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ON SOME FAMILIES OF INTEGRALS CONNECTED TO THE HURWITZ ZETA FUNCTION

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Abstract: Expressions for a family of integrals involving the Hurwitz zeta function are established using standard properties of the Fourier transform.

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1. Introduction

The Hurwitz zeta function is defined by

$$\zeta(s,a)=\sum_{n\geqslant 0}\frac{1}{(a+n)^s}$$

for $s\in\mathbb{C}$, $\Re(s)>1$, and a is chosen appropriately so there are no singularities in the series. $\zeta(s,a)$ admits the integral representation

$$\zeta(s,a)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{e^{-at}}{1-e^{-t}}t^{s-1}dt, \tag{1.1}$$

where $\Gamma(s)$ is Euler's gamma function, which is valid for $\Re(s)>1$ and $\Re(a)>0$. Hermite proved an interesting integral representation, which actually provides an explicit realization of the analytic continuation to $\mathbb{C}-\{1\}$ and $\Re(a)>0:$

$$\zeta(s,a)=\frac{a^{-s}}{2}+\frac{a^{1-s}}{s-1}+2\int_0^{\infty}\frac{\sin(s\tan^{-1}(t/a))dt}{(a^2+t^2)^{s/2}(e^{2\pi t}-1)}. \tag{1.2}$$

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The function $\zeta(s,a)$ is analytic for $s \neq 1$, and direct differentiation of (1.2) yields

$$\begin{aligned}\zeta'(s,a) = &-\frac{a^{-s}\ln a}{2} - \frac{a^{1-s}\ln a}{s-1} - \frac{a^{1-s}}{(s-1)^2}\\ &- 2a^{1-s}\ln a\int_0^\infty \frac{\sin(s\tan^{-1}(t))dt}{(1+t^2)^{s/2}(e^{2a\pi t}-1)}\\ &+ 2a^{1-s}\int_0^\infty \frac{\cos(s\tan^{-1}(t))\tan^{-1}(t)dt}{(1+t^2)^{s/2}(e^{2a\pi t}-1)}\\ &- a^{1-s}\int_0^\infty \frac{\sin(s\tan^{-1}(t))\ln(t^2+1)dt}{(1+t^2)^{s/2}(e^{2a\pi t}-1)},\end{aligned}$$

where $\zeta'(s,a)$ denotes $\partial\zeta(s,a)/\partial s$.

The work presently discussed is a continuation of [2, 4, 7] where these integral representations have been employed to evaluate interesting definite integrals. General information about $\zeta(s,a)$ can be found in [1], [5] and [6].

The main result is presented next.

Theorem 1. *Let $n \in \mathbb{N}_0$. For $\Re(a) > 0$ and $0 \leqslant 2n < \Re(s)$, define*

$$S_n(a,s) := \int_0^\infty \frac{t^{2n}\sin(s\tan^{-1}(t/a))dt}{(a^2+t^2)^{s/2}(e^{2\pi t}-1)}.$$

Then

$$S_n(a,s) = \frac{1}{2}\sum_{m=0}^{2n}(-1)^{m+n}\binom{2n}{m}a^m P_1(a, m+s-2n), \tag{1.3}$$

where

$$P_1(a,s) = \zeta(s,a) - \frac{a^{-s}}{2} - \frac{a^{1-s}}{s-1}.$$

Observe that (1.2) corresponds to the special case $n=0$ in (1.3). Here we note that $S_n(a,s)$ is analytic in the set $\{n \in \mathbb{N}_0, 0 \leqslant 2n < \Re(s) : s-2n \neq 1\}$.

The proof of Theorem 1 is based on identifying the Fourier sine transform of two special functions and then apply the corresponding Parseval identity. Recall that for a function defined on the half-line, the Fourier sine transform is

$$\mathfrak{F}(f)(w) := \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin(wt)dt,$$

provided the integral converges. The corresponding Parseval identity states that

$$\int_0^\infty \mathfrak{F}(f)(w)\mathfrak{F}(g)(w)dw = \int_0^\infty f(t)g(t)dt. \tag{1.4}$$

Theorem 1 is a direct consequence of Parseval's relation applied to the functions

$$f(t) = 1/(e^{2\pi t}-1) \qquad \text{and} \qquad g(t) = \frac{t^{2n}\sin(s\tan^{-1}(t/a))}{(a^2+t^2)^{s/2}}.$$

The Fourier sine transform $f(t)$ comes from entry 3.951.12 of [3]. It states an equivalent form of the identity

$$\int_0^\infty \frac{\sin(wt)dt}{e^{2\pi t}-1} = \frac{1}{2}\left(\frac{1}{e^w-1}+\frac{1}{2}-\frac{1}{w}\right). \tag{1.5}$$

The Fourier sine transform of $g(t)$ is given in terms of the *associated Laguerre polynomials* $L_n^k(x)$ defined by the Rodrigues representation

$$L_n^k(x) = \frac{e^x x^{-k}}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+k}\right), \tag{1.6}$$

for $n \in \mathbb{N}\cup\{0\}$.

Theorem 1 is extended in Section 3 to include integrals in which the kernel $1/(e^{2\pi t}-1)$ is replaced by

$$1/(e^{\pi t}+1), \qquad 1/\sinh(\pi t), \qquad 1/\cosh(\pi t).$$

Consider the families of integrals

$$\begin{aligned} I_k(q) &= \int_0^\infty \frac{tdt}{(1+t^2)^{k+1}(e^{2\pi qt}-1)}, \\ T_k(q) &= \int_0^\infty \frac{t^k \tan^{-1} tdt}{(e^{2\pi qt}-1)}, \\ L_k(q) &= \int_0^\infty \frac{t^k \ln(1+t^2)dt}{(e^{2\pi qt}-1)}. \end{aligned}$$

The reader will find in [2] explicit expression for $I_k(q)$ in terms of the derivatives of the polygamma function and for $T_{2k}(q)$ and $L_{2k+1}(q)$ remains an open problem. It would be of interest to analyze the evaluations discussed here in relation to this open problem.

2. The proof

The proof of Theorem 1 is based on the computation of two Fourier sine transforms. Formula 3.951.12 in [3] states an equivalent form of the identity (1.5), which gives the sine transform of

$$f(t) = \frac{1}{e^{2\pi t}-1},$$

as

$$\mathfrak{F}(f)(w) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{e^w-1}+\frac{1}{2}-\frac{1}{w}\right).$$

The second Fourier sine transform is that of the associated Laguerre polynomials (1.6). The explicitly formula

$$L_n^k(x) = \sum_{j=0}^n \frac{(-1)^j(n+k)!}{(n-j)!(k+j)!j!}x^j \tag{2.1}$$

is employed in the derivation.

Formula 3.769.4 of [3] contains the integral representation

$$\int_0^\infty t^{2n}\left((a-it)^{-s}-(a+it)^{-s}\right)\sin(wt)dt=\frac{(-1)^n i\pi(2n)!}{\Gamma(s)e^{aw}w^{2n+1-s}}L_{2n}^{s-2n-1}(aw), \tag{2.2}$$

for $w>0$, $\Re(a)>0$ and $0\leqslant 2n<\Re(s)$. The integrand can be simplified using

$$(a-it)^{-s}-(a+it)^{-s}=\frac{2i\sin(s\tan^{-1}(t/a))}{(a^2+t^2)^{s/2}}.$$

Therefore (2.2) can be written as

$$\int_0^\infty t^{2n}\frac{\sin(s\tan^{-1}(t/a))}{(a^2+t^2)^{s/2}}\sin(wt)dt=\frac{(-1)^n\pi(2n)!}{2\Gamma(s)e^{aw}w^{2n+1-s}}L_{2n}^{s-2n-1}(aw). \tag{2.3}$$

Or equivalently we may state that the Fourier sine transform of

$$g(t)=\frac{t^{2n}\sin(s\tan^{-1}(t/a))}{(a^2+t^2)^{s/2}},$$

is given by

$$\mathfrak{F}(g)(w)=\frac{(-1)^n\pi(2n)!}{2\Gamma(s)e^{aw}w^{2n+1-s}}L_{2n}^{s-2n-1}(aw).$$

Parseval's identity (1.4) gives the next result.

Lemma 2. *For* $\Re(a),\Re(s)>0$,

$$\begin{aligned}\int_0^\infty &\frac{t^{2n}\sin(s\tan^{-1}(t/a))dt}{(a^2+t^2)^{s/2}(e^{2\pi t}-1)}\\ &=\frac{(-1)^n(2n)!}{2\Gamma(s)}\int_0^\infty e^{-aw}w^{-2n-1+s}L_{2n}^{s-2n-1}(aw)\left(\frac{1}{e^w-1}+\frac{1}{2}-\frac{1}{w}\right)dw.\end{aligned}$$

The explicit formula (2.1) for the Laguerre polynomials is now employed to evaluate the integral on the right side of Lemma 2.

$$\begin{aligned}\int_0^\infty &\frac{e^{-aw}w^{-2n-1+s}L_{2n}^{s-2n-1}(aw)}{e^w-1}dw\\ &=\sum_{j=0}^{2n}\frac{(-1)^j(s-1)!a^j}{(2n-j)!(s-2n-1+j)!j!}\int_0^\infty\frac{w^{s-2n-1+j}e^{-(a+1)w}dw}{1-e^{-w}}\\ &=\sum_{j=0}^{2n}\frac{(-1)^j(s-1)!a^j}{(2n-j)!j!}\zeta(s-2n+j,a+1).\end{aligned}$$

In the last step we have employed the integral representation for the Hurwitz zeta function (1.1). For the desired formula we must write $1/(e^w-1)=e^w/(e^w-1)-1$. The remaining integrals corresponding to the terms $1/2$ and $1/w$ are elementary, and so are omitted.

3. Related integrals

In this section we produce results similar to Theorem 1 for a family of integrals of the form

$$\int_0^\infty f(t)K(t)dt,$$

where the kernel $1/(e^{2\pi t}-1)$ in Theorem 1 is replaced by

$$1/(e^{\pi t}+1), \qquad 1/\sinh(\pi t), \quad \text{or} \quad 1/\cosh(\pi \mathrm{t}).$$

The next lemma will be needed for future computations corresponding to these kernels.

Lemma 3. *Assume* $\Re(s)>1$ *and* $\Re(a)\geqslant 0$. *Then*

$$\int_0^\infty \frac{t^{s-1}e^{-at}}{\sinh(t)}dt = \Gamma(s)\left(\zeta(s,a)-2^{-s}\zeta(s,a/2)\right).$$

If $\Re(a)>0$, *then*

$$\int_0^\infty \frac{t^{s-1}e^{-at}}{1+e^{-t}}dt = \Gamma(s)\left(-\zeta(s,a)+2^{1-s}\zeta(s,a/2)\right),$$

and

$$\int_0^\infty \frac{t^{s-1}e^{-at}}{\cosh(t)}dt = \Gamma(s)2^{-2s}\left(\zeta(s,\frac{1+a}{4})-\zeta(s,\frac{a+3}{4})\right).$$

These integrals are well-known variations of (1.1). Details are in [2].

Theorem 4. *Let* $n\in\mathbb{N}_0$. *For* $\Re(a)>0$ *and* $0\leqslant 2n<\Re(s)$, *define*

$$SH_n(a,s) := \int_0^\infty \frac{t^{2n}\sin(s\tan^{-1}(t/a))dt}{(a^2+t^2)^{s/2}\sinh(\pi t)}.$$

Then

$$SH_n(a,s) = \frac{1}{2}\sum_{m=0}^{2n}(-1)^{m+n}\binom{2n}{m}a^m P_2(a,m+s-2n),$$

where

$$P_1(a,s) = 2^{2-s}\zeta(s,a/2)-2\zeta(s,a)-a^{-s}.$$

Proof. The identity

$$\int_0^\infty \frac{\sin(wt)}{\sinh(\beta t)} = \frac{\pi}{2\beta}\tanh\frac{\pi w}{2\beta} \tag{3.1}$$

appears as entry 3.981.1 in [3].

The value $\beta=\pi$ in (3.1) shows that the Fourier sine transform of $1/\sinh(\pi t)$ is $\frac{1}{2}\tanh(w/2)$. Then (1.4) and (2.3) give

$$\int_0^\infty \frac{t^{2n}\sin(s\tan^{-1}(t/a))dt}{(a^2+t^2)^{s/2}\sinh(\pi t)}$$
$$=\frac{(-1)^n\pi(2n)!}{4\Gamma(s)}\int_0^\infty \tanh(\frac{w}{2})e^{-aw}w^{-2n-1+s}L_{2n}^{s-2n-1}(aw)\left(\frac{1}{e^w-1}+\frac{1}{2}-\frac{1}{w}\right)dw.$$

Now use

$$\tanh(w/2)=\frac{2}{1+e^{-w}}-1$$

and proceed as in the proof of Theorem 1. ■

The next results are established along similar lines of the proof presented above. The details are omitted. Entries 3.911.1 and 3.981.3 in [3] are

$$\int_0^\infty \frac{\sin(wt)}{e^{\beta t}+1}=\frac{1}{2w}-\frac{\pi}{2\beta\sinh\frac{\pi w}{\beta}},$$

and

$$\int_0^\infty \frac{\cos(wt)}{\cosh(\beta t)}=\frac{\pi}{2\beta\cosh\frac{\pi w}{2\beta}},$$

respectively. These are used instead of (3.1) in the proofs.

Theorem 5. *Let $n\in\mathbb{N}_0$. For $\Re(a)>0$ and $0\leqslant 2n<\Re(s)$, define*

$$EP_n(a,s):=\int_0^\infty \frac{t^{2n}\sin(s\tan^{-1}(t/a))dt}{(a^2+t^2)^{s/2}(e^{2\pi t}+1)}.$$

Then

$$EP_n(a,s)=\frac{1}{2}\sum_{m=0}^{2n}(-1)^{m+n}\binom{2n}{m}a^mP_1(a,m+s-2n),$$

where

$$P_3(a,s)=\frac{a^{1-s}}{s-1}-\zeta(s,a)-2^{-s}\zeta(s,a/2).$$

Theorem 6. *Let $n\in\mathbb{N}_0$. For $\Re(a)>0$ and $0\leqslant 2n<\Re(s)$, define*

$$CH_n(a,s):=\int_0^\infty \frac{t^{2n}\sin(s\tan^{-1}(t/a))dt}{(a^2+t^2)^{s/2}\cosh(\pi t/2)}.$$

Then

$$CH_n(a,s)=\frac{1}{2}\sum_{m=0}^{2n}(-1)^{m+n}\binom{2n}{m}a^mP_1(a,m+s-2n),$$

where

$$P_4(a,s)=\frac{1}{2^{2s}}\left(\zeta(s,\frac{a+1}{4})-\zeta(s,\frac{a+3}{4})\right).$$

The final result describes integrals containing odd powers of t in the integrand. As before, the proofs are similar to that of Theorem 1, so they are omitted.

Theorem 7. *Let $n \in \mathbb{N}_0$. For $\Re(a) > 0$ and $-1 \leqslant 2n+1 < \Re(s)$, then*

$$\int_0^\infty \frac{t^{2n+1}\cos(s\tan^{-1}(t/a))dt}{(a^2+t^2)^{s/2}(e^{2\pi t}-1)} = \frac{1}{2}\sum_{m=0}^{2n+1}(-1)^{m+n}\binom{2n+1}{m}a^m P_1(a, m+s-2n-1),$$

$$\int_0^\infty \frac{t^{2n+1}\cos(s\tan^{-1}(t/a))dt}{(a^2+t^2)^{s/2}\sinh(\pi t)} = \frac{1}{2}\sum_{m=0}^{2n+1}(-1)^{m+n}\binom{2n+1}{m}a^m P_2(a, m+s-2n-1),$$

$$\int_0^\infty \frac{t^{2n+1}\cos(s\tan^{-1}(t/a))dt}{(a^2+t^2)^{s/2}(e^{\pi t}+1)} = \frac{1}{2}\sum_{m=0}^{2n+1}(-1)^{m+n}\binom{2n+1}{m}a^m P_3(a, m+s-2n-1),$$

and if $\Re(a) > 0$ and $0 \leqslant 2n < \Re(s) - 1$,

$$\int_0^\infty \frac{t^{2n+1}\sin(s\tan^{-1}(t/a))dt}{(a^2+t^2)^{s/2}\cosh(\pi t/2)} = \frac{1}{2}\sum_{m=0}^{2n+1}(-1)^{m+n}\binom{2n+1}{m}a^m P_4(a, m+s-2n-1).$$

References

[1] G. Andrews, R. Askey, and R. Roy, *Special Functions, volume 71 of Encyclopedia of Mathematics and its Applications*, Cambridge University Press, New York, 1999.

[2] G. Boros, O. Espinosa, and V. Moll, *On some families of integrals solvable in terms of polygamma and negapolygamma functions*, Integrals Transforms and Special Functions **14** (2003), 187–203.

[3] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, edited by A. Jeffrey and D. Zwillinger, Academic Press, New York, 7th edition, 2007.

[4] J. Zhang, S. Kanemitsu, Y. Tanigawa, *Evaluation of Spanen integrals of the product of zeta functions*, Integrals Transforms and Special Functions **19** (2008), 115–128.

[5] H.M. Srivastava and J. Choi, *Series associated with the zeta and related functions*, Kluwer Academic Publishers, 1st edition, 2001.

[6] E.T. Whittaker and G.N. Watson, *Modern Analysis*, Cambridge University Press, 1962.

[7] N.Y. Zhang and K.S. Williams, *Special values of the Lerch zeta function and the evaluation of certain integrals.* Proc. Amer. Math. Soc. **119** (1993), 35–49.

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