Functiones et Approximatio
50.2 (2014), 283–296
doi: 10.7169/facm/2014.50.2.6

A GENERALIZATION OF DREWNOWSKI'S RESULT ON THE CANTOR-BERNSTEIN TYPE THEOREM FOR A CLASS OF NONSEPARABLE BANACH SPACES

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Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: Let X_a denote the order continuous part of a Banach lattice X, and let Γ be an uncountable set. We extend Drewnowski's theorem on the comparison of linear dimensions between Banach spaces having uncountable symmetric bases to a class of discrete Banach lattices, the so-called D-spaces.

We show that if X and Y are two D-spaces and there are continuous linear injections (not necessarily embeddings) from X into Y and vice versa, then X and Y are order-topologically isomorphic. In the proof we apply a theorem on the extension of an order isomorphism from X_a onto Y_a to an order isomorphism from X onto Y, the classical Drewnowski's theorem, and a supplement of Troyanski's theorem on embeddings of $\ell_1(\Gamma)$ spaces into a Banach space with an uncountable symmetric basis.

Our result applies to the class of Orlicz spaces $\ell_\varphi(\Gamma)$, where φ is an Orlicz function.

Keywords: linear dimension, discrete Banach lattice, Levi norm, uncountable symmetric basis, Orlicz space.

1. Introduction

We use standard notations and for notions undefined below we refer the reader to the next section and to the monographs [4, 19, 23]. In what follows, Γ and Θ denote *fixed* uncountable sets, all Banach spaces are of infinite dimension, and all operators are linear and continuous. For X a Banach lattice, the symbol X_a stands for the order continuous part of X.

Following Banach [5, Chap. XII, p. 193], for two Banach spaces X and Y, we write $\dim_\ell(X) \leqslant \dim_\ell(Y)$ if X embeds isomorphically into Y. If both $\dim_\ell(X) \leqslant \dim_\ell(Y)$ and $\dim_\ell(Y) \leqslant \dim_\ell(X)$, we write $\dim_\ell(X) = \dim_\ell(Y)$ and say that X *and Y have the same linear dimension.*

2010 Mathematics Subject Classification: primary: 46B42; secondary: 46B26, 46B45, 47B37

In this paper, we consider theorems of the following form, which we call Cantor-Bernstein type theorems:

Let $\mathcal{B}$ be a fixed class of Banach spaces, and let $X, Y \in \mathcal{B}$ be such that there are continuous linear injections from X into Y and from Y into X (in particular, this is so if $\dim_\ell(X) = \dim_\ell(Y)$). *Then X and Y are isomorphic.*

In Sections 3, 4, and 5 we extend remarkable results obtained in 1987 by L. Drewnowski (Propositions 1.2 and 1.4 below) concerning the comparison of linear dimensions between nonseparable Banach spaces having uncountable symmetric bases. Notice that, if X is such a *real* space, it can be considered as a discrete Banach lattice with $X = X_a$ (see [16, p. 2]). In our main result, stated in Theorem 4.2, we show that the series of the equivalences in the Drewnowski's theorems remain true also in a large class of discrete Banach lattices X with $X \neq X_a$.

In 1932, Banach and Mazur [6] proved that $\dim_\ell(C[0,1]) = \dim_\ell(C[0,1] \oplus \ell_1)$, yet the spaces $C[0,1]$ and $C[0,1] \oplus \ell_1$ are not isomorphic because $\dim_\ell(C[0,1]^*) \neq \dim_\ell((C[0,1] \oplus \ell_1)^*)$.

It is interesting to note that Banach obtained the following positive result [5, Lemme and Théorème 4–6 on pp. 202–203]:

Proposition 1.1. *If $p, q \in (1, \infty)$, then* $\dim_\ell(L_p[0,1]) = \dim_\ell(L_q[0,1])$ *if and only if $p = q$.*

Cf. [18]; see also [14] for details of the many contributors to this achievement and an account of its proof.

In 1987, Drewnowski [8, Theorem] extended the above Banach's result to a class of nonseparable Banach spaces as follows:

Proposition 1.2. *Let X and Y be nonseparable Banach spaces having uncountable symmetric bases $(x_\gamma)_{\gamma\in\Gamma}$ and $(y_t)_{t\in\Theta}$, respectively. Then the following conditions are equivalent:*

(i) *X and Y have the same linear dimension.*
(ii) *X and Y are isomorphic.*
(iii) *There is a bijection $f\colon \Gamma \to \Theta$ and an isomorphism $T\colon X \to Y$ such that $T(x_\gamma) = y_{f(\gamma)}$, for every $\gamma \in \Gamma$.*

The proof uses the result below, due to Troyanski [20, Corollary 2], which, roughly speaking, asserts that ℓ_1-bases have maximal linear dimension in the family of uncountable symmetric bases (recall that a symmetric basis $(x_\gamma)_{\gamma\in\Gamma}$ for some Banach space is said to be an ℓ_1-*basis*, whenever the basis $(x_\gamma)_{\gamma\in\Gamma}$ is equivalent to the unit vector basis $(e_\gamma)_{\gamma\in\Gamma}$ of $\ell_1(\Gamma)$).

Proposition 1.3. *Let X be a (nonseparable) Banach space with an uncountable symmetric basis $(x_t)_{t\in\Theta}$. If X contains an isomorphic copy of $\ell_1(\Gamma)$, then (x_t) is an ℓ_1-basis.*

(Note that a similar result [20, Corollary 1] holds for $c_0(\Gamma)$ instead of $\ell_1(\Gamma)$. See also [9, 10] and the monograph [11] for other results in this direction.)

In the same paper [8], Drewnowski observes that in some cases the assumptions in Proposition 1.2 can be weakened: he indicates how to adapt its proof to obtain the following result [8, Corollary on p. 161], which we extend in Theorem 4.2 to a larger class of Banach spaces:

Proposition 1.4. *Let X and Y be nonseparable Banach spaces having uncountable symmetric bases $(x_\gamma)_{\gamma\in\Gamma}$ and $(y_t)_{t\in\Theta}$, respectively. Assume further that both the bases are non-ℓ_1 bases. Then conditions* (i)–(iii) *in Proposition 1.2 are equivalent to the condition*

(iv) *There are continuous linear injections $X \to Y$ and $Y \to X$.*

This result readily applies to the class of Orlicz spaces $\ell_\varphi(\Gamma)$, where φ is an Orlicz function fulfilling the Δ_2-condition at zero. Then the space $\ell_\varphi(\Gamma) = (\ell_\varphi(\Gamma))_a$ has an uncountable symmetric basis, and the basis is an ℓ_1-basis if and only if φ is equivalent at zero to the identity.

From our Theorem 4.2 it follows that the equivalence $(i) \iff (ii)$ in Proposition 1.2 holds true in the *whole* class of (real) spaces of the form $\ell_\varphi(\Gamma)$, where $\mathrm{card}(\Gamma) \geqslant 2^{\aleph_0}$ and the Orlicz function φ is arbitrary (see Section 5).

Here one should note that there are separable Orlicz sequence spaces with uncountably non-equivalent symmetric bases [15, Theorem 4.b.9]. This rules out the equivalence of conditions (ii) and (iii) in Proposition 1.2 for the separable case. On the other hand, as far as we know, it is an open question of whether or not conditions (i) and (ii) in Proposition 1.2 are equivalent for sequence spaces with a symmetric basis.

The paper is organized as follows. In the next section, we fix notation and present the basic facts needed for the proof of our main results. In Section 3, we address the 'critical' ℓ_1-case : (*) in Theorem 3.1 we supplement Troyanski's Proposition 1.3, changing the role of the spaces involved and considering continuous injections instead of isomorphic embeddings, and (**) in Theorem 3.3 we show that, for every pair G, D of infinite subsets of the interval $[0,1]$, there exist a continuous injection from $\ell_1(D)$ into $\ell_1(G)$. These results suggest it makes sense to restrict our studies to (nonseparable) discrete Banach lattices with a set of atoms of the cardinality at least continuum. In Section 4, we introduce the notion of a D-space and prove our main theorem for such spaces. In Section 5 we illustrate how our results work for a class of "big" discrete Orlicz spaces.

2. Preliminaries

Throughout this section, X denotes either an Archimedean vector lattice or a Banach space; in the latter case $\| \ \|$ denotes a norm on X, and if, additionally, X is a Banach lattice, the norm is a lattice norm (i.e., the condition $|x| \leqslant |y|$ implies that $\|x\| \leqslant \|y\|$; here the symbol $|x|$ denotes the modulus of $x \in X$).

Let X be a Banach space, and let J be an infinite set.

A family $(x_j)_{j\in J}$ of elements in X is said to be an unconditional basis of X if, for every $x \in X$ there is a unique family of scalars $(a_j)_{j\in J}$ such that the series

$$x = \sum_{j\in J} a_j x_j \tag{$*$}$$

converges unconditionally in X, i.e., for every $\varepsilon > 0$ there is a finite subset F of J such that $\left\| x - \sum_{j\in F'} a_j x_j \right\| \leqslant \varepsilon$ for every finite subset F' of J such that $F' \supset F$. In particular, for every $x \in X$, at most countably many coordinates a_j are nonzero. Then the support of $x \in X$ is defined as $s(x) := \{j \in J : a_j \neq 0\}$, and $s(x)$ is a countable subset of J.

The uniqueness of the representation $(*)$ of x by means of the family of scalars $(a_j)_{j\in J}$ allows us to define a family $(x_j^*)_{j\in J}$ of (continuous) functionals, associated to the basis $(x_j)_{j\in J}$, by the formula $x_j^*(x) = a_j$, $j \in J$. Hence $x_j^*(x_k) = \delta_{jk}$, where δ_{jk} is the Kronecker symbol, and $x = \sum_{j\in J} x_j^*(x) x_j$ for every $x \in X$.

The basis $(x_j)_{j\in J}$ is called *symmetric* if, for every sequence (j_n) in J the basic sequence (x_{j_n}) is symmetric in the usual sense [15, p. 113]: for any sequence (r_n) in J the basic sequences (x_{j_n}) and (x_{r_n}) are equivalent. For details about uncountable unconditional and symmetric bases, we refer the reader to the monographs [11, 19].

Now let X denote a vector lattice, and let X^+ denote the cone of non-negative elements of X. The lattice X is said to be $[\sigma-]$ Dedekind complete if every nonempty [countable, resp.] subset A of X, with $a \leqslant b$ for all $a \in A$ and some $b \in X$, has a supremum $\sup A$ in X.

Two vector lattices X, Y are said to be *order-isomorphic* if there is a linear bijection R from X onto Y preserving finite suprema: $R(\sup\{x_1, x_2\}) = \sup\{R(x_1), R(x_2)\}$ for all $x_1, x_2 \in X$; then R is referred to as an order isomorphism. If, additionally, X, Y are Banach lattices and R is a topological isomorphism, then we say that these spaces are *order-topologically isomorphic*.

An element $0 < e \in X^+$ is said to be *discrete* (or an atom) if, for every $x \in X$ such that $0 < x \leqslant e$, there exists a real number $\lambda > 0$ such that $x = \lambda a$. The symbol A_X stands for the set of all discrete elements of X. A vector lattice X is said to be *discrete* (or *atomic*) whenever A_X is a maximal disjoint system in X (i.e., the condition $|x| \wedge e = 0$ for all $e \in A_X$ implies $x = 0$). This is equivalent to the requirement that X is order-isomorphic to an order dense (i.e., cofinal) sublattice of the vector lattice of the form $\mathbb{R}^\Gamma$. Here, Γ can be chosen so that $A_X = \{e_\gamma : \gamma \in \Gamma\}$ (see [3, Theorem 2.17, pp. 17–18]).

An order ideal J of X is said to be [super-] order dense in X if, for every $x \in X^+$, there is a [countable, resp.] subset A of J such that $x = \sup A$; then $x = \sup([0, x] \cap J)$. Without loss of generality we may assume that A is directed upward, and if $A = \{u_n\}_{n=1}^\infty$, then $u_n \leqslant u_{n+1} \leqslant x$ for all n.

An element x in a Banach lattice X is said to be *order-continuous* if for every net (x_α) in X with $|x| \geqslant |x_\alpha| \downarrow 0$, we have that $\|x_\alpha\| \to 0$. The order ideal X_a, of all order-continuous elements of X, is referred to as *the order-continuous part of* X, and X_a is norm-closed in X (see e.g. [22, p. 60]). The ideal X_a is Dedekind complete (with the ordering inherited form X). If $X = X_a$, we say that X has order continuous norm, or that X *is order continuous* (o.c.).

It is easy to check that, for X a discrete Banach lattice, the order continuous part X_a is an order dense ideal in X. In this case, we have

$$X_a = \overline{\operatorname{lin}(A_X)} \qquad \text{(norm closure)},$$

and A_X is an unconditional basis for X_a.

The classical Banach lattices $X = \ell_p(\Gamma)$, $1 \leqslant p < \infty$ are o.c., and for $Y = \ell_\infty(\Gamma)$ we have $Y_a = c_0(\Gamma)$. Here the set of the standard unit vectors $e_\gamma : \gamma \in \Gamma$, forms a symmetric basis for the respective spaces X and Y_a. For other examples see Section 5.

The norm $\| \ \|$ on a Banach lattice X is said to be *Levi* [resp., σ-Levi] if every norm-bounded increasing net [resp., sequence] in X^+ has a supremum (cf., [3, Definition 9.3, p. 61]). In this case, we also say that X *has the Levi* [*resp.,* σ-*Levi*] *property.* As it is pointed out in [2] (see also [1]), this property appears in the literature under many different names. For example, the terms *monotone complete norm* and σ-*Levi norm* are identical.

It is known that, for every Banach lattice X, its dual X^* has the Levi property [17, Theorems 2.4.19 and 2.4.21], and it is obvious that every Banach lattice X with the Levi [resp., σ-Levi] property is also [resp., σ-]Dedekind-complete.

The following extension theorem will be applied in the proof of Theorem 4.2.

Theorem 2.1. *Let X, Y be two Banach lattices with the* [σ-]*Levi property, and let J_1, J_2 be two* [*super-*]*order dense and norm-closed ideals of X and Y, respectively. Then every order isomorphism R from J_1 onto J_2 can be extended to an order isomorphism $\widetilde{R}$ from X onto Y. The extension is of the form*

$$\widetilde{R}(x) = \sup R([0,x] \cap J_1), \ \ x \in X^+. \tag{2.1}$$

Proof. By following the lines of the proof of Veksler's Theorem [4, Theorem 4.12], if we establish that, for every $x \in X^+$, the supremum $\widetilde{R}(x) := \sup R([0,x] \cap J_1)$ exists in Y, then we can conclude $\widetilde{R}$ is an additive and positively homogeneous operator on X^+ which extends to a positive operator on X (denoted also by $\widetilde{R}$) by the formula $\widetilde{R}(x) = \widetilde{R}(x^+) - \widetilde{R}(x^-)$. Let us note that then $\widetilde{R}(x^+) \wedge \widetilde{R}(x^-) = 0$, whence $|\widetilde{R}(x)| = \widetilde{R}(|x|)$, thus $\widetilde{R}$ is an order homomorphism.

The proof of (2.1) will be given only for the case when X and Y are σ-Levi and the ideals J_1, J_2 are super-order dense in X and Y, respectively, as the proof of the other case is similar. Let us fix $x \in X^+$.

Claim. *If $x = \sup\{u_n \colon n \in \mathbb{N}\}$ for a nondecreasing sequence (u_n) in J_1^+, then $\sup R([0,x] \cap J_1) = \sup_{n \geqslant 1} R(u_n)$.*

The set $A = \{u_n \colon \ n \in \mathbb{N}\}$ is topologically bounded in X (i.e. $\alpha_n u_n \to 0$ for every sequence $(\alpha_n) \in c_0$), thus from the continuity of R (see [4, Theorem 4.3]) it follows that the set $\{Ru_n \colon \ n \in \mathbb{N}\}$ is topologically bounded and directed upward in Y, too. Since Y is σ-Levi, the element $b := \sup\{Ru_n \colon \ n \in \mathbb{N}\}$ exists in Y^+. Moreover, for every $u \in [0,x] \cap J_1$ we have $u = u \wedge x = \sup\{u \wedge u_n \colon \ n \in \mathbb{N}\}$, thus from the order continuity of R we obtain $Ru = \sup\{R(u \wedge u_n) \colon n \in \mathbb{N}\} \leqslant \sup_{n\geqslant 1} Ru_n = b$. It follows that b is the least upper bound of the set $R([0,x] \cap J_1)$, thus $b = \sup R([0,x] \cap J_1)$. This proves our Claim, which shows that formula (2.1) is well defined.

From the order density of J_1 in X we immediately obtain that the operator $\widetilde{R}$ is injective. Moreover, $\widetilde{R}$ is surjective. Indeed, let $Y^+ \ni y = \sup\{w_n \colon n \in \mathbb{N}\}$ for a sequence (w_n) in J_2^+, and set $u_n = R^{-1}(w_n)$, $n = 1, 2, \dots$. Then $0 \leqslant u_n \leqslant u_{n+1}$ and $\|u_n\| \leqslant \|R^{-1}\| \cdot \|w_n\| \leqslant \|R^{-1}\| \cdot \|y\|$ for all $n \geqslant 1$. Since X is σ-Levi, there exists $x := \sup_{n\geqslant 1} u_n$ in X. Now the Claim implies that $\widetilde{R}(x) = \sup_{n\geqslant 1} Tu_n = \sup_{n\geqslant 1} w_n = y$. Thus $\widetilde{R}$ maps X onto Y. ■

3. The ℓ_1-case

The following property of $\ell_1(\Gamma)$-spaces will be applied in the proof of our main result. It supplements Troyanski's result stated in Proposition 1.3 and seems to be new.

Theorem 3.1. *Let Γ and Θ be two uncountable sets, and let X be a nonseparable real Banach space with a symmetric basis $\{x_t \colon t \in \Theta\}$. If X can be injected continuously into $\ell_1(\Gamma)$, then the basis of X is equivalent to the standard basis of $\ell_1(\Theta)$.*

The proof of the above theorem is based on the following combinatorial lemma [8, Lemma 2]:

Lemma 3.2. *Let $\{S_t\}_{t\in\Theta}$ be an uncountable family of (at most) countable subsets of a set S such that, for each $s \in S$, the set $\{t \in \Theta \colon s \in S_t\}$ is (at most) countable. Then there exists a subset Θ' of Θ such that* $\operatorname{card}(\Theta') = \operatorname{card}(\Theta)$ *and $\{S_t\}_{t\in\Theta'}$ is a disjoint family.*

Proof of Theorem 3.1. Let $(e_\gamma)_{\gamma\in\Gamma}$ denote the standard symmetric basis of $\ell_1(\Gamma)$, and let $(e^*_\gamma)_{\gamma\in\Gamma}$ be the dual family in $\ell_1(\Gamma)^*$, biorthogonal to $(e_\gamma)_{\gamma\in\Gamma}$, and let $R : X \to \ell_1(\Gamma)$ be an injective and continuous mapping. Since the basis $\{x_t \colon t \in \Theta\}$ is symmetric, we have that $0 < A := \inf_{t\in\Theta} \|x_t\| \leqslant \sup_{t\in\Theta} \|x_t\| =: B < \infty$ (see [12, Theorem 5], cf. [15, p. 114]). Set $y_t = Rx_t$, $t \in \Theta$. Let us consider two cases:

(a) there is $\gamma_0 \in \Gamma$ such that $e^*_{\gamma_0}(y_t) \neq 0$ for uncountably many $t \in \Theta$,
(b) for every $\gamma \in \Gamma$, $e^*_\gamma(y_t) \neq 0$ only for countably many $t \in \Theta$.

In case (a), there is an uncountable subset Θ_0 of Θ and a positive number ε_0 such that $|e^*_{\gamma_0}(y_t)| \geqslant \varepsilon_0$ for all $t \in \Theta_0$. It follows that, for every family $(a_t)_{t\in\Theta_0}$ of real numbers, the convergence of a series $\sum_{t\in\Theta_0} a_t x_t$ implies the convergence of the series $\sum_{t\in\Theta_0} |a_t|$. Since the elements of the symmetric basis $\{x_t : t \in \Theta\}$ are norm-bounded (by B), the basic set $\{x_t\}_{t\in\Theta_0}$, and hence the basis $\{x_t\}_{t\in\Theta}$, is an ℓ_1-basis.

In case (b), we have the uncountable family $\mathcal{F} := \{s(y_t) : t \in \Theta\}$ of (countable) supports of y_t such that each subfamily $\mathcal{F}'$ of $\mathcal{F}$ with $\bigcap \mathcal{F}' \neq \emptyset$ is countable. Now Lemma 3.2 implies that there is an uncountable subset Θ' of Θ, with $\mathrm{card}(\Theta') = \mathrm{card}(\Theta)$, such that the elements of the subfamily $\{s(y_t) : t \in \Theta'\}$ of $\mathcal{F}$ are pairwise disjoint. Since $\mathrm{card}(\Theta') > \aleph_0$, there is an uncountable subset Θ'' of Θ' such that $C := \inf_{t\in\Theta''} \|y_t\| > 0$. Hence the family $\{y_t : t \in \Theta''\}$ is norm-bounded from above (by $B \cdot \|R\|$) and form below (by C) and consists of pairwise disjoint elements of $\ell_1(\Gamma)$, thus it is equivalent to the standard basis of $\ell_1(\Theta'')$. It follows that the three series: $\sum_{t\in\Theta''} a_t x_t$, $\sum_{t\in\Theta''} a_t y_t$ and $\sum_{t\in\Theta''} |a_t|$ converge simultaneously. But the basis $\{x_t : t \in \Theta\}$ is symmetric, so the latter property implies the basis is equivalent to the standard basis of $\ell_1(\Theta)$.

The proof is complete. ■

The next result shows that Proposition 1.4 cannot be extended to "small" ℓ_1-spaces, and that the restriction in our Theorem 4.2 to discrete Banach lattices X with $\mathrm{card}(A_X) \geqslant 2^{\aleph_0}$ is, in a sense, natural.

Theorem 3.3. *Let D, G be two infinite sets of the cardinality $\leqslant 2^{\aleph_0}$. Then there is a continuous injection from (real or complex) $\ell_1(D)$ into $\ell_1(G)$. The result is independent on the Continuum Hypothesis.*

Proof. We assume without loss of generality that $D \subset G \subset K$, where K denotes the unit interval.

Let (x_n) denote a fixed Schauder basis for $C(K)$. Let us consider a linear continuous operator R from c_0 into $C(K)$ of the form

$$R((t_n)) = \sum_{n=1}^{\infty} (t_n/n^2) x_n.$$

The range of R is norm dense in $C(K)$, thus its conjugate R^* is a continuous linear injection from $C(K)^*$ into $\ell_1 = c_0^*$.

Now let S denote the restriction of R^* to the atomic part of $C(K)^*$, which is isometrically isomorphic to $\ell_1(K)$ (see e.g. [13, Theorem 7 on p. 50, and Theorem 9 on p. 52]). Hence S is a continuous injection S from $\ell_1(K)$ into ℓ_1.

Since the restriction S_1 of S to $\ell_1(G)$ is injective, and the natural embedding J of ℓ_1 into $\ell_1(D)$ is injective too, the composition JS_1 is a continuous injection from $\ell_1(G)$ into $\ell_1(D)$, as required. ■

4. The main result

Throughout this section, X denotes a discrete real Banach lattice with a fixed uncountable maximal set of discrete and pairwise disjoint elements $A_X = \{x_\gamma : \gamma \in \Gamma\}$. Recall that X_a denotes the order continuous part of X.

To shorten the text, we shall use the following notion.

Definition 4.1. We say that X is a *Drewnowski space*, or a *D-space*, if it fulfils the following three conditions:

(D1) either X is order continuous (i.e., $X = X_a$) or X has the σ-Levi property,
(D2) the ideal X_a is super-order dense in X,
(D3) the set A_X is a symmetric basis for X_a.

By [16, p. 2], every nonseparable real Banach space X with an uncountable symmetric basis, endowed with the coordinatewise ordering, may be regarded as an o.c. Drewnowski space. The name *Drewnowski space* is justified by the Drewnowski's results cited in Section 1.

Now we can present our main result in a concise form.

Theorem 4.2. *Let X and Y be two discrete Banach lattices such that both the sets, A_X and A_Y, are symmetric bases for X_a and Y_a, respectively, and that A_X and A_Y have the cardinality at least continuum.*

If there are continuous linear injections from X into Y and from Y into X (in particular, if X and Y are of the same linear dimension), then:

(1.) *the bases A_X and A_Y are permutatively equivalent: there is a bijection $f : \Gamma \to \Theta$ and an isomorphism R from X_a onto Y_a such that $R(x_\gamma) = y_{f(\gamma)}$, for every $\gamma \in \Gamma$; hence*
(2.) *X_a and Y_a are order-topologically isomorphic.*
(3.) *Moreover, if X and Y are non-order continuous D-spaces, then X and Y are order-topologically isomorphic: the above operator R extends to an order-topological isomorphism from X onto Y.*

It should be noted that the isomorphism of the spaces X and Y in Theorem 4.2 is obtained when X and Y are of the same order-continuity type:

(a) From part (1.) (for the case $X = X_a$ and $Y = Y_a$) we immediately obtain a partial strengthening of Drewnowski's Proposition 1.4: *if the symmetric bases of X and Y are of the cardinality at least continuum, then the conditions $(i) - (iv)$ are equivalent also in the case when one of the bases is of ℓ_1-type.*

(b) In the proof of part (3.) of the theorem we essentially apply the extension theorem 2.1 for X and Y non-order continuous. We do not know, however, whether the final conclusion of this part holds true also when X is o.c. and Y is not.

Before proceeding to the proof of the above theorem we shall present below two, somewhat surprising, properties of D-spaces. Notice first that the classical (real) space $c_0(\Gamma)$ is an o.c. D-space, and that, for every D-space X, with $A_X = \{x_\gamma : \gamma \in \Gamma\}$, the mapping $X_a \ni \sum_{\gamma\in\Gamma} a_\gamma x_\gamma \mapsto (a_\gamma)_{\gamma\in\Gamma} \in c_0(\Gamma)$ is well defined (because the basis A_X is norm bounded from below and from above) and it is a continuous injection. Moreover, the (real) space $\ell_\infty^c(\Gamma)$, of the elements of $\ell_\infty(\Gamma)$ with a countable support, is a D-space, too, with $(\ell_\infty^c(\Gamma))_a = c_0(\Gamma)$.

Corollary 4.3. *Let X be a D-space with* $\operatorname{card}(A_X) = \operatorname{card}(\Gamma)$. *Then:*

(i) *If the spaces X_a and $c_0(\Gamma)$ are not isomorphic (as Banach spaces), then there is no continuous injection from $c_0(\Gamma)$ into X_a;*
(ii) *If X is not o.c. and the spaces X_a and $c_0(\Gamma)$ are isomorphic, then the Banach lattices X and $\ell_\infty^c(\Gamma)$ are order-topologically isomorphic.*

Parts (*i*) and (*ii*) of the corollary are direct consequences of Theorem 4.2 and the extension theorem 2.1, respectively.

Proof of Theorem 4.2. *Part (1.).* Let $T\colon X \to Y$ and $S\colon Y \to X$ be continuous linear injections.

Step 1. Set $f_\gamma := Tx_\gamma$, $\gamma \in \Gamma$. Since T is injective, $f_\gamma \neq 0$ for all γ. Let y_t^* denote the functional associated to y_t, $t \in \Theta$.

We consider two cases:

(i) there is $t_0 \in \Theta$ such that $y_{t_0}^*(f_\gamma) \neq 0$ for uncountably many γ;
(ii) for every $t \in \Theta$, $y_t^*(f_\gamma) \neq 0$ only for countably many γ.

In case (i), repeating the proof of case (a) in the proof of Theorem 3.1 we obtain that the basis A_X of X_a is equivalent to the standard basis of $\ell_1(\Gamma)$, i.e., X_a is order-topologically isomorphic to $\ell_1(\Gamma)$. In particular, X_a is a KB-space (i.e., every increasing norm bounded sequence in X^+ is norm convergent to an element of X). Since X_a is super-order dense in X, the latter implies that X_a is topologically dense in X, whence $X = X_a$, because X_a is norm-closed in X.

Thus, in case (i), X is order-topologically isomorphic to $\ell_1(\Gamma)$, and so, by the hypothesis of our theorem, we may assume without loss of generality that S maps Y, whence Y_a too, into $\ell_1(\Gamma)$ injectively. By Theorem 3.1, Y_a is order isomorphic to $\ell_1(\Theta)$, and by the arguments as above, $Y = Y_a$; thus we may also assume that $Y = \ell_1(\Theta)$.

Now the injectivity of T and S along with the hypothesis that $\operatorname{card}(A_X)$, $\operatorname{card}(A_Y) \geqslant 2^{\aleph_0}$ imply that $\operatorname{card}(\Gamma) = \operatorname{card}(\Theta)$. Hence the Banach lattices $X = X_a$ and $Y = Y_a$ are order-topologically isomorphic (to a "big" ℓ_1-space).

Similar arguments employed to the case

(i') there is $\gamma_0 \in \Gamma$ such that $x_{\gamma_0}^*(g_t) \neq 0$ for uncountably many $t \in \Theta$, where $g_t = Sy_t$, $t \in \Theta$,

also give us that $X = X_a$ and $Y = Y_a$ are order-topologically isomorphic to an ℓ_1-space.

It thus remains to consider case (ii) along with its counterpart:

(ii') for every $\gamma \in \Gamma$, $x_\gamma^*(g_t \neq 0$ only for countably many $t \in \Theta$, where $g_t = S(y_t)$.

Step 2. Assume (ii) and (ii'). By Lemma 3.2, there are uncountable subsets Γ_0 and Θ_0 of Γ and Θ, respectively, such that the elements of either of the sets $\{T(x_\gamma)\colon \gamma \in \Gamma_0\}$ and $\{S(y_t)\colon t \in \Theta_0\}$ are pairwise disjoint. Now we mimic a part of Drewnowski's proof of Case 2 of [8, Theorem]. Let a series $\sum_{\gamma\in\Gamma_0} a_\gamma x_\gamma$ be convergent in X. Then the series $\sum_{\gamma\in\Gamma_0} a_\gamma T(x_\gamma)$ is convergent in Y; equivalently (as the elements of $\{T(x_\gamma) : \gamma \in \Gamma_0\}$ are pairwise disjoint),

$$\text{the series} \quad \sum_{\gamma\in\Gamma_0} |a_\gamma| \cdot |T(x_\gamma)| \quad \text{is convergent in } Y. \tag{4.1}$$

We have that for every $u_\gamma := |T(x_\gamma)| \in Y^+ \setminus \{0\}$, where $\gamma \in \Gamma_0$, there is $y^*_{t(\gamma)}$ such that $y^*_{t(\gamma)}(u_\gamma) > 0$ (i.e., $t(\gamma) \in s(u_\gamma)$) and $u_\gamma \geqslant y^*_{t(\gamma)}(y_\gamma) \cdot y_{t(\gamma)}$. Now from (4.1) it follows that the series

$$\sum_{\gamma\in\Gamma_0} |a_\gamma| \cdot y^*_{t(\gamma)}(u_\gamma) \cdot y_{t(\gamma)} \tag{4.2}$$

converges in Y, too. But the set Γ_0 is uncountable, so we may assume that the number $\inf_{\gamma\in\Gamma_0} y^*_{t(\gamma)}(u_\gamma)$ is positive, thus the convergence of the series (4.2) implies the convergence of the series $\sum_{\gamma\in\Gamma_0} |a_\gamma| y_{t(\gamma)}$. Since the elements $y_{t(\gamma)}$, $\gamma \in \Gamma_0$, are pairwise disjoint we have that the basic set $\{x_\gamma : \gamma \in \Gamma_0\}$ dominates the basic set $\{y_{t(\gamma)} : \gamma \in \Gamma_0\} \subset \{y_t : t \in \Theta\} = A_Y$, with $\mathrm{card}(\Gamma) = \mathrm{card}(\Gamma_0) \leqslant \mathrm{card}(\Theta)$. But the bases are symmetric, so the latter implies that for every two sequences $\{\gamma_n\} \subset \Gamma$ and $\{t_n\} \subset \Theta$ the basic sequence $\{x_{\gamma_n}\}$ dominates the basic sequence $\{y_{t_n}\}$.

Similarly, there is an uncountable basic subset of $A_Y = \{y_t \colon t \in \Theta\}$ of the same cardinality as Θ, dominating an uncountable basic subset of $A_Y = \{x_\gamma : \gamma \in \Gamma\}$, and $\mathrm{card}(\Theta) \leqslant \mathrm{card}(\Gamma)$. Hence (similarly as above), the basic sequence $\{y_{t_n}\}$ dominates the basic sequence $\{x_{\gamma_n}\}$, so that the sequences are equivalent.

We thus have obtained that the sets Γ and Θ are of the same cardinality and that the bases A_X and A_Y of X_a and Y_a, respectively, are equivalent. This means that there is a bijection f from Γ onto Θ such that, for every family $(a_\gamma)_{\gamma\in\Gamma}$ of real numbers, both the series, $\sum_{\gamma\in\Gamma} a_\gamma x_\gamma$ and $\sum_{\gamma\in\Gamma} a_\gamma y_f(\gamma)$, converge (topologically) simultaneously.

It follows that the linear operator R from X_a onto Y_a such that $R(x_\gamma) = y_f(\gamma)$, for all $\gamma \in \Gamma$, is well defined and that R is a bijection. Because the orderings of X_a and Y_a are coordinatewise, the operators R and R^{-1} are positive, whence continuous [4, Theorem 12.3]. The proof of part (1.) of our theorem is complete.

Part (2.). Both the operators R and R^{-1}, defined above, are positive, hence order isomorphisms [4, Theorem 7.3]. Thus R is an order-topological isomorphism from X_a onto Y_a.

Part (3.). This part follows from Theorem 2.1 applied to part (2.). ∎

Remark 4.4. An inspection of Step 1 of the proof of Theorem 4.2 shows that if X_a (or Y_a) is not an ℓ_1-space, then we may replace the request for A_X and A_Y to be of the cardinality at least continuum by the weaker hypothesis: $\mathrm{card}(A_X)$, $\mathrm{card}(A_X) > \aleph_0$.

5. Applications to Orlicz spaces $\ell\varphi(\Gamma)$

In this section, we present a method of constructing D-spaces from "small" Banach sequence lattices with the σ-Levi property. The construction is similar to Drewnowski's construction of order continuous D-spaces [8, p. 158].

Let $W = (W, \| \ \|_W)$ be a σ-Dedekind complete Banach sequence lattice, i.e., W is an order ideal of the lattice $\mathbb{R}^{\mathbb{N}}$ such that, if $x = (a_n) \in W$ and $|b_n| \leqslant |a_n|$ for all n's, then $y = (b_n) \in W$ and $\|y\|_W \leqslant \|x\|_W$. Let us assume that W is a *symmetric sequence space*: for every $x = (a_n) \in W$ and every permutation π of the integers, we have

(s_1) the element $\widehat{\pi}(x) := (a_{\pi(n)})$ lies in W, and

(s_2) $\|x\| = \|\widehat{\pi}(x)\|$.

These two conditions imply that the sequence (e_n) of the standard unit vectors is a symmetric basis of the order continuous part W_a of W, and that, if $x = (a_n) \in W$, then

$$\left\| \sum_{n=1}^{\infty} a_n e_n \right\|_W = \left\| \sum_{n=1}^{\infty} |a_n| e_{\pi(n)} \right\|_W = \left\| \sum_{n=1}^{\infty} |a_\pi(n)| e_{k_n} \right\|_W \tag{5.1}$$

for every permutation π and every sequence (k_n) of pairwise distinct positive integers [15, pp. 114-115].

Now we set

$$W(\Gamma) := \{x \in \mathbf{R}^{\Gamma} : s(x) \text{ is either finite, or countable infinite and } x_{|s(x)} \in W\}$$

(recall that $s(x)$ denotes the support of x). Then $W(\Gamma)$, endowed with the coordinatewise ordering, is a σ-Dedekind complete linear lattice.

If $x = (a_\gamma)_{\gamma \in \Gamma} \in W(\Gamma)$ has an infinite support, we set $\|x\| := \|x_{|s(x)}\|_W$. If $s(x) = \emptyset$, we set $\|x\| = 0$, and for $s(x)$ finite (nonempty): $s(x) = \{\gamma_1, \dots, \gamma_k\}$, we complete $s(x)$ with any infinite sequence (of pairwise distinct elements) $\{\gamma_{k+1}, \dots\} \subset \Gamma$ and we set $\|x\| := \|(a_{\gamma_n})\|_W$, where $a_{\gamma_n} = 0$ for $n \geqslant k+1$. By property 5.1, $(W(\Gamma), \| \ \|)$ is a Banach lattice, and it is easy to check that its order continuous part $(W(\Gamma))_a$ equals $W_a(\Gamma)$. Moreover,

(p_1) *if W is not o.c. and has the σ-Levi property, then $W(\Gamma)$ is also of this type* (obvious);

(p_2) *if U is another symmetric sequence space and the spaces $W_a(\Gamma)$ and $U_a(\Gamma)$ are isomorphic, then the* (*symmetric*) *bases of W_a and U_a are equivalent* (by Proposition 1.2).

In particular, for $W = \ell_\infty$ we obtain the D-space $\ell_\infty^c(\Gamma)$ with $(W(\Gamma))_a = W_a(\Gamma) = c_0(\Gamma)$.

Now we come to a larger class of non-o.c. symmetric sequence spaces with the Levi property.

An *Orlicz function* φ is a non-negative, non-decreasing, convex function defined on $[0,\infty)$ and satisfying $\varphi(0) = 0$. We define the convex modular functional $\varrho_\varphi\colon \mathbb{R}^{\mathbb{N}} \to [0,\infty]$ by the formula $\varrho_\varphi(x) = \sum_{n=1}^{\infty} \varphi(|a_n|)$, $x = (a_n)$. The *Orlicz space* ℓ_φ is a Banach sequence lattice of the form

$$\ell_\varphi = \{x \in \mathbb{R}^{\mathbb{N}}\colon\ \varrho_\varphi(x/\lambda) < \infty,\ \text{for some } \lambda > 0\}$$

endowed with the norm

$$\|x\|_\varphi = \inf\{\lambda > 0\colon \varrho_\varphi(x/\lambda) \leqslant 1\}.$$

It is obvious that ℓ_φ is a symmetric sequence space, and it is known [21] that the space

$$h_\varphi := \{x \in \mathbb{R}^{\mathbb{N}}\colon\ \varrho_\varphi(x/\lambda) < \infty,\ \text{for all } \lambda > 0\}$$

is the order continuous part of ℓ_φ. We have that $\ell_\varphi = h_\varphi$ if and only if φ fulfils the so-called Δ_2-condition at 0: there are positive numbers t_0 and K such that $\varphi(2t) \leqslant K\varphi(t)$ for all $t \in [0, t_0]$ (see e.g., [15, Proposition 4.a.4]); then we write $\varphi \in \Delta_2$.

It is easy to verify that the Orlicz function φ of the form $\varphi(x) = x^2 e^{-1/x}$, for $x > 0$, and $\varphi(0) = 0$, does not satisfy the Δ_2-condition at 0. For other examples of Orlicz functions φ such that $\varphi \notin \Delta_2$ see e.g. [7]. In the latter case, ℓ_φ is a dual Banach lattice [15, Proposition 4.b.1], thus it has the Levi property [17, Theorems 2.4.19 and 2.4.21]. Summing up,

if $\varphi \notin \Delta_2$, then $W := \ell_\varphi$ is a symmetric non-o.c. sequence space with the Levi property.

By property (p_1), if $\varphi \notin \Delta_2$ then $\ell_\varphi(\Gamma)$ is a non-o.c. D-space and its order continuous part $(\ell_\varphi(\Gamma))_a$ equals $h_\varphi(\Gamma)$. Hence, from Theorem 4.2, property (p_2), and [15, Proposition 4.a.5] we obtain

Corollary 5.1. *Let Γ, Θ be two sets of the cardinality at least continuum, and let φ, ψ be two Orlicz functions not fulfilling the Δ_2-condition at 0. Then the following four conditions are equivalent:*

(i) *There are continuous linear injections from $\ell_\varphi(\Gamma)$ into $\ell_\psi(\Theta)$ and vice versa.*
(ii) *The Banach lattices $\ell_\varphi(\Gamma)$ and $\ell_\psi(\Theta)$ (as well as $h_\varphi(\Gamma)$ and $h_\psi(\Theta)$) are order-topologically isomorphic.*
(iii) *The natural unit vector bases of h_φ and h_ψ are equivalent.*
(iv) *The functions φ and ψ are equivalent at 0: there exist constants $k, K > 0$ and $t_0 > 0$ such that for all $t \in [0, t_0]$*

$$K^{-1}\varphi(k^{-1}t) \leqslant \psi(t) \leqslant K\varphi(kt).$$

We end this section with the following observation. If φ is an Orlicz function with $\varphi \notin \Delta_2$, then ℓ_φ contains an isomorphic copy of ℓ_∞ (see [15, Proposition 4.a.4]), so $\ell_\varphi(\mathbb{R})$ does. Moreover, the identity embedding of $\ell_\varphi(\mathbb{R})$ into the D-space $\ell_\infty^c(\mathbb{R})$ is continuous (e.g., because positive). We claim that:

If φ is nondegenerate (i.e., φ does not vanish on an interval $[0,s]$ for some $s>0$), then $\ell_\varphi(\mathbb{R})$ contains only isomorphic copies of ℓ_∞ and not $\ell^c_\infty(\mathbb{R})$. In particular, there is no extension of any isomorphic embedding $\ell_\infty \to \ell_\varphi(\mathbb{R})$ to an isomorphism (and even to a continuous injection) $\ell_\infty(\mathbb{R}) \to \ell_\varphi(\mathbb{R})$.

Indeed, if there was a continuous injection from $\ell^c_\infty(\mathbb{R})$ into $\ell_\varphi(\mathbb{R})$ then, by Theorem 4.2, the bases of $c_0(\mathbb{R}) = (\ell^c_\infty(\mathbb{R}))_a$ and $h_\varphi(\mathbb{R}) = (\ell_\varphi(\mathbb{R}))_a$ would be equivalent. By property (p_2), we would have $h_\varphi = c_0$, thus φ would be degenerate; a contradiction.

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Received: 4 October 2013; **revised:** 2 March 2014