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A BUILDING-THEORETIC APPROACH TO RELATIVE TAMAGAWA NUMBERS OF SEMISIMPLE GROUPS OVER GLOBAL FUNCTION FIELDS

Rony A. Bitan, Ralf Köhl (né Gramlich)

Abstract: Let G be a semisimple almost simple group defined over a global function field K, not anisotropic of type A_n. We express the (relative) Tamagawa number of G in terms of local data, including the number $t_\infty(G)$ of types in one orbit of a special vertex in the Bruhat–Tits building of $G_\infty(\hat{K}_\infty)$ for some place ∞ and the class number $h_\infty(G)$ of G at ∞.

Keywords: relative Tamagawa number, Bruhat-Tits building, class number, special vertex.

1. Introduction

Let C be a smooth, projective and irreducible algebraic curve defined over the finite field $\mathbb{F}_q$ and let $K = \mathbb{F}_q(C)$ be its function field. Let G be a (connected) semisimple group defined and almost simple over K. The Tamagawa number $\tau(G)$ of G is defined as the covolume of the group $G(K)$ of K-rational points in the adelic group $G(\mathbb{A})$ (embedded diagonally as a discrete subgroup) with respect to the volume induced by the Tamagawa measure on $G(\mathbb{A})$ (see [Weil], [Clo] and Section 4 below). Let $\pi : G^{\mathrm{sc}} \to G$ be the universal covering and let $F = \ker(\pi)$ be the fundamental group. We assume that G is not anisotropic of type A_n and that $(\mathrm{char}(K), |F|) = 1$.

Weil's conjecture states that $\tau(G^{\mathrm{sc}}) = 1$. By [Har1] the Weil conjecture is known to hold for split G^{sc}. A geometric proof of Weil's conjecture by Gaitsgory–Lurie has been announced in [Gai, 1.2.3], see also: [Lur].

In the present article we investigate the relative Tamagawa number $\frac{\tau(G)}{\tau(G^{\mathrm{sc}})}$ from a building-theoretic point of view – in the situation in which G is *locally isotropic* everywhere. Let ∞ be some non-constant closed point of K and let $\mathbb{A}_\infty := \hat{K}_\infty \times \prod_{\mathfrak{p} \neq \infty} \hat{\mathcal{O}}_\mathfrak{p}$ be the subring of $\{\infty\}$-adèles in the adèle ring $\mathbb{A}$. Defining some local integral models of G, we may refer to the set of double cosets $\mathrm{Cl}_\infty(G) :=$

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$G(\mathbb{A}_\infty)\backslash G(\mathbb{A})/G(K)$ which is finite (4.6). This allows us to split off the class number $h_\infty(G) = |\mathrm{Cl}_\infty(G)|$ and proceed by computing the co-volume of $G(K)$ in the trivial coset $G(\mathbb{A}_\infty)G(K)$ w.r.t. the Tamagawa measure by considering the natural action of $G(\mathbb{A}_\infty)$ on the Bruhat–Tits building of $G_\infty(\hat{K}_\infty)$, resulting in formula (14) below

$$\tau(G) = q^{-(g-1)\dim(G)} \cdot h_\infty(G) \cdot i_\infty(G) \cdot \prod_{\mathfrak{p}} \omega_{\mathfrak{p}}(\underline{G}^0_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}))$$

where for each $\mathfrak{p}$, $\omega_{\mathfrak{p}}$ is some multiplicative local Haar-measure, $\underline{G}^0_{\mathfrak{p}}$ stands for the connected component of the Bruhat–Tits $\hat{\mathcal{O}}_{\mathfrak{p}}$-model at some special point and $i_\infty(G)$ is an arithmetic invariant related to $G_\infty := G \otimes_K \hat{K}_\infty$. The key obstruction for using this formula is to determine a fundamental domain for the action of the discrete subgroup $\mathcal{G}^0(\mathcal{O}_{\{\infty\}})$ on a $G_\infty(\hat{K}_\infty)$-orbit of the Bruhat–Tits building of $G_\infty(\hat{K}_\infty)$ where $\mathcal{G}^0$ is a flat connected smooth and finite type model of G defined over the ring $\mathcal{O}_{\{\infty\}}$ of $\{\infty\}$-integers in K.

In Proposition 5.3 below, we will see that for computing the *relative* local volumes $\frac{i_\infty(G)}{i_\infty(G^{\mathrm{sc}})}$ it suffices to compare orbits under $G^{\mathrm{sc}}_\infty(\hat{K}_\infty)$ and $G_\infty(\hat{K}_\infty)$, whose behavior is controlled by the number $t_\infty(G)$ of types in the $G_\infty(\hat{K}_\infty)$-orbit of the fundamental special vertex and by the number $j_\infty(G)$ expressing the comparison between the fundamental domains of $\mathcal{G}^{\mathrm{sc}}(\mathcal{O}_{\{\infty\}})$ and $\mathcal{G}^0(\mathcal{O}_{\{\infty\}})$.

We then arrive at the following main result of our article. By K^s_∞ we denote the separable closure of $\hat{K}_\infty$ with Galois group $\mathfrak{g}_\infty = \mathrm{Gal}(K^s_\infty/\hat{K}_\infty)$ and inertia subgroup $I_\infty = \mathrm{Gal}(K^s_\infty/K^{\mathrm{un}}_\infty)$. Moreover, σ_∞ denotes a generator of $\mathfrak{g}_\infty/I_\infty$, i.e., the map $\sigma_\infty : x \mapsto x^{|k_\infty|}$ where k_∞ is the residue field of $\hat{K}_\infty$. Let $F_\infty := \ker[G^{\mathrm{sc}}_\infty \to G_\infty]$ and $\widehat{F_\infty} := \mathrm{Hom}(F_\infty \otimes \hat{K}^s_\infty, \mathbb{G}_{m,\hat{K}^s_\infty})$.

Main Theorem. *With these notations and assuming the validity of the Weil conjecture, one has*

$$\tau(G) = h_\infty(G) \cdot \frac{t_\infty(G)}{j_\infty(G)}.$$

The number $t_\infty(G)$ *satisfies*

$$t_\infty(G) = |H^1(I_\infty, F_\infty(\hat{K}^s_\infty))^{\sigma_\infty}| = |\widehat{F_\infty}^{\mathfrak{g}_\infty}|$$

and:

$$j_\infty(G) = \frac{|H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{F})|}{|\mathcal{F}(\mathcal{O}_{\{\infty\}})|}$$

where $\mathcal{F} := \ker[\mathcal{G}^{\mathrm{sc}} \to \mathcal{G}^0]$.

In particular, if G is quasi-split and the separable closure $\overline{\mathcal{O}}^G_{\{\infty\}}$ of $\mathcal{O}_{\{\infty\}}$ in the splitting field of G is a UFD (see Remark 1.1), then $j_\infty(G) = 1$ and so

$$\tau(G) = h_\infty(G) \cdot t_\infty(G) = h_\infty(G) \cdot |\widehat{F_\infty}^{\mathfrak{g}_\infty}|.$$

If, in addition, G is split or adjoint then $h_\infty(G) = 1$ and so $\tau(G) = t_\infty(G) = |\widehat{F_\infty}^{\mathfrak{g}_\infty}|$ and just $|F|$ in the split case (see Corollaries 7.10 and 7.14 below).

Remark 1.1. Such $\overline{\mathcal{O}}^G_{\{\infty\}}$ being a UFD arises in cases the underlying extended curve $\overline{C^{\mathrm{af}}}$ is of genus zero, and $\infty \in \overline{C^{\mathrm{af}}}(\mathbb{F}_q)$ or it admits no $\mathbb{F}_q$-rational point (see [Sam, Theorem 5.1]).

Our method of proof is a combination of geometric group theory and cohomology. Our approach is independent of Prasad's covolume formula described in [Pra2], but it is likely that with some effort it can be used to deduce our Main Theorem.

As an application in case the group G is quasi-split and $\overline{\mathcal{O}}^G_{\{\infty\}}$ is a UFD, we combine our result with [Ono1, Formula (3.9.1')] and the techniques from [PR, § 8.2] in order to relate the cokernels of Bourqui's degree maps $\deg_{T^{\mathrm{sc}}}$ and $\deg_T$ from [Bou, Section 2.2], where T^{sc} and T denote suitable maximal tori of G^{sc} and G respectively; cf. Proposition 7.8. These concrete computations allow us to also provide a wealth of examples in Section 6 for which we compute the relative Tamagawa numbers. We also demonstrate the result in a case of a split group defined over the function field of an elliptic curve (Remark 7.13).

This article is organized as follows: In the preliminary Section 2, we fix the relevant notions from Bruhat–Tits theory. In Section 3, we compute volumes of parahoric subgroups over local fields, their maximal unramified extensions, and their valuation rings. In Section 4, we revise the definition of the Tamagawa number of semisimple K-groups and establish a decomposition of $G(\mathbb{A})/G(K)$ enabling us to express $\tau(G)$ in terms of some global and local invariants. In Section 5, we compute cohomology groups over rings of S-integers with $|S| = 1$, use Bruhat–Tits theory and Serre's formula ([Ser1, p. 84], [BL, Corollary 1.6]) in order to derive the above-mentioned formula (14) for computing the Tamagawa number. In Section 6, we express the number $t_\infty(G)$ of types in the orbit of a special point in terms of F_∞, accomplishing the proof of our Main Theorem. The final Section 7 addresses the above-mentioned application and examples.

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2. Basic notions from Bruhat-Tits theory

We retain the notation from Section 1, only here we assume G to be quasi-split, allowing us to consider a maximal torus T of G, being the centralizer of a maximal split subtorus of G. At the end of Section 3, however, this assumption will be dropped. Since K is a function field, all valuations defined on K are non-Archimedean. For any prime $\mathfrak{p}$ of K let $v_\mathfrak{p}$ be the induced discrete valuation on K. Let $\mathcal{O}_\mathfrak{p} := \{x \in K : v_\mathfrak{p}(x) \geqslant 0\}$ and let $K_\mathfrak{p}$ be its fraction field. Let $\hat{K}_\mathfrak{p}$ be the completion of $K_\mathfrak{p}$ w.r.t. $v_\mathfrak{p}$ and let $\hat{\mathcal{O}}_\mathfrak{p}$ be its ring of integers. Let $k_\mathfrak{p} = \hat{\mathcal{O}}_\mathfrak{p}/\mathfrak{p}$ be the corresponding (finite) residue field. Then $G_\mathfrak{p} = G \otimes_K \hat{K}_\mathfrak{p}$ is semisimple. The assumption $(\mathrm{char}(K), |F|) = 1$ says that π is separable. $T_\mathfrak{p} = T \otimes_K \hat{K}_\mathfrak{p}$ is a maximal torus in $G_\mathfrak{p}$. Let $(X^*(T_\mathfrak{p}), \Phi, X_*(T_\mathfrak{p}), \Phi^\vee)$ be the root datum of $(G_\mathfrak{p}, T_\mathfrak{p})$ and let W be the associated constant Weyl group. Let $\mathcal{B}_\mathfrak{p}$ be the Bruhat–Tits building associated to the adjoint group of $G_\mathfrak{p}$ (cf. [BT1, Section 7], also [AB, Chapter 11]) and let $\mathcal{A}$ be the apartment in $\mathcal{B}_\mathfrak{p}$ corresponding to $T_\mathfrak{p}$.

We fix a special vertex $x \in \mathcal{A}$, i.e., a vertex whose isotropy group in the setwise stabilizer of $\mathcal{A}$ is isomorphic to W. Since the Bruhat–Tits building $\mathcal{B}_\mathfrak{p}$ is locally finite, the stabilizer P_x of x in $G_\mathfrak{p}(\hat{K}_\mathfrak{p})$ is a compact subgroup of $G_\mathfrak{p}(\hat{K}_\mathfrak{p})$. In fact, P_x is a profinite group. As the Bruhat–Tits building is non-positively curved ([BT1, Section 2.5]; also [AB, Theorem 11.16]), any compact subgroup of $G_\mathfrak{p}(\hat{K}_\mathfrak{p})$ necessarily fixes a simplex of $\mathcal{B}_\mathfrak{p}$ by the Bruhat–Tits fixed point lemma ([BT1, Lemma 3.2.3]; also [AB, Theorem 11.23], [BH, Section II.2, p. 178]) and, hence, P_x has finite index in a maximal compact subgroup of $G_\mathfrak{p}(\hat{K}_\mathfrak{p})$. A detailed discussion of maximal parahoric subgroups, maximal compact subgroups, and conjugacy classes thereof can be found in [BT1, Section 3.3] (see also [IM, Tables (I), (II), p. 41], [Tit, p. 51]). Let $\underline{G}_x$ be the Bruhat–Tits model associated to P_x, i.e., such that $\underline{G}_x(\hat{\mathcal{O}}_\mathfrak{p}) = P_x$. Denote by $\overline{G}_x$ the reduction modulo $\mathfrak{p}$ of $\underline{G}_x$ and by $\underline{G}_x^0$ the open subscheme of $\underline{G}_x$ whose reduction is the identity component $\overline{G}_x^0$ of $\overline{G}_x$. Let $\underline{T}_\mathfrak{p}$ be the Néron–Raynaud $\hat{\mathcal{O}}_\mathfrak{p}$-model (shortly referred as NR-model) of $T_\mathfrak{p}$ which is of finite type, i.e., such that $\underline{T}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p})$ is the maximal compact subgroup of $T_\mathfrak{p}(\hat{K}_\mathfrak{p})$ (see Theorem 2 in [BLR, § 10.2] and [CY, § 3.2] for an explicit construction). Denote by $\underline{T}_\mathfrak{p}^0$ its subscheme having a connected special fiber. $\underline{T}_\mathfrak{p}^0(\hat{\mathcal{O}}_\mathfrak{p})$ is the pointwise stabilizer of $\mathcal{A}$ and is a subgroup of $\underline{G}_x^0(\hat{\mathcal{O}}_\mathfrak{p})$. Since $G_\mathfrak{p}$ is semisimple and the residue field $k_\mathfrak{p}$ is finite, the adjoint group of $G_\mathfrak{p}(\hat{K}_\mathfrak{p})$ permutes transitively the special vertices (see [Tit, § 2.5]). If Φ is not reduced, we adapt the convention of Prasad in [Pra2, § 1.2] and of Gross in [Gro, § 4]: for each component of the local Dynkin diagram of the type

$$\bullet \Longleftarrow \bullet - \bullet \cdots \bullet - \bullet \Longleftarrow \bullet$$

we choose the special vertex at the right end of the diagram. Now $\underline{G}_x$ is well-defined up to isomorphism and is denoted from now on by $\underline{G}_\mathfrak{p}$. x is called the *fundamental special vertex* of $\mathcal{B}_\mathfrak{p}$.

Remark 2.1. If G is either simply connected or adjoint, being also almost simple, it is one copy of restriction of scalars $R_{K'/K}(G')$ where K' is a separable extension of K and G' is split and simple. If G is also quasi-split, its maximal torus T being the centralizer of a maximal split subtorus of G', is maximal (see in the proof of [Spr, Prop. 16.2.2]) and quasi-trivial, i.e. a Weil torus $R_{K'/K}(\mathbb{G}_m^d)$. In this case, $\underline{T}_\mathfrak{p}$ is connected for any $\mathfrak{p}$ (see [NX, Prop. 2.4]).

3. Volumes of parahoric subgroups

As $\hat{K}_\mathfrak{p}$ is locally compact, its underlying additive group admits a Haar measure $dx_\mathfrak{p}$ which is unique up to a scalar multiple, determined by fixing the value of $dx_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p})$. Such a normalization induces a multiplicative Haar measure $\varpi_\mathfrak{p}$ on the locally compact group $G_{\mathfrak{p}}(\hat{K}_{\mathfrak{p}})$ such that for any $U \subset \hat{K}_{\mathfrak{p}}^{\times}$, $\varpi_{\mathfrak{p}}(U) = dx_{\mathfrak{p}}(\mathrm{Lie}(U))$. Our choice of the Bruhat–Tits model in the preceding section allows us easily to determine the volume of the fundamental parahoric subgroup with respect to this Haar measure; since $\underline{G}_{\mathfrak{p}}^0$ is smooth and connected we have (see [Oes, Thm. 2.5]):

$$\varpi_{\mathfrak{p}}(\underline{G}_{\mathfrak{p}}) = dx_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}})^d \cdot |k_{\mathfrak{p}}|^{-d} \cdot |\overline{\underline{G}}_{\mathfrak{p}}^0(k_{\mathfrak{p}})|, \qquad d = \dim(\underline{G}_{\mathfrak{p}}). \tag{1}$$

Remark 3.1. [BT2, 4.6.22] If $G_{\mathfrak{p}}$ splits over an unramified extension, then $\underline{G}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}) = \underline{G}_{\mathfrak{p}}^0(\hat{\mathcal{O}}_{\mathfrak{p}})$.

Let $\pi_{\mathfrak{p}} : G_{\mathfrak{p}}^{\mathrm{sc}} \to G_{\mathfrak{p}}$ be the universal covering of $G_{\mathfrak{p}}$. According to [BT2, 4.4.18(VI)], the cover $\pi_{\mathfrak{p}}$ restricted to $T_{\mathfrak{p}}^{\mathrm{sc}}$ extends to a homomorphism $\underline{T^{\mathrm{sc}}}_{\mathfrak{p}} \to \underline{T}_{\mathfrak{p}}$ over $\mathrm{Spec}\,\hat{\mathcal{O}}_{\mathfrak{p}}$. Together with the associated root subgroups $\hat{\mathcal{O}}_{\mathfrak{p}}$-scheme $\mathfrak{X}$, which is equal for both $G_{\mathfrak{p}}^{\mathrm{sc}}$ and $G_{\mathfrak{p}}$, this homomorphism over $\mathrm{Spec}\,\hat{\mathcal{O}}_{\mathfrak{p}}$ extends to a homomorphism $\underline{G^{\mathrm{sc}}}_{\mathfrak{p}} \to \underline{G}_{\mathfrak{p}}$ of the Bruhat–Tits schemes. Let $\underline{F}_{\mathfrak{p}} := \ker[\underline{G^{\mathrm{sc}}}_{\mathfrak{p}} \to \underline{G}_{\mathfrak{p}}]$. It is finite, flat and smooth, due to our assumption that $(\mathrm{char}(K), F) = 1$. It is also central, thus embedded in $\underline{T^{\mathrm{sc}}}_{\mathfrak{p}}$.

Let $\hat{K}_{\mathfrak{p}}^{\mathrm{un}}$ be the maximal unramified extension of $\hat{K}_{\mathfrak{p}}$, i.e., the strict henselization of $\hat{K}_{\mathfrak{p}}$ with ring of integers $\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}}$ and algebraically closed residue field $k_{\mathfrak{p}}^s$. Let $\hat{K}_{\mathfrak{p}}^s$ be a separable closure of $\hat{K}_{\mathfrak{p}}$ containing $\hat{K}_{\mathfrak{p}}^{\mathrm{un}}$ and let $I_{\mathfrak{p}} = \mathrm{Gal}(\hat{K}_{\mathfrak{p}}^s/\hat{K}_{\mathfrak{p}}^{\mathrm{un}})$ be the inertia subgroup of $\mathfrak{g}_{\mathfrak{p}} = \mathrm{Gal}(\hat{K}_{\mathfrak{p}}^s/\hat{K}_{\mathfrak{p}})$. Let $\sigma_{\mathfrak{p}}$ be a generator of $\mathfrak{g}_{\mathfrak{p}}/I_{\mathfrak{p}}$, i.e., the map $\sigma_{\mathfrak{p}} : x \mapsto x^{|k_{\mathfrak{p}}|}$ where as above $k_{\mathfrak{p}}$ is the residue field of $\hat{K}_{\mathfrak{p}}$.

Proposition 3.2. *Any separable isogeny $\pi_{\mathfrak{p}} : T_{\mathfrak{p}} \to T'_{\mathfrak{p}}$ of $\hat{K}_{\mathfrak{p}}$-tori can be extended to an isogeny $\underline{\pi} : \underline{T^{\mathrm{un}}}_{\mathfrak{p}} \to \underline{(T')^{\mathrm{un}}}_{\mathfrak{p}}$ over $\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}}$, inducing a surjection $\underline{T^{\mathrm{un}}}^0_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}}) \to \underline{(T')^{\mathrm{un}}}^0_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}})$.*

Proof. Any $\hat{K}_{\mathfrak{p}}^{\mathrm{un}}$-torus $T_{\mathfrak{p}}$ admits a decomposition, i.e., an exact sequence of $\hat{K}_{\mathfrak{p}}^{\mathrm{un}}$-tori

$$1 \to T_{I,\mathfrak{p}} \to T_{\mathfrak{p}} \to T_{a,\mathfrak{p}} \to 1 \tag{2}$$

on which $T_{I,\mathfrak{p}}$ is the maximal subtorus of $T_{\mathfrak{p}}$ splitting over $\hat{K}_{\mathfrak{p}}^{\mathrm{un}}$ and $T_{a,\mathfrak{p}}$ is $I_{\mathfrak{p}}$-anisotropic, i.e., such that $X^*(T_{a,\mathfrak{p}})^{I_{\mathfrak{p}}} = \{0\}$.

We denote by $\underline{T}_{\mathfrak{p}}^{\mathrm{lft}}$ the locally of finite type (lft) NR-model of $T_{\mathfrak{p}}$ defined over $\mathrm{Spec}\,\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}}$ (see 2). Let j_* be the functor taking algebraic $\hat{K}_{\mathfrak{p}}^{\mathrm{un}}$-tori to their lft-Néron models. Since $T_{I,\mathfrak{p}}^{\mathrm{un}}$ is $\hat{K}_{\mathfrak{p}}^{\mathrm{un}}$-split, we have $R^1 j_* = 0$ (cf. the beginning of the proof of III.C.10 in [Mil2]). Thus the exact sequence (2) can be extended to

$$1 \to \underline{T}_{I,\mathfrak{p}}^{\mathrm{lft}} \to \underline{T}_{\mathfrak{p}}^{\mathrm{lft}} \to \underline{T}_{a,\mathfrak{p}}^{\mathrm{lft}} \to 1. \tag{3}$$

According to [LL, Proposition 4.2(b)], the groups of $k_{\mathfrak{p}}^s$-points of the connected components of the reductions of these models fit into the exact sequence

$$1 \to \overline{T_{I,\mathfrak{p}}}^0(k_{\mathfrak{p}}^s) \to \overline{T}_{\mathfrak{p}}^0(k_{\mathfrak{p}}^s) \to \overline{T_{a,\mathfrak{p}}}^0(k_{\mathfrak{p}}^s) \to 1.$$

As $k_\mathfrak{p}^s$ is algebraically closed, this sequence implies the corresponding exact sequence of $k_\mathfrak{p}^s$-schemes

$$1 \to \overline{T_{I,\mathfrak{p}}}^0 \to \overline{T}_\mathfrak{p}^0 \to \overline{T_{a,\mathfrak{p}}}^0 \to 1.$$

Notice that the identity components of the lft NR-models coincide with the ones of the finite type models. Thus the reduction preimages of the latter $k_\mathfrak{p}^s$-schemes, embedded in the $\hat{\mathcal{O}}_\mathfrak{p}^{\mathrm{sh}}$-schemes in sequence (3), yield the exact sequence of the identity components over $\hat{\mathcal{O}}_\mathfrak{p}^{\mathrm{sh}}$

$$1 \to \underline{T_I}_\mathfrak{p}^0 \to \underline{T}_\mathfrak{p}^0 \to \underline{T_a}_\mathfrak{p}^0 \to 1. \tag{4}$$

Now let $\pi_\mathfrak{p} : T_\mathfrak{p} \to T'_\mathfrak{p}$ be an isogeny of $\hat{K}_\mathfrak{p}$-tori. Denote by $T_\mathfrak{p}^{\mathrm{un}}$ and $(T'_\mathfrak{p})^{\mathrm{un}}$ these tori tensored with $\hat{K}_\mathfrak{p}^{\mathrm{un}}$. Then applying the decomposition (4) on both $T_\mathfrak{p}^{\mathrm{un}}$ and $(T'_\mathfrak{p})^{\mathrm{un}}$ results in the exact sequences

$$\begin{gathered} 1 \to \underline{T_I^{\mathrm{un}}}_\mathfrak{p}^0 \to \underline{T^{\mathrm{un}}}_\mathfrak{p}^0 \to \underline{T_a^{\mathrm{un}}}_\mathfrak{p}^0 \to 1, \\ 1 \to \underline{(T'_I)^{\mathrm{un}}}_\mathfrak{p}^0 \to \underline{(T')^{\mathrm{un}}}_\mathfrak{p}^0 \to \underline{(T'_a)^{\mathrm{un}}}_\mathfrak{p}^0 \to 1. \end{gathered} \tag{5}$$

If we show that the left-hand and right-hand groups in the upper sequence surject onto the corresponding groups in the lower one, then the surjection of the middle groups will follow. On the left hand side, $T_{I,\mathfrak{p}}^{\mathrm{un}}$ and $(T'_{I,\mathfrak{p}})^{\mathrm{un}}$ are isogenous and $\hat{K}_\mathfrak{p}^{\mathrm{un}}$-split. Then $\pi_I := \ker[T_I^{\mathrm{un}} \twoheadrightarrow (T'_I)^{\mathrm{un}}]$ is a finite $\hat{K}_\mathfrak{p}^{\mathrm{un}}$-split group of multiplicative type. Thus, the Kummer exact sequence of $\hat{K}_\mathfrak{p}^{\mathrm{un}}$-schemes

$$1 \to \pi_I \to T_{I,\mathfrak{p}}^{\mathrm{un}} \to (T'_{I,\mathfrak{p}})^{\mathrm{un}} \to 1$$

extends to the exact sequence of corresponding schemes over $\hat{\mathcal{O}}_\mathfrak{p}^{\mathrm{sh}}$

$$1 \to \underline{\pi_I} \to \underline{T_I^{\mathrm{un}}}_\mathfrak{p} \to \underline{(T'_I)^{\mathrm{un}}}_\mathfrak{p} \to 1,$$

showing the desired surjection on the left-hand side (notice that both $\underline{T_I^{\mathrm{un}}}_\mathfrak{p}$ and $\underline{(T'_I)^{\mathrm{un}}}_\mathfrak{p}$ split over $\hat{\mathcal{O}}_\mathfrak{p}^{\mathrm{sh}}$ and, thus, are connected, i.e. coincide with their identity component; see Remark 3.1).

Both groups $T_{a,\mathfrak{p}}^{\mathrm{un}}$ and $(T'_{a,\mathfrak{p}})^{\mathrm{un}}$ on the right-hand side of sequences (5) are $I_\mathfrak{p}$-anisotropic. Therefore their NR-models coincide with the finite type (classical) Néron model. In that case, according to [BLR, Section 7.3, Proposition 6], the $\hat{K}_\mathfrak{p}^{\mathrm{un}}$-isogeny $T_{a,\mathfrak{p}}^{\mathrm{un}} \to (T'_{a,\mathfrak{p}})^{\mathrm{un}}$ extends to a $\hat{\mathcal{O}}_\mathfrak{p}^{\mathrm{sh}}$-isogeny $\underline{T_a^{\mathrm{un}}}_\mathfrak{p} \to \underline{(T'_a)^{\mathrm{un}}}_\mathfrak{p}$, such that the surjection holds for the identity components, see Definition 4 of *loc. cit.* Hence we deduce the surjection $\underline{T^{\mathrm{un}}}_\mathfrak{p}^0 \twoheadrightarrow \underline{(T')^{\mathrm{un}}}_\mathfrak{p}^0$.

Further, as the degree of the latter $\hat{\mathcal{O}}_\mathfrak{p}^{\mathrm{sh}}$-isogeny is prime to $\mathrm{char}(\hat{K}_\mathfrak{p})$, its kernel $\underline{F^{\mathrm{un}}}_\mathfrak{p}$ has a smooth reduction as well. Thus the exact sequence of the reduction groups over the algebraically closed residue field $k_\mathfrak{p}^s$

$$1 \to \overline{F^{\mathrm{un}}}_\mathfrak{p}(k_\mathfrak{p}^s) \to \overline{T^{\mathrm{un}}}_\mathfrak{p}^0(k_\mathfrak{p}^s) \to \overline{(T')^{\mathrm{un}}}_\mathfrak{p}^0(k_\mathfrak{p}^s) \to 1$$

implies the exactness of the reduction preimage groups of $\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}}$-points

$$1 \to \underline{F^{\mathrm{un}}}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}}) \to \underline{T^{\mathrm{un}}}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}}) \to \underline{(T')^{\mathrm{un}}}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}}) \to 1. \qquad \blacksquare$$

Corollary 3.3. *The homomorphism of $\hat{\mathcal{O}}_{\mathfrak{p}}$-schemes $\underline{G^{\mathrm{sc}}}_{\mathfrak{p}} \to \underline{G}_{\mathfrak{p}}^{0}$ is surjective.*

Proof. Our assumption $(\mathrm{char}(K), F) = 1$ in Section 1 implies that the isogeny $\pi_{\mathfrak{p}} : T_{\mathfrak{p}}^{\mathrm{sc}} \to T_{\mathfrak{p}}$ is separable at any $\mathfrak{p}$. As $\underline{G}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}}) = \underline{T}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}})\mathfrak{X}(\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}})$, the surjection of groups of $\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}}$-points in Proposition 3.2 can be extended to $\underline{\pi} : \underline{G^{\mathrm{sc}}}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}}) \twoheadrightarrow \underline{G}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}})$. As $G_{\mathfrak{p}}^{\mathrm{sc}}$ is simply connected, $\underline{G^{\mathrm{sc}}}_{\mathfrak{p}}$ has a connected special fiber (see [Tit, § 3.5.2]). By [BT2, Proposition 1.7.6], we know that the coordinate ring representing $\underline{G}_{\mathfrak{p}}$ is

$$\hat{\mathcal{O}}_{\mathfrak{p}}[\underline{G}_{\mathfrak{p}}] = \left\{ f \in \hat{K}_{\mathfrak{p}}[G_{\mathfrak{p}}] : f(\underline{G}_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}}^{\mathrm{sh}})) \subset \mathcal{O}_{\mathfrak{p}}^{\mathrm{sh}} \right\} \subset \hat{K}_{\mathfrak{p}}[G_{\mathfrak{p}}].$$

As $\underline{\pi}(\underline{G^{\mathrm{sc}}}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}})) = \underline{G}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}})$, any function $f \in \hat{\mathcal{O}}_{\mathfrak{p}}[\underline{G}_{\mathfrak{p}}^{0}]$ satisfies

$$f \circ \underline{\pi}(\underline{G^{\mathrm{sc}}}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}})) \subset f(\underline{G}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}})) \subset \hat{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{sh}},$$

thus $f \circ \underline{\pi} \in \hat{\mathcal{O}}_{\mathfrak{p}}[\underline{G^{\mathrm{sc}}}_{\mathfrak{p}}]$ yielding the surjection of the contravariant functor of schemes. $\blacksquare$

Lemma 3.4. $\varpi_{\mathfrak{p}}(\underline{G^{\mathrm{sc}}}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}})) = \varpi_{\mathfrak{p}}(\underline{G}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}}))$.

Proof. Consider the following exact sequences, obtained by the reduction of groups of points

$$1 \to \underline{G^{\mathrm{sc}}}_{\mathfrak{p}}^{1}(\hat{\mathcal{O}}_{\mathfrak{p}}) \to \underline{G^{\mathrm{sc}}}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}}) \xrightarrow{\mathrm{red}} \overline{G^{\mathrm{sc}}}_{\mathfrak{p}}^{0}(k_{\mathfrak{p}}) \to 1,$$
$$1 \longrightarrow \underline{G}_{\mathfrak{p}}^{1}(\hat{\mathcal{O}}_{\mathfrak{p}}) \longrightarrow \underline{G}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}}) \xrightarrow{\mathrm{red}} \overline{G}_{\mathfrak{p}}^{0}(k_{\mathfrak{p}}) \to 1.$$

Since $T_{\mathfrak{p}}$ is maximal and $G_{\mathfrak{p}}$ is quasi-split, by [BT2, Corollary 4.6.7] $\underline{G}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}}) = \underline{T}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}})\mathfrak{X}(\hat{\mathcal{O}}_{\mathfrak{p}})$ where $\mathfrak{X}(\hat{\mathcal{O}}_{\mathfrak{p}})$ is the group generated by the root subgroups each fixing an half apartment containing x. Let $d = \dim(\underline{G}_{\mathfrak{p}})$. The preimage of 1_d in $\underline{G}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}})/\underline{T}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}})$ is homeomorphic to the additive group $\mathfrak{p}^{|\Phi|}$. The preimage of 1_d in $\underline{T}_{\mathfrak{p}}^{0}(\hat{\mathcal{O}}_{\mathfrak{p}})$ is isomorphic to $(1+\mathfrak{p})^{\dim T_{\mathfrak{p}}}$, being homeomorphic to $\mathfrak{p}^{\dim T_{\mathfrak{p}}}$. Together, $\mathrm{Lie}(\underline{G}_{\mathfrak{p}}^{1}(\hat{\mathcal{O}}_{\mathfrak{p}})) \cong \mathfrak{p}^{\dim T_{\mathfrak{p}} + |\Phi|} = \mathfrak{p}^{d}$. The same is true for $\underline{G^{\mathrm{sc}}}_{\mathfrak{p}}^{1}(\hat{\mathcal{O}}_{\mathfrak{p}})$. Thus the two kernels above share the same volume with respect to $\varpi_{\mathfrak{p}}$. Further, as the residue field $k_{\mathfrak{p}}$ is finite and the reductions $\overline{G^{\mathrm{sc}}}_{\mathfrak{p}}^{0} = \overline{G^{\mathrm{sc}}}_{\mathfrak{p}}$ and $\overline{G}_{\mathfrak{p}}^{0}$ are connected and $k_{\mathfrak{p}}$-isogeneous, they also share the same number of rational $k_{\mathfrak{p}}$-points (see [Bor, § 16.8]). Now the claim follows from equality (1). $\blacksquare$

Remark 3.5. The following argument for generalizing the above results for a non quasi-split group can be found in [Gro, Prop. 4.7]. At any prime $\mathfrak{p}$, if $G_{\mathfrak{p}}$ is not quasi-split, then it is an inner form of a quasi-split group $H_{\mathfrak{p}}$. Let $\underline{G^{\mathrm{sc}}}_{\mathfrak{p}}$ and $\underline{G}_{\mathfrak{p}}^{0}$ be the underlying group schemes stabilizing respectively the images of $\underline{H^{\mathrm{sc}}}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}})$ and $\underline{H}^0_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}})$ in the twisted building. These schemes are connected and the surjection $\underline{G^{\mathrm{sc}}}_{\mathfrak{p}} \to \underline{G}^0_{\mathfrak{p}}$ as shown in Corollary 3.3 is preserved. Applying the twisted measure of $\varpi_{\mathfrak{p}}$ on $H_{\mathfrak{p}}$ – being again some scalar multiple of $\varpi_{\mathfrak{p}}$ – results in Lemma 3.4 equality for $H_{\mathfrak{p}}$ and therefore for $G_{\mathfrak{p}}$ as well. And so, a posteriori, the restriction to quasi-split groups throughout this section is redundant for Corollary 3.3 and Lemma 3.4.

4. The Tamagawa number of semisimple groups

We return to the definition of G over the global field K as introduced in Section 1. Let ω be a non-zero left-invariant differential K-form on G of highest degree. It induces a Haar measure on the adelic group $G(\mathbb{A})$ of G, which is unique up to a scalar multiplication. Let $\omega_{\mathfrak{p}}$ be the multiplicative Haar measure induced locally by ω at $\mathfrak{p}$. The *Tamagawa measure* on $G(\mathbb{A})$ is defined as

$$\tau = q^{-(g-1)\dim G} \prod_{\mathfrak{p}} \omega_{\mathfrak{p}}$$

where g is the genus of C.

Due to the product formula, the measure τ does not depend on the choice of ω, i.e., for each $\lambda \in K^\times$ the volume forms ω and $\lambda\omega$ yield identical Haar measures (cf. [Weil, 2.3.1]). Therefore τ is wel- defined. Identifying K with its diagonal embedding in $\mathbb{A}$ and consequently $G(K)$ with its diagonal embedding in $G(\mathbb{A})$, we consider the following arithmetic invariant of G:

Definition 4.1. The *Tamagawa number* $\tau(G)$ of G is the volume of $G(\mathbb{A})/G(K)$ with respect to the Tamagawa measure τ.

Remark 4.2. Since the multiplicative Haar measure on $G_{\mathfrak{p}}(\hat{K}_{\mathfrak{p}})$ at any $\mathfrak{p}$ is unique up to a scalar multiplication, there exists $\lambda_{\mathfrak{p}} \in \hat{K}_{\mathfrak{p}}^\times$ such that $\omega_{\mathfrak{p}} = \lambda_{\mathfrak{p}}\varpi_{\mathfrak{p}}$ (see notation in Section 3) and so Lemma 3.4 and Remark 3.5 remain true for any $\mathfrak{p}$ after replacing $\varpi_{\mathfrak{p}}$ with $\omega_{\mathfrak{p}}$.

Recall that all discrete valuations of K are non-archimedean. For any finite set S of primes of K, we define:

$$\mathbb{A}_S := \left\{(x_{\mathfrak{p}})_{\mathfrak{p}\notin S} : x_{\mathfrak{p}} \in \hat{\mathcal{O}}_{\mathfrak{p}} \text{ for almost all } \mathfrak{p}\right\} \subset \prod_{\mathfrak{p}\notin S} \hat{K}_{\mathfrak{p}}.$$

We also define the ring of S-*adèles* as:

$$\mathbb{A}(S) := \prod_{\mathfrak{p}\in S} \hat{K}_{\mathfrak{p}} \times \prod_{\mathfrak{p}\notin S} \hat{\mathcal{O}}_{\mathfrak{p}}.$$

Note that

$$\mathbb{A} = \bigcup_S \mathbb{A}(S).$$

For any prime $\mathfrak{p}$, let $\underline{G}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p})$ be the maximal compact subgroup of $G_\mathfrak{p}(\hat{K}_\mathfrak{p})$ w.r.t. some special point x as defined in (2). Moreover, we define

$$G_S := \prod_{\mathfrak{p}\in S} G_\mathfrak{p}(\hat{K}_\mathfrak{p}), \qquad G(\mathbb{A}(S)) := G_S \times \prod_{\mathfrak{p}\notin S} \underline{G}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}).$$

Definition 4.3 ([Kne, p. 187], [Pla]). We say that G satisfies the *strong approximation* property w.r.t. a finite set of primes S, if the diagonal embedding $G(K) \hookrightarrow G(\mathbb{A}_S)$ is dense, or, equivalently, if $G_S \cdot G(K)$ is dense in $G(\mathbb{A})$. If $|S| = 1$, we call it the *absolute strong approximation* property.

Theorem 4.4 ([Pra1, Theorem A]). *Let G be a connected, simply connected, semisimple linear algebraic group defined and almost simple over K. If the topological group G_S is non-compact w.r.t. to a finite set of primes S, then $G_S \cdot G(K)$ is dense in $G(\mathbb{A})$.*

Theorem 4.5 ([Tha, Thm. 3.2 3)], [PR, Prop. 8.8] in the number field case). *Let G be a connected reductive K-group such that the simply connected covering of the derived subgroup of G has the strong approximation property w.r.t. a finite set of primes S. Then $G(\mathbb{A}(S))G(K)$ is a normal subgroup of $G(\mathbb{A})$ with finite abelian quotient, the S*-class group *$\mathrm{Cl}_S(G) = G(\mathbb{A})/G(\mathbb{A}(S))G(K)$ of cardinality $h_S(G) = |\mathrm{Cl}_S(G)|$.*

We choose an arbitrary closed point ∞ of C to be the point at infinity, and in accordance to Section 1 define:

$$\mathbb{A}_\infty := \mathbb{A}(\{\infty\}), \qquad G(\mathbb{A}_\infty) := G_\infty(\hat{K}_\infty) \times \prod_{\mathfrak{p}\neq\infty} \underline{G}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}).$$

The following facts are now deduced from the preceding theorems in the case of $S = \{\infty\}$:

Definition 4.6. There exists a finite set $\{x_1, ..., x_h\} \subset G(\mathbb{A})$ such that

$$G(\mathbb{A}) = \bigsqcup_{i=1}^{h} G(\mathbb{A}_\infty) x_i G(K).$$

The finite number $h = h_\infty(G)$ is called the *class number* of G (see [Beh, Satz 7], [BP, Prop. 3.9], also [BW, proof of Theorem 2.1]).

Remark 4.7. Our restriction of G as mentioned in Section 1, of not being anisotropic of type A_n, implies that for any prime $\mathfrak{p}$, $G_\mathfrak{p}$ is $K_\mathfrak{p}$-isotropic (see [BT3, 4.3 and 4.4]). Hence by Theorem 4.4 in the case of $S = \{\infty\}$, G^{sc} admits the absolute strong approximation property implying $h_\infty(G^{\mathrm{sc}}) = 1$.

According to Theorem 4.5 together with Remark 4.7, $G(\mathbb{A}_\infty)G(K)$ is a normal subgroup of $G(\mathbb{A})$ and we may consider the natural epimorphism:

$$\varphi : G(\mathbb{A})/G(K) \twoheadrightarrow G(\mathbb{A})/G(\mathbb{A}_\infty)G(K) : \ \forall x \in G(\mathbb{A}) : xG(K) \mapsto xG(\mathbb{A}_\infty)G(K)$$

for which

$$\begin{aligned}\ker(\varphi) &= \{xG(K) : x \in G(\mathbb{A}_\infty)G(K)\} = G(\mathbb{A}_\infty)G(K)/G(K)\\ &\cong G(\mathbb{A}_\infty)/G(\mathbb{A}_\infty) \cap G(K).\end{aligned}$$

Since all fibers of φ are isomorphic to $\ker(\varphi)$, we get a bijection of measure spaces

$$\begin{aligned}G(\mathbb{A})/G(K) &\cong \mathrm{im}(\varphi) \times \ker(\varphi) \\ &= (G(\mathbb{A})/G(\mathbb{A}_\infty)G(K)) \times (G(\mathbb{A}_\infty)/G(\mathbb{A}_\infty) \cap G(K)) \\ &\overset{4.5}{\cong} \mathrm{Cl}_\infty(G) \times (G(\mathbb{A}_\infty)/\Gamma)\end{aligned} \tag{6}$$

on which the first factor cardinality is $h_\infty(G)$ and $\Gamma := G(\mathbb{A}_\infty) \cap G(K)$. We will next study the volume of the second factor.

5. On the cohomology of $\mathcal{O}_{\{\infty\}}$-schemes and relative local covolumes

The discrete group $\Gamma = G(K) \cap G(\mathbb{A}_\infty)$ consists only of elements over the ring of $\{\infty\}$-integers of K, namely:

$$\mathcal{O}_{\{\infty\}} = \{a \in K \mid v_\mathfrak{p}(a) \geqslant 0 \ \ \forall \mathfrak{p} \neq \infty\} = \bigcap_{\mathfrak{p} \neq \infty} \mathcal{O}_\mathfrak{p}.$$

So it would be natural to describe it using an $\mathcal{O}_{\{\infty\}}$-scheme. Consider the following construction: For any $\mathfrak{p}$ let $\widetilde{G}_\mathfrak{p}$ be the Bruhat-Tits model of $G_\mathfrak{p}$ defined over $\mathcal{O}_\mathfrak{p}$, i.e., such that:

1. $\widetilde{G}_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} \hat{K}_\mathfrak{p} = G_\mathfrak{p}$, and
2. $\widetilde{G}_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} \hat{\mathcal{O}}_\mathfrak{p} = \underline{G}_\mathfrak{p}$.

According to Proposition D.4(a) in [BLR, § 6.2] the patch $(G_\mathfrak{p}, \underline{G}_\mathfrak{p}, \tau)$, where τ is the canonical isomorphism $G_\mathfrak{p} \otimes_{K_\mathfrak{p}} \hat{K}_\mathfrak{p} \cong \underline{G}_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} \hat{K}_\mathfrak{p}$, corresponds uniquely to the $\mathcal{O}_\mathfrak{p}$-module $\widetilde{G}_\mathfrak{p}$, in the sense that it covers it with a canonical descent datum. Now since C is one dimensional, for any two distinct primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$, the product $\mathcal{O}_{\mathfrak{p}_1} \otimes \mathcal{O}_{\mathfrak{p}_2}$ is isomorphic to K. Thus we may Zariski-glue all geometric fibers $\{\mathrm{Spec}\,\mathcal{O}_\mathfrak{p} : \mathfrak{p} \neq \infty\}$ along the generic point $\mathrm{Spec}\,K$, resulting in $\mathrm{Spec}\,\mathcal{O}_{\{\infty\}}$. Then the aforementioned patches cover (with descent datum) a unique group scheme $\mathcal{G}$ over $\mathrm{Spec}\,\mathcal{O}_{\{\infty\}}$. Moreover, for any $\mathfrak{p}$, the localization $(\mathcal{O}_{\{\infty\}})_\mathfrak{p}$ is a base change of $\mathcal{O}_\mathfrak{p}$. Thus the bijection $\mathrm{Spec}\,(\mathcal{O}_{\{\infty\}})_\mathfrak{p} \to \mathrm{Spec}\,\mathcal{O}_\mathfrak{p}$ is faithfully flat (see [Liu, Thm. 3.16]). Hence $\mathcal{G}$ extended to $\mathrm{Spec}\,\hat{\mathcal{O}}_\mathfrak{p}$ is smooth by construction so that $\mathcal{G}$ is smooth at $\mathfrak{p}$ by faithfully flat descent, see [EGAIV, 17.7.3]. Its generic fiber is G and it satisfies:

$$\mathcal{G}(\mathcal{O}_{\{\infty\}}) = G(K) \cap \prod_{\mathfrak{p} \neq \infty} \underline{G}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}) = G(K) \cap G(\mathbb{A}_\infty).$$

We denote by $\mathcal{G}^0$ the subscheme of $\mathcal{G}$ whose geometric fibers are $\underline{G}_\mathfrak{p}^0$. The same construction for G^{sc} is denoted by $\mathcal{G}^{\mathrm{sc}}$. The surjectivity at the geometric fibers

$\underline{G^{\mathrm{sc}}}_{\mathfrak{p}} \to \underline{G}^0_{\mathfrak{p}}$ (see Lemma 3.3 and Remark 3.5) leads to the surjection $\pi_{\mathcal{O}_{\{\infty\}}} : \mathcal{G}^{\mathrm{sc}} \to \mathcal{G}^0$ over $\operatorname{Spec} \mathcal{O}_{\{\infty\}}$ as étale sheaves. Since the morphism of the geometric fibers $\underline{G^{\mathrm{sc}}}_{\mathfrak{p}} \to \underline{G^0}_{\mathfrak{p}}$ is smooth for any $\mathfrak{p}$ (due to our assumption that $\mathrm{char}(K)$ is prime to $|F|$), according to Proposition 17.8.2. in [EGAIV] the global morphism $\pi_{\mathcal{O}_{\{\infty\}}}$ is also smooth, as well as its kernel denoted by $\mathcal{F}$. We have an exact sequence of $\mathcal{O}_{\{\infty\}}$-models:

$$1 \to \mathcal{F} \to \mathcal{G}^{\mathrm{sc}} \to \mathcal{G}^0 \to 1. \tag{7}$$

Let $\mathcal{T}$ be the subscheme of $\mathcal{G}$ whose generic fiber is T, let $\mathcal{T}^0 := \mathcal{T} \cap \mathcal{G}^0$ and let $\mathcal{T}^{\mathrm{sc}}$ be its preimage under $\pi_{\mathcal{O}_{\{\infty\}}}$ in $\mathcal{G}^{\mathrm{sc}}$. Being central (as all its geometric fibers), $\mathcal{F}$ is equal to the kernel of the corresponding $\mathcal{O}_{\{\infty\}}$-tori-models, fitting into the following exact sequence of $\mathcal{O}_{\{\infty\}}$-schemes

$$1 \to \mathcal{F} \to \mathcal{T}^{\mathrm{sc}} \to \mathcal{T}^0 \to 1. \tag{8}$$

Lemma 5.1. $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{G}^{\mathrm{sc}}) = 1$.

Proof. According to Nisnevitch ([Nis, 3.6.2]), we have the following exact sequence

$$1 \to \mathrm{Cl}_\infty(G^{\mathrm{sc}}) \to H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{G}^{\mathrm{sc}}) \to H^1(K, G^{\mathrm{sc}}(K^s))$$

on which in our case $\mathrm{Cl}_\infty(G^{\mathrm{sc}})$ is trivial (see Remark 4.7), and the latter group in the sequence is trivial as well due to Harder's result (see [Har2, Satz A]). The claim follows. ∎

Lemma 5.2. *Let $\pi_{\mathcal{O}_{\{\infty\}}} : \mathcal{G}^{\mathrm{sc}}(\mathcal{O}_{\{\infty\}}) \to \mathcal{G}^0(\mathcal{O}_{\{\infty\}})$. Then:*

$$j_\infty(G) := \frac{|\operatorname{coker}(\pi_{\mathcal{O}_{\{\infty\}}})|}{|\ker(\pi_{\mathcal{O}_{\{\infty\}}})|} = \frac{|H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{F})|}{|\mathcal{F}(\mathcal{O}_{\{\infty\}})|}.$$

If in particular G is quasi-split and $\overline{\mathcal{O}}^G_{\{\infty\}}$ is a UFD, then $j_\infty(G) = 1$ which means that the discrete groups Γ^{sc} and Γ^0 are bijective.

Proof. Since $\mathcal{F}$ is smooth we have: $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{F}) = H^1_{\mathrm{fppf}}(\mathcal{O}_{\{\infty\}}, \mathcal{F})$. Due to Lemma 5.1, flat cohomology applied on sequence (7) gives rise to the following sequence of groups of $\mathcal{O}_{\{\infty\}}$-points:

$$1 \to \mathcal{F}(\mathcal{O}_{\{\infty\}}) \to \mathcal{G}^{\mathrm{sc}}(\mathcal{O}_{\{\infty\}}) \xrightarrow{\pi_{\mathcal{O}_{\{\infty\}}}} \mathcal{G}^0(\mathcal{O}_{\{\infty\}}) \to H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{F}) \to 1. \tag{9}$$

This gives us the first assertion.

If G is quasi-split, then T^{sc} is a Weil-torus (see Remark 2.1). Consequently, its $\mathcal{O}_{\{\infty\}}$-model $\mathcal{T}^{\mathrm{sc}}$ is isomorphic to $R_{\mathcal{O}'_{\{\infty\}}, \mathcal{O}_{\{\infty\}}}(\mathbb{G}_m^d)$ where $\overline{\mathcal{O}}^G_{\{\infty\}}$ stands for the integral closure of $\mathcal{O}_{\{\infty\}}$ in the splitting field of T. By Shapiro's formula for the flat topology, we have:

$$\begin{aligned} H^1_{\mathrm{fppf}}(\mathcal{O}_{\{\infty\}}, \mathcal{T}^{\mathrm{sc}}) &\cong H^1_{\mathrm{fppf}}(\mathcal{O}_{\{\infty\}}, R_{\overline{\mathcal{O}}^G_{\{\infty\}}/\mathcal{O}_{\{\infty\}}}(\mathbb{G}_m^d)) \cong H^1_{\mathrm{fppf}}(\overline{\mathcal{O}}^G_{\{\infty\}}, \mathbb{G}_m^d) \\ &= \bigoplus \operatorname{Pic}\,(\overline{\mathcal{O}}^G_{\{\infty\}}). \end{aligned}$$

Hence given that $\overline{\mathcal{O}}_{\{\infty\}}^G$ is a UFD, being a Dedekind domain Pic $(\overline{\mathcal{O}}_{\{\infty\}}^G) = 0$, and so flat cohomology applied on sequence (8) gives rise to the following sequence of multiplicative groups of $\mathcal{O}_{\{\infty\}}$-points

$$1 \to \mathcal{F}(\mathcal{O}_{\{\infty\}}) \to \mathcal{T}^{\mathrm{sc}}(\mathcal{O}_{\{\infty\}}) \xrightarrow{\pi} \mathcal{T}^0(\mathcal{O}_{\{\infty\}}) \to H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{F}) \to 1. \tag{10}$$

Recall that $\mathcal{O}_{\{\infty\}} = K \cap \bigcap_{\mathfrak{p}\neq\infty} \mathcal{O}_{\mathfrak{p}}$, i.e., $\mathcal{O}_{\{\infty\}}$ consists of exactly those elements of K that do not have poles at any place $\mathfrak{p} \neq \infty$. If $x \in \mathcal{O}_{\{\infty\}}$ has a proper pole at ∞, then it has a proper zero at some place $\mathfrak{p} \neq \infty$. Hence its inverse $x^{-1} \in K$ has a proper pole at that place and, thus, $x^{-1} \in K \backslash \mathcal{O}_{\{\infty\}}$. We conclude that the only invertible elements of $\mathcal{O}_{\{\infty\}}$ are the constants. In other words, since the curve C is projective, its regular functions are exactly the constants. This means that $\mathcal{T}^{\mathrm{sc}}(\mathcal{O}_{\{\infty\}}) = \mathcal{T}^{\mathrm{sc}}(\mathbb{F}_q)$ and $\mathcal{T}(\mathcal{O}_{\{\infty\}}) = \mathcal{T}(\mathbb{F}_q)$ are finite groups.

As the reduction of all geometric fibers of $\mathcal{T}^{\mathrm{sc}}$ and $\mathcal{T}^0$ are smooth and connected, the specializations $\overline{\mathcal{T}^{\mathrm{sc}}} = \mathcal{T}^{\mathrm{sc}} \otimes_{\operatorname{Spec} \mathcal{O}_{\{\infty\}}} \operatorname{Spec} \mathbb{F}_q$ and $\overline{\mathcal{T}}^0 = \mathcal{T}^0 \otimes_{\operatorname{Spec} \mathcal{O}_{\{\infty\}}} \operatorname{Spec} \mathbb{F}_q$ are connected $\mathbb{F}_q$-schemes, where $\operatorname{Spec} \mathbb{F}_q \to \operatorname{Spec} \mathcal{O}_{\{\infty\}}$ is the closed immersion of the special point. Thus the exact sequence (10) can be rewritten as:

$$1 \to \mathcal{F}(\mathcal{O}_{\{\infty\}}) \to \overline{\mathcal{T}^{\mathrm{sc}}}(\mathbb{F}_q) \xrightarrow{\pi} \overline{\mathcal{T}}^0(\mathbb{F}_q) \to H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{F}) \to 1. \tag{11}$$

The surjectivity of $\mathcal{T}^{\mathrm{sc}} \twoheadrightarrow \mathcal{T}^0$ implies the one of $\overline{\mathcal{T}^{\mathrm{sc}}} \twoheadrightarrow \overline{\mathcal{T}}^0$. These schemes are isogenous, connected and defined over $\mathbb{F}_q$, so they share the same number of $\mathbb{F}_q$-points. Then the exactness of (11) implies that $|\mathcal{F}(\mathcal{O}_{\{\infty\}})| = |H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{F})|$. Returning back to the exact sequence (9), we get the claim. ■

The group $G(\mathbb{A})$ admits a natural action on the product $\mathcal{B} = \prod_{\mathfrak{p}} \mathcal{B}_{\mathfrak{p}}$ of the Bruhat–Tits buildings, and its subgroup $G(\mathbb{A}_\infty)$ fixes the fundamental special vertex of each building $\mathcal{B}_{\mathfrak{p}}$ with $\mathfrak{p} \neq \infty$. Identifying $\mathcal{B}_\infty$ with its product with these fundamental special vertices therefore yields an action of $G(\mathbb{A}_\infty)$ on $\mathcal{B}_\infty$. Let:

$$G^0(\mathbb{A}_\infty) = G_\infty(\hat{K}_\infty) \times \prod_{\mathfrak{p}\neq\infty} \underline{G}^0_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}),\ \Gamma^0 = G^0(\mathbb{A}_\infty) \cap G(K) \subset \Gamma.$$

Notice that as G^{sc} is simply connected $\Gamma^{\mathrm{sc}} := G^{\mathrm{sc}}(\mathbb{A}_\infty) \cap G(K) = (\Gamma^{\mathrm{sc}})^0$. Consider the following compact subgroups:

$$U^{\mathrm{sc}} = \prod_{\mathfrak{p}} \underline{G}^{\mathrm{sc}}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}) \subset G^{\mathrm{sc}}(\mathbb{A}_\infty), \qquad U = \prod_{\mathfrak{p}} \underline{G}^0_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}) \subset G(\mathbb{A}_\infty).$$

Let Y^{sc} and Y be the sets of representatives respectively for the double cosets sets:

$$\Gamma^{\mathrm{sc}} \backslash G^{\mathrm{sc}}(\mathbb{A}_\infty)/U^{\mathrm{sc}} \cong (\Gamma^{\mathrm{sc}} \cap G^{\mathrm{sc}}(\hat{K}_\infty)) \backslash G^{\mathrm{sc}}_\infty(\hat{K}_\infty)/\underline{G}^{\mathrm{sc}}_\infty(\hat{\mathcal{O}}_\infty), \tag{12}$$
$$\Gamma^0 \backslash G(\mathbb{A}_\infty)/U \cong (\Gamma^0 \cap G(\hat{K}_\infty)) \backslash G_\infty(\hat{K}_\infty)/\underline{G}^0_\infty(\hat{\mathcal{O}}_\infty).$$

For any $y \in Y, yUy^{-1}$ is compact and Γ^0 is discrete thus their intersection is finite. More precisely, by the isomorphism above any such y may represents a non-trivial double coset only by its ∞-component, whence $yUy^{-1} \subset G(\mathbb{A}_\infty)$ and therefore:

$$yUy^{-1} \cap \Gamma^0 = yUy^{-1} \cap (G(K) \cap G(\mathbb{A}_\infty)) = yUy^{-1} \cap G(K).$$

But conjugation by y on the ∞-component of U is a shift to the stabilizer of yx in $G_\infty(\hat{K}_\infty)$:

$$yUy^{-1} = \underline{G}_{\infty,yx}(\hat{\mathcal{O}}_\infty) \times \prod_{\mathfrak{p}\neq\infty} \underline{G}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}).$$

Thus $yUy^{-1} \cap G(K)$ admits an underlying group scheme $\widehat{G}_{\infty,y}$ having only global sections on K, i.e., defined over $\operatorname{Spec} \mathbb{F}_q$ (recall that C is projective). We denote by $\widehat{G}^{\mathrm{sc}}_{\infty,y'}$ the resulting $\mathbb{F}_q$-group for the same construction for G^{sc} with $y' \in Y^{\mathrm{sc}}$ s.t. $\pi(y') = y$. The surjectivity of $\underline{G}^{\mathrm{sc}}_\mathfrak{p} \twoheadrightarrow \underline{G}^0_\mathfrak{p}$ for all $\mathfrak{p}$ and of $G^{\mathrm{sc}} \twoheadrightarrow G$ imply the one of $\widehat{G}^{\mathrm{sc}}_{\infty,y'} \twoheadrightarrow \widehat{G}_{\infty,y}$ having a finite kernel as well. So the groups $\widehat{G}^{\mathrm{sc}}_{\infty,y}$ and $\widehat{G}_{\infty,y}$, being isogeneous, connected and of finite dimension, defined over the finite field $\mathbb{F}_q$, share the same finite number of $\mathbb{F}_q$-points, i.e.:

$$\forall y' \in Y^{\mathrm{sc}} : |y'U^{\mathrm{sc}}y'^{-1} \cap \Gamma^{\mathrm{sc}}| = |\pi(y')U\pi(y')^{-1} \cap \Gamma^0|. \tag{13}$$

As $G(\mathbb{A}_\infty)$ is unimodular by [Mar, Corollary I.2.3.3], we get to Serre's formula ([Ser1, p. 84],[BL, Corollary 1.6]) :

$$\begin{aligned}
\tau(G) &\overset{(6)}{=} h_\infty(G) \cdot \tau(G(K)\backslash G(\mathbb{A}_\infty)) = h_\infty(G) \cdot \sum_{y \in Y} \tau(yU) \qquad (14)\\
&= h_\infty(G) \cdot \sum_{y\in Y} \frac{\tau(U)}{|yUy^{-1} \cap \Gamma^0|}\\
&= q^{-(g-1)\dim(G)} \cdot h_\infty(G) \cdot \prod_\mathfrak{p} \omega_\mathfrak{p}(\underline{G}^0_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p})) \cdot \sum_{y\in Y} \frac{1}{|yUy^{-1} \cap \Gamma^0|}\\
&= q^{-(g-1)\dim(G)} \cdot h_\infty(G) \cdot i_\infty(G) \cdot \prod_\mathfrak{p} \omega_\mathfrak{p}(\underline{G}^0_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}))
\end{aligned}$$

where:

$$i_\infty(G) = \sum_{y\in Y} \frac{1}{|yUy^{-1} \cap \Gamma^0|}. \tag{15}$$

Lemma 5.3. *With the previously introduced notations, one has*

$$\frac{i_\infty(G)}{i_\infty(G^{\mathrm{sc}})} = \frac{t_\infty(G)}{j_\infty(G)}.$$

Proof. Let us regard the double cosets groups in formulas 12. The representatives of $G^{\mathrm{sc}}(\mathbb{A}_\infty)/U^{\mathrm{sc}} \cong G^{\mathrm{sc}}_\infty(\hat{K}_\infty)/\underline{G}^{\mathrm{sc}}_\infty(\hat{\mathcal{O}}_\infty)$ and of $G(\mathbb{A}_\infty)/U \cong G_\infty(\hat{K}_\infty)/\underline{G}^0_\infty(\hat{\mathcal{O}}_\infty)$ correspond to vertices in the orbits of x in $\mathcal{B}_\infty$ obtained by the actions of $G^{\mathrm{sc}}_\infty(\hat{K}_\infty)$

and $G(\hat{K}_\infty)$, respectively (the Iwahori subgroup is the kernel of this action in each case). These actions are transitive on the alcoves in $\mathcal{B}_\infty$, thus it is sufficient to compare between the orbits inside one alcove, in which there are $t_\infty(G)$ (special) points in the orbit of x, while simply-connected groups are type preserving, thus $t_\infty(G^{\mathrm{sc}}) = 1$ (see [Tit, § 2.5]). So the orbit of x under the $G(\hat{K}_\infty)$-action is bijective to $t_\infty(G)$-times the orbit under the $G^{\mathrm{sc}}_\infty(\hat{K}_\infty)$-one.

To accomplish the comparison between Y^{sc} and Y, the above right quotients, taken modulo the discrete subgroups Γ^{sc} and Γ^0 (from the left), respectively, correspond to vertices in some fundamental domains of the aforementioned orbits of x. In Lemma 5.2, we compared between these subgroups and got that Γ^0 is bijective to $j_\infty(G)$-times Γ^{sc}. Moreover, along any orbit the Bruhat-Tits schemes are isomorphic (see [Tit, § 2.5.,p. 47]) and have isomorphic reductions. Thus the cardinality $|yUy^{-1} \cap \Gamma^0|$ is the same for all $y \in Y$. We get:

$$\begin{aligned} i_\infty(G) &\overset{(15)}{=} \sum_{y \in Y} \frac{1}{|yUy^{-1} \cap \Gamma^0|} \\ &\overset{(13)}{=} \frac{t_\infty(G)}{j_\infty(G)} \cdot \sum_{y' \in Y^{\mathrm{sc}}} \frac{1}{|y'U^{\mathrm{sc}}y'^{-1} \cap \Gamma^{\mathrm{sc}}|} \\ &= \frac{t_\infty(G)}{j_\infty(G)} \cdot i_\infty(G^{\mathrm{sc}}). \end{aligned}$$
■

Recall that

$$\tau(G) \overset{(14)}{=} q^{-(g-1)\dim G} \cdot h_\infty(G) \cdot i_\infty(G) \cdot \prod_{\mathfrak{p}} \omega_{\mathfrak{p}}(\underline{G}^0_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}})).$$

Clearly the invariant $q^{-(g-1)\dim G}$ is the same for both G and G^{sc}, as well as the volume of the compact subgroups U^{sc} and U (see Lemma 3.4 and Remark 4.2). We conclude that:

$$\frac{\tau(G)}{\tau(G^{\mathrm{sc}})} = \frac{h_\infty(G)}{h_\infty(G^{\mathrm{sc}})} \cdot \frac{i_\infty(G)}{i_\infty(G^{\mathrm{sc}})}.$$

Now assuming the validity of the Weil Conjecture: $\tau(G^{\mathrm{sc}}) = 1$ and due to the strong approximation related to G^{sc} for which $h_\infty(G^{\mathrm{sc}}) = 1$ (see Remark 4.7), plus Lemma 5.3 we finally deduce:

Corollary 5.4.

$$\tau(G) = h_\infty(G) \cdot \frac{t_\infty(G)}{j_\infty(G)}.$$

If G is quasi-split and $\overline{\mathcal{O}}^G_{\{\infty\}}$ is a UFD, according to Lemma 5.2 this formula simplifies to:

$$\tau(G) = h_\infty(G) \cdot t_\infty(G).$$

Remark 5.5. From a geometric point of view, the key issue of computing the Tamagawa number $\tau(G)$ therefore is to describe a fundamental domain for the action of $\mathcal{G}(\mathcal{O}_{\{\infty\}})$ on the $G_\infty(\hat{K}_\infty)$-orbit of the fundamental special vertex x of the Bruhat–Tits building $\mathcal{B}_\infty$. This is not surprising at all via the following consideration: If G is simply connected, by the absolute strong approximation property (see Remark 4.7 above) the group $G(K)$ is dense in $\prod_{\mathfrak{p}\neq\infty} G_{\mathfrak{p}}(K_{\mathfrak{p}})$ so that it has the same orbits on the product of buildings $\mathcal{B} = \prod_{\mathfrak{p}\neq\infty} \mathcal{B}_{\mathfrak{p}}$ as $\prod_{\mathfrak{p}\neq\infty} G_{\mathfrak{p}}(K_{\mathfrak{p}})$. Hence any element of $\mathcal{B}$ in the $G(\mathbb{A})$-orbit of the tuple consisting of all the fundamental special vertices of the Bruhat–Tits buildings $\mathcal{B}_{\mathfrak{p}}$ in fact lies is the same $G(K)$-orbit with a tuple consisting of a special vertex of $\mathcal{B}_\infty$ and all the fundamental special vertices of the Bruhat–Tits buildings $\mathcal{B}_{\mathfrak{p}}$ for $\mathfrak{p} \neq \infty$.

The situation of a general G can then be analyzed by studying how a $G(K)$-orbit changes under an isogeny.

6. Number of types in the orbit of a special point

We retain the notation and terminology introduced in the preceding sections.

Lemma 6.1. *For any prime $\mathfrak{p}$, one has $H^1(\langle\sigma_{\mathfrak{p}}\rangle, \pi(G^{\mathrm{sc}}_{\mathfrak{p}}(\hat{K}^{\mathrm{un}}_{\mathfrak{p}}))) = 1$.*

Proof. At any prime $\mathfrak{p}$, we may consider the following exact sequence of $\hat{K}_{\mathfrak{p}}$-groups:

$$1 \to F_{\mathfrak{p}} \to G^{\mathrm{sc}}_{\mathfrak{p}} \to \pi(G^{\mathrm{sc}}) \to 1.$$

Due to Harder [Har2, Satz A], we know that $H^1(\langle\sigma_{\mathfrak{p}}\rangle, G^{\mathrm{sc}}_{\mathfrak{p}}(\hat{K}^{\mathrm{un}}_{\mathfrak{p}})) = 1$, hence $\langle\sigma_{\mathfrak{p}}\rangle$-cohomology gives rise to the exact sequence:

$$1 \to H^1(\langle\sigma_{\mathfrak{p}}\rangle, \pi(G^{\mathrm{sc}}_{\mathfrak{p}}(\hat{K}^{\mathrm{un}}_{\mathfrak{p}})) \to H^2(\langle\sigma_{\mathfrak{p}}\rangle, F_{\mathfrak{p}}(\hat{K}^{\mathrm{un}}_{\mathfrak{p}}))$$

on which the right term is trivial as $F_{\mathfrak{p}}(\hat{K}^{\mathrm{un}}_{\mathfrak{p}})$ is finite. This gives the required result. ■

Lemma 6.2. *The number $t_\infty(G)$ of (special) types in the $\underline{G}_\infty(\hat{K}_\infty)$-orbit of the fundamental special vertex x in $\mathcal{B}_\infty$ is given by*

$$t_\infty(G) = |H^1(I_\infty, F_\infty(\hat{K}^s_\infty))^{\sigma_\infty}| = |\widehat{F_\infty}^{\mathfrak{g}_\infty}|.$$

Proof. Galois I_∞ and $\mathfrak{g}_\infty$-cohomology yield the exact diagram

$$\begin{tikzcd}
1 \arrow[r] & F_\infty(\hat{K}^{\mathrm{un}}_\infty) \arrow[r] & G^{\mathrm{sc}}_\infty(\hat{K}^{\mathrm{un}}_\infty) \arrow[r, "\pi"] & G_\infty(\hat{K}^{\mathrm{un}}_\infty) \arrow[r] & H^1(I_\infty, F_\infty(\hat{K}^s_\infty)) \arrow[r] & 1 \\
1 \arrow[r] & F_\infty(\hat{K}_\infty) \arrow[r] & G^{\mathrm{sc}}_\infty(\hat{K}_\infty) \arrow[r, "\pi"] \arrow[u, hook] & G_\infty(\hat{K}_\infty) \arrow[r] \arrow[u, hook] & H^1(\mathfrak{g}_\infty, F_\infty(\hat{K}^s_\infty)) \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

The group $\pi_\infty(G^{\mathrm{sc}}_\infty(\hat{K}^{\mathrm{un}}_\infty)) \cap G_\infty(\hat{K}_\infty)$ is the largest type-preserving subgroup of $G_\infty(\hat{K}_\infty)$. By the classification of affine Dynkin diagrams an automorphism of

$\mathcal{B}_\infty$ preserves the types of special vertices in $\mathcal{B}_\infty$ if and only if it preserves types of arbitrary vertices. Therefore the cosets of $\pi_\infty(G^{\mathrm{sc}}_\infty(\hat{K}^{\mathrm{un}}_\infty)) \cap G_\infty(\hat{K}_\infty)$ in $G_\infty(\hat{K}_\infty)$ are in 1-to-1 correspondence with the types of special vertices in the $\underline{G}_\infty(\hat{K}_\infty)$-orbit. We conclude that

$$t_\infty(G) = \left| G_\infty(\hat{K}_\infty) / \left(\pi(G^{\mathrm{sc}}_\infty(\hat{K}^{\mathrm{un}}_\infty)) \cap G_\infty(\hat{K}_\infty) \right) \right|.$$

The exact sequence

$$1 \to F_\infty(\hat{K}^{\mathrm{un}}_\infty) \to G^{\mathrm{sc}}_\infty(\hat{K}^{\mathrm{un}}_\infty) \xrightarrow{\pi} G_\infty(\hat{K}^{\mathrm{un}}_\infty) \to H^1(I_\infty, F_\infty(\hat{K}^s_\infty)) \to 1$$

can be shortened to

$$1 \to \pi(G^{\mathrm{sc}}_\infty(\hat{K}^{\mathrm{un}}_\infty)) \to G_\infty(\hat{K}^{\mathrm{un}}_\infty) \to H^1(I_\infty, F_\infty(\hat{K}^s_\infty)) \to 1.$$

Applying $\langle \sigma_\infty \rangle$-cohomology on this exact sequence gives the exact sequence

$$\begin{aligned} 1 \to \pi(G^{\mathrm{sc}}_\infty(\hat{K}^{\mathrm{un}}_\infty)) \cap G_\infty(\hat{K}_\infty) \to G_\infty(\hat{K}_\infty) \\ \to H^1(I_\infty, F_\infty(\hat{K}^s_\infty))^{\sigma_\infty} \to H^1(\langle \sigma_\infty \rangle, \pi(G^{\mathrm{sc}}_\infty(\hat{K}^{\mathrm{un}}_\infty))) \end{aligned}$$

on which the right-hand group is trivial by Lemma 6.1. Hence $t_\infty(G) = |H^1(I_\infty, F_\infty(\hat{K}^s_\infty))^{\sigma_\infty}|$.

More explicitly, the Kottwitz epimorphism together with Galois descent, yields an epimorphism $T_\infty(\hat{K}_\infty) \to X_*(T_\infty)^{\sigma_\infty}_{I_\infty}$ whose kernel is the Iwahori subgroup $\underline{T}^0_\infty(\hat{\mathcal{O}}_\infty)$ (see [Bit, Corollary 3.2]). We get the following exact diagram

$$\begin{tikzcd}
1 \arrow[r] & \underline{F}_\infty(\hat{\mathcal{O}}_\infty) \arrow[r] \arrow[d, equal] & \underline{T^{\mathrm{sc}}}_\infty(\hat{\mathcal{O}}_\infty) \arrow[r, "\pi_\infty"] \arrow[d] & \underline{T}^0_\infty(\hat{\mathcal{O}}_\infty) \arrow[r] \arrow[d] & H^1(\langle \sigma_\infty \rangle, \underline{F}_\infty(\hat{\mathcal{O}}^{\mathrm{sh}}_\infty)) \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & F_\infty(\hat{K}_\infty) \arrow[r] \arrow[d] & T^{\mathrm{sc}}_\infty(\hat{K}_\infty) \arrow[r, "\pi_\infty"] \arrow[d] & T_\infty(\hat{K}_\infty) \arrow[r] \arrow[d] & H^1(\mathfrak{g}_\infty, F_\infty(\hat{K}^s_\infty)) \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & X_*(T^{\mathrm{sc}}_\infty)^{\sigma_\infty}_{I_\infty} \arrow[r, "\pi^\vee_{I_\infty}"] & X_*(T_\infty)^{\sigma_\infty}_{I_\infty} \arrow[r] & H^1(I_\infty, F_\infty(\hat{K}^s_\infty))^{\sigma_\infty} \arrow[r] & 0
\end{tikzcd}$$

on which the lower row can be also obtained by the following steps: applying the contravariant left-exact functor $\mathrm{Hom}(-, \mathbb{Z})$ on the exact sequence of character $\mathfrak{g}_\infty$-modules

$$0 \to X^*(T_\infty) \to X^*(T^{\mathrm{sc}}_\infty) \to \widehat{F_\infty} \to 0,$$

on which $\widehat{F_\infty} = \mathrm{Hom}(F_\infty \otimes \hat{K}^s_\infty, \mathbb{G}_{m,\hat{K}^s_\infty})$, gives the exact sequence

$$\begin{aligned} 0 \to 0 = \mathrm{Hom}(\widehat{F_\infty}, \mathbb{Z}) \to X_*(T^{\mathrm{sc}}_\infty) \xrightarrow{\pi^\vee} X_*(T_\infty) \\ \to \mathrm{Ext}^1(\widehat{F_\infty}, \mathbb{Z}) \to \mathrm{Ext}^1(X^*(T^{\mathrm{sc}}_\infty), \mathbb{Z}) = 0. \quad (16) \end{aligned}$$

Applying the functor $\mathrm{Hom}(\widehat{F_\infty}, -)$ on the resolution

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

gives rise to a long exact sequence on which as $\widehat{F_\infty}$ is finite, $\mathrm{Hom}(\widehat{F_\infty}, \mathbb{Q}) = 0$ and $\mathrm{Ext}^1(\widehat{F_\infty}, \mathbb{Q}) = 0$, showing the existence of an isomorphism

$$\mathrm{Ext}^1(\widehat{F_\infty}, \mathbb{Z}) \cong \mathrm{Hom}(\widehat{F_\infty}, \mathbb{Q}/\mathbb{Z}) = \widehat{F_\infty}^*$$

where $\widehat{F_\infty}^*$ is the Pontryagin dual of $\widehat{F_\infty}$, i.e., the group of finite order characters of $\widehat{F_\infty}$, see also [Mil2, p. 23]. Being finite, these duals are isomorphic. So sequence (16) can be rewritten as

$$0 \to X_*(T^{\mathrm{sc}}_\infty) \xrightarrow{\pi^\vee} X_*(T_\infty) \to \widehat{F_\infty}^* \to 0. \tag{17}$$

The I_∞-coinvariants functor is in general only right exact, but here as $\underline{T^{\mathrm{sc}}}_\infty$ is connected, $X_*(T^{\mathrm{sc}}_\infty)_{I_\infty}$ is free (see [Bit, Formula (3.1)]) and embedded into $X_*(T_\infty)_{I_\infty}$. Thus applying this functor on

$$0 \to X_*(T^{\mathrm{sc}}_\infty) \xrightarrow{\pi^\vee} X_*(T_\infty) \to \widehat{F_\infty}^* \cong \widehat{F_\infty} \to 0$$

also leaves the left hand side exact

$$0 \to X_*(T^{\mathrm{sc}}_\infty)_{I_\infty} \xrightarrow{\pi^\vee} X_*(T_\infty)_{I_\infty} \to \widehat{F_\infty}^*_{I_\infty} \to 0.$$

Now applying the Galois $\langle\sigma_\infty\rangle$-cohomology gives the exact lower row on the above diagram

$$0 \to X_*(T^{\mathrm{sc}}_\infty)^{\sigma_\infty}_{I_\infty} \xrightarrow{\pi^\vee_{I_\infty}} X_*(T_\infty)^{\sigma_\infty}_{I_\infty} \to (\widehat{F_\infty}^*_{I_\infty})^{\sigma_\infty} \to H^1(\langle\sigma_\infty\rangle, X_*(T^{\mathrm{sc}}_\infty)) = 0. \tag{18}$$

Returning to the diagram, as $\widehat{F_\infty}^*$ being finite is isomorphic as a $\mathfrak{g}_\infty$-module to $\widehat{F_\infty}$, we finally get

$$t_\infty(G) = |H^1(I_\infty, F_\infty(\hat{K}^s_\infty))^{\sigma_\infty}| = |\operatorname{coker}(\pi^\vee_{I_\infty})| = |\widehat{F_\infty}^{\mathfrak{g}_\infty}|. \qquad \blacksquare$$

Remark 6.3.

1. The sequence (18) illustrates the fact that the number $t_\infty(G)$ of types in the orbit of x depends only on the embedding of $X_*(T^{\mathrm{sc}}_\infty)$ in $X_*(T_\infty)$.
2. By the geometric version of Čebotarev's density theorem (see in [Jar]), one may choose the point ∞ such that G_∞ is split. In this case $t_\infty(G) = |F_\infty|$.

Now Corollary 5.4 together with Lemma 6.2 lead to the Main Theorem.

Main Theorem. *Assuming the Weil conjecture validity one has:*

$$\tau(G) = h_\infty(G) \cdot \frac{t_\infty(G)}{j_\infty(G)}.$$

The number $t_\infty(G)$ satisfies

$$t_\infty(G) = |H^1(I_\infty, F_\infty(\hat{K}_\infty^s))^{\sigma_\infty}| = |\widehat{F_\infty}^{\mathfrak{g}_\infty}|$$

and

$$j_\infty(G) = \frac{|H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{F})|}{|\mathcal{F}(\mathcal{O}_{\{\infty\}})|}.$$

If in particular G is quasi-split and $\overline{\mathcal{O}}_{\{\infty\}}^G$ is a UFD then $j_\infty(G) = 1$ and so

$$\tau(G) = h_\infty(G) \cdot t_\infty(G) = h_\infty(G) \cdot |\widehat{F_\infty}^{\mathfrak{g}_\infty}|.$$

7. Application and examples

In this section, we describe an application of our Main Theorem in case G is *quasi-split* and $\overline{\mathcal{O}}_{\{\infty\}}^G$ is a UFD (see Remark 1.1). We combine our result with [Ono1, Formula (3.9.1')] and the techniques from [PR, § 8.2] in order to relate the cokernels of Bourqui's degree maps $\deg_{T^{\text{sc}}}$ and $\deg_T$ from [Bou, Section 2.2], where T^{sc} and T denote suitable maximal tori of G^{sc} and G, respectively; cf. Proposition 7.8 below. These concrete computations will allow us to also provide a wealth of examples for which we compute the relative Tamagawa numbers. Ono's formula was originally designed for groups over number fields and was generalized to the function field case in [BD, Theorem 6.1]. We will use freely the notation concerning algebraic tori introduced in [Ono1]. In this section, we will usually assume that Weil's conjecture $\tau(G^{\text{sc}}) = 1$ holds.

Remark 7.1. According to the Bruhat–Tits construction, $\underline{G}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}) = \underline{T}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p})\mathfrak{X}(\hat{\mathcal{O}}_\mathfrak{p})$. As $G_\mathfrak{p}$ is quasi-split, one has (see [BT2, Corollary 4.6.7]) $\underline{G}^0_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}) = \underline{T}^0_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p})\mathfrak{X}(\hat{\mathcal{O}}_\mathfrak{p})$ and so

$$[\underline{G}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}) : \underline{G}^0_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p})] = [\underline{T}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}) : \underline{T}^0_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p})].$$

Definition 7.2. The finite group $W(T) = T(K) \cap T^c(\mathbb{A}) = \mathcal{T}(\mathbb{F}_q)$ is the *group of units* of T and its cardinality is denoted by $w(T)$.

Lemma 7.3.

$$\frac{w(T)}{w(T^{\text{sc}})} = \frac{|\mathcal{T}(\mathbb{F}_q)|}{|\mathcal{T}^{\text{sc}}(\mathbb{F}_q)|} = \frac{|\mathcal{T}(\mathbb{F}_q)|}{|\mathcal{T}^0(\mathbb{F}_q)|} = [\mathcal{T}(\mathbb{F}_q) : \mathcal{T}^0(\mathbb{F}_q)].$$

Proof. Under the assumptions of G being quasi-split and $\overline{\mathcal{O}}_{\{\infty\}}^G$ is a UFD, the finite groups $\mathcal{T}^{\text{sc}}(\mathbb{F}_q)$ and $\mathcal{T}^0(\mathbb{F}_q)$ are of the same cardinality (see in the proof of Lemma 5.2). The assertion follows. ∎

For an algebraic K-torus T, we set the following subgroup of the adelic group $T(\mathbb{A})$

$$T^1(\mathbb{A}) := \{x \in T(\mathbb{A}) : ||\chi(x)|| = 1 \ \forall \chi \in X^*(T)_K\}. \tag{19}$$

Let $\mathfrak{g} = \mathrm{Gal}(K^s/K)$. Following J. Oesterlé in [Oes, I.5.5], D. Bourqui defines in [Bou, §2.2.1] the morphism

$$\deg_T : T(\mathbb{A}) \to \mathrm{Hom}(X^*(T)^{\mathfrak{g}}, \mathbb{Z})$$

with $\ker(\deg_T) = T^1(\mathbb{A})$ and a finite cokernel (see [Bou, Proposition 2.21]). The maximal compact subgroup of $T(\mathbb{A})$ is denoted by

$$T^c(\mathbb{A}) := \prod_{\mathfrak{p}} \underline{T}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}).$$

Definition 7.4. The *class number* of T is $h(T) := [T^1(\mathbb{A}) : T^c(\mathbb{A})T(K)]$.

By [Ono1, Formula (3.9.1')] for a K-isogeny $\pi : T \to T'$ of tori T, T' defined over K, one has

$$\tau(\pi) := \frac{\tau(T')}{\tau(T)} = \frac{w(T)}{w(T')} \frac{h(T')}{h(T)} \prod_{\mathfrak{p}} \frac{L_{\mathfrak{p}}(1, \chi_{T'_{\mathfrak{p}}}) \cdot \omega_{\mathfrak{p}}(\underline{T'}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}))}{L_{\mathfrak{p}}(1, \chi_{T_{\mathfrak{p}}}) \cdot \omega_{\mathfrak{p}}(\underline{T}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}))}. \tag{20}$$

We shall need the following

Lemma 7.5. *Let $\underline{H}_{\mathfrak{p}}$ be an affine, smooth and connected group scheme defined over $\mathcal{O}_{\mathfrak{p}}$. Then $H^1(\langle \sigma_{\mathfrak{p}} \rangle, \underline{H}_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}}^{\mathrm{sh}})) = 1$.*

Proof. As $\mathcal{O}_{\mathfrak{p}}$ is Henselian, we have $H^1(\langle \sigma_{\mathfrak{p}} \rangle, \underline{H}_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}}^{\mathrm{sh}})) \cong H^1(\langle \sigma_{\mathfrak{p}} \rangle, \overline{H}_{\mathfrak{p}}(k_{\mathfrak{p}}^s))$ (see Remark 3.11(a) in [Mil1, Chapter III, §3]). The group on the right hand side is trivial by Lang's Theorem (see [Lan] and [Ser2, Chapter VI, Proposition 5]). ■

Remark 7.6. As $G_{\mathfrak{p}}^{\mathrm{sc}}$ is quasi-split, simply connected and almost simple, its maximal torus $T_{\mathfrak{p}}^{\mathrm{sc}}$ is a quasi-trivial torus (i.e. a Weil tori). Thus not only $H^1(\mathfrak{g}_{\mathfrak{p}}, G_{\mathfrak{p}}^{\mathrm{sc}}(K_{\mathfrak{p}}^s)) = 1$ (which is due to Harder as mentioned above), but also $H^1(\mathfrak{g}_{\mathfrak{p}}, T_{\mathfrak{p}}^{\mathrm{sc}}(K_{\mathfrak{p}}^s)) = 1$ as well as $H^1(\mathfrak{g}, G^{\mathrm{sc}}(K^s)) = 1$ and $H^1(\mathfrak{g}, T^{\mathrm{sc}}(K^s)) = 1$.

Lemma 7.7. *The map $\pi_K^\vee : Hom(X^*(T^{\mathrm{sc}})^{\mathfrak{g}}, \mathbb{Z}) \to Hom(X^*(T)^{\mathfrak{g}}, \mathbb{Z})$ is injective. One has*

$$h_\infty(G) \cdot \frac{h(T^{\mathrm{sc}})}{h(T)} = \frac{|\operatorname{coker}(\pi_K^\vee)|}{t_\infty(G) \cdot |\mathcal{D}|} \cdot \frac{\prod_{\mathfrak{p}} [\underline{T}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}) : \underline{T}_{\mathfrak{p}}^0(\hat{\mathcal{O}}_{\mathfrak{p}})]}{[\mathcal{T}(\mathbb{F}_q) : \mathcal{T}^0(\mathbb{F}_q)]}.$$

Proof. Since G is of non-compact type, the exact sequence of K-groups

$$1 \to F \to G^{\mathrm{sc}} \xrightarrow{\pi} G \to 1$$

induces the exactness over the adelic ring $\mathbb{A}$

$$1 \to F(\mathbb{A}) \to G^{\mathrm{sc}}(\mathbb{A}) \xrightarrow{\pi_{\mathbb{A}}} G(\mathbb{A}) \overset{\psi_{\mathbb{A}}}{\twoheadrightarrow} \operatorname{coker}(\pi_{\mathbb{A}}) \subset \prod_{\mathfrak{p}} H^1(\mathfrak{g}_{\mathfrak{p}}, F_{\mathfrak{p}}(\hat{K}_{\mathfrak{p}}^s))$$

where $\mathfrak{g}_\mathfrak{p} := \mathrm{Gal}(\hat{K}_\mathfrak{p}^s/\hat{K}_\mathfrak{p})$ – see [PR, § 8.2] and 3) in the proof of Thm. 3.2. in [Tha] for the function field case. According to [PR, Proposition 8.8], one has

$$h_\infty(G) = [\psi_\mathbb{A}(G(\mathbb{A})) : \psi_\mathbb{A}(G(\mathbb{A}_\infty)G(K))].$$

Denote $G^0(\mathbb{A}_\infty) = G_\infty(\hat{K}_\infty) \times \prod_{\mathfrak{p}\neq\infty} \underline{G}_\mathfrak{p}^0(\hat{\mathcal{O}}_\mathfrak{p})$. Define the finite set $S := \{\mathfrak{p} \mid \mathfrak{p} \text{ ramified}\}$. If $S = \emptyset$, then $G^0(\mathbb{A}_\infty) = G(\mathbb{A}_\infty)$ (see Remark 3.1). Otherwise, by the Borel density theorem (e.g. in the guise of [CM, Thm. 2.4, Prop. 2.8]), $\mathcal{G}(\mathcal{O}_{\{\infty\cup S\}})$ is Zariski-dense in $\prod_{\mathfrak{p}\in S\setminus\{\infty\}} \underline{G}_\mathfrak{p}$. This implies the equality $G(\mathbb{A}_\infty)G(K) = G^0(\mathbb{A}_\infty)G(K)$, and so

$$h_\infty(G) = [\psi_\mathbb{A}(G(\mathbb{A})) : \psi_\mathbb{A}(G^0(\mathbb{A}_\infty)G(K))]. \tag{21}$$

Since F is central in G^{sc}, it is embedded in T^{sc}. The corresponding exact sequence of K-groups of multiplicative type

$$1 \to F \to T^{\mathrm{sc}} \xrightarrow{\pi} T \to 1$$

induces by $\mathfrak{g}$-cohomology the exact sequences over K (see Remark 7.6):

$$\begin{aligned} 1 \to F(K) \to G^{\mathrm{sc}}(K) \xrightarrow{\pi} G(K) \xrightarrow{\delta_K} H^1(\mathfrak{g}, F(K^s)) \to 1 \\ 1 \to F(K) \to T^{\mathrm{sc}}(K) \xrightarrow{\pi} T(K) \xrightarrow{\delta_K} H^1(\mathfrak{g}, F(K^s)) \to 1 \end{aligned}$$

showing that $\delta_K(G(K)) = \delta_K(T(K))$. At any $\mathfrak{p}$, as $\underline{G^{\mathrm{sc}}}_\mathfrak{p}$ is connected, by Lemma 7.5 and Remark 7.6 one has

$$\begin{aligned} \mathrm{coker}[\underline{G^{\mathrm{sc}}}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}) \to \underline{G}_\mathfrak{p}^0(\hat{\mathcal{O}}_\mathfrak{p})] = \mathrm{coker}[\underline{T^{\mathrm{sc}}}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}) \to \underline{T}_\mathfrak{p}^0(\hat{\mathcal{O}}_\mathfrak{p})] = H^1(\langle\sigma_\mathfrak{p}\rangle, \underline{F}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}^{\mathrm{sh}})), \\ \mathrm{coker}[G_\mathfrak{p}^{\mathrm{sc}}(\hat{K}_\mathfrak{p}) \to G_\mathfrak{p}(\hat{K}_\mathfrak{p})] = \mathrm{coker}[T_\mathfrak{p}^{\mathrm{sc}}(\hat{K}_\mathfrak{p}) \to T_\mathfrak{p}(\hat{K}_\mathfrak{p})] = H^1(\mathfrak{g}_\mathfrak{p}, F_\mathfrak{p}(\hat{K}_\mathfrak{p}^s)). \end{aligned}$$

Thus together with $[\underline{G}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}) : \underline{G}_\mathfrak{p}^0(\hat{\mathcal{O}}_\mathfrak{p})] = [\underline{T}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}) : \underline{T}_\mathfrak{p}^0(\hat{\mathcal{O}}_\mathfrak{p})]$ (see Remark 7.1), we may infer that

$$\psi_\mathbb{A}(G(\mathbb{A})) = \mathrm{coker}[G^{\mathrm{sc}}(\mathbb{A}) \to G(\mathbb{A})] = \mathrm{coker}[T^{\mathrm{sc}}(\mathbb{A}) \to T(\mathbb{A})] = \psi_\mathbb{A}(T(\mathbb{A})).$$

In particular, over $\mathbb{A}_\infty$, due to Corollary 3.3 Galois cohomology yields an exact sequence

$$\begin{aligned} 1 \to F(\mathbb{A}_\infty) \to G^{\mathrm{sc}}(\mathbb{A}_\infty) \xrightarrow{\pi_\mathbb{A}} G^0_\infty(\mathbb{A}_\infty) \\ \xrightarrow{\psi_\mathbb{A}} H^1(\mathfrak{g}_\infty, F_\infty(\hat{K}_\infty^s)) \times \prod_{\mathfrak{p}\neq\infty} H^1(\langle\sigma_\mathfrak{p}\rangle, \underline{F}_\mathfrak{p}(\hat{\mathcal{O}}_\mathfrak{p}^{\mathrm{sh}})) \to 1 \end{aligned}$$

and similarly for the tori, showing that $\psi_\mathbb{A}(G^0(\mathbb{A}_\infty) = \psi_\mathbb{A}(T^0(\mathbb{A}_\infty))$. These cokernel equalities enable us to express $h_\infty(G)$ as given in (21) via T, namely

$$h_\infty(G) = [\psi_\mathbb{A}(T(\mathbb{A})) : \psi_\mathbb{A}(T^0(\mathbb{A}_\infty)T(K))]. \tag{22}$$

Applying the Snake Lemma on its two middle rows, we get the exactness of the diagram

$$\begin{tikzcd}
1 \arrow[r] & F(\mathbb{A}) \arrow[r] \arrow[d, equal] & (T^{\mathrm{sc}})^1(\mathbb{A}) \arrow[r, "\pi_{\mathbb{A}}"] \arrow[d, hook] & T^1(\mathbb{A}) \arrow[r, "\psi_{\mathbb{A}}"] \arrow[d, hook] & \psi_{\mathbb{A}}(T^1(\mathbb{A})) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & F(\mathbb{A}) \arrow[r] \arrow[d] & T^{\mathrm{sc}}(\mathbb{A}) \arrow[r, "\pi_{\mathbb{A}}"] \arrow[d, "\deg_{T^{\mathrm{sc}}}"] & T(\mathbb{A}) \arrow[r, "\psi_{\mathbb{A}}"] \arrow[d, "\deg_T"] & \psi_{\mathbb{A}}(T(\mathbb{A})) \arrow[r] \arrow[d] & 1 \\
 & 0 \arrow[r] & \mathrm{Hom}(X^*(T^{\mathrm{sc}})^{\mathfrak{g}}, \mathbb{Z}) \arrow[r, "\pi_K^\vee"] \arrow[d] & \mathrm{Hom}(X^*(T)^{\mathfrak{g}}, \mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{coker}(\pi_K^\vee) \arrow[r] \arrow[d] & 0 \\
 & & \mathrm{coker}(\deg_{T^{\mathrm{sc}}}) \arrow[r, "\overline{\pi}_K^\vee"] & \mathrm{coker}(\deg_T) \arrow[r] & \mathcal{D} \arrow[r] & 0
\end{tikzcd}$$

(note that the elements in $\ker(\pi_{\mathbb{A}})$ are units, and so belong to $T^1(\mathbb{A})$) from which we see that:

$$[\psi_{\mathbb{A}}(T(\mathbb{A})) : \psi_{\mathbb{A}}(T^1(\mathbb{A}))] = |\,\mathrm{coker}(\pi_K^\vee)|/|\mathcal{D}|. \tag{23}$$

Furthermore, from the following exact diagram

$$\begin{tikzcd}
1 \arrow[r] & F(\mathbb{A}) \arrow[r] \arrow[d, equal] & (T^{\mathrm{sc}})^c(\mathbb{A})T^{\mathrm{sc}}(K) \arrow[r, "\pi_{\mathbb{A}}"] \arrow[d, hook] & (T^c)^0(\mathbb{A})T(K) \arrow[r, "\psi_{\mathbb{A}}"] \arrow[d, hook] & \psi_{\mathbb{A}}((T^c)^0(\mathbb{A})T(K)) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & F(\mathbb{A}) \arrow[r] \arrow[d] & (T^{\mathrm{sc}})^1(\mathbb{A}) \arrow[r, "\pi_{\mathbb{A}}"] \arrow[d] & T^1(\mathbb{A}) \arrow[r, "\psi_{\mathbb{A}}"] \arrow[d] & \psi_{\mathbb{A}}(T^1(\mathbb{A})) \arrow[r] \arrow[d] & 1 \\
 & 1 \arrow[r] & \mathrm{Cl}(T^{\mathrm{sc}}) \arrow[r] & \mathrm{Cl}(T^0) \arrow[r] & \psi_{\mathbb{A}}(T^1(\mathbb{A}))/\psi_{\mathbb{A}}((T^c)^0(\mathbb{A})T(K)) \arrow[r] & 1
\end{tikzcd}$$

with $(T^c)^0(\mathbb{A}) := \prod_{\mathfrak{p}} \underline{T}^0_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}})$ one can see that

$$\frac{h(T)}{h(T^{\mathrm{sc}})} = \frac{h(T^0)/h(T^{\mathrm{sc}})}{[T^c(\mathbb{A})T(K) : (T^c)^0(\mathbb{A})T(K)]} = \frac{[\psi_{\mathbb{A}}(T^1(\mathbb{A})) : \psi_A((T^c)^0(\mathbb{A})T(K))]}{[T^c(\mathbb{A})T(K) : (T^c)^0(\mathbb{A})T(K)]}. \tag{24}$$

Using the Third and Second Isomorphism Theorems one has

$$\begin{aligned}
T^c(\mathbb{A})T(K)\Big/(T^c)^0(\mathbb{A})T(K) &\cong T^c(\mathbb{A})T(K)/T(K)\Big/(T^c)^0(\mathbb{A})T(K)/T(K) \\
&\cong T^c(\mathbb{A})/\mathcal{T}(\mathbb{F}_q)\Big/(T^c)^0(\mathbb{A})/\mathcal{T}^0(\mathbb{F}_q)
\end{aligned}$$

whence

$$[T^c(\mathbb{A})T(K) : (T^c)^0(\mathbb{A})T(K)] = \frac{\prod_{\mathfrak{p}}[\underline{T}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}) : \underline{T}^0_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}})]}{[\mathcal{T}(\mathbb{F}_q) : \mathcal{T}^0(\mathbb{F}_q)]}. \tag{25}$$

Similarly,

$$\begin{aligned}
T^0(\mathbb{A}_\infty)T(K)\Big/(T^c)^0(\mathbb{A})T(K) &\cong T^0(\mathbb{A}_\infty)T(K)/T(K)\Big/(T^c)^0(\mathbb{A})T(K)/T(K) \\
&\cong T^0(\mathbb{A}_\infty)/\mathcal{T}^0(\mathbb{F}_q)\Big/(T^c)^0/\mathcal{T}^0(\mathbb{F}_q) \qquad (26)\\
&\cong T^0(\mathbb{A}_\infty)/(T^c)^0(\mathbb{A}).
\end{aligned}$$

In order to compute the cardinality of the latter ratio image under ψ, we may use cohomology again. Fix a separable closure $\hat{K}_\infty^s$ of $\hat{K}_\infty$ containing the maximal unramified extension $\hat{K}_\infty^{\mathrm{un}}$ of $\hat{K}_\infty$ with absolute Galois group $\mathfrak{g}_\infty$ and inertia subgroup $I_\infty = \mathrm{Gal}(\hat{K}_\infty^s/\hat{K}_\infty^{\mathrm{un}})$. The spectral sequence then induces the exact sequence (see [Ser3, I.2.6(b)])

$$\begin{aligned}
0 \to H^1(\langle\sigma_\infty\rangle, F_\infty(\hat{K}_\infty^{\mathrm{un}})) &\overset{\mathrm{inf}}{\to} H^1(\mathfrak{g}_\infty, F_\infty(\hat{K}_\infty^s)) \\
&\overset{\mathrm{res}}{\to} H^1(I_\infty, F_\infty(\hat{K}_\infty^s))^{\sigma_\infty} \to H^2(\langle\sigma_\infty\rangle, F_\infty(\hat{K}_\infty^{\mathrm{un}})) = 0
\end{aligned}$$

which shows that

$$\begin{aligned}
[\psi_{\mathbb{A}}(T^0(\mathbb{A}_\infty)) : \psi_{\mathbb{A}}((T^c)^0(\mathbb{A}))] &= \frac{|H^1(\mathfrak{g}_\infty, F_\infty(\hat{K}_\infty^s))|}{|H^1(\langle\sigma_\infty\rangle, \underline{F}_\infty(\hat{\mathcal{O}}_\infty^{\mathrm{sh}}))|} \qquad (27)\\
&= |H^1(I_\infty, F_\infty(\hat{K}_\infty^s))^{\sigma_\infty}| \overset{(6.2)}{=} t_\infty(G).
\end{aligned}$$

All together, we finally get

$$\begin{aligned}
h_\infty(G)\cdot\frac{h(T^{\mathrm{sc}})}{h(T)} &\overset{(22),(24),(25)}{=} \frac{[\psi_{\mathbb{A}}(T(\mathbb{A})) : \psi_{\mathbb{A}}(T^0(\mathbb{A}_\infty)T(K))]}{[\psi_{\mathbb{A}}(T^1(\mathbb{A})) : \psi_{\mathbb{A}}((T^c)^0(\mathbb{A})T(K))]} \\
&\qquad \times \frac{\prod_{\mathfrak{p}}[\underline{T}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}) : \underline{T}_{\mathfrak{p}}^0(\hat{\mathcal{O}}_{\mathfrak{p}})]}{[\mathcal{T}(\mathbb{F}_q) : \mathcal{T}^0(\mathbb{F}_q)]} \\
&= \frac{[\psi_{\mathbb{A}}(T(\mathbb{A})) : \psi_{\mathbb{A}}(T^1(\mathbb{A}))]}{[\psi_{\mathbb{A}}(T^0(\mathbb{A}_\infty)T(K)) : \psi_{\mathbb{A}}((T^c)^0(\mathbb{A})T(K))]} \\
&\qquad \times \frac{\prod_{\mathfrak{p}}[\underline{T}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}) : \underline{T}_{\mathfrak{p}}^0(\hat{\mathcal{O}}_{\mathfrak{p}})]}{[\mathcal{T}(\mathbb{F}_q) : \mathcal{T}^0(\mathbb{F}_q)]} \\
&\overset{(26)}{=} \frac{[\psi_{\mathbb{A}}(T(\mathbb{A})) : \psi_{\mathbb{A}}(T^1(\mathbb{A}))]}{[\psi_{\mathbb{A}}(T^0(\mathbb{A}_\infty)) : \psi_{\mathbb{A}}((T^c)^0(\mathbb{A}))]} \cdot \frac{\prod_{\mathfrak{p}}[\underline{T}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}) : \underline{T}_{\mathfrak{p}}^0(\hat{\mathcal{O}}_{\mathfrak{p}})]}{[\mathcal{T}(\mathbb{F}_q) : \mathcal{T}^0(\mathbb{F}_q)]} \\
&\overset{(23),(27)}{=} \frac{|\operatorname{coker}(\pi_K^\vee)|}{|\mathcal{D}|\cdot t_\infty(G)} \cdot \frac{\prod_{\mathfrak{p}}[\underline{T}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}) : \underline{T}_{\mathfrak{p}}^0(\hat{\mathcal{O}}_{\mathfrak{p}})]}{[\mathcal{T}(\mathbb{F}_q) : \mathcal{T}^0(\mathbb{F}_q)]}. \qquad \blacksquare
\end{aligned}$$

The following proposition now is an immediate consequence of the Main Theorem, Lemma 7.7.

Proposition 7.8. $|\mathcal{D}| = |\operatorname{coker}(\overline{\pi}_K^\vee) = 1$.

Proof. Following [Ono3] by the proof of Theorem 6.1 in [BD], one has

$$\tau(G) = \tau(G^{\mathrm{sc}}) \cdot \frac{\tau(T)}{\tau(T^{\mathrm{sc}})} \cdot |\operatorname{coker}(\widehat{\pi}_K)|. \tag{28}$$

Applying the functor $\mathrm{Hom}(-, \mathbb{Z})$ on the sequence:

$$0 \to X^*(T)^{\mathfrak{g}} \xrightarrow{\widehat{\pi}_K} X^*(T^{\mathrm{sc}})^{\mathfrak{g}} \to M := \operatorname{coker}(\widehat{\pi}_K) \to 0 \tag{29}$$

gives rise to the exact sequence

$$\begin{aligned} 0 \to 0 = \mathrm{Hom}(M, \mathbb{Z}) &\to \mathrm{Hom}(X^*(T^{\mathrm{sc}})^{\mathfrak{g}}, \mathbb{Z}) \\ &\xrightarrow{\pi_K^\vee} \mathrm{Hom}(X^*(T)^{\mathfrak{g}}, \mathbb{Z}) \to \mathrm{Ext}^1(M, \mathbb{Z}) \cong \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}) \to 0 \end{aligned}$$

which shows that $\operatorname{coker}(\pi_K^\vee)$ is the Pontryagin dual of $\operatorname{coker}(\widehat{\pi}_K)$. As both groups are finite, they therefore have the same cardinality. Hence from formula (28) we get

$$\begin{aligned} \tau(G) &= \tau(G^{\mathrm{sc}}) \cdot |\operatorname{coker}(\pi_K^\vee)| \cdot \frac{\tau(T)}{\tau(T^{\mathrm{sc}})} \\ &\overset{(20)}{=} \tau(G^{\mathrm{sc}}) \cdot |\operatorname{coker}(\pi_K^\vee)| \cdot \frac{h(T)}{h(T^{\mathrm{sc}})} \cdot \frac{w(T^{\mathrm{sc}})}{w(T)} \prod_{\mathfrak{p}} L_{\mathfrak{p}}(1, \chi_{T_{\mathfrak{p}}}) \cdot \omega_{\mathfrak{p}}(\underline{T}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}})) \\ &\overset{[\text{Bit, } 3.2]}{=} \tau(G^{\mathrm{sc}}) \cdot |\operatorname{coker}(\pi_K^\vee)| \cdot \frac{h(T)}{h(T^{\mathrm{sc}})} \cdot \frac{w(T^{\mathrm{sc}})}{w(T)} \prod_{\mathfrak{p}} [\underline{T}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}) : \underline{T}_{\mathfrak{p}}^0(\hat{\mathcal{O}}_{\mathfrak{p}})] \\ &\overset{7.3}{=} \tau(G^{\mathrm{sc}}) \cdot |\operatorname{coker}(\pi_K^\vee)| \cdot \frac{h(T)}{h(T^{\mathrm{sc}})} \cdot \frac{\prod_{\mathfrak{p}} [\underline{T}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}) : \underline{T}_{\mathfrak{p}}^0(\hat{\mathcal{O}}_{\mathfrak{p}})]}{[\mathcal{T}(\mathbb{F}_q) : \mathcal{T}^0(\mathbb{F}_q)]} \\ &\overset{7.7}{=} \tau(G^{\mathrm{sc}}) \cdot h_\infty(G) \cdot t_\infty(G) \cdot |\mathcal{D}| \\ &\overset{1}{=} \tau(G^{\mathrm{sc}}) \cdot \tau(G) \cdot |\mathcal{D}|. \end{aligned}$$

This implies $|D| = \frac{1}{\tau(G^{\mathrm{sc}})} = 1$, according to the Weil conjecture. ■

Remark 7.9. Any isogenous K-tori T^{sc} and T with T^{sc} quasi-trivial can be realized as maximal tori of semisimple and quasi-split groups G^{sc} and G, respectively, with G^{sc} simply connected. E.g., given the isogeny $\pi : T^{\mathrm{sc}} \to T$, then each factor $R_{L/K}(\mathbb{G}_m^d)$ in T^{sc} is a maximal torus of the quasi-split and simply connected group $G^{\mathrm{sc}} = R_{L/K}(\mathbf{SL}_{d+1})$, and T is a maximal torus of $G = G^{\mathrm{sc}}/\ker(\pi)$, cf. Examples 7.16 – 7.18 below. Hence we may generalize Proposition 7.8 to the statement that for any isogeny $T^{\mathrm{sc}} \to T$, the induced map $\operatorname{coker}(\deg_{T^{\mathrm{sc}}}) \to \operatorname{coker}(\deg_T)$ is surjective.

Quite naturally, our Main Theorem reproduces the following well-known facts. Recall that in this section we assume the validity of the Weil Conjecture.

Corollary 7.10. *If G is K-split and $\mathcal{O}_{\{\infty\}}$ is a UFD, then $h_\infty(G) = 1$ and $\tau(G) = t_\infty(G) = |F|$.*

Proof. If G is K-split, then T^{sc} and T are K-split thus having connected reduction everywhere and $h(T) = h(T^{\mathrm{sc}})$. Furthermore, $|\operatorname{coker}(\pi_K^\vee)| = |F| = |F_\infty| = t_\infty(G)$ whence by Lemma 7.7 $h_\infty(G) = 1$. Hence according to the Main Theorem 1, we get $\tau(G) = t_\infty(G) = |F_\infty| = |F|$. ∎

Example 7.11. (Type A_2) Let $G = \mathrm{PGL}_3$ with $G^{\mathrm{sc}} = \mathrm{SL}_3$ defined over the rational function field $K = \mathbb{F}_q(x)$. Since G is split, by removing one point ∞ we get that $h_\infty(G) = 1$ (Cor. 7.10). Let T_∞ be the diagonal maximal torus in G_∞ and let $T_\infty^{\mathrm{sc}} = \pi^{-1}(T_\infty)$ in G_∞^{sc}. We have the following $\mathbb{Z}$-lattices:

$$X^*(T^{\mathrm{sc}}) = P = \left\langle \varepsilon_1 - \frac{1}{3}\sum_{i=1}^{3}\varepsilon_i, \varepsilon_1 + \varepsilon_2 - \frac{1}{3}\sum_{i=1}^{3}\varepsilon_i \right\rangle,$$
$$X^*(T) = Q = \langle \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3 \rangle$$

where ε_i is the projection of the i-entry in the diagonal matrix and:

$$X_*(T_\infty^{\mathrm{sc}}) = P^\vee = \langle e_{12} = \lambda_1 - \lambda_2, e_{23} = \lambda_2 - \lambda_3 \rangle \quad \text{embedded in:}$$
$$X_*(T_\infty) = Q^\vee = \left\langle \lambda_1 - \frac{1}{3}\sum_{i=1}^{3}\lambda_i, \lambda_1 + \lambda_2 - \frac{2}{3}\sum_{i=1}^{3}\lambda_i \right\rangle$$

where for any $t \in K_\infty^\times$, $\lambda_i(t)$ is the diagonal matrix with t in the i-th entry and 1 everywhere else. The fundamental alcove associated to the root basis $\Delta = \{\alpha_{12} := \varepsilon_1 - \varepsilon_2, \alpha_{23} := \varepsilon_2 - \varepsilon_3\}$ with highest root $\alpha_{13} := \varepsilon_1 - \varepsilon_3$ is bounded by the walls $H_{\alpha_{12},0}, H_{\alpha_{23},0}$ and $H_{\alpha_{13},-1}$. Its special vertices are:

$$o,\ x = \frac{e_{12} + 2e_{23}}{3}, \qquad y = \frac{2e_{12} + e_{23}}{3}$$

of types denoted by filled circle, empty circle and filled rectangle, respectively in Figure 4.12. Then: $x, y, x - y \in X_*(T_\infty) - X_*(T_\infty^{\mathrm{sc}})$ and therefore $\tau(\mathrm{PGL}_3) = t_\infty(\mathrm{PGL}_3) = 3$.

Example 7.12. (Type $B_2 \cong C_2$) Consider the special unitary group of the quadratic form $f(\bar{x}) = x_1x_5 + x_2x_4 + x_3^2$, $G = \mathrm{SU}(f) = \mathrm{SO}_5$ with $G^{\mathrm{sc}} = \mathrm{Spin}_5$ defined over $K = \mathbb{F}_q(x)$. Over K_∞, let $T_\infty = \{\mathrm{diag}(t, t', 1, t'^{-1}, t^{-1})\} \subset G_\infty$ and $T_\infty^{\mathrm{sc}} = \pi^{-1}(T_\infty) \subset G_\infty^{\mathrm{sc}}$. We have:

$$X^*(T^{\mathrm{sc}}) = P = \left\langle \varepsilon_1, \frac{1}{2}(\varepsilon_1 + \varepsilon_2) \right\rangle, \qquad X^*(T) = Q = \langle \alpha = \varepsilon_1 - \varepsilon_2, \beta = \varepsilon_2 \rangle.$$

and:

$$X_*(T_\infty^{\mathrm{sc}}) = P^\vee = \langle \lambda_1 - \lambda_2, 2\lambda_2 \rangle \quad \text{embedded in:} \quad X_*(T_\infty) = Q^\vee = \langle \lambda_1, \lambda_1 + \lambda_2 \rangle.$$

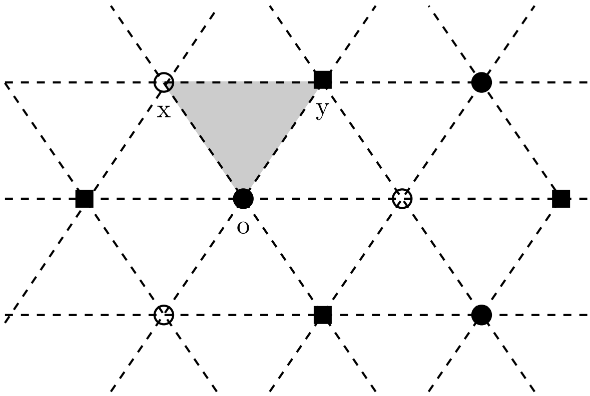

Figure 4.12. Affine apartment of type A_2

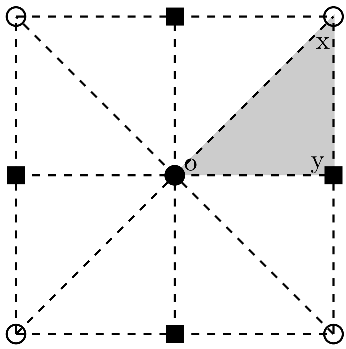

Figure 4.13. Affine apartment of type B_2

The fundamental alcove associated to the basis: $\Delta = \{\alpha := \varepsilon_2, \beta := \varepsilon_1 - \varepsilon_2\}$ with highest root: $\gamma := 2\alpha + \beta = \varepsilon_1 + \varepsilon_2$ is bounded by the walls $H_{\alpha,0}, H_{\beta,0}$ and $H_{\gamma,-1}$. Its vertices are

$$o, \; x = \lambda_1, \quad y = \frac{1}{2}(\lambda_1 + \lambda_2)$$

of types denoted by filled circle, empty circle and filled rectangle respectively in Figure 4.13 among which only o, x are special as the isotropic group of y is not maximal (lying on the intersection of only two walls out of three possible). Since: $x - o = x = \lambda_1 \in X_*(T) - X_*(T^{\mathrm{sc}})$ we get that (G is split again): $\tau(\mathrm{SO}_5) = t_\infty(\mathrm{SO}_5) = 2$.

Remark 7.13. We have assumed in this section that $\overline{\mathcal{O}}^G_{\{\infty\}}$ is a UFD. Otherwise, $h_\infty(G)$ does not need to be 1, though G splits over K. For example, let $G = \mathbf{PGL}_n$ defined over $K = \mathbb{F}_q(C)$ where C is an elliptic curve ($g = 1$). Let ∞ be a $\mathbb{F}_q$-rational point and let $\mathcal{G}$ be an affine, smooth, flat, connected and of finite type model of G defined over $\operatorname{Spec} \mathcal{O}_{\{\infty\}}$ as was constructed above. Let $\mathcal{GL}_n$ be a similar construction for $\mathbf{GL}_n$ and $\mathcal{G}_m$ for $\mathbb{G}_m$. According to Nisnevich's exact sequence (see [Nis, 3.5.2] and also [Gon, Thm. 3.4]), since the Shafarevich-Tate group w.r.t. $S = \{\infty\}$ is trivial in this split case, we have:

$$\mathrm{Cl}_\infty(G) \cong H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{G}).$$

The exact sequence of smooth $\mathcal{O}_{\{\infty\}}$-groups

$$1 \to \mathcal{G}_m \to \mathcal{GL}_n \to \mathcal{G} \to 1$$

gives rise by flat cohomology to the following exact sequence

$$\mathrm{Pic}\ (C^{\mathrm{af}}) \xrightarrow{\partial} H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{GL}_n) \xrightarrow{\delta} H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{G}) \to H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathbb{G}_m)$$

on which $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{GL}_n)$ classifies the rank n vector bundles defined over $C^{\mathrm{af}} := C - \{\infty\}$. Every rank-$n$ vector bundle over a Dedekind domain is a direct sum $\mathcal{O}^{n-1}_{C^{\mathrm{af}}} \oplus \mathcal{L}$, where $[\mathcal{L}] \in \mathrm{Pic}\ (C^{\mathrm{af}})$ and $\mathcal{O}_{C^{\mathrm{af}}}$ is the trivial line bundle. As $\partial : [\mathcal{L}] \mapsto n\mathcal{L}$ we have:

$$\begin{aligned} \mathrm{im}(\delta) &\cong H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{GL}_n)/\ker(\delta) \\ &= H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{GL}_n)/\,\mathrm{im}(\partial) \cong \mathrm{Pic}\ (C^{\mathrm{af}})[n] := \mathrm{Pic}\ (C^{\mathrm{af}})/n\mathrm{Pic}\ (C^{\mathrm{af}}). \end{aligned}$$

Moreover, as C^{af} is smooth, one has (see [Mil1, Prop. 2.15]): $H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{G}_m) = \mathrm{Br}(\mathcal{O}_{\{\infty\}})$, classifying Azumaya $\mathcal{O}_{\{\infty\}}$-algebras (see [Mil1, § 2]). At each prime $\mathfrak{p}$: $\mathrm{Br}(\mathcal{O}_{\{\infty\}}) \subseteq \mathrm{Br}((\mathcal{O}_{\{\infty\}})_\mathfrak{p}) \subseteq \mathrm{Br}(\hat{\mathcal{O}}_\mathfrak{p})$. As $\hat{\mathcal{O}}_\mathfrak{p}$ is complete, the latter group is isomorphic to $\mathrm{Br}(k_\mathfrak{p})$ (see [AG, Thm. 6.5]). But $k_\mathfrak{p}$ is a finite field, thus $\mathrm{Br}(k_\mathfrak{p})$ is trivial as well as $H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathbb{G}_m)$ and δ is surjective. We get that $h_\infty(G) = |\mathrm{Cl}_\infty(G)| = |H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mathcal{G})| = |\mathrm{Pic}\ (C^{\mathrm{af}})[n]|$. In order to compute this group, we obtain the following isomorphism: Since $\{\infty\}$ is an irreducible subset of co-dimension 1 in C, the restriction of C to C^{af} gives rise to an exact sequence (see [Hart, Cha.II, Prop.6.5(c)]):

$$0 \to \mathbb{Z} \to \mathrm{Pic}\ (C) \to \mathrm{Pic}\ (C^{\mathrm{af}}) \to 0$$

on which the first map $1 \mapsto 1 \cdot \{\infty\}$ is injective because the degree of a curve's divisor is well defined. As we assumed ∞ is $\mathbb{F}_q$-rational, this sequence splits as abelian groups. Moreover, the embedding of the identity component of $\mathrm{Pic}\ (C)$ yields another exact sequence:

$$0 \to \mathrm{Pic}\ ^0(C) \to \mathrm{Pic}\ (C) \to \mathbb{Z} \to 0$$

in which the right term is the Néron-Severi group. It also splits as abelian groups and so we get an isomorphism of summands $\mathrm{Pic}\ ^0(C) \cong \mathrm{Pic}\ (C^{\mathrm{af}})$. Together with another isomorphism of abelian groups: $C(\mathbb{F}_q) \cong \mathrm{Pic}\ ^0(C); P \mapsto [P] - [\infty]$ we may deduce that:

$$C(\mathbb{F}_q) \cong \mathrm{Pic}\ (C^{\mathrm{af}}).$$

Now it is easy to find an elliptic curve C for which $h_\infty(G) = |C(\mathbb{F}_q)[n]| > 1$.

Applying flat cohomolopgy on Kummer's exact sequence of $\mathcal{O}_{\{\infty\}}$-schemes:

$$1 \to \mu_n \to \mathcal{G}_m \overset{x \mapsto x^n}{\longrightarrow} \mathcal{G}_m$$

gives rise to the exact sequence of groups of $\mathcal{O}_{\{\infty\}}$-points:

$$(\mathcal{O}_{\{\infty\}})^\times \overset{x \mapsto x^n}{\longrightarrow} (\mathcal{O}_{\{\infty\}})^\times \to H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mu_n) \to \text{Pic }(C^{\text{af}}) \overset{z \mapsto nz}{\longrightarrow} \text{Pic }(C^{\text{af}})$$

which in light of the proof of Lemma 5.2 can we rewritten as

$$1 \to \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^n \to H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mu_n) \to \text{Pic }(C^{\text{af}})[n] \to 0.$$

We deduce that $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mu_n)$ is an extension of $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^n$ by Pic $(C^{\text{af}})[n]$ and so

$$|H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mu_n)| = |\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^n| \cdot |\text{Pic }(C^{\text{af}})[n]| = |H^1(\mathbb{F}_q, \mu_n)| \cdot |\text{Pic }(C^{\text{af}})[n]|.$$

Consequently,

$$j_\infty(G) = \frac{|H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mu_n)|}{|\mu_n(\mathbb{F}_q)|} = \frac{|H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mu_n)|}{|H^1(\mathbb{F}_q, \mu_n)|} = |\text{Pic }(C^{\text{af}})[n]| = h_\infty(G)$$

and finally:

$$\tau(G) = h_\infty(G) \cdot \frac{t_\infty(G)}{j_\infty(G)} = |\widehat{F_\infty}^{\mathfrak{g}_\infty}| = |F| = n.$$

Corollary 7.14. *If G is adjoint (not necessarily split) and $\overline{\mathcal{O}}^G_{\{\infty\}}$ is a UFD, then $h_\infty(G) = 1$ and $\tau(G) = t_\infty(G) = |\widehat{F}^{\mathfrak{g}}|$, where $\widehat{F} := \text{Hom}(F(K^s), \mathbb{G}_{m,K^s})$ and $\mathfrak{g} := \text{Gal}(K^s/K)$.*

Proof. According to Ono's formula (3.9.11′) in [Ono1], considering the isogeny of class groups of T^{sc} and T, there exists a finite set of primes S for which

$$\frac{h(T)}{h(T^{\text{sc}})} = \left(\frac{q(\alpha^1_S)}{\prod_{\mathfrak{p} \in S} q(\alpha_{\mathcal{O}_\mathfrak{p}})}\right) \Big/ \left(\frac{q(\alpha^S_K)}{q(\alpha_W)}\right)$$

where for any isogeny α, $q(\alpha)$ stands for $|\operatorname{coker}(\alpha)|/|\ker(\alpha)|$ and (see notation in Section 4):

$$T^1_S(\mathbb{A}) := T^1(\mathbb{A}) \cap T_S, \quad T_S(K) := T(\mathbb{A}(S)) \cap T(K),$$

$$\alpha^1_S := (T^{\text{sc}}_S)^1(\mathbb{A}) \to T^1_S(\mathbb{A}), \quad \alpha^S_K := T^{\text{sc}}_S(K) \to T_S(K), \quad \alpha_W := W(T^{\text{sc}}) \to W(T).$$

As G^{sc} is simply-connected and G is adjoint, both quasi-split, their maximal tori T^{sc} and T are quasi-trivial and their integral models are connected everywhere (see Remark 2.1). In this case, the quantities $q(\alpha)$ related to α^S and α^S_K are equal to the ones obtained in the split case on which the class group of each $\mathbb{G}_m$ is the class group of K (see Formulas (3.1.7) and (3.1.8) in [Ono1]), thus equal to 1. Also by

Lemma 3.4 one may deduce that $q(\alpha_{\mathcal{O}_\mathfrak{p}}) = 1$ at each $\mathfrak{p}$ and by Lemma 7.3 (recall $\overline{\mathcal{O}}_{\{\infty\}}^G$ is a UFD), this can be deduced also for α_W. Hence T^{sc} and T share the same class number and so by Lemma 7.7, Proposition 7.8 and our Main Theorem $\tau(G) = h_\infty(G) \cdot t_\infty(G) = |\operatorname{coker}(\pi_K^\vee)|$ (see in the proof of 7.8). But as T is quasi-trivial, $X^*(T) = \bigoplus_{i=1}^n \operatorname{Ind}_{\{id\}}^{H_i}(\mathbb{Z})$ where H_i are some finite subgroups of $\mathfrak{g}$, thus by Shapiro's lemma $H^1(\mathfrak{g}, X^*(T)) \cong \bigoplus H^1(H_i, \mathbb{Z}) = 0$. Consequently, the exact sequence of character groups (considered as $\mathfrak{g}$-modules):

$$0 \to X^*(T) \to X^*(T^{\mathrm{sc}}) \to \widehat{F} \to 0$$

gives rise by $\mathfrak{g}$-cohomology to the exact sequence:

$$0 \to X^*(T)^\mathfrak{g} \xrightarrow{\widehat{\pi}_K} X^*(T^{\mathrm{sc}})^\mathfrak{g} \to \widehat{F}^\mathfrak{g} \to H^1(\mathfrak{g}, X^*(T)) = 0$$

from which we can see that $\tau(G) = |\operatorname{coker}(\pi_K^\vee)| = |\operatorname{coker}(\widehat{\pi}_K)| = |\widehat{F}^\mathfrak{g}|$. This also shows by Ono's formula [Ono3, Main Theorem] (see Cor. 7.15 below) that $Ш^1(\widehat{F}) = 1$. ■

More generally, our Main Theorem leads us to the following more general result obtained by Ono in 1965 (see Main Theorem in [Ono3]). It was designed for groups over number fields and has been generalized by Behrend and Dhillon at 2009 to the function field case in [BD, Theorem 6.1].

Corollary 7.15 (Ono's formula). *One has*

$$\tau(G) = \frac{|\widehat{F}^\mathfrak{g}|}{|Ш^1(\widehat{F})|}$$

where the denominator is the first Shafarevitch–Tate group assigned to $\widehat{F}$.

Proof. Applying Galois $\mathfrak{g}$-cohomology to the sequence of groups of characters

$$0 \to X^*(T) \xrightarrow{\widehat{\pi}} X^*(T^{\mathrm{sc}}) \to \widehat{F} \to 0$$

where $\widehat{F} := \operatorname{Hom}(F \otimes_K K^s, \mathbb{G}_{m,K^s})$ yields the relation

$$|\operatorname{coker}(\widehat{\pi}_K)| = \frac{|\widehat{F}^\mathfrak{g}|}{|H^1(\mathfrak{g}, X^*(T))|}. \tag{30}$$

The following formula for the Tamagawa number of a torus is taken from [Ono2, Main Theorem], [Oes, Corollary 3.3]

$$\tau(T) = \frac{|H^1(\mathfrak{g}, X^*(T))|}{|Ш^1(T)|}. \tag{31}$$

Together with $|\text{Ш}^1(T)| = |\text{Ш}^1(\widehat{F})|$ ([Ono3, p. 102]), we conclude

$$\begin{aligned}\tau(G) &\overset{(28)}{=} \tau(G^{\mathrm{sc}}) \cdot \tau(T) \cdot |\operatorname{coker}(\widehat{\pi}_K)| \\ &\overset{(30)}{=} \tau(G^{\mathrm{sc}}) \cdot \tau(T) \cdot \frac{|\widehat{F}^{\mathfrak{g}}|}{|H^1(\mathfrak{g}, X^*(T))|} \\ &\overset{(31)}{=} \tau(G^{\mathrm{sc}}) \cdot \frac{|\widehat{F}^{\mathfrak{g}}|}{|\text{Ш}^1(\widehat{F})|}. \qquad \blacksquare\end{aligned}$$

In the following examples, we refer to a construction which was demonstrated by Ono over number fields, in [Ono2, Example 6.3]. Our ground field is $K = \mathbb{F}_q(t)$ with odd characteristic and ∞ is chosen to correspond to the pole of t. At each example we consider another extension L of K. We denote $\mathfrak{g} = \mathrm{Gal}(L/K)$. The group $G^{\mathrm{sc}} = R_{L/K}(\mathbf{SL}_2)$ is the universal cover of the semisimple and quasi-split K-group $G = G^{\mathrm{sc}}/F$ where $F := R^{(1)}_{L/K}(\mu_n) = \ker[R_{L/K}(\mu_n) \to \mu_n]$. Let S be the diagonal K-split maximal torus in G. Then $T = \mathrm{Cent}_G(S)$ is a maximal torus of G and is isomorphic as a $\mathfrak{g}$-module to the K-torus $\mathbb{G}_m \times R^{(1)}_{L/K}(\mathbb{G}_m)$ where the right hand factor is the norm torus, namely the kernel of the norm map (see [San, Example 5.6])

$$R^{(1)}_{L/K}(\mathbb{G}_m) := \ker\left[R_{L/K}(\mathbb{G}_m) \xrightarrow{N_{L/K}} \mathbb{G}_m\right].$$

Its preimage in G^{sc} is the Weil torus $T^{\mathrm{sc}} = R_{L/K}(\mathbb{G}_{m,L})$, fitting into the exact sequence

$$1 \to F \to T^{\mathrm{sc}} \xrightarrow{\pi} T \to 1.$$

Over any $\hat{\mathcal{O}}_{\mathfrak{p}}$, the norm torus is $\mathrm{Spec}\,\hat{\mathcal{O}}_\infty[a,b]/(a^2 - \mathfrak{p}b^2 - 1)$. Its reduction provides at each place $\mathfrak{p}$, $e_{\mathfrak{p}}$ connected components, where $e_{\mathfrak{p}}$ stands for the ramification index there (see [Bit, Example 3.3]), i.e. $[\underline{T}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}) : \underline{T}^0_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}})] = e_{\mathfrak{p}}$. In this construction $|\operatorname{coker}(\widehat{\pi}_K)| = 1$.

Example 7.16. We start by $L = \mathbb{F}_{q^2}(t)$ obtained by extending the field of constants of K. Since the extension is quadratic, $F_\infty = \mu_2$ is $\hat{K}_\infty$-split whence $t_\infty(G) = |\widehat{F_\infty}| = |F_\infty| = 2$. Moreover, as L/K is imaginary and totally unramified we have $h(T)/h(T^{\mathrm{sc}}) = 2$ (see [Mor, Example 1]). Thus by Lemma 7.7, $h_\infty(G) = 1$ whence according to our Main Theorem $\tau(G) = h_\infty(G) \cdot t_\infty(G) = 2$.

Example 7.17. Now let $L = K(\sqrt{d})$ where d is a product of m distinct finite primes $\mathfrak{p}_i$. As before, $F_\infty = \mu_2$ and $t_\infty(G) = 2$. Recall that the norm torus is the only factor in T^{sc} and T which might have a disconnected reduction. This time, since each $\mathfrak{p}_i$, as well as ∞, ramifies in L with $e_{\mathfrak{p}} = 2$ we have

$$\prod_{\mathfrak{p}} [\underline{T}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}) : \underline{T}^0_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}})] = 2^{m+1}$$

while:

$$[\mathcal{T}(\mathbb{F}_q) : \mathcal{T}^0(\mathbb{F}_q)] = |\{x \in \mathbb{F}_q : x^2 = 1\}| = |\{\pm 1\}| = 2.$$

Moreover, as $h(T^{\mathrm{sc}})/h(T) = 2^{m-1}$ (see [Mor, Example 1]) and $\mathrm{coker}(\pi_K^\vee) = 1$, by Lemma 7.7 we get $h_\infty(G) = 1$. All together, we see by the Main Theorem that $\tau(G)$ remains equal to 2, independently of m. Both this result and the one of the previous example agree with Ono's formula 7.15; indeed, as L/K is cyclic, $Ш^1(\widehat{F}) = 1$ and $\tau(G) = |\widehat{F}^{\mathfrak{g}}| = |F| = 2$.

Example 7.18. Let $L = K(\Lambda_f)$ be the f-cyclotomic extension where f is an irreducible polynomial of degree d. Then $\mathfrak{g}$ is cyclic of order $n = q^d - 1$. We still have $h(T)/h(T^{\mathrm{sc}}) = 1$ ([Mor, Example 2]). The only places which ramify in L are ∞ with $e_\infty = q - 1$ and (f) which is totally ramified (see [Hay, Theorem 3.2]). Therefore $[\underline{T}_{(\infty)}(\mathcal{O}_{(\infty)}) : \underline{T}^0_{(\infty)}(\mathcal{O}_{(\infty)})] = q - 1$ and $[\underline{T}_{(f)}(\mathcal{O}_{(f)}) : \underline{T}^0_{(f)}(\mathcal{O}_{(f)})] = n$. On the units group, since $q - 1 | n$ we have

$$[\mathcal{T}(\mathbb{F}_q) : \mathcal{T}^0(\mathbb{F}_q)] = |\{x \in \mathbb{F}_q : x^n = 1\}| = |\mathbb{F}_q^\times| = q - 1.$$

Moreover, $t_\infty(G) = |\widehat{F_\infty}^{\mathfrak{g}_\infty}| = |\mu_n| = n$ and as before $\mathrm{coker}(\widehat{\pi}_K) = 1$. So by Lemma 7.7, we get

$$h_\infty(G) = \frac{\prod_{\mathfrak{p}} [\underline{T}_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}}) : \underline{T}^0_{\mathfrak{p}}(\hat{\mathcal{O}}_{\mathfrak{p}})]}{t_\infty(G) \cdot [\mathcal{T}(\mathbb{F}_q) : \mathcal{T}^0(\mathbb{F}_q)]} = \frac{(q-1) \cdot n}{n \cdot (q-1)} = 1.$$

Thus by our Main Theorem we conclude that $\tau(G) = t_\infty(G) = n$. Indeed, as L/K is cyclic, $Ш^1(\widehat{F}) = 1$ and $\tau(G) = |\widehat{F}^{\mathfrak{g}}| = |\mu_n| = n$, which agrees again with Ono's formula 7.15.

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Addresses: Rony A. Bitan: Institut Camille Jordan, Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France;
Ralf Köhl: JLU Gießen, Mathematisches Institut, Arndtstraße 2, 35392 Gießen, Germany.

E-mail: rony.bitan@gmail.com, ralf.koehl@math.uni-giessen.de

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