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ON THE IDEAL THEOREM FOR NUMBER FIELDS

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To the memory of Patrick Sargos

Abstract: Let K be an algebraic number field and ν_K be the ideal-counting function of K. Many authors have estimated the remainder term $\Delta_n(x, K)$ in the asymptotic formula of the average order of ν_K. The purpose of this work is twofold: we first generalize Müller's method to the n-dimensional case and improve on Nowak's result. A key part in the proof is played by a profound result on a triple exponential sum recently derived by Robert & Sargos.

Keywords: ideal theorem, Voronoi-Atkinson type formula, exponential sums of type I and II.

1. Introduction and result

Let K be an algebraic number field of fixed degree $n \geqslant 2$ and, for all $m \in \mathbb{Z}_{\geqslant 1}$, let $\nu_K(m)$ be the number of non-zero integral ideals of $\mathcal{O}_K$ of norm m. The Ideal Theorem is the investigation of the error term $\Delta_n(x, K)$ defined by

$$\Delta_n(x, K) := \sum_{m \leqslant x} \nu_K(m) - \kappa_K x$$

where κ_K is the residue at the point $s = 1$ of the Dedekind zeta-function attached to K. It is customary to set α_n to be the infimum of the positive numbers a_n such that

$$\Delta_n(x, K) \ll_K x^{a_n}.$$

The first non-trivial result in this problem is attributed to Weber [17] who showed circa 1895 that

$$\alpha_n \leqslant 1 - \frac{1}{n}.$$

With the dazzling progress of complex analysis and methods of contour integration, Weber's result was quickly superseded by Landau who first noticed

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[9, pp. 106–107] that the Schnee–Landau theorem implies

$$\alpha_n \leqslant 1 - \frac{2}{n+2}$$

and next using a more precise contour and some averaging arguments he was able to prove [10, Satz 210] that

$$\alpha_n \leqslant 1 - \frac{2}{n+1}.$$

In the opposite direction, Landau [10, Satz 211] also proved that

$$\alpha_n > \frac{n-1}{2n}.$$

As usual, it is surmised that this omega-result is in fact the right order of magnitude, but this is still an open problem.

Landau's result remains unbeaten for more than eighty years. The first improvements came in the quadratic field case, since the evaluation of the average order of ν_K can be reduced to a two-dimensional divisor problem. Using the recent discrete Hardy-Littlewood's circle method developed by Bombieri, Iwaniec, Mozzochi, Huxley and Watt, the authors in [6] proved that, if K is a quadratic field, then

$$\left| \sum_{m \leqslant N} \nu_K(m) - \kappa_K N \right| \leqslant B_K \left(d_K N\right)^{23/73} (\log N)^{315/146}$$

where N is a positive integer satisfying $N \gg d_K^C$, $C > 0$ is an effectively computable constant and $B_K > 0$ is a constant depending essentially on d_K. In fact, applying Huxley's latest estimates for exponential sums [5], the slight improvement

$$\alpha_2 \leqslant \frac{131}{416}$$

holds.

The cubic field case was investigated by Müller [11] by taking the analogy of this problem with the Dirichlet three-dimensional divisor problem into account. Following an idea developed by Atkinson [1], Müller showed that the error-term can be expressed as a double exponential sums of type I in Vaughan's terminology. Then using very fine estimates of this type of exponential sums due to Kolesnik [7], he proved that one has exactly the same value for the error-term than that of the sum $\sum_{m \leqslant x} \tau_3(m)$, where τ_3 is the third Dirichlet-Piltz divisor function, and hence

$$\alpha_3 \leqslant \frac{43}{96}.$$

when K is a cubic field.

The general case was treated by Nowak [12] who proved, using a completely different method, that, for any $n \geqslant 3$, we have

$$\sum_{m \leqslant x} \nu_K(m) = \kappa_K x + O_{K,\varepsilon}\left(x^{\theta_n+\varepsilon}\right)$$

where

$$\theta_n := \begin{cases} 1 - \dfrac{2}{n} + \dfrac{8}{n(5n+2)}, & \text{if } 3 \leqslant n \leqslant 6 \\[2ex] 1 - \dfrac{2}{n} + \dfrac{3}{2n^2}, & \text{if } n \geqslant 7. \end{cases}$$

The purpose of this paper is to generalize Müller's work to the n-dimensional case and improve on Nowak's result. We first prove the analogue of the Atkinson-Voronoi type formula (see Proposition 2.1) and then establish some estimates of exponential sums of type I and II (Propositions 3.2–3.5) needed in the proof of our main result. A crucial part is played by a recent estimate of a triple exponential sum derived by Robert & Sargos [13].

Theorem 1.1. *Let K be an algebraic number field of degree $n \geqslant 4$ and $x \geqslant d_K^{1/2}$ be a large real number. For any $\varepsilon \in]0,1[$, we have*

$$\sum_{m \leqslant x} \nu_K(m) = \kappa_K x + O_{K,\varepsilon}\left(x^{\lambda_n+\varepsilon}\right)$$

where $\lambda_4 = \dfrac{41}{72}$ and $\lambda_n = 1 - \dfrac{4}{2n+1}$ whenever $n \geqslant 5$.

Notation. In this paper, K is an algebraic number field of degree $n \geqslant 3$, signature (r_1, r_2), absolute value of the discriminant d_K, class number h_K, regulator $\mathcal{R}_K$ and w_K is the number of roots of unity lying in K. We denote by ζ_K the Dedekind zeta-function attached to K with residue at the point $s = 1$ denoted by κ_K. Recall that the so-called analytic class number formula states that

$$\kappa_K = \frac{2^{r_1}(2\pi)^{r_2} h_K \mathcal{R}_K}{w_K d_K^{1/2}}.$$

Let $\nu_K(m)$ be the mth coefficient ζ_K, i.e. the number of non-zero integral ideals of $\mathcal{O}_K$ of norm m. Finally, τ_n is the nth Dirichlet-Piltz divisor function and $e(x) = e^{2\pi i x}$ $(x \in \mathbb{R})$.

The well-known functional equation of ζ_K may be written as [10, Satz 156]

$$\zeta_K(1-s) = \gamma_K(s)\zeta_K(s) \tag{1}$$

with

$$\gamma_K(s) = d_K^{s-1/2}\left(\cos \tfrac{\pi s}{2}\right)^{r_1+r_2}\left(\sin \tfrac{\pi s}{2}\right)^{r_2}\left(\pi^{-s}\, 2^{1-s}\Gamma(s)\right)^n. \tag{2}$$

2. A Voronoi-Atkinson type formula

This section is devoted to the proof of the following proposition generalizing Müller's work.

Proposition 2.1. *Let K be a number field of degree $n \geqslant 3$, $x \geqslant d_K^{1/2}$ such that $x \notin \mathbb{Z}$ and let $\max\left(1, d_K x^{-1}\right) \leqslant R \leqslant (2\pi)^{-n} x$ be any real parameter. Then, for all $\varepsilon \in]0,1[$, we have*

$$\sum_{m \leqslant x} \nu_K(m) = \kappa_K x + O_{K,\varepsilon}\left(x^{\frac{n-1}{2n}} \left| \sum_{m \leqslant R} \frac{\nu_K(m)}{m^{\frac{n+1}{2n}}}\, e\left(c_n (xm)^{1/n}\right) \right| + \frac{x^{1-1/n+\varepsilon}}{R^{1/n}} \right)$$

where $c_n := 2\pi n d_K^{-1/n}$.

The proof is the consequence of the following technical lemmas.

Lemma 2.2. *Let $n \in \mathbb{Z}_{\geqslant 1}$, $s = \sigma + it \in \mathbb{C}$ with $\sigma > 0$ and $t \in \mathbb{R}$. Then*

$$\Gamma(s)^{n-1}\Gamma(s-1) = (2\pi)^{(n-1)/2}\, n^{n/2+1-ns}\, \Gamma\left(ns - \tfrac{n+1}{2}\right) \left\{1 + O_n\left(|s|^{-1}\right)\right\}.$$

Proof. Follows from a straightforward application of Stirling's formula. We leave the details to the reader. ■

Lemma 2.3. *Let $y > 0$, $1 < A \leqslant U$ and $\Phi \in \{\cos, \sin\}$. Set*

$$I := \frac{1}{2\pi i} \int_{A-iU}^{A+iU} \Gamma(s)\, \Phi\left(\tfrac{\pi s}{2}\right) y^{-s}\, \mathrm{d}s.$$

- ▷ *If $y \leqslant U$, then*

$$I = \Phi(y) + O\left\{ y^{-1/2} \min\left(\left(\log \frac{U}{y}\right)^{-1}, U^{1/2} \right) + y^{-A} U^{A-1/2} + y^{-1/2} \right\}.$$

- ▷ *If $y > U$, then*

$$I = O\left\{ y^{-A} \left[U^{A-1/2} \min\left(\left(\log \frac{y}{U}\right)^{-1}, U^{1/2} \right) + A^{A+1/2} \right] \right\}.$$

Proof. This is [1, Lemmas 1 and 2]. ■

We now are in a position to prove Proposition 2.1.

Let $x \geqslant 4$ such that $x \notin \mathbb{Z}$, $1 \leqslant T \leqslant x$ be real numbers and assume that $x \geqslant (d_K T^n)^{1/2}$. Set $\delta = \delta(x) := (\log x)^{-1}$. From Perron's formula we get

$$\sum_{m \leqslant x} \nu_K(m) = \frac{1}{2\pi i} \int_{1+\delta-iT}^{1+\delta+iT} \frac{\zeta_K(s)}{s}\, x^s\, \mathrm{d}s + O_{n,\varepsilon}\left(x^{1+\varepsilon} T^{-1}\right)$$

and shifting the integration to the parallel segment with $\operatorname{Re} s = -\delta$ and taking the poles into account, changing the variable s into $1-s$ and applying the functional equation (1), we obtain

$$\sum_{m \leqslant x} \nu_K(m) = \kappa_K x + \frac{x}{2\pi i} \int_{1+\delta-iT}^{1+\delta+iT} \frac{\zeta_K(s)\gamma_K(s)}{1-s}\, x^{-s}\, \mathrm{d}s + O_{n,\varepsilon}\left(x^{1+\varepsilon}\, T^{-1}\right)$$

where γ_K is defined in (2), the contribution of the integral over the horizontal lines being absorbed by the term $x^{1+\varepsilon}\, T^{-1}$ since $x \geqslant (d_K T^n)^{1/2}$. Using Lemma 2.2, the reflection formula implying

$$\Gamma(s)^n = -(1-s)\Gamma(s)^{n-1}\Gamma(s-1)$$

and the identity

$$2^{n-1}\left(\cos \tfrac{\pi s}{2}\right)^{r_1+r_2}\left(\sin \tfrac{\pi s}{2}\right)^{r_2} = \pm\Phi\left(\tfrac{n\pi s}{2}\right)\left\{1 + O_n\left(e^{-\pi|t|}\right)\right\}$$

where $\Phi \in \{\cos, \sin\}$, we get for $\sigma > 1$

$$\begin{aligned}\frac{\gamma_K(s)}{1-s} &= \pm 2^n \left(\tfrac{\pi}{2}\right)^{(n-1)/2} n^{1+n/2} d_K^{-1/2}\, \Gamma\left(ns - \tfrac{n+1}{2}\right)\, \Phi\left(\tfrac{n\pi s}{2}\right)\, \left(2\pi n d_K^{-1/n}\right)^{-ns} \\ &\quad \times \left(1 + O_n\left(e^{-\pi|t|}\right)\right)\left(1 + O_n\left(|s|^{-1}\right)\right)\end{aligned}$$

where $\Phi \in \{\cos, \sin\}$. If $s = 1 + \delta + it$ with $0 < \delta \leqslant 1$ and $|t| \geqslant 1$, then

$$\left|ns - \tfrac{n+1}{2}\right|^2 = \left(\tfrac{n-1}{2} + n\delta\right)^2 + (nt)^2 < \left(\tfrac{3n}{2}\right)^2 + (nt)^2 < 4(nt)^2$$

so that

$$\left|\Gamma\left(ns - \tfrac{n+1}{2}\right)\right| \ll \left|ns - \tfrac{n+1}{2}\right|^{n\sigma - 1 - n/2} e^{-\pi|t|n/2} \ll_n |t|^{-1+n/2+n\delta} e^{-\pi|t|n/2}$$

and hence, for $s = 1 + \delta + it$ with $0 < \delta \leqslant 1$ and $|t| \geqslant 1$, we get

$$\begin{aligned}\frac{\gamma_K(s)}{1-s} &= \pm 2^n \left(\tfrac{\pi}{2}\right)^{(n-1)/2} n^{1+n/2} d_K^{-1/2}\, \Gamma\left(ns - \tfrac{n+1}{2}\right)\, \Phi\left(\tfrac{n\pi s}{2}\right)\, \left(2\pi n d_K^{-1/n}\right)^{-ns} \\ &\quad + O_K\left(|t|^{-2+n/2+n\delta}\right)\end{aligned}$$

and since

$$\left|\zeta_K\left(1 + \delta + it\right)\right| \leqslant \zeta(1+\delta)^n \ll (\log x)^n$$

we obtain

$$\begin{aligned}
&\sum_{m \leqslant x} \nu_K(m) \\
&= \kappa_K x \pm 2^n \left(\tfrac{\pi}{2}\right)^{(n-1)/2} n^{1+n/2} d_K^{-1/2} \\
&\quad \times \frac{x}{2\pi i} \int_{1+\delta-iT}^{1+\delta+iT} \zeta_K(s)\, \Gamma\left(ns - \tfrac{n+1}{2}\right)\, \Phi\left(\tfrac{n\pi s}{2}\right) \left(2\pi n\, x^{1/n} d_K^{-1/n}\right)^{-ns} \mathrm{d}s \\
&\quad + O_K\left(x^{-\delta} (\log x)^n \int_1^T t^{-2+n/2+n\delta}\, \mathrm{d}t\right) + O_{n,\varepsilon}\left(x^{1+\varepsilon}\, T^{-1}\right) \\
&= \kappa_K x \\
&\quad + C_K \frac{x^{\frac{n-1}{2n}}}{2\pi i} \int_{(n-1)/2+n\delta-niT}^{(n-1)/2+n\delta+niT} \zeta_K\left(\tfrac{s}{n} + \tfrac{n+1}{2n}\right) \Gamma(s)\, \Phi\left(\tfrac{\pi s}{2} + \tfrac{\pi(n+1)}{4}\right) \left(c_n x^{1/n}\right)^{-s} \mathrm{d}s \\
&\quad + O_{K,\varepsilon}\left(T^{n/2-1} (\log x)^n + x^{1+\varepsilon}\, T^{-1}\right)
\end{aligned}$$

with

$$C_K := \pm \frac{d_K^{1/(2n)}}{n^{1/2}\pi}, \tag{3}$$

c_n is given in Proposition 2.1 and where we used $T^{n\delta} \leqslant x^{n\delta} = e^n$. Note that the hypothesis $x \geqslant (d_K T^n)^{1/2}$ implies that $T^{n/2-1}(\log x)^n \ll x^{1+\varepsilon}\, T^{-1}$, and hence

$$\begin{aligned}
\sum_{m \leqslant x} \nu_K(m) &= \kappa_K x \\
&\quad + C_K \frac{x^{\frac{n-1}{2n}}}{2\pi i} \int_{(n-1)/2+n\delta-niT}^{(n-1)/2+n\delta+niT} \zeta_K\left(\tfrac{s}{n} + \tfrac{n+1}{2n}\right) \Gamma(s)\, \Phi\left(\tfrac{\pi s}{2} + \tfrac{\pi(n+1)}{4}\right) \left(c_n x^{1/n}\right)^{-s} \mathrm{d}s \\
&\quad + O_{K,\varepsilon}\left(x^{1+\varepsilon}\, T^{-1}\right)
\end{aligned}$$

where $\Phi \in \{\cos, \sin\}$ and C_K is given in (3).

For any $m \in \mathbb{Z}_{\geqslant 1}$, we set

$$I_m(x) := \frac{1}{2\pi i} \int_{(n-1)/2+n\delta-niT}^{(n-1)/2+n\delta+niT} \Gamma(s)\, \Phi\left(\tfrac{\pi s}{2} + \tfrac{\pi(n+1)}{4}\right) \left(c_n (mx)^{1/n}\right)^{-s} \mathrm{d}s \tag{4}$$

where $\Phi \in \{\cos, \sin\}$. Replacing the function ζ_K by its Dirichlet series, we get

$$\sum_{m \leqslant x} \nu_K(m) = \kappa_K x + C_K\, x^{\frac{n-1}{2n}} \sum_{m=1}^{\infty} \frac{\nu_K(m)}{m^{\frac{n+1}{2n}}}\, I_m(x) + O_{K,\varepsilon}\left(x^{1+\varepsilon}\, T^{-1}\right).$$

We then choose

$$T = 2\pi d_K^{-1/n} (Rx)^{1/n} := c_n n^{-1} (Rx)^{1/n}$$

and apply Lemma 2.3 to the integral (4) with $y = c_n(mx)^{1/n}$, $A = \frac{n-1}{2} + n\delta$ and $U = nT$, noticing that

$$\Phi\left(\tfrac{\pi s}{2} + \tfrac{\pi(n+1)}{4}\right) = \pm \begin{cases} \Phi_1\left(\frac{\pi s}{2}\right), & \text{if } n \text{ odd} \\ 2^{-1/2}\left(\Phi_2\left(\frac{\pi s}{2}\right) \mp \Phi_3\left(\frac{\pi s}{2}\right)\right), & \text{if } n \text{ even} \end{cases}$$

where $\Phi_j \in \{\cos, \sin\}$ and $\Phi_2 \neq \Phi_3$. Thus, it is natural to define

$$\varphi_n(z) := \pm \begin{cases} \Phi_1(z), & \text{if } n \text{ odd} \\ \Phi_2(z) \mp \Phi_3(z), & \text{if } n \text{ even} \end{cases}$$

where $\Phi_j \in \{\cos, \sin\}$ and $\Phi_2 \neq \Phi_3$.

Also note that the constraint $A \leqslant U$ is fullfiled whenever $T \geqslant 2$. Taking the choice of T into account, it is necessary to assume that $R \geqslant d_K x^{-1}$.

Setting $\xi_n = 2^{-1/2}$ if n is even and 1 otherwise, Lemma 2.3 provides

$$\begin{aligned} \sum_{m \leqslant x} \nu_K(m) = \kappa_K x &+ C_K\, \xi_n\, x^{\frac{n-1}{2n}} \sum_{m \leqslant R} \frac{\nu_K(m)}{m^{\frac{n+1}{2n}}}\, \varphi_n\left(c_n(mx)^{1/n}\right) \\ &+ O_K\left\{x^{\frac{n-2}{2n}} \sum_{m \leqslant R} \frac{\nu_K(m)}{m^{\frac{n+2}{2n}}} \min\left\{(\log(R/m))^{-1}, (Rx)^{\frac{1}{2n}}\right\}\right\} \\ &+ O_K\left\{x^{\frac{n-2}{2n}} \sum_{m \leqslant R} \frac{\nu_K(m)}{m^{\frac{n+2}{2n}}} \left(\left(Rm^{-1}\right)^{\frac{n-2}{2n}} + 1\right)\right\} \\ &+ O_K\left\{(Rx)^{\frac{n-2}{2n}} \sum_{m > R} \frac{\nu_K(m)}{m^{1+\delta}} \min\left\{(\log(m/R))^{-1}, (Rx)^{\frac{1}{2n}}\right\}\right\} \\ &+ O_{K,\varepsilon}\left(x^{1-1/n+\varepsilon} R^{-1/n}\right). \end{aligned}$$

Since $R \leqslant x$, the 1st error term contributes

$$\begin{aligned} &\ll x^{\frac{n-2}{2n}} \left(\sum_{m \leqslant R/2} + \sum_{R/2 < m \leqslant \lfloor R \rfloor - 1}\right) \frac{\nu_K(m)}{m^{\frac{n+2}{2n}}} (\log(R/m))^{-1} \\ &\quad + x^{\frac{n-1}{2n}} R^{\frac{1}{2n}} \sum_{\lfloor R \rfloor - 1 < m \leqslant R} \frac{\nu_K(m)}{m^{\frac{n+2}{2n}}} \\ &\ll x^{\frac{n-2}{2n}} \sum_{m \leqslant R/2} \frac{\nu_K(m)}{m^{\frac{n+2}{2n}}} + x^{\frac{n-2}{2n}} R \sum_{R/2 < m \leqslant \lfloor R \rfloor - 1} \frac{\nu_K(m)}{m^{\frac{n+2}{2n}}} \frac{1}{R-m} + x^{\frac{n-1}{2n}} R^{-\frac{n+1}{2n}+\varepsilon} \\ &\ll (Rx)^{\frac{n-2}{2n}} R^{\varepsilon} + x^{\frac{n-1}{2n}} R^{-\frac{n+1}{2n}+\varepsilon} \ll x^{1-1/n+\varepsilon} R^{-1/n}. \end{aligned}$$

Similarly, the 2nd error term contributes

$$\ll (Rx)^{\frac{n-2}{2n}} \sum_{m \leqslant R} \frac{\nu_K(m)}{m} \ll (Rx)^{\frac{n-2}{2n}} \log R \ll x^{1-1/n+\varepsilon} R^{-1/n}$$

and the 3rd error term contributes

$$\begin{aligned}
&\ll (Rx)^{\frac{n-2}{2n}} \left(\sum_{R<m\leqslant \lfloor R \rfloor +1} + \sum_{\lfloor R \rfloor +1 < m \leqslant 2R} + \sum_{m>2R} \right) \frac{\nu_K(m)}{m^{1+\delta}} \\
&\quad \times \min \left\{ \left(\log (m/R) \right)^{-1}, (Rx)^{\frac{1}{2n}} \right\} \\
&\ll (Rx)^{\frac{n-1}{2n}} R^{-1+\varepsilon} + (Rx)^{\frac{n-2}{2n}} \left(\sum_{\lfloor R \rfloor +1 < m \leqslant 2R} + \sum_{m>2R} \right) \frac{\nu_K(m)}{m^{1+\delta}} \left(\log (m/R) \right)^{-1} \\
&\ll x^{\frac{n-1}{2n}} R^{-\frac{n+1}{2n}+\varepsilon} + (Rx)^{\frac{n-2}{2n}} R \sum_{\lfloor R \rfloor +1 < m \leqslant 2R} \frac{\nu_K(m)}{m^{1+\delta}} \, \frac{1}{m-R} \\
&\quad + (Rx)^{\frac{n-2}{2n}} \sum_{m>2R} \frac{\nu_K(m)}{m^{1+\delta}} \\
&\ll x^{\frac{n-1}{2n}} R^{-\frac{n+1}{2n}+\varepsilon} + (Rx)^{\frac{n-2}{2n}} R^{\varepsilon} \ll x^{1-1/n+\varepsilon} R^{-1/n}.
\end{aligned}$$

Proposition 2.1 then follows.

3. Exponential sums

In this section, let $F := \nu_K \star \mu$ be the Eratosthenes transform of ν_K. Since $\nu_K(m) \leqslant \tau_n(m)$, we get the bound $|F(m)| \leqslant \tau_{n+1}(m) \ll_{n,\varepsilon} m^{\varepsilon}$. We first prove a more convenient version of Proposition 2.1.

Proposition 3.1. *Let $1 \leqslant T \leqslant R \leqslant x$ and define $E_K(x,T,R)$ to be the error term in the asymptotic formula*

$$\sum_{m \leqslant x} \nu_K(m) = \kappa_K x + O_{n,\varepsilon} \left(E_K(x,T,R) \, x^{\varepsilon} + x^{1-1/n+\varepsilon} R^{-1/n} \right).$$

Then

$$E_K(x,T,R) \ll (xT)^{\frac{n-1}{2n}} + \max_{T<S\leqslant R} \mathcal{S}_K(x,S)$$

where

$$\begin{aligned}
&\mathcal{S}_K(x,S) := x^{\frac{n-1}{2n}} S^{-\frac{n+1}{2n}} \\
&\qquad \times \max_{S \leqslant S_1 \leqslant 2S} \; \max_{\substack{M,N \leqslant S_1 \\ MN \asymp S}} \; \max_{\substack{M<M_1 \leqslant 2M \\ N<N_1 \leqslant 2N}} \left| \sum_{M<m\leqslant M_1} F(m) \sum_{N<d\leqslant N_1} e \left(c_n (xmd)^{1/n} \right) \right| .
\end{aligned}$$

Proof. From Proposition 2.1, we have

$$E_n(x,K) = x^{\frac{n-1}{2n}} \left| \left(\sum_{k \leqslant T} + \sum_{T<k\leqslant R} \right) \frac{\nu_K(k)}{k^{\frac{n+1}{2n}}}\, e\left(c_n (xk)^{1/n}\right) \right|$$

$$\ll x^{\frac{n-1}{2n}} \left\{ \sum_{k\leqslant T} \frac{\nu_K(k)}{k^{\frac{n+1}{2n}}} + \max_{T<S\leqslant R} \left| \sum_{S<k\leqslant 2S} \frac{\nu_K(k)}{k^{\frac{n+1}{2n}}}\, e\left(c_n(xk)^{1/n}\right) \right| \log R \right\}$$

$$\ll x^{\frac{n-1}{2n}} \left\{ T^{\frac{n-1}{2n}} + \max_{T<S\leqslant R} S^{-\frac{n+1}{2n}} \max_{S\leqslant S_1\leqslant 2S} \left| \sum_{S<k\leqslant S_1} \nu_K(k)\, e\left(c_n(xk)^{1/n}\right) \right| \log R \right\}$$

$$\ll x^{\frac{n-1}{2n}} \left\{ T^{\frac{n-1}{2n}} + \max_{T<S\leqslant R} S^{-\frac{n+1}{2n}} \max_{S\leqslant S_1\leqslant 2S} \left| \sum_{S<md\leqslant S_1} F(m)\, e\left(c_n(xmd)^{1/n}\right) \right| \log R \right\}$$

$$\ll x^{\frac{n-1}{2n}} \Bigg\{ T^{\frac{n-1}{2n}} + \max_{T<S\leqslant R} S^{-\frac{n+1}{2n}}$$

$$\times \max_{S\leqslant S_1\leqslant 2S} \max_{\substack{M,N\leqslant S_1 \\ MN\asymp S}} \max_{\substack{M<M_1\leqslant 2M \\ N<N_1\leqslant 2N}} \left| \sum_{M<m\leqslant M_1} F(m) \sum_{N<d\leqslant N_1} e\left(c_n(xmd)^{1/n}\right) \right| R^{\varepsilon} \Bigg\}$$

as asserted. ■

Thus, Proposition 3.1 reduces the initial problem to an exponential sums problem. In the literature, a lot of authors have established non-trivial bounds for so-called sums of type I and II (see [8, 13, 14, 18, 19] for instance). The following first bound is a simplified version of [8, Proposition 5].

Proposition 3.2. *Let $X \geqslant 1$ be a real number, $1 \leqslant M < M_1 \leqslant 2M$ and $1 \leqslant N < N_1 \leqslant 2N$ be integers, $\alpha \in]0,1[$ and $\beta \in \mathbb{R}^* \setminus \{1\}$. Let $(a_m),(b_n) \in \mathbb{C}$ such that $|a_m| \leqslant 1$ and $|b_n| \leqslant 1$. Then*

$$(MN)^{-\varepsilon} \sum_{M<m\leqslant M_1} a_m \sum_{N<n\leqslant N_1} b_n\, e\left(X\left(\frac{m}{M}\right)^{\alpha}\left(\frac{n}{N}\right)^{\beta}\right)$$
$$\ll \left(XM^3N^2\right)^{1/4} + M^{3/4}N + MN^{1/2} + X^{-1/4}MN.$$

Proof. Let S be the sum at the left-hand side. From the Cauchy-Schwarz inequality and [13, Lemma 8], we have

$$|S|^2 \leqslant \sum_{M<m\leqslant 2M} |a_m|^2 \sum_{M<m\leqslant 2M} \left| \sum_{N<n\leqslant N_1} b_n\, e\left(X\left(\frac{m}{M}\right)^{\alpha}\left(\frac{n}{N}\right)^{\beta}\right) \right|^2$$

$$\ll M \sum_{M<m\leqslant 2M} \sum_{n_1}\sum_{n_2} b_{n_1}\overline{b_{n_2}}\, e\left(X\left(\frac{m}{M}\right)^{\alpha} \frac{n_1^{\beta}-n_2^{\beta}}{N^{\beta}}\right)$$

$$\ll MX^{1/2}\mathcal{B}_1^{1/2}\mathcal{B}_2^{1/2}$$

where

$$\mathcal{B}_1 := \sum_{\substack{M<m_1,m_2\leqslant 2M \\ |m_1^\alpha - m_2^\alpha|\leqslant M^\alpha X^{-1}}} 1 \qquad \text{and} \qquad \mathcal{B}_2 := \sum_{\substack{N<n_1,\dots,n_4\leqslant 2N \\ \left|n_1^\beta - n_2^\beta - n_3^\beta + n_4^\beta\right|\leqslant N^\beta X^{-1}}} 1.$$

Now from [2, Lemma 1] we infer

$$M^{-\varepsilon}\mathcal{B}_1 \ll M + M^2 X^{-1}$$

and from [13, Theorem 2] we have

$$N^{-\varepsilon}\mathcal{B}_2 \ll N^2 + N^4 X^{-1}$$

so that

$$|S|^2 \ll M^{1+\varepsilon} N^\varepsilon X^{1/2}\left(M^{1/2} + MX^{-1/2}\right)\left(N + N^2 X^{-1/2}\right)$$

implying the asserted result. ■

The second estimate is a consequence of van der Corput's second derivative test for exponential sums.

Proposition 3.3. *Let $X \geqslant 1$ be a real number, $1 \leqslant M < M_1 \leqslant 2M$ and $1 \leqslant N < N_1 \leqslant 2N$ be integers, $\alpha \in \left]0, \frac{1}{2}\right]$ and $\beta \in \mathbb{R}\setminus\mathbb{Z}_{\geqslant 0}$. Let $(a_m) \in \mathbb{C}$ such that $|a_m| \ll m^\varepsilon$. Then*

$$M^{-\varepsilon} \sum_{M<m\leqslant M_1} a_m \sum_{N<n\leqslant N_1} e\left(X\left(\frac{m}{M}\right)^\alpha \left(\frac{n}{N}\right)^\beta\right) \ll MX^{1/2} + X^{-1}MN.$$

Proof. Let S be the sum at the left-hand side. As in Proposition 3.2, using the Cauchy-Schwarz inequality we get

$$|S|^2 \ll M^{1+2\varepsilon} \sum_{M<m\leqslant M_1} \left| \sum_{N<n\leqslant N_1} e\left(Y_m n^\beta\right)\right|^2$$

where $Y_m := X\left(M^{-1}m\right)^\alpha N^{-\beta}$. Now we use van der Corput's second derivative test in the following shape [3, Theorem 2.9]: *let $z \in \mathbb{R}\setminus\{0\}$, $N < N_1 \leqslant 2N$ be integers and $a \in \mathbb{R}\setminus\mathbb{Z}_{\geqslant 0}$. Then*

$$\sum_{N<n\leqslant N_1} e\left(zn^a\right) \ll \left(|z|N^a\right)^{1/2} + N^{1-a}|z|^{-1}.$$

This gives

$$\begin{aligned} |S|^2 &\ll M^{1+2\varepsilon}\left\{\frac{X}{M^\alpha}\sum_{M<m\leqslant M_1} m^\alpha + \left(\frac{N}{X}\right)^2 M^{2\alpha}\sum_{M<m\leqslant M_1}\frac{1}{m^{2\alpha}}\right\} \\ &\ll M^{2\varepsilon}\left(M^2 X + X^{-2}(MN)^2\right) \end{aligned}$$

achieving the proof. ■

Proposition 3.4. *Let $X \geqslant 1$ be a real number, $1 \leqslant M < M_1 \leqslant 2M$ and $1 \leqslant N < N_1 \leqslant 2N$ be integers, $\alpha, \beta \in \mathbb{R}$ such that $\alpha\beta(\alpha-1)(\beta-1) \neq 0$. Let $(a_m), (b_n) \in \mathbb{C}$ such that $|a_m| \leqslant 1$ and $|b_n| \leqslant 1$ and set $\mathcal{L} := \log(XMN+2)$. Then*

$$\mathcal{L}^{-2} \sum_{M<m\leqslant M_1} a_m \sum_{N<n\leqslant N_1} b_n\, e\left(X\left(\frac{m}{M}\right)^{\alpha}\left(\frac{n}{N}\right)^{\beta}\right)$$
$$\ll \left(XM^3N^4\right)^{1/5} + \left(X^4M^{10}N^{11}\right)^{1/16} + \left(XM^7N^{10}\right)^{1/11} + MN^{1/2} + X^{-1/2}MN.$$

Proof. This is [18, (2.2)]. See also [19, (2.16)]. ∎

The last result plays a crucial part in the proof of Theorem 1.1. To this end, we use one of the sharpest estimate for triple exponential sums established by Robert & Sargos in [13].

Proposition 3.5. *Let $X \geqslant 1$ be a real number, $1 \leqslant M < M_1 \leqslant 2M$ and $1 \leqslant N < N_1 \leqslant 2N$ be integers, $\alpha, \beta \in \mathbb{R}$ such that $(\alpha-1)(\alpha-2)\alpha\beta \neq 0$. Let $(a_m), (b_n) \in \mathbb{C}$ such that $|a_m| \leqslant 1$ and $|b_n| \leqslant 1$. If $M \gg X$, then*

$$(MN)^{-\varepsilon} \sum_{M<m\leqslant M_1} a_m \sum_{N<n\leqslant N_1} b_n e\left(X\left(\frac{m}{M}\right)^{\alpha}\left(\frac{n}{N}\right)^{\beta}\right)$$
$$\ll \left(XM^5N^7\right)^{1/8} + N\left(X^{-2}M^{11}\right)^{1/12} + \left(X^{-3}M^{21}N^{23}\right)^{1/24} + M^{3/4}N + X^{-1/4}MN.$$

Proof. Let S be the sum at the left-hand side. By the Cauchy-Schwarz inequality, we have

$$|S|^2 \leqslant N \sum_{N<n\leqslant 2N} \left|\sum_{M<m\leqslant M_1} a_m e\left(Y_n m^{\alpha}\right)\right|^2$$

where $Y_n := XM^{-\alpha}N^{-\beta}n^{\beta}$, and using van der Corput's A-process [3, (2.3.5)], we also have for any integer $1 \leqslant H \leqslant M$

$$\left|\sum_{M<m\leqslant M_1} a_m e\left(Y_n m^{\alpha}\right)\right|^2$$
$$\leqslant \frac{2M^2}{H} + \frac{4M}{H} \operatorname{Re}\left\{\sum_{h<H}\left(1-\frac{h}{H}\right) \sum_{M<m\leqslant M_1-h} \overline{a_m} a_{m+h} e\left(Y_n\left[(m+h)^{\alpha}-m^{\alpha}\right]\right)\right\}.$$

Now by Taylor's formula, we have

$$(m+h)^{\alpha} - m^{\alpha} = \alpha h m^{\alpha-1} + \int_0^h (h-t) f''(t+m)\mathrm{d}t := \alpha h m^{\alpha-1} + r(h,m)$$

so that

$$|S|^2 \ll \frac{(MN)^2}{H} + \frac{MN}{H} \left| \sum_{h<H} \sum_{M<m\leqslant M_1-h} \sum_{N<n\leqslant 2N} \left(1-\frac{h}{H}\right) \overline{a_m} a_{m+h} e\left(Y_n \alpha h m^{\alpha-1}\right) e(u(n)) \right|$$

where $u(n) := Y_n r(h,m)$. The total variation of the function $e(u(n))$ does not exceed

$$\begin{aligned}\sum_{N<n\leqslant 2N} |e(u(n+1)) - e(u(n))| &\leqslant 2\pi \sum_{N<n\leqslant 2N} |u(n+1) - u(n)| \\ &= 2\pi X M^{-\alpha} N^{-\beta} |r(h,m)| \sum_{N<n\leqslant 2N} \left\{(n+1)^\beta - n^\beta\right\} \\ &\ll X\left(HM^{-1}\right)^2\end{aligned}$$

so that by partial summation we get, setting $Z := \alpha X H_1 M^{-1}$

$$\begin{aligned}|S|^2 \ll \frac{(MN)^2}{H} &+ \frac{MN}{H}\left\{1 + X\left(\frac{H}{M}\right)^2\right\} \\ &\times \max_{H_1 \leqslant H} \sum_{H_1<h\leqslant 2H_1} \sum_{M<m\leqslant 2M} \left| \sum_{N<n\leqslant 2N} e\left(Z \frac{h}{H_1}\left(\frac{m}{M}\right)^{\alpha-1}\left(\frac{n}{N}\right)^{\beta}\right)\right| \log H.\end{aligned}$$

From [13, Theorem 3] we obtain

$$\begin{aligned}(HMN)^{-\varepsilon/2} |S|^2 \ll \frac{(MN)^2}{H} &+ \left(XM^5N^7\right)^{1/4} + H^2\left(X^5M^{-3}N^7\right)^{1/4} \\ &+ X(NH)^2M^{-1/2} + HMN^2 + M^{3/2}N^2 + X^{-1}H^{-1}M^3N^2\end{aligned}$$

and hence

$$\begin{aligned}(HMN)^{-\varepsilon/4} |S| \ll \frac{MN}{H^{1/2}} &+ \left(XM^5N^7\right)^{1/8} + H\left(X^5M^{-3}N^7\right)^{1/8} + X^{1/2}HM^{-1/4}N \\ &+ (HM)^{1/2}N + M^{3/4}N + X^{-1/2}H^{-1/2}M^{3/2}N\end{aligned}$$

and Srinivasan's optimization lemma [15, Lemma 4] gives the asserted result with the supplementary terms

$$\left(X^5M^{-3}N^7\right)^{1/8} + N\left(X^2M^7\right)^{1/12} + N\left(X^2M^{-1}\right)^{1/4} + \left(X^5M^{13}N^{23}\right)^{1/24}$$

which are all dominated by the sum of Proposition 3.5 since $M \gg X$. ■

4. Proof of Theorem 1.1

In what follows, $X \geqslant 1$ is a real number, $M, N, S \in \mathbb{Z}_{\geqslant 1}$ such that $MN \asymp S$. We also use the notation $\mathcal{S}_K(x,S)$ of Proposition 3.1.

1st case: $S^{1/2-1/(2n)} \ll N \ll S^{1/2}$.

Proposition 3.2 with $X = (xMN)^{1/n}$ gives

$$S^{-\varepsilon}\mathcal{S}_K(x,S) \ll x^{\frac{2n-1}{4n}} S^{\frac{n-1}{8n}} + x^{\frac{n-1}{2n}} S^{\frac{3n-4}{8n}} + (Sx)^{\frac{2n-3}{4n}}. \tag{5}$$

2nd case: $S^{1/2} \ll N \ll S^{1/2+1/(2n)}$.

We use Proposition 3.2 again reversing the roles of M and N, which gives

$$S^{-\varepsilon}\mathcal{S}_K(x,S) \ll x^{\frac{2n-1}{4n}} S^{\frac{n-1}{8n}} + x^{\frac{n-1}{2n}} S^{\frac{3n-4}{8n}} + (Sx)^{\frac{2n-3}{4n}}. \tag{6}$$

3rd case: $N \gg S^{1/2+1/(2n)}$.

We use Proposition 3.3 and get

$$S^{-\varepsilon}\mathcal{S}_K(x,S) \ll x^{1/2} S^{-\frac{1}{2n}} + (Sx)^{\frac{n-3}{2n}}. \tag{7}$$

4th case: $M^{1/3} \ll N \ll S^{1/2-1/(2n)}$.

We use Proposition 3.4, noticing that the hypothesis $N \gg M^{1/3}$ implies that $MN^{1/2} \ll S^{7/8}$, so that

$$\begin{aligned} S^{-\varepsilon}\mathcal{S}_K(x,S) \ll\ & x^{\frac{5n-3}{10n}} S^{\frac{n-2}{5n}} + x^{\frac{2n-1}{4n}} S^{\frac{5n-9}{32n}} + x^{\frac{11n-9}{22n}} S^{\frac{3n-6}{11n}} \\ & + x^{\frac{n-1}{2n}} S^{\frac{3n-4}{8n}} + x^{\frac{11n-12}{22n}} S^{\frac{15n-33}{44n}} + (Sx)^{\frac{n-2}{2n}}. \end{aligned} \tag{8}$$

5th case: $N \ll M^{1/3}$.

We apply Proposition 3.5. First note that $S \asymp MN \ll M^{4/3}$ so that $M \gg S^{3/4}$. Thus, with $X = (xMN)^{1/n} \asymp (Sx)^{1/n}$, the constraint $M \gg X$ is fullfiled as soon as $S^{3/4} \gg (Sx)^{1/n}$, i.e. $S \gg x^{\frac{4}{3n-4}}$. Since $S > T$, it is sufficient to choose $T = x^{\frac{4}{3n-4}}$ to ensure the validity of this constraint.

We then get

$$\begin{aligned} S^{-\varepsilon}\mathcal{S}_K(x,S) \ll\ & x^{\frac{4n-3}{8n}} S^{\frac{5n-9}{24n}} + x^{\frac{3n-4}{6n}} S^{\frac{4n-6}{9n}} + x^{\frac{4n-5}{8n}} S^{\frac{29n-45}{72n}} \\ & + x^{\frac{n-1}{2n}} S^{\frac{2n-3}{6n}} + (Sx)^{\frac{2n-3}{4n}}. \end{aligned} \tag{9}$$

Let $F_K(x,T,R) := E_K(x,T,R)\, x^{\varepsilon} + x^{1-1/n+\varepsilon} R^{-1/n}$ be the error term in Proposition 3.1. Putting (5) – (9) all together, we derive

$$\begin{aligned} (Rx)^{-\varepsilon} F_K(x,T,R) \ll\ & (xT)^{\frac{n-1}{2n}} + x^{\frac{2n-1}{4n}} R^{\frac{n-1}{8n}} + x^{\frac{n-1}{2n}} R^{\frac{3n-4}{8n}} + (Rx)^{\frac{2n-3}{4n}} \\ & + x^{1/2} T^{-\frac{1}{2n}} + (Rx)^{\frac{n-2}{2n}} + x^{\frac{5n-3}{10n}} R^{\frac{n-2}{5n}} + x^{\frac{2n-1}{4n}} R^{\frac{5n-9}{32n}} \\ & + x^{\frac{11n-9}{22n}} R^{\frac{3n-6}{11n}} + x^{\frac{n-1}{2n}} R^{\frac{3n-4}{8n}} + x^{\frac{11n-12}{22n}} R^{\frac{15n-33}{44n}} \\ & + x^{\frac{4n-3}{8n}} R^{\frac{5n-9}{24n}} + x^{\frac{3n-4}{6n}} R^{\frac{4n-6}{9n}} + x^{\frac{4n-5}{8n}} R^{\frac{29n-45}{72n}} \\ & + x^{\frac{n-1}{2n}} R^{\frac{2n-3}{6n}} + x^{1-1/n} R^{-1/n}. \end{aligned}$$

Taking $T = x^{\frac{4}{3n-4}}$ and noticing that the sum of the 6th and 15th term is dominated by the sum of the 4th and 10th, we deduce that the error term in Proposition 3.1 does not exceed

$$\begin{aligned}
&\ll x^{\frac{3n-3}{6n-8}} + x^{\frac{2n-1}{4n}} R^{\frac{n-1}{8n}} + x^{\frac{n-1}{2n}} R^{\frac{3n-4}{8n}} + (Rx)^{\frac{2n-3}{4n}} + x^{\frac{5n-3}{10n}} R^{\frac{n-2}{5n}} \\
&\quad + x^{\frac{2n-1}{4n}} R^{\frac{5n-9}{32n}} + x^{\frac{11n-9}{22n}} R^{\frac{3n-6}{11n}} + x^{\frac{11n-12}{22n}} R^{\frac{15n-33}{44n}} \\
&\quad + x^{\frac{4n-3}{8n}} R^{\frac{5n-9}{24n}} + x^{\frac{3n-4}{6n}} R^{\frac{4n-6}{9n}} + x^{\frac{4n-5}{8n}} R^{\frac{29n-45}{72n}} + x^{1-1/n} R^{-1/n}.
\end{aligned}$$

Theorem 1.1 follows by choosing $R = x^{\frac{2n-1}{2n+1}}$.

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$$\lambda_n = \frac{n-1}{n+2} \qquad \bigl(n \geqslant 4,\ n = p \ \text{ or } \ n = p^2,\ p \text{ prime}\bigr).$$

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References

[1] F.V. Atkinson, *A divisor problem*, Quart. J. Oxford **12** (1941), 193–200.

[2] E. Fouvry, H. Iwaniec, *Exponential sums with monomials*, J. Number Theory **33** (1989), 311–333.

[3] S.W. Graham, G. Kolesnik, *Van der Corput's Method of Exponential Sums*, Cambridge University Press, 1991.

[4] G.H. Hardy, J.E. Littlewood, *The approximate functional equation in the theory of the zeta-function, with applications to the divisor-problems of Dirichlet and Piltz*, Proc. London Math. Soc. **2**(21) (1922), 39–74.

[5] M.N. Huxley, *Exponential sums and lattice points III*, Proc. London Math. Soc. **87** (2003), 591–609.

[6] M.N. Huxley, N. Watt, *The number of ideals in a quadratic field II*, Israel J. Math. **120** (2002), 125–153.

[7] G. Kolesnik, *The estimation of multiple exponential sums*, in: Recent Progress in Analytic Number Theory, Vol. I, (Halberstam, H., Hooley, L.; eds.). London: Academic Press. (1981), 231–246.

[8] E. Kowalski, O. Robert, J. Wu, *Small gaps in coefficients of L-functions and $\mathfrak{B}$-free numbers in short intervals*, Rev. Mat. Iberoam **23** (2007), 281–326.

[9] E. Landau, *Über die Heckesche Funktionalgleichung*, Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen (1917), 102–111.

[10] E. Landau, *Einführung in die Elementare und Analytische Theorie der Algebraischen Zahlen und der Ideale*, AMS Chelsea Publishing, 2005, reprint of the 1949 Edition.

[11] W. Müller, *On the distribution of ideals in cubic number fields*, Monatsh. Math. **106** (1988), 211–219.

[12] W.G. Nowak, *On the distribution of integer ideals in algebraic number fields*, Math. Nachr. **161** (1993), 59–74.

[13] O. Robert, P. Sargos, *Three-dimensional exponential sums with monomials*, J. Reine Angew. Math. **591** (2006), 1–20.

[14] P. Sargos, J. Wu, *Multiple exponential sums with monomials and their applications in number theory*, Acta Math. Hungar. **87** (2000), 333–354.

[15] B.R. Srinivasan, *On the Van der Corput's and Nieland's results on the Dirichlet's divisor problem and the circle problem*, Proc. Natl. Inst. Sci. India, part. A **28** (1962), 732–742.

[16] E.C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Oxford Science Publications, 1986, revised by Heath-Brown.

[17] H. Weber, *Lehrbuch der Algebra*, vol II, Braunschweig, 1896.

[18] J. Wu, *On the average number of unitary factors of finite abelian groups*, Acta Arith. **84** (1998), 17–29.

[19] J. Wu, *On the primitive circle problem*, Monatsh. Math. **135** (2002), 69–89.

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