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EXPLICIT EVALUATION OF CERTAIN SUMS OF MULTIPLE ZETA-STAR VALUES

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Abstract: Bowman and Bradley proved an explicit formula for the sum of multiple zeta values whose indices are the sequence $(3,1,3,1,\dots,3,1)$ with a number of 2's inserted. Kondo, Saito and Tanaka considered the similar sum of multiple zeta-star values and showed that this value is a rational multiple of a power of π. In this paper, we give an explicit formula for the rational part. In addition, we interpret the result as an identity in the harmonic algebra.

Keywords: multiple zeta values, multiple zeta-star values, harmonic algebra, Bowman–Bradley theorem, Kondo–Saito–Tanaka theorem.

1. Introduction

Let us consider the multiple zeta values (MZV, for short)

$$\zeta(k_1,\dots,k_n)=\sum_{m_1>\cdots>m_n>0}\frac{1}{m_1^{k_1}\cdots m_n^{k_n}}.$$

In some cases, explicit evaluations are known for these values or sums of them. For example, there are the formulas

$$\zeta(\{2\}^q)=\frac{\pi^{2q}}{(2q+1)!},\tag{1.1}$$

$$\zeta(\{3,1\}^p)=\frac{\pi^{4p}}{(2p+1)(4p+1)!}\tag{1.2}$$

(the notation $\{\ \ \}^p$ means that the sequence in the bracket is repeated p-times). In fact, these values are the special cases $s(0,q)$ and $s(p,0)$ of the following sums

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$$s(p,q)=\sum_{\substack{j_0,j_1,\dots,j_{2p}\geqslant 0\\ j_0+j_1+\dots+j_{2p}=q}}\zeta(\{2\}^{j_0},3,\{2\}^{j_1},1,\{2\}^{j_2},3,\dots,3,\{2\}^{j_{2p-1}},1,\{2\}^{j_{2p}}),$$

for which an explicit formula was given by Bowman-Bradley [BB]:

$$s(p,q)=\binom{2p+q}{q}\frac{\pi^{4p+2q}}{(2p+1)(4p+2q+1)!}. \tag{1.3}$$

On the other hand, we may also consider the multiple zeta-star values (MZSV for short)

$$\zeta^\star(k_1,\dots,k_n)=\sum_{m_1\geqslant\dots\geqslant m_n\geqslant 1}\frac{1}{m_1^{k_1}\cdots m_n^{k_n}}.$$

As an analogue of $s(p,q)$, we put

$$s^\star(p,q)=\sum_{\substack{j_0,j_1,\dots,j_{2p}\geqslant 0\\ j_0+j_1+\dots+j_{2p}=q}}\zeta^\star(\{2\}^{j_0},3,\{2\}^{j_1},1,\{2\}^{j_2},3,\dots,3,\{2\}^{j_{2p-1}},1,\{2\}^{j_{2p}}).$$

Then the theorem of Kondo-Saito-Tanaka [KST] states that $s^\star(p,q)\in\mathbb{Q}\pi^{4p+2q}$ (see also [T]). The rational part, however, has not been given explicitly except for the cases $p=0$ (Zlobin [Z]) and $q=0,1$ (Muneta [M]). The formula for $p=0$ is

$$s^\star(0,q)=\zeta^\star(\{2\}^q)=(2^{2q}-2)\frac{(-1)^{q-1}B_{2q}}{(2q)!}\pi^{2q} \tag{1.4}$$

(B_{2q} is the $2q$-th Bernoulli number).

In this paper, we prove the following relation between $s(p,q)$ and $s^\star(p,q)$:

Theorem 1.1. *For any $p,q\geqslant 0$, we have*

$$s^\star(p,q)=\sum_{\substack{2i+k+u=2p\\ j+l+v=q}}(-1)^{j+k}\binom{k+l}{k}\binom{u+v}{u}s(i,j)\,\zeta^\star(\{2\}^{k+l})\,\zeta^\star(\{2\}^{u+v}). \tag{1.5}$$

By substituting (1.3) and (1.4) into (1.5), we obtain an explicit formula for the value of $s^\star(p,q)$:

$$\frac{s^\star(p,q)}{\pi^{4p+2q}}=\sum_{\substack{2i+k+u=2p\\ j+l+v=q}}(-1)^{j+k}\binom{k+l}{k}\binom{u+v}{u}\binom{2i+j}{j}\frac{\beta_{k+l}\beta_{u+v}}{(2i+1)(4i+2j+1)!},$$

where

$$\beta_r=(2^{2r}-2)\frac{(-1)^{r-1}B_{2r}}{(2r)!}.$$

In particular, when $q = 0$, we can reproduce Muneta's expression for $s^\star(p,0)$ [M, Theorem B]. When $q = 1$, however, our result appears different from his formula for $s^\star(p,1)$ [M, Theorem C].

In fact, our result is slightly more general than Theorem 1.1, namely, the numbers $3,1,2$ are replaced by arbitrary positive integers a,b,c such that $a+b=2c$ and $a \geqslant 2$. Moreover, it is shown as a corollary of the corresponding identity between finite partial sums of multiple zeta series (see Theorem 2.1). In §3, we also give an interpretation as an identity in the harmonic algebra.

2. Generating series of truncated sums

For an integer $m \geqslant 0$ and an index $\mathbf{k} = (k_1,\dots,k_n)$ $(k_1,\dots,k_n \geqslant 1)$, we define finite sums $\zeta_m(\mathbf{k})$ and $\zeta^\star_m(\mathbf{k})$ by truncating the series for $\zeta(\mathbf{k})$ and $\zeta^\star(\mathbf{k})$, respectively:

$$\zeta_m(\mathbf{k}) = \sum_{m\geqslant m_1>\cdots>m_n>0} \frac{1}{m_1^{k_1}\cdots m_n^{k_n}}, \qquad \zeta^\star_m(\mathbf{k}) = \sum_{m\geqslant m_1\geqslant\cdots\geqslant m_n\geqslant 1} \frac{1}{m_1^{k_1}\cdots m_n^{k_n}}.$$

Here an empty sum is read as 0. When $n = 0$, we denote by $\varnothing$ the unique index of length zero, and put $\zeta_m(\varnothing) = \zeta^\star_m(\varnothing) = 1$ for all $m \geqslant 0$.

In the following, we fix positive integers a, b and c satisfying $a+b=2c$. For integers $p,q \geqslant 0$, let $I_{p,q} = I^{a,b,c}_{p,q}$ denote the set of all indices obtained by shuffling two sequences $(\{a,b\}^q)$ and $(\{c\}^p)$. For example,

$$I_{0,0} = \{\varnothing\}, \qquad I_{1,1} = \{(a,b,c),(a,c,b),(c,a,b)\},$$
$$I_{1,2} = \{(a,b,c,c),(a,c,b,c),(a,c,c,b),(c,a,b,c),(c,a,c,b),(c,c,a,b)\}.$$

Let us consider the sums of truncated MZVs and MZSVs analogous to $s(p,q)$ and $s^\star(p,q)$ in the introduction:

$$s_m(p,q) = \sum_{\mathbf{k}\in I_{p,q}} \zeta_m(\mathbf{k}), \qquad s^\star_m(p,q) = \sum_{\mathbf{k}\in I_{p,q}} \zeta^\star_m(\mathbf{k}).$$

Then Theorem 1.1 is obtained from the following identity by putting $(a,b,c) = (3,1,2)$ and letting $m \to \infty$:

Theorem 2.1. *For any $p,q \geqslant 0$ and $m \geqslant 0$, we have*

$$s^\star_m(p,q) = \sum_{\substack{2i+k+u=2p\\ j+l+v=q}} (-1)^{j+k}\binom{k+l}{k}\binom{u+v}{u} s_m(i,j)\,\zeta^\star_m(\{c\}^{k+l})\,\zeta^\star_m(\{c\}^{u+v}). \tag{2.1}$$

If we put

$$F_m(x,y) = \sum_{p,q\geqslant 0} s_m(p,q)x^{2p}y^q, \qquad H_m(z) = \sum_{r\geqslant 0} \zeta_m(\{c\}^r)z^r,$$
$$F^\star_m(x,y) = \sum_{p,q\geqslant 0} s^\star_m(p,q)x^{2p}y^q, \qquad H^\star_m(z) = \sum_{r\geqslant 0} \zeta^\star_m(\{c\}^r)z^r,$$

then it is not difficult to see that Theorem 2.1 is equivalent to the following generating series identity:

Theorem 2.2. *We have*

$$F_m^\star(x,y) = F_m(x,-y)H_m^\star(y-x)H_m^\star(y+x). \tag{2.2}$$

Remark 2.3. Prof. Kaneko pointed out that, since

$$H_m^\star(z) = \prod_{l=1}^{m}\left(1-\frac{z}{l^c}\right)^{-1} = H_m(-z)^{-1},$$

(2.2) can be written more symmetrically as

$$\frac{F_m^\star(x,y)}{H_m^\star(x+y)} = \frac{F_m(x,-y)}{H_m(x-y)}.$$

To prove the identity (2.2), we introduce another kind of sums and their generating series. We define $J_{p,q} = J_{p,q}^{a,b,c}$ as the set of all shuffles of $(b,\{a,b\}^q)$ and $(\{c\}^p)$, e.g.

$$J_{0,0} = \{(b)\}, \qquad J_{1,1} = \{(b,a,b,c),(b,a,c,b),(b,c,a,b),(c,b,a,b)\},$$

and put

$$t_m(p,q) = \sum_{\mathbf{k}\in J_{p,q}} \zeta_m(\mathbf{k}), \qquad G_m(x,y) = \sum_{p,q\geqslant 0} t_m(p,q)x^{2p+1}y^q,$$
$$t_m^\star(p,q) = \sum_{\mathbf{k}\in J_{p,q}} \zeta_m^\star(\mathbf{k}), \qquad G_m^\star(x,y) = \sum_{p,q\geqslant 0} t_m^\star(p,q)x^{2p+1}y^q.$$

Lemma 2.4. *For $m \geqslant 0$, we have*

$$\begin{pmatrix} F_m(x,y) \\ G_m(x,y) \end{pmatrix} = U_m U_{m-1}\cdots U_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.3}$$

$$\begin{pmatrix} F_m^\star(x,y) \\ G_m^\star(x,y) \end{pmatrix} = V_m V_{m-1}\cdots V_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.4}$$

where

$$U_l = \begin{pmatrix} 1+\frac{y}{l^c} & \frac{x}{l^a} \\ \frac{x}{l^b} & 1+\frac{y}{l^c} \end{pmatrix},$$
$$V_l = \frac{1}{\left(1-\frac{y-x}{l^c}\right)\left(1-\frac{y+x}{l^c}\right)} \begin{pmatrix} 1-\frac{y}{l^c} & \frac{x}{l^a} \\ \frac{x}{l^b} & 1-\frac{y}{l^c} \end{pmatrix}.$$

Proof. For $m=0$, both (2.3) and (2.4) are obvious. For $m \geqslant 1$, we write

$$F_m(x,y) = \sum_{p,q\geqslant 0}\sum_{\mathbf{k}\in I_{p,q}} \zeta_m(\mathbf{k})\, x^{2p}y^q$$
$$= \sum_{p,q\geqslant 0}\ \sum_{(k_1,\dots,k_{2p+q})\in I_{p,q}}\ \sum_{m\geqslant m_1>\cdots>m_{2p+q}\geqslant 1} \frac{x^{2p}y^q}{m_1^{k_1}\cdots m_{2p+q}^{k_{2p+q}}}.$$

We decompose this series into three partial sums, each consisting of the terms such that (i) $m_1 < m$, (ii) $m_1 = m$ and $k_1 = a$, or (iii) $m_1 = m$ and $k_1 = c$, respectively. Then we obtain the equality

$$F_m(x,y) = F_{m-1}(x,y) + \frac{x}{m^a} G_{m-1}(x,y) + \frac{y}{m^c} F_{m-1}(x,y).$$

Similarly, we also have

$$G_m(x,y) = G_{m-1}(x,y) + \frac{x}{m^b} F_{m-1}(x,y) + \frac{y}{m^c} G_{m-1}(x,y).$$

Combining them together, we get

$$\begin{pmatrix} F_m(x,y) \\ G_m(x,y) \end{pmatrix} = U_m \begin{pmatrix} F_{m-1}(x,y) \\ G_{m-1}(x,y) \end{pmatrix},$$

and hence (2.3) by induction.

In a similar way, we can show that

$$\begin{aligned} F^\star_m(x,y) &= F^\star_{m-1}(x,y) + \frac{x}{m^a} G^\star_m(x,y) + \frac{y}{m^c} F^\star_m(x,y), \\ G^\star_m(x,y) &= G^\star_{m-1}(x,y) + \frac{x}{m^b} F^\star_m(x,y) + \frac{y}{m^c} G^\star_m(x,y), \end{aligned}$$

that is,

$$\begin{pmatrix} 1 - \frac{y}{m^c} & -\frac{x}{m^a} \\ -\frac{x}{m^b} & 1 - \frac{y}{m^c} \end{pmatrix} \begin{pmatrix} F_m(x,y) \\ G_m(x,y) \end{pmatrix} = \begin{pmatrix} F_{m-1}(x,y) \\ G_{m-1}(x,y) \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 - \frac{y}{m^c} & -\frac{x}{m^a} \\ -\frac{x}{m^b} & 1 - \frac{y}{m^c} \end{pmatrix}^{-1} = V_m$$

under the assumption $a + b = 2c$, we obtain (2.4) by induction. ■

Now it is easy to prove Theorem 2.2. Indeed, the identities (2.3) and (2.4) imply that

$$\begin{aligned} \begin{pmatrix} F^\star_m(x,y) \\ G^\star_m(x,y) \end{pmatrix} &= \prod_{l=1}^{m} \left\{ \left(1 - \frac{y-x}{l^c}\right)\left(1 - \frac{y+x}{l^c}\right) \right\}^{-1} \cdot \begin{pmatrix} F_m(x,-y) \\ G_m(x,-y) \end{pmatrix} \\ &= H^\star_m(y-x) H^\star_m(y+x) \begin{pmatrix} F_m(x,-y) \\ G_m(x,-y) \end{pmatrix}. \end{aligned}$$

Remark 2.5. In the above proof, it is also shown that

$$t^\star_m(p,q) = \sum_{\substack{2i+k+u=2p \\ j+l+v=q}} (-1)^{j+k} \binom{k+l}{k} \binom{u+v}{u} t_m(i,j)\, \zeta^\star_m(\{c\}^{k+l})\, \zeta^\star_m(\{c\}^{u+v}). \tag{2.5}$$

3. Identities in the harmonic algebra

In this section, we give algebraic interpretations of identities (2.1) and (2.5). First we recall the setup of harmonic algebra (see [IKOO] for a more general discussion).

Let $\mathfrak{H}^1 = \mathbb{Q}\langle z_k \mid k \geqslant 1\rangle$ be the free $\mathbb{Q}$-algebra generated by countable number of variables z_k ($k = 1, 2, 3, \ldots$). The harmonic product $*$ is the $\mathbb{Q}$-bilinear product on $\mathfrak{H}^1$ defined by

$$w * 1 = 1 * w = w,$$
$$z_k w * z_l w' = z_k(w * z_l w') + z_l(z_k w * w') + z_{k+l}(w * w')$$

for $k, l \geqslant 1$ and $w, w' \in \mathfrak{H}^1$. It is known that $\mathfrak{H}^1$ equipped with the product $*$ becomes a unitary commutative $\mathbb{Q}$-algebra, denoted by $\mathfrak{H}^1_*$.

For an integer $m \geqslant 0$, we define a $\mathbb{Q}$-linear map $Z_m\colon \mathfrak{H}^1 \longrightarrow \mathbb{Q}$ by

$$Z_m(1) = 1, \qquad Z_m(z_{k_1} \cdots z_{k_n}) = \zeta_m(k_1, \ldots, k_n).$$

In fact, Z_m is a $\mathbb{Q}$-algebra homomorphism from $\mathfrak{H}^1_*$ to $\mathbb{Q}$. Moreover, we define a $\mathbb{Q}$-linear transformation on $\mathfrak{H}^1$ by

$$S(1) = 1, \qquad S(z_k) = z_k, \qquad S(z_k z_l w) = z_k S(z_l w) + z_{k+l} S(w)$$

and put $Z^\star_m = Z_m \circ S$, so that

$$Z^\star_m(z_{k_1} \cdots z_{k_n}) = \zeta^\star_m(k_1, \ldots, k_n)$$

holds for any $k_1, \ldots, k_n \geqslant 1$.

Now let us put

$$\mathfrak{s}_{p,q} = \sum_{(k_1,\ldots,k_{2p+q}) \in I_{p,q}} z_{k_1} \cdots z_{k_{2p+q}}, \qquad \mathfrak{t}_{p,q} = \sum_{(k_1,\ldots,k_{2p+q+1}) \in J_{p,q}} z_{k_1} \cdots z_{k_{2p+q+1}}.$$

Then the fact that the identities (2.1) and (2.5) hold for *all* $m \geqslant 0$ suggests that the identities

$$S(\mathfrak{s}_{p,q}) = \sum_{\substack{2i+k+u=2p \\ j+l+v=q}} (-1)^{j+k} \binom{k+l}{k} \binom{u+v}{u} \mathfrak{s}_{i,j} * S(z_c^{k+l}) * S(z_c^{u+v}), \tag{3.1}$$

$$S(\mathfrak{t}_{p,q}) = \sum_{\substack{2i+k+u=2p \\ j+l+v=q}} (-1)^{j+k} \binom{k+l}{k} \binom{u+v}{u} \mathfrak{t}_{i,j} * S(z_c^{k+l}) * S(z_c^{u+v}) \tag{3.2}$$

hold in $\mathfrak{H}^1$. Indeed, this speculation is justified by the following theorem:

Theorem 3.1. *For $w \in \mathfrak{H}^1$, denote the rational sequence $\big\{Z_m(w)\big\}_{m\geqslant 0}$ by $\mathcal{Z}(w)$. Then the resulting $\mathbb{Q}$-algebra homomorphism $\mathcal{Z}\colon \mathfrak{H}^1_* \longrightarrow \mathbb{Q}^{\mathbb{N}}$ is injective.*

If we put $\mathfrak{H}^1_{>0} = \bigoplus_{k\geqslant 1} z_k\mathfrak{H}^1$, it is obvious from the definition of Z_m that $\mathfrak{H}^1_{>0} = \operatorname{Ker} Z_0$. Hence it suffices to consider the map

$$\mathfrak{H}^1_{>0} \longrightarrow \mathbb{Q}^{\mathbb{Z}_{>0}};\ w \longmapsto \bigl\{Z_m(w)\bigr\}_{m>0}.$$

The injectivity of this map is an immediate consequence of the following theorem:

Theorem 3.2 ([MPH, Theorem 4], [U, Theorem 3]). *The multiple polylogarithm functions*

$$Li_{\mathbf{k}}(t) = \sum_{m_1>\cdots>m_n>0} \frac{t^{m_1}}{m_1^{k_1}\cdots m_n^{k_n}} = \sum_{m>0}\bigl(\zeta_m(\mathbf{k}) - \zeta_{m-1}(\mathbf{k})\bigr)t^m,$$

for $\mathbf{k} = (k_1,\dots,k_n) \in (\mathbb{Z}_{>0})^n$ *and* $n \geqslant 1$, *are linearly independent over* $\mathbb{C}$.

Remark 3.3. It is also possible to prove the identities (3.1) and (3.2) directly, by making computations similar to the proof of Proposition 4 in [IKOO], in the matrix algebra $M_2\bigl(\mathfrak{H}^1_*[[x,y]]\bigr)$.

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