4	−1	0	−4	−3	4	8	−4	0	−1	−3/2	6	−3	−3	1/2	2
4	−2	0	−3	0	6	1	−2	1/4	−5/4	−9/4	9	−9/4	−9/4	1/4	1
5	−4	−3	0	6	0	0	0	0	0	2	−8	−1	−2	0	0
6	−7	−6	2	15	0	−6	0	1	−2	−8	32	1	2	0	0
4	−3	0	2	−1	0	2	0	0	0	3/2	−6	0	0	−1/2	−2
5	−6	−3	4	8	0	−4	0	1	−2	−6	24	0	0	0	0
4	−5	0	8	1	−4	−4	4	1	−1	−3/2	6	0	−3	1/2	2
4	−4	0	5	−2	−2	5	−2	0	−1	−3	12	0	3	−1	−4
4	−5	0	6	1	0	−2	0	1	−2	−9/2	18	0	0	−1/2	−2
4	−8	0	15	2	−6	−5	2	4	−5	−9	36	0	−9	1	4
4	−3	0	1	3	−1	−3	3	1/2	−1/2	−1	4	1/2	−1/2	0	0
4	−2	0	−2	0	1	6	−3	0	−1	−2	8	−2	−1	0	0
4	−3	0	−1	3	3	−1	−1	1/2	−3/2	−3	12	−3/2	−3/2	0	0
4	−6	0	8	4	−3	−4	1	2	−3	−6	24	0	−3	0	0
3	−1	3	−1	−3	0	3	0	0	0	1	−4	1	2	−1	−4
4	−4	0	1	6	0	−3	0	1	−2	−4	16	−1	−2	0	0
3	−3	3	3	−1	0	−1	0	1	−2	−3	12	0	0	−1	−4
2	−1	6	0	−3	0	0	0	1	−2	−2	8	1	2	−2	−8
3	2	−1	−3	−2	1	3	1	0	0	0	0	3/2	3/2	−3/2	−2
3	3	−1	−6	−5	3	12	−5	0	−1	0	0	−6	−6	3	4
3	2	−1	−5	−2	5	5	−3	0	−1	0	0	−9/2	−9/2	3/2	2
4	−1	−4	−1	7	1	−3	1	1/2	−1/2	0	0	−3/2	−3/2	2	0
3	0	−1	3	0	−3	−3	5	1/2	−1/2	0	0	0	−3	−1/2	2
3	1	−1	0	−3	−1	6	−1	0	0	0	0	0	3	1	−4
3	0	−1	1	0	1	−1	1	1/2	−1/2	0	0	0	0	1/2	−2
4	−2	−4	2	6	−1	0	−1	0	−1	0	0	0	−3	−4	0
3	−1	−1	4	−1	−1	2	−1	0	−1	0	0	0	0	−3	−4
3	−3	−1	10	1	−5	−4	3	2	−2	0	0	0	−9	−1	4
3	1	−1	−3	1	2	3	−2	0	−1	0	0	−3	−3	0	0
3	−1	−1	3	3	−2	−3	2	1	−1	0	0	0	−3	0	0
2	2	2	−2	−2	1	0	1	1/2	−1/2	0	0	3/2	3/2	−1	−4

r_1	r_2	r_3	r_4	r_6	r_8	r_{12}	r_{24}	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
2	1	2	1	−3	−1	3	−1	0	−1	0	0	0	3	−2	−8
2	4	−2	−2	2	−1	−2	3	1/2	−1/2	0	0	3/2	−3/2	−2	0
2	5	−2	−5	−1	1	7	−3	0	−1	0	0	−6	−3	4	0
2	4	−2	−4	2	3	0	−1	1/2	−3/2	0	0	−9/2	−9/2	2	0
2	1	−2	5	3	−3	−3	1	2	−3	0	0	0	−9	−4	0
1	6	1	−4	−4	0	4	0	0	0	0	0	3	6	−2	−8
2	3	−2	−2	5	0	−2	0	1	−2	0	0	−3	−6	0	0
1	4	1	0	−2	0	0	0	1	−2	0	0	0	0	−2	−8
0	6	4	−3	−4	0	1	0	1	−2	0	0	3	6	−4	−16
0	9	0	−5	−3	1	1	1	1/2	−1/2	0	0	9/2	9/2	−4	−8
−2	13	2	−6	−5	0	2	0	1	−2	0	0	9	18	−8	−32
6	−3	−2	0	1	0	−2	4	−1/4	1/4	3/4	−3	3/4	3/4	−5/4	−1
6	−2	−2	−3	−2	2	7	−2	0	0	3/2	−6	−3	−3	5/2	2
6	−3	−2	−2	1	4	0	0	−1/4	1/4	9/4	−9	−9/4	−9/4	5/4	1
7	−5	−5	−1	7	2	1	−2	0	−1	−6	24	3	3	−4	0
6	−4	−2	3	0	−2	1	2	0	0	−3/2	6	0	3	3/2	−2
6	−3	−2	0	−3	0	10	−4	0	−1	−3	12	0	−3	−3	4
6	−4	−2	1	0	2	3	−2	0	−1	−9/2	18	0	0	−3/2	2
7	−7	−5	5	9	−2	−5	2	−1	1	6	−24	0	−3	−4	0
6	−6	−2	7	2	−2	−3	2	−1	1	9/2	−18	0	0	−5/2	−2
6	−7	−2	10	1	−4	0	0	0	−1	−9	36	0	9	3	−4
6	−4	−2	0	4	1	−2	1	−1/2	1/2	3	−12	−3/2	−3/2	0	0
6	−5	−2	3	3	−1	1	−1	0	−1	−6	24	0	3	0	0
5	−2	1	−2	−2	2	4	−2	0	−1	−3	12	−3	−3	1	4
5	−4	1	4	0	−2	−2	2	−1	1	3	−12	0	3	−1	−4
4	1	0	−1	−1	−1	−1	3	−1/2	1/2	0	0	3/2	9/2	−1	−4
4	2	0	−4	−4	1	8	−3	0	−1	0	0	−6	−9	2	8
4	1	0	−3	−1	3	1	−1	−1/2	−1/2	0	0	−9/2	−9/2	1	4
4	−2	0	6	0	−3	−2	1	−2	1	0	0	0	9	−2	−8
3	5	−1	−5	−3	2	5	−2	0	−1	0	0	−9	−9	6	8
3	3	−1	1	−1	−2	−1	2	−1	1	0	0	0	9	2	−8
8	−5	−4	1	5	−1	−3	3	1/2	−1/2	−3	12	−3/2	3/2	4	0
8	−4	−4	−2	2	1	6	−3	0	−1	−6	24	6	3	−8	0
8	−5	−4	−1	5	3	−1	−1	1/2	−3/2	−9	36	9/2	9/2	−4	0
8	−8	−4	8	6	−3	−4	1	2	−3	−18	72	0	9	8	0
7	−3	−1	−1	−1	0	3	0	0	0	3	−12	−3	−6	1	4
8	−6	−4	1	8	0	−3	0	1	−2	−12	48	3	6	0	0
7	−5	−1	3	1	0	−1	0	1	−2	−9	36	0	0	1	4
6	−3	2	0	−1	0	0	0	1	−2	−6	24	−3	−6	2	8
6	0	−2	−2	0	1	0	1	1/2	−1/2	0	0	−9/2	−9/2	5	4
6	−1	−2	1	−1	−1	3	−1	0	−1	0	0	0	−9	−6	8
4	4	0	−3	−2	0	1	0	1	−2	0	0	−9	−18	4	16
9	−4	−3	−2	0	2	4	−2	0	−1	−9	36	9	9	−9	−4
9	−6	−3	4	2	−2	−2	2	−1	1	9	−36	0	−9	−7	4
10	−5	−2	0	1	0	0	0	1	−2	−18	72	9	18	−2	−8

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