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ON THE FOUR-DIMENSIONAL DIVISOR PROBLEM OF (a, b, c, c) TYPE

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Abstract: In this paper we study the four-dimensional divisor problem of (a, b, c, c) type, where $1 \leqslant a \leqslant b < c$ are fixed integers. Our theorems improve some classical results. As an application, we study the error term of the summatory function of the exponential totient function.

Keywords: divisor function, exponential divisor, error term, exponential sum.

1. Introduction and main results

Let $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$ be fixed positive integers. Without the loss of generality, we suppose $(a_1, a_2, a_3, a_4) = 1$. The divisor function $d(a_1, a_2, a_3, a_4; n)$ denotes the number of ways n can be written as a product $n_1^{a_1} n_2^{a_2} n_3^{a_3} n_4^{a_4} (n_j \in \mathbb{N}, j = 1, 2, 3, 4)$. The four-dimensional divisor problem is to study the properties of the error function

$$\Delta(a_1, a_2, a_3, a_4; x) := \sum_{n \leqslant x} d(a_1, a_2, a_3, a_4; n) - \sum_{j=1}^{4} \operatorname*{Res}_{s=a_j} \prod_{l=1}^{4} \zeta(a_l s) \frac{x^s}{s}. \tag{1.1}$$

For the history and classical results about $\Delta(a_1, a_2, a_3, a_4; x)$, see for example, M. Vogts [28], E. Krätzel [14, 15, 16], A. Ivić [10, 11] and a recent survey article of A. Ivić, E. Krätzel, M. Kühleitner and W.G. Nowak [12].

The study of the four-dimensional divisor problem is very important in the analytic number theory. For example, the case $(a_1, a_2, a_3, a_4) = (1, 1, 1, 1)$ is the well-known Piltz divisor problem of dimension 4, the case $(a_1, a_2, a_3, a_4) = (1, 2, 3, 4)$ is closely related to the number of finite abelian groups, the case $(a_1, a_2, a_3, a_4) = (1, 1, 2, 2)$ is closely related to the number of direct factors of finite abelian groups, etc.

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The aim of this paper is to study the four-dimensional divisor problem for the case $(a_1, a_2, a_3, a_4) = (a, b, c, c)$, where a, b, c are fixed positive integers such that $a \leqslant b < c$ and $(a, b, c) = 1$. Let

$$A(a,b,c,c;x) = \sum_{n\leqslant x} d(a,b,c,c;n), \tag{1.2}$$

$$\begin{aligned} H(a,b,c,c;x) = &\operatorname*{Res}_{s=\frac{1}{a}} \zeta(as)\zeta(bs)\zeta^2(cs)\frac{x^s}{s} + \operatorname*{Res}_{s=\frac{1}{b}} \zeta(as)\zeta(bs)\zeta^2(cs)\frac{x^s}{s} \\ &+ \operatorname*{Res}_{s=\frac{1}{c}} \zeta(as)\zeta(bs)\zeta^2(cs)\frac{x^s}{s} \end{aligned} \tag{1.3}$$

if $a \neq b$, otherwise an appropriate limit approach should be taken in the above sum.

From A. Ivić [11] we have

$$\Delta(a,b,c,c;x) = \Omega\left(x^{\max\left(\frac{1}{2(a+b)},\frac{1}{a+b+c}\right)}\right). \tag{1.4}$$

One may conjecture that

$$\Delta(a,b,c,c;x) = O\left(x^{\max\left(\frac{1}{2(a+b)},\frac{1}{a+b+c}\right)+\varepsilon}\right). \tag{1.5}$$

Now we introduce some notations for later use. Let $d(n)$ denote the Dirichlet divisor function. Let $x \geqslant 1$ be real and $\mathcal{L} = \log x$, $\{t\}$ denotes the fractional part of $t, \psi(t) = t - [t] - 1/2$, $e(t) = \exp(2\pi i t)$. ε is a fixed positive constant, not necessarily the same in all occurrences. $m \sim M$ means that $cM < m \leqslant CM$ for some constants $0 < c < C$. $d(a,b;n) = \sum_{m_1^a m_2^b = n} 1$ and $A(a,b;x) = \sum_{n\leqslant x} d(a,b;n)$. Let $\Delta(a,b;x)$ denote the error term of the asymmetric two-dimensional divisor problem,

$$\Delta(a,b;x) := \sum_{m^a n^b \leqslant x} 1 - \zeta\left(\frac{b}{a}\right)x^{\frac{1}{a}} - \zeta\left(\frac{a}{b}\right)x^{\frac{1}{b}} := A(a,b;x) - H(a,b;x), \tag{1.6}$$

if $a \neq b$. If $a = b$, then an appropriate limit approach should be taken in the above sum. Hence $H(1,1;x) = x\log x + (2\gamma - 1)x$, where γ is Euler's constant. For convenience, we also use notations $A(x)$ and $\Delta(x)$ to denote $A(1,1;x)$ and $\Delta(1,1;x)$, respectively. Krätzel [14] gives a series of classical results about the upper bound of $\Delta(a,b;x)$. Recently, the authors [31] studies the mean square of $\Delta(a,b;x)$ and obtains its asymptotic formula.

Theorem 6.8 of Krätzel [14] is an important theorem for the many-dimensional divisor problems. We first prove the following Theorem 1, which is a refined version of this theorem in our case.

Theorem 1. *Suppose a,b and c are fixed positive integers such that $1\leqslant a<b<c$, $1<y<x$ are two large real numbers, $0<\alpha(a,b)<1/c$ is a real number such that the estimate $\Delta(a,b;x)\ll x^{\alpha(a,b)}$ holds, then*

$$\Delta(a,b,c,c;x)=\sum_{m\leqslant y}d(a,b;m)\Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right)+\sum_{n\leqslant(\frac{x}{y})^{\frac{1}{c}}}d(n)\Delta\left(a,b;\frac{x}{n^c}\right) \tag{1.7}$$
$$+O\left(x^{\frac{1}{c}}y^{-\frac{1}{c}-\frac{1}{2(a+b)}}\log x+x^{-\frac{1}{4c}}y^{\frac{1}{a}+\frac{1}{4c}}+y^{\alpha(a,b)}\left(\frac{x}{y}\right)^{\frac{131}{416c}}\log^4 x\right).$$

Using the bounds $\Delta(t)\ll t^{131/416}\log^{26947/8320}t$ (see Huxley [9]) and $\Delta(a,b;x)\ll x^{\alpha(a,b)}$ in Theorem 1 directly, we get immediately the following Corollary 1.1.

Corollary 1.1. *Under the conditions of Theorem 1, we have*

$$\Delta(a,b,c,c;x)\ll x^{\frac{416-131a\alpha(a,b)}{285a+416c(1-a\alpha(a,b))}}(\log x)^4. \tag{1.8}$$

Remark 1. When we use Theorem 6.8 of Krätzel [14] to estimate $\Delta(a,b,c,c;x)$, we encounter some four-dimensional exponential sums, however if we use Theorem 1, we only need to estimate some three-dimensional exponential sums, which is much easier to handle. When the differences between the numbers a,b,c are comparatively large, the upper bound results obtained through Theorem 6.8 of Krätzel [14] are usually weak. In this paper, we shall get the following Theorem 2 through Theorem 1 with the help of an approach due to Heath-Brown [7]. We note that Theorem 2 can not be covered by Krätzel's results in [14, 16].

Theorem 2. *Suppose a,b,c are fixed positive integers such that $1\leqslant a<b$ and $a+b\leqslant c<2(a+b)$, $0<\alpha(a,b)<1/c$ is a real number such that the estimate $\Delta(a,b;x)\ll x^{\alpha(a,b)}$ holds. If the condition*

$$\alpha(a,b)\geqslant\frac{9c-4a-2b}{c(23a+25b)-18(a+b)^2} \tag{1.9}$$

is true, then

$$\Delta(a,b,c,c;x)\ll x^{\theta(a,b,c)+\varepsilon}, \tag{1.10}$$

where

$$\theta(a,b,c)=\frac{3-3\alpha(a,b)(a+b)}{3c+2a-c(5a+3b)\alpha(a,b))}. \tag{1.11}$$

Remark 2. When dealing with exponential sums in the proof of Theorem 2, the main approach is due to Heath-Brown [7]. It is not difficult to improve Theorem 2 by using more precise estimates of the exponential sums(see for example, [1], [6], [8], [18], [23], [30]).

Corollary 1.2. *Suppose* $1 \leqslant a < b$, $a+b \leqslant c < 2(a+b)$, *and define*

$$\vartheta(a,b,c) = \begin{cases} \frac{15(a+b)}{29a^2+29ab-4ac+15bc} & \text{if } 11a \geqslant 8b \text{ and } \\ & \qquad \frac{113a^2+255ab+142b^2}{88a+107b} \leqslant c < \frac{29(a+b)}{19}, \\ \frac{21a+12b}{34a^2+28ab+ac+12bc} & \text{if } 2a \geqslant b > \frac{11}{8}a \text{ and } \\ & \qquad \frac{112a^2+270ab+152b^2}{77a+124b} \leqslant c < \frac{17a+14b}{10}, \\ \frac{9a+3b}{13a^2+9ab+2ac+3bc} & \text{if } 4a \geqslant b > 2a \text{ and } \\ & \qquad \frac{37a^2+95ab+54b^2}{22a+47b} \leqslant c < \frac{39a+27b}{21}. \end{cases}$$

Then

$$\Delta(a,b,c,c;x) \ll x^{\vartheta(a,b,c)+\varepsilon}. \tag{1.12}$$

From Corollary 1.2 we get immediately that

$$\Delta(1,3,5,5;x) \ll x^{18/95+\varepsilon}, \tag{1.13}$$

which will be used to prove the following Theorem 3.

Subbarao[24] introduced the definition of exponential convolution, which is closely related to the exponential divisors of natural numbers. Let $n > 1$ be an integer of canonical form $n = p_1^{a_1} \cdots p_r^{a_r}$. An integer d is called an exponential divisor of n if $d = p_1^{b_1} \cdots p_r^{b_r}$ satisfies $b_j | a_j (1 \leqslant j \leqslant r)$, denoted by $d|_e n$. For convenience let $1|_e 1$. The exponential convolution is an analogue of the classical Dirichlet convolution, which is studied by several authors, see for example, [5, 13, 19, 20, 21, 22, 29]. The exponential totient function $\phi^{(e)}(n)$ denotes the number of divisors d of n such that d and n are exponentially coprime, namely, they don't have common exponential divisors. The function $\phi^{(e)}(n)$ was studied in J. Sándor[21], Tóth[25, 26, 27], Pétermann[20]. Tóth[25] proved that

$$\sum_{n \leqslant x} \phi^{(e)}(n) = C_1 x + C_2 x^{1/3} + O(x^{1/5+\varepsilon}), \tag{1.14}$$

where C_1, C_2 are computable constants. The estimate $O(x^{1/5+\varepsilon})$ was improved to $O(x^{1/5})$ in Pétermann[20], which is the best result up to date.

In this paper we prove the following

Theorem 3.

(i) *The asymptotic formula*

$$\sum_{n \leqslant x} \phi^{(e)}(n) = C_1 x + C_2 x^{1/3} + D_1 x^{1/5} \log x + D_2 x^{1/5} + O(x^{18/95+\varepsilon}) \tag{1.15}$$

holds, where D_1, D_2 *are computable constants.*

(ii) *We have*

$$\sum_{n \leqslant x} \phi^{(e)}(n) = C_1 x + C_2 x^{1/3} + D_1 x^{1/5} \log x + D_2 x^{1/5} + \Omega\left(x^{1/8}\right). \tag{1.16}$$

Remark 3. The asymptotic formula (1.15) is a substantial improvement to the formula (1.14). Numerically we have $\frac{18}{95} = 0.18947\ldots < \frac{1}{5}$.

2. The proof of Theorem 1 and Corollary 1.1

Lemma 2.1. *Let $x \geqslant 1$ and $\Delta(a, b; x)$ be defined by (1.6). We let*

$$E(a, b; x, s) := \Delta(a, b; x)x^{-s} - s\int_x^{\infty} \Delta(a, b; t)t^{-s-1}\,\mathrm{d}t \tag{2.1}$$

(i) *If $a < b$, $\Delta(a, b; x) \ll x^{\alpha(a,b)}$ such that $\alpha(a, b) < \frac{1}{b}$, then*

$$\sum_{n\leqslant x} d(a, b; n)n^{-s} = \frac{\zeta\left(\frac{b}{a}\right)}{1 - as}x^{\frac{1}{a}-s} + \frac{\zeta\left(\frac{a}{b}\right)}{1 - bs}x^{\frac{1}{b}-s} + \zeta(as)\zeta(bs) + E(a, b; x, s) \tag{2.2}$$

holds for $s > \alpha(a, b)$. For $s = \frac{1}{a}$ or $s = \frac{1}{b}$ we take the limiting values.

(ii) *If $\Delta(x) \ll x^{\alpha}$ such that $\alpha < \frac{1}{3}$, then*

$$\sum_{n\leqslant x} d(n)n^{-s} = \left(\frac{\log x + 2\gamma - 1}{1 - s} - \frac{s}{(1 - s)^2}\right)x^{1-s} + \zeta^2(s) + E(1, 1; x, s) \tag{2.3}$$

holds for $s > \alpha$. For $s = 1$ we take the limiting value.

Proof. By partial summation formula and (1.6), we obtain

$$\begin{aligned}\sum_{n\leqslant x} \frac{d(a, b; n)}{n^s} &= A(a, b; x)x^{-s} + s\int_1^x A(a, b; t)t^{-s-1}\,\mathrm{d}t \qquad (2.4)\\ &= \frac{H(a, b; x) + \Delta(a, b; x)}{x^s} + s\int_1^x \frac{(H(a, b; t) + \Delta(a, b; t))}{t^{s+1}}\,\mathrm{d}t\\ &= \frac{\zeta\left(\frac{b}{a}\right)}{1 - as}x^{\frac{1}{a}-s} + \frac{\zeta\left(\frac{a}{b}\right)}{1 - bs}x^{\frac{1}{b}-s} - \frac{as\zeta\left(\frac{b}{a}\right)}{1 - as} - \frac{bs\zeta\left(\frac{a}{b}\right)}{1 - bs}\\ &\quad + s\int_1^{\infty} \Delta(a, b; t)t^{-s-1}\,\mathrm{d}t + \Delta(a, b; x)x^{-s}\\ &\quad - s\int_x^{\infty} \Delta(a, b; t)t^{-s-1}\,\mathrm{d}t.\end{aligned}$$

Clearly

$$E(a, b; x, s) \ll x^{\alpha(a,b)-s}. \tag{2.5}$$

Suppose that $s > 1$, we have from (2.4) and the condition $\alpha(a, b) < \frac{1}{b}$, when $x \to \infty$

$$\zeta(as)\zeta(bs) = -\frac{as\zeta\left(\frac{b}{a}\right)}{1 - as} - \frac{bs\zeta\left(\frac{a}{b}\right)}{1 - bs} + s\int_1^{\infty} \Delta(a, b; t)t^{-s-1}\,\mathrm{d}t.$$

By analytic continuation this equation also holds for $\Re s > \alpha(a, b)$. Now we substitute this into (2.4), and this completes the proof of (2.2). The proof of (2.3) is similar to that of (2.2), we omit the details. ■

Lemma 2.2. *Let $x \geqslant 1$ and $\Delta(a,b;x)$ be defined by (1.6). Let*

$$E_1(a,b;x,s) := \Delta(a,b;x)x^{-s}\log x - \int_x^{\infty} \Delta(a,b;t)(s\log t - 1)t^{-s-1}\,\mathrm{d}t \tag{2.6}$$

If $a < b$, $\Delta(a,b;x) \ll x^{\alpha(a,b)}$ such that $\alpha(a,b) < \frac{1}{b}$, then

$$\begin{aligned}\sum_{n\leqslant x} d(a,b;n)n^{-s}\log n = {}& \zeta\left(\frac{b}{a}\right)\left(\frac{\log x}{1-as} - \frac{a}{(1-as)^2}\right)x^{\frac{1}{a}-s} \\ & + \zeta\left(\frac{a}{b}\right)\left(\frac{\log x}{1-bs} - \frac{b}{(1-bs)^2}\right)x^{\frac{1}{b}-s} \\ & - (\zeta(as)\zeta(bs))' + E_1(a,b;x,s)\end{aligned} \tag{2.7}$$

holds for $s > \alpha(a,b)$. For $s = \frac{1}{a}$ or $s = \frac{1}{b}$ we take the limiting values.

Proof. Similar to the proof of (2.4) we have

$$\begin{aligned}\sum_{n\leqslant x} \frac{d(a,b;n)}{n^s}\log n &= A(a,b;x)\frac{\log x}{x^s} + \int_1^x A(a,b;t)\frac{(s\log t - 1)}{t^{s+1}}\,\mathrm{d}t \\ &= \zeta\left(\frac{b}{a}\right)\left(\frac{\log x}{1-as} - \frac{a}{(1-as)^2}\right)x^{\frac{1}{a}-s} \\ &\quad + \zeta\left(\frac{a}{b}\right)\left(\frac{\log x}{1-bs} - \frac{b}{(1-bs)^2}\right)x^{\frac{1}{b}-s} \\ &\quad + \frac{a\zeta\left(\frac{b}{a}\right)}{(1-as)^2} + \frac{b\zeta\left(\frac{a}{b}\right)}{(1-bs)^2} + \Delta(a,b;x)x^{-s}\log x \\ &\quad + \int_1^{\infty} \Delta(a,b;t)\frac{(s\log t - 1)}{t^{s+1}}\,\mathrm{d}t \\ &\quad - \int_x^{\infty} \Delta(a,b;t)(s\log t - 1)t^{-s-1}\,\mathrm{d}t.\end{aligned} \tag{2.8}$$

Assuming that $s > 1$, we have, as $x \to \infty$

$$-(\zeta(as)\zeta(bs))' = \frac{a\zeta\left(\frac{b}{a}\right)}{(1-as)^2} + \frac{b\zeta\left(\frac{a}{b}\right)}{(1-bs)^2} + \int_1^{\infty} \Delta(a,b;t)\frac{(s\log t - 1)}{t^{s+1}}\,\mathrm{d}t. \tag{2.9}$$

By analytic continuation (2.9) also holds for $\Re s > \alpha(a,b)$. Substituting (2.9) into (2.8) completes the proof of Lemma 2.2. ∎

Lemma 2.3. *Let $a<b<c$ and $1<y<x$. If $\Delta(a,b;x)\ll x^{\alpha(a,b)}$ such that $\alpha(a,b)<\frac{1}{c}$, then*

$$\begin{aligned}\Delta(a,b,c,c;x)=&\sum_{m\leqslant y}d(a,b;m)\Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right)+\sum_{n\leqslant(\frac{x}{y})^{\frac{1}{c}}}d(n)\Delta\left(a,b;\frac{x}{n^c}\right)\\&-\frac{x^{\frac{1}{c}}}{c}\int_y^{\infty}\left(\frac{\log x-\log t}{c}+2\gamma\right)\Delta(a,b;t)t^{-\frac{1}{c}-1}\,\mathrm{d}t\\&-\frac{c}{a}\zeta\left(\frac{b}{a}\right)x^{\frac{1}{a}}\int_{(\frac{x}{y})^{\frac{1}{c}}}^{\infty}\Delta(t)t^{-\frac{c}{a}-1}\,\mathrm{d}t\\&-\frac{c}{b}\zeta\left(\frac{a}{b}\right)x^{\frac{1}{b}}\int_{(\frac{x}{y})^{\frac{1}{c}}}^{\infty}\Delta(t)t^{-\frac{c}{b}-1}\,\mathrm{d}t-\Delta(a,b;y)\Delta\left(\left(\frac{x}{y}\right)^{\frac{1}{c}}\right).\end{aligned}\tag{2.10}$$

Proof. By (1.2) and Dirichlet's splitting argument we write

$$A(a,b,c,c;x)=\sum_{mn^c\leqslant x}d(a,b;m)d(n)=S_1+S_2-S_3,\tag{2.11}$$

where

$$S_1=\sum_{m\leqslant y}d(a,b;m)\sum_{n^c\leqslant x/m}d(n),\qquad S_2=\sum_{n^c\leqslant x/y}d(n)\sum_{m\leqslant x/n^c}d(a,b;m),$$
$$S_3=\sum_{m\leqslant y}d(a,b;m)\sum_{n^c\leqslant x/y}d(n).$$

By (1.6), (2.2) of Lemma 2.1 and Lemma 2.2 we get

$$\begin{aligned}S_1=&\sum_{m\leqslant y}d(a,b;m)\left\{\left(\frac{\log x}{c}+(2\gamma-1)-\frac{\log m}{c}\right)\frac{x^{\frac{1}{c}}}{m^{\frac{1}{c}}}+\Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right)\right\}\\=&\left(\frac{\log x}{c}+(2\gamma-1)\right)x^{\frac{1}{c}}\sum_{m\leqslant y}\frac{d(a,b;m)}{m^{\frac{1}{c}}}-\frac{x^{\frac{1}{c}}}{c}\sum_{m\leqslant y}\frac{d(a,b;m)\log m}{m^{\frac{1}{c}}}\\&+\sum_{m\leqslant y}d(a,b;m)\Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right)\\=&\left(\frac{\log x}{c}+(2\gamma-1)\right)x^{\frac{1}{c}}\left\{\frac{\zeta\left(\frac{b}{a}\right)}{1-\frac{a}{c}}y^{\frac{1}{a}-\frac{1}{c}}+\frac{\zeta\left(\frac{a}{b}\right)}{1-\frac{b}{c}}y^{\frac{1}{b}-\frac{1}{c}}+\zeta\left(\frac{a}{c}\right)\zeta\left(\frac{b}{c}\right)\right.\\&\left.+E\Big(a,b;y,\frac{1}{c}\Big)\right\}-\frac{x^{\frac{1}{c}}}{c}\left\{\zeta\left(\frac{b}{a}\right)\left(\frac{\log y}{1-\frac{a}{c}}-\frac{a}{(1-\frac{a}{c})^2}\right)y^{\frac{1}{a}-\frac{1}{c}}\right.\\&\left.+\zeta\left(\frac{a}{b}\right)\left(\frac{\log x}{1-\frac{b}{c}}-\frac{b}{(1-\frac{b}{c})^2}\right)y^{\frac{1}{b}-\frac{1}{c}}\right\}\\&-\frac{x^{\frac{1}{c}}}{c}\left(-\left(\zeta(as)\zeta(bs)\right)'_{s=\frac{1}{c}}+E_1\Big(a,b;y,\frac{1}{c}\Big)\right)+\sum_{m\leqslant y}d(a,b;m)\Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right).\end{aligned}\tag{2.12}$$

By (1.6) and (2.3) of Lemma 2.1 we obtain

$$\begin{aligned}
S_2 = &\sum_{n\leqslant (\frac{x}{y})^{\frac{1}{c}}} d(n)\left\{\zeta\left(\frac{b}{a}\right)\frac{x^{\frac{1}{a}}}{n^{\frac{c}{a}}}+\zeta\left(\frac{a}{b}\right)\frac{x^{\frac{1}{b}}}{n^{\frac{c}{b}}}+\Delta\left(a,b;\frac{x}{n^c}\right)\right\} \\
&= \zeta\left(\frac{b}{a}\right)x^{\frac{1}{a}}\Bigg\{\left(\frac{\frac{1}{c}\log\frac{x}{y}+2\gamma-1}{1-\frac{c}{a}}-\frac{ac}{(c-a)^2}\right)\left(\frac{x}{y}\right)^{\frac{1}{c}-\frac{1}{a}} \\
&+\zeta^2\left(\frac{c}{a}\right)+E\left(1,1;\left(\frac{x}{y}\right)^{\frac{1}{c}},\frac{c}{a}\right)\Bigg\} \\
&+\zeta\left(\frac{a}{b}\right)x^{\frac{1}{b}}\Bigg\{\left(\frac{\frac{1}{c}\log\frac{x}{y}+2\gamma-1}{1-\frac{c}{b}}-\frac{bc}{(c-b)^2}\right)\left(\frac{x}{y}\right)^{\frac{1}{c}-\frac{1}{b}} \\
&+\zeta^2\left(\frac{c}{b}\right)+E\left(1,1;\left(\frac{x}{y}\right)^{\frac{1}{c}},\frac{c}{b}\right)\Bigg\}+\sum_{n\leqslant (\frac{x}{y})^{\frac{1}{c}}} d(n)\Delta\left(a,b;\frac{x}{n^c}\right).
\end{aligned} \tag{2.13}$$

From (1.6) we also have

$$\begin{aligned}
S_3 = &\left\{\zeta\left(\frac{b}{a}\right)y^{\frac{1}{a}}+\zeta\left(\frac{a}{b}\right)y^{\frac{1}{b}}+\Delta(a,b;y)\right\} \\
&\times\left\{\left(\frac{1}{c}\log\frac{x}{y}+2\gamma-1\right)\left(\frac{x}{y}\right)^{\frac{1}{c}}+\Delta\left(\left(\frac{x}{y}\right)^{\frac{1}{c}}\right)\right\}.
\end{aligned} \tag{2.14}$$

Combining (2.11)–(2.14), (2.1) and (2.6) we get

$$\begin{aligned}
A(a,b,c,c;x) = &\zeta\left(\frac{b}{a}\right)\zeta^2\left(\frac{c}{a}\right)x^{\frac{1}{a}}+\zeta\left(\frac{a}{b}\right)\zeta^2\left(\frac{c}{b}\right)x^{\frac{1}{b}} \\
&+\left(\left(\frac{\log x}{c}+(2\gamma-1)\right)\zeta\left(\frac{a}{c}\right)\zeta\left(\frac{b}{c}\right)+\frac{1}{c}\left(\zeta(as)\zeta(bs)\right)'_{s=\frac{1}{c}}\right)x^{\frac{1}{c}} \\
&+\sum_{m\leqslant y} d(a,b;m)\Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right)+\sum_{n\leqslant (\frac{x}{y})^{\frac{1}{c}}} d(n)\Delta\left(a,b;\frac{x}{n^c}\right) \\
&-\frac{x^{\frac{1}{c}}}{c}\int_y^{\infty}\left(\frac{\log x-\log t}{c}+2\gamma\right)\Delta(a,b;t)t^{-\frac{1}{c}-1}\,\mathrm{d}t \\
&-\frac{c}{a}\zeta\left(\frac{b}{a}\right)x^{\frac{1}{a}}\int_{(\frac{x}{y})^{\frac{1}{c}}}^{\infty}\Delta(t)t^{-\frac{c}{a}-1}\,\mathrm{d}t \\
&-\frac{c}{b}\zeta\left(\frac{a}{b}\right)x^{\frac{1}{b}}\int_{(\frac{x}{y})^{\frac{1}{c}}}^{\infty}\Delta(t)t^{-\frac{c}{b}-1}\,\mathrm{d}t-\Delta(a,b;y)\Delta\left(\left(\frac{x}{y}\right)^{\frac{1}{c}}\right).
\end{aligned} \tag{2.15}$$

Finally, Lemma 2.3 follows from (1.1)–(1.3) and (2.15) at once. ■

Proof of Theorem 1. Applying Lemma 5.7 of Krätzel [14] (or Theorem 3 of Cao [3]) and integrating by parts we have

$$x^{\frac{1}{c}}\int_y^\infty\left(\frac{\log x-\log t}{c}+2\gamma\right)\Delta(a,b;t)t^{-\frac{1}{c}-1}\,\mathrm{d}t\ll x^{\frac{1}{c}}y^{-\frac{1}{c}-\frac{1}{2(a+b)}}\log x. \tag{2.16}$$

Similarly, we also have

$$x^{\frac{1}{a}}\int_{(\frac{x}{y})^{\frac{1}{c}}}^\infty\Delta(t)t^{-\frac{c}{a}-1}\,\mathrm{d}t\ll x^{-\frac{1}{4c}}y^{\frac{1}{a}+\frac{1}{4c}}, \tag{2.17}$$

and

$$x^{\frac{1}{b}}\int_{(\frac{x}{y})^{\frac{1}{c}}}^\infty\Delta(t)t^{-\frac{c}{b}-1}\,\mathrm{d}t\ll x^{-\frac{1}{4c}}y^{\frac{1}{b}+\frac{1}{4c}}. \tag{2.18}$$

M. N. Huxley[9] showed that $\Delta(1,1;x)\ll x^{\frac{131}{416}}(\log x)^{\frac{26947}{8320}}$. Now combining Lemma 2.3, (2.16)–(2.18) finishes the proof of Theorem 1. ■

The proof of Corollary 1.1. By Theorem 1 and the estimate $\Delta(1,1;x)\ll x^{\frac{131}{416}}(\log x)^4$, then applying partial summation formula and (1.6) we have

$$\begin{aligned}\Delta(a,b,c,c;x)x^{\varepsilon}&\ll\mathcal{L}^4\sum_{m\leqslant y}d(a,b;m)\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right)^{\frac{131}{416}}+\sum_{n\leqslant(\frac{x}{y})^{\frac{1}{c}}}d(n)\left(\frac{x}{n^c}\right)^{\alpha(a,b)}\\&\ll x^{\frac{131}{416c}}y^{\frac{1}{a}-\frac{131}{416c}}\mathcal{L}^4+x^{\frac{1}{c}}y^{\alpha(a,b)-\frac{1}{c}}\mathcal{L}.\end{aligned} \tag{2.19}$$

Taking $y=x^{\frac{285a}{285a+416c(1-a\alpha(a,b))}}$, we obtain $\Delta(a,b,c,c;x)\ll x^{\frac{416-131a\alpha(a,b)}{285a+416c(1-a\alpha(a,b))}}\mathcal{L}^4$, and this completes the proof of Corollary 1.1. ■

3. The proof of Theorem 2

In the proof of Theorem 2 we will use the following lemmas. (3.1) in Lemma 3.1 is well-known, (3.2) is Theorem 5.1 in [14]. Lemma 3.2 is Lemma 4 of Cao [4] (also see (2.1) in Wu [30], Heath-Brown's method), Lemma 3.3 is Lemma 10 in [4] (Process B, then Heath-Brown's method).

Lemma 3.1. *Let $\varepsilon>0$ and $\Delta(a,b;x)$ be defined by (1.6). Then for $1\leqslant Y<x$*

$$\Delta(x)=\frac{x^{\frac{1}{4}}}{\sqrt{2}\pi}\sum_{m\leqslant Y}\frac{\tau(m)}{m^{\frac{3}{4}}}\cos(4\pi\sqrt{mx}-\frac{\pi}{4})+O\left(x^{\frac{1}{2}+\varepsilon}Y^{-\frac{1}{2}}+x^{\varepsilon}\right) \tag{3.1}$$

and

$$\Delta(a,b;x)=-\sum_{n^{a+b}\leqslant x}\psi\left(\left(\frac{x}{x^b}\right)^{\frac{1}{a}}\right)+\psi\left(\left(\frac{x}{x^a}\right)^{\frac{1}{b}}\right)+O(1). \tag{3.2}$$

Lemma 3.2. *Let $x \geqslant 2$, α, β, γ be given real numbers with $\alpha(\alpha-1)\beta\gamma \neq 0$, $|a(m)| \leqslant 1$, $|b(n_1, n_2)| \leqslant 1$. Suppose $G = xM^\alpha N_1^\beta N_2^\gamma$, (κ, λ) is an exponent pair and*

$$T(M, N_1, N_2) = \sum_{m\sim M} \sum_{n_1\sim N_1} \sum_{n_2\sim N_2} a(m)b(n_1, n_2)e(xm^\alpha n_1^\beta n_2^\gamma). \tag{3.3}$$

Then

$$\begin{aligned} T(M, N_1, N_2)\mathcal{L}^{-2} \ll \left(G^\kappa M^{1+\lambda+\kappa}(N_1N_2)^{2+\kappa}\right)^{\frac{1}{2+2\kappa}} + M^{\frac{1}{2}}N_1N_2 \\ + M(N_1N_2)^{\frac{1}{2}} + G^{-\frac{1}{2}}MN_1N_2. \end{aligned} \tag{3.4}$$

Lemma 3.3. *Let u, v, w be positive numbers. Suppose $x \geqslant 2$, $|a(m)| \leqslant 1$, $G = \left(\frac{x}{M^uN^v}\right)^{\frac{1}{w}}$ and*

$$T(M, N) = \sum_{m\sim M} \sum_{\substack{n\sim N \\ m^un^{v+w}\leqslant x}} a(m)\psi\left(\left(\frac{x}{m^un^v}\right)^{\frac{1}{w}}\right). \tag{3.5}$$

Then

$$\begin{aligned} T(M, N)\mathcal{L}^{-6} \ll \left(G^{1+\kappa}M^{2+\lambda+\kappa}N^{1+\kappa}\right)^{\frac{1}{3+2\kappa}} + G^{\frac{1}{3}}M^{\frac{2}{3}}N^{\frac{1}{3}} \\ + MN^{\frac{1}{2}} + G^{-\frac{1}{2}}MN. \end{aligned} \tag{3.6}$$

Lemma 3.4. *Let $u > 0$, $v > 0$ and $\max(u, v) \leqslant c$, define*

$$T_{(u,v)}(x; N, M) := \sum_{N<n\leqslant 2N} d(n) \sum_{\substack{m\sim M \\ m^{u+v}n^c\leqslant x}} \psi\left(\left(\frac{x}{m^un^c}\right)^{\frac{1}{v}}\right). \tag{3.7}$$

(i) *If $u \geqslant v$ and $MN \gg x^\theta$, then*

$$x^{-\varepsilon}T_{(u,v)}(x; N, M) \ll \left(x^{1-(u-v)\theta}N^{-c+u+v}\right)^{\frac{3}{8v}} + x^{\frac{1}{2(u+v)}}N^{1-\frac{c}{2(u+v)}}. \tag{3.8}$$

(ii) *If $u < v$, then*

$$x^{-\varepsilon}T_{(u,v)}(x; N, M) \ll x^{\frac{3}{4(u+v)}}N^{\frac{3}{4}\left(1-\frac{c}{u+v}\right)} + x^{\frac{1}{2(u+v)}}N^{1-\frac{c}{2(u+v)}}. \tag{3.9}$$

Proof. By Lemma 3.3 with $(u, v, w) = (c, u, v)$, we get

$$x^{-\varepsilon}T_{(u,v)}(x; N, M) \ll \left(\left(\frac{x}{M^u N^c}\right)^{\frac{1+\kappa}{v}} N^{2+\lambda+\kappa} M^{1+\kappa}\right)^{\frac{1}{3+2\kappa}} \tag{3.10}$$
$$+ \left(\frac{x}{M^u N^c}\right)^{\frac{1}{3v}} N^{\frac{2}{3}} M^{\frac{1}{3}} + NM^{\frac{1}{2}} + \left(\frac{x}{M^u N^c}\right)^{-\frac{1}{2v}} NM$$
$$\ll x^{\frac{1+\kappa}{v(3+2\kappa)}} N^{\frac{2v-c+v\lambda+(v-c)\kappa}{v(3+2\kappa)}} M^{\frac{(v-u)(1+\kappa)}{v(3+2\kappa)}} + x^{\frac{1}{3v}} N^{\frac{2v-c}{3v}} M^{\frac{v-u}{3v}}$$
$$+ NM^{\frac{1}{2}} + x^{-\frac{1}{2v}} N^{1+\frac{c}{2v}} M^{1+\frac{u}{2v}}.$$

If $u \geqslant v$, taking $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$, then applying $MN \gg x^{\theta}$ and $N^c M^{u+v} \ll x$ it follows from (3.10) that

$$x^{-\varepsilon}T_{(u,v)}(x; N, M) \ll \left(\frac{x}{N^{c-u-v}} \frac{1}{(MN)^{u-v}}\right)^{\frac{3}{8v}} + \left(\frac{x}{N^{c-u-v}(NM)^{u-v}}\right)^{\frac{1}{3v}} \tag{3.11}$$
$$+ N^{1-\frac{c}{2(u+v)}} \left(N^c M^{u+v}\right)^{\frac{1}{2(u+v)}}$$
$$+ x^{-\frac{1}{2v}} N^{1-\frac{c}{2(u+v)}} \left(N^c M^{u+v}\right)^{\frac{u+2v}{2v(u+v)}}$$
$$\ll \left(\frac{x^{1-(u-v)\theta}}{N^{c-u-v}}\right)^{\frac{3}{8v}} + \left(\frac{x^{1-(u-v)\theta}}{N^{c-u-v}}\right)^{\frac{1}{3v}} + x^{\frac{1}{2(u+v)}} N^{1-\frac{c}{2(u+v)}}.$$

Since $\frac{3}{8} > \frac{1}{3}$, then the second term in the above expression is less than the first term, hence (3.8) holds.

If $u \leqslant v$, similar to the proof of (3.11) we also have

$$x^{-\varepsilon}T_{(u,v)}(x; N, M) \ll \left(xN^{2v\left(1-\frac{c}{u+v}\right)} \left(N^c M^{u+v}\right)^{\frac{v-u}{u+v}}\right)^{\frac{3}{8v}} \tag{3.12}$$
$$+ \left(xN^{2v\left(1-\frac{c}{u+v}\right)} \left(N^c M^{u+v}\right)^{\frac{v-u}{u+v}}\right)^{\frac{1}{3v}}$$
$$+ N^{1-\frac{c}{2(u+v)}} \left(N^c M^{u+v}\right)^{\frac{1}{2(u+v)}}$$
$$+ x^{-\frac{1}{2v}} N^{1-\frac{c}{2(u+v)}} \left(N^c M^{u+v}\right)^{\frac{u+2v}{2v(u+v)}}$$
$$\ll x^{\frac{3}{4(u+v)}} N^{\frac{3}{4}\left(1-\frac{c}{u+v}\right)} + x^{\frac{2}{3(u+v)}} N^{\frac{2}{3}\left(1-\frac{c}{u+v}\right)}$$
$$+ x^{\frac{1}{2(u+v)}} N^{1-\frac{c}{2(u+v)}}.$$

Since $\frac{3}{4} > \frac{2}{3}$, and this completes the proof of (3.9). ∎

Lemma 3.5. *Let $a \leqslant b \leqslant c$, $M \leqslant \frac{x}{4}$ and define*

$$S_{(a,b)}(x; M) := \sum_{M < m \leqslant 2M} d(a, b; m) \Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right). \tag{3.13}$$

If $M \leqslant \frac{1}{3} x^{\frac{a}{a+c}}$, then

$$x^{-\varepsilon} S_{(a,b)}(x; M) \ll x^{\frac{1}{3c}} M^{\frac{5c-2a}{6ac}} + x^{\frac{1}{4c}} M^{\frac{1}{a}-\frac{1}{4c}}. \tag{3.14}$$

Proof. From (3.1) and (1.6), we have for any $1 \leqslant Y \leqslant \left(\frac{x}{4M}\right)^{\frac{1}{c}}$

$$\begin{aligned} S_{(a,b)}(x;M) \ll x^{\frac{1}{4c}} &\left| \sum_{m\sim M} \frac{d(a,b;m)}{m^{\frac{1}{4c}}} \sum_{n\leqslant Y} \frac{\tau(n)}{n^{\frac{3}{4}}} e\left(2\left(\frac{n^c x}{m}\right)^{\frac{1}{2c}}\right)\right| \\ &+ \sum_{M<m\leqslant 2M} d(a,b;m)\left(\left(\frac{x}{m}\right)^{\frac{1}{2c}+\varepsilon} Y^{-\frac{1}{2}} + x^{\varepsilon}\right) \\ &\ll x^{\frac{1}{4c}} \left| \sum_{m_1^a m_2^b\sim M} \frac{1}{(m_1^a m_2^b)^{\frac{1}{4c}}} \sum_{n\leqslant Y} \frac{\tau(n)}{n^{\frac{3}{4}}} e\left(2\left(\frac{n^c x}{m_1^a m_2^b}\right)^{\frac{1}{2c}}\right)\right| \\ &+ x^{\frac{1}{2c}+\varepsilon} M^{\frac{1}{a}-\frac{1}{2c}} Y^{-\frac{1}{2}} + M^{\frac{1}{a}} x^{\varepsilon}. \end{aligned} \tag{3.15}$$

By Lemma 3.2 with $(\alpha,\beta,\gamma) = (\frac{1}{2}, -\frac{a}{2c}, -\frac{b}{2c})$, we have

$$\begin{aligned} &x^{-\varepsilon} \sum_{\substack{m_1\sim M_1, m_2\sim M_2 \\ m_1^a m_2^b \sim M}} \frac{1}{(m_1^a m_2^b)^{\frac{1}{4c}}} \sum_{n\sim N} \frac{\tau(n)}{n^{\frac{3}{4}}} e\left(2\left(\frac{n^c x}{m_1^a m_2^b}\right)^{\frac{1}{2c}}\right) \\ &\ll M^{-\frac{1}{4c}} N^{-\frac{3}{4}} \left\{ \left(\left(\frac{N^c x}{M}\right)^{\frac{\kappa}{2c}} N^{1+\lambda+\kappa} (M_1M_2)^{2+\kappa}\right)^{\frac{1}{2+2\kappa}} + N^{\frac{1}{2}} M_1 M_2 \right. \\ &\left. + N(M_1M_2)^{\frac{1}{2}} + \left(\frac{N^c x}{M}\right)^{-\frac{1}{4c}} N M_1 M_2 \right\} \\ &\ll x^{\frac{\kappa}{4c(1+\kappa)}} N^{\frac{2\lambda-1}{4(1+\kappa)}} M^{\frac{4c+2(c-a)\kappa-a}{4ac(1+\kappa)}} + N^{-\frac{1}{4}} M^{\frac{1}{a}-\frac{1}{4c}} + N^{\frac{1}{4}} M^{\frac{1}{2a}-\frac{1}{4c}} + x^{-\frac{1}{4c}} M^{\frac{1}{a}}, \end{aligned} \tag{3.16}$$

here we use $M_1M_2 \ll (M_1^a M_2^b)^{\frac{1}{a}} \ll M^{\frac{1}{a}}$.

Now we choose $Y = x^{\frac{1}{3c}} M^{\frac{2}{3a}-\frac{1}{3c}}$ and $(\kappa,\lambda) = (\frac{1}{2},\frac{1}{2})$. Note that if $M \leqslant \frac{1}{3} x^{\frac{a}{a+c}}$, then $1 \leqslant Y \leqslant \left(\frac{x}{4M}\right)^{\frac{1}{c}}$. By a simple splitting argument, $M^{\frac{1}{a}} \ll x^{\frac{1}{4c}} M^{\frac{1}{a}-\frac{1}{4c}}$ and $x^{\frac{1}{3c}} M^{\frac{2}{3a}-\frac{1}{3c}} \ll x^{\frac{1}{3c}} M^{\frac{5c-2a}{6ac}}$, Lemma 3.5 follows from (3.15) and (3.16) at once. ■

Proof of Theorem 2. We choose $y = x^{\frac{4a(a+b)(1-c\alpha)}{(2a+2b-c)(3c+2a-c(5a+3b)\alpha)}}$, and let $N^* = x^{\frac{3-(5a+3b)\alpha}{3c+2a-c(5a+3b)\alpha}}$, where $\theta = \theta(a,b,c)$ and $\alpha = \alpha(a,b)$ are defined by Theorem 2.

It is easy to verify that if $\alpha(a,b) \geqslant \frac{3c-2b}{c(9a+7b)-6(a+b)^2}$, then $M \leqslant \frac{y}{2} \leqslant \frac{1}{3} x^{\frac{a}{a+c}}$. Applying a splitting argument and Lemma 3.5, we get

$$x^{-\varepsilon} \sum_{m\leqslant y} d(a,b;m) \Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right) \ll x^{\frac{1}{3c}} y^{\frac{5c-2a}{6ac}} + x^{\frac{1}{4c}} y^{\frac{1}{a}-\frac{1}{4c}}. \tag{3.17}$$

By the condition $\alpha(a,b) \geqslant \frac{9c-4a-2b}{c(23a+25b)-18(a+b)^2}$, one can check

$$x^{\frac{1}{4c}} y^{\frac{1}{a}-\frac{1}{4c}} \ll x^{\theta+\varepsilon}. \tag{3.18}$$

Similarly, by the conditions $a+b \leqslant c < 2(a+b)$ and $\alpha(a,b) \geqslant \frac{9c-4a-2b}{c(23a+25b)-18(a+b)^2}$, we may also check that both $\alpha(a,b) \geqslant \frac{3c-2b}{c(9a+7b)-6(a+b)^2}$ and

$$x^{\frac{1}{3c}} y^{\frac{5c-2a}{6ac}} \ll x^{\theta+\varepsilon} \tag{3.19}$$

hold.

From (3.17)–(3.19) we obtain

$$\sum_{m \leqslant y} d(a,b;m)\Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right) \ll x^{\theta+\varepsilon}. \tag{3.20}$$

Now we write

$$T_{(a,b)}(x;N) := \sum_{N<n\leqslant 2N} d(n)\Delta\left(a,b;\frac{x}{n^c}\right). \tag{3.21}$$

For $1 \ll N \ll \left(\frac{x}{y}\right)^{\frac{1}{c}}$, combining Theorem 1 and (3.20), an estimate

$$T_{(a,b)}(x;N) \ll x^{\theta+\varepsilon} \tag{3.22}$$

would suffice to complete the proof Theorem 2.

Now we consider two cases.

Case i. If $1 \ll N \ll N^*$, by the condition $\Delta(a,b;x) \ll x^{\alpha(a,b)}$ we easily check

$$T_{(a,b)}(x;N) \ll \sum_{N<n\leqslant 2N} d(n)\left(\frac{x}{n^c}\right)^{\alpha(a,b)} \ll x^{\alpha(a,b)} N^{1-\alpha(a,b)c}\log x \ll x^{\theta+\varepsilon}. \tag{3.23}$$

Case ii. If $N^* \ll N \ll \left(\frac{x}{y}\right)^{\frac{1}{c}}$, from (3.2) we have

$$\begin{aligned} T_{(a,b)}(x;N) &= -\sum_{N<n\leqslant 2N} d(n) \sum_{m^{a+b}\leqslant \frac{x}{n^c}} \psi\left(\left(\frac{x}{m^b n^c}\right)^{\frac{1}{a}}\right) \\ &\quad - \sum_{N<n\leqslant 2N} d(n) \sum_{m^{a+b}\leqslant \frac{x}{n^c}} \psi\left(\left(\frac{x}{m^a n^c}\right)^{\frac{1}{b}}\right) + O(N) \\ &:= -T^{(1)}_{(a,b)}(x;N) - T^{(2)}_{(a,b)}(x;N) + O(N). \end{aligned} \tag{3.24}$$

Applying a splitting argument and (3.8) in Lemma 3.4 with $(u,v)=(b,a)$, we get

$$\begin{aligned} x^{-\varepsilon} T^{(1)}_{(a,b)}(x;N) &\ll \left(\frac{x^{1-(b-a)\theta}}{N^{*(c-a-b)}}\right)^{\frac{3}{8a}} + x^{\frac{1}{2(a+b)}}(xy^{-1})^{\frac{1}{c}(1-\frac{c}{2(a+b)})} + x^{\theta} \\ &\ll x^{\theta} + x^{\frac{1}{c}} y^{-\frac{1}{c}+\frac{1}{2(a+b)}} = 2x^{\theta}. \end{aligned} \tag{3.25}$$

Here if $MN \ll x^{\theta}$, we use a trivial estimate.

Similarly, from (3.9) with $(u,v)=(a,b)$ and the condition $\alpha(a,b)<\frac{1}{c}\leqslant\frac{1}{a+b}$, we also have

$$x^{-\varepsilon}T^{(2)}_{(a,b)}(x;N)\ll x^{\frac{3}{4(a+b)}}N^{*\frac{3}{4}\left(1-\frac{c}{a+b}\right)}+x^{\frac{1}{2(a+b)}}(xy^{-1})^{\frac{1}{c}\left(1-\frac{c}{2(a+b)}\right)}\ll x^{\theta}. \tag{3.26}$$

Finally, from (3.24)–(3.26), (3.22) holds in this case. This completes the proof of Theorem 2. ■

Proof of Corollary 1.2. From Theorem 5.12 of Krätzel [14], we have

$$\Delta(a,b;x)\ll\begin{cases} x^{\frac{19}{29(a+b)}}\log^2 x, & \text{if } 11a\geqslant 8b\ ,\\ x^{\frac{10}{17a+14b}}\log^2 x, & \text{if } 2a\geqslant b>\frac{11}{8}a,\\ x^{\frac{21}{39a+27b}}\log^2 x, & \text{if } 4a\geqslant b>2a\end{cases} \tag{3.27}$$

By (3.27), the condition $\alpha(a,b)\geqslant\frac{9c-4a-2b}{c(23a+25b)-18(a+b)^2}$ of Theorem 2 is equivalent to

$$\begin{cases} c\geqslant\frac{113a^2+255ab+142b^2}{88a+107b}, & \text{if } 11a\geqslant 8b\ ,\\ c\geqslant\frac{112a^2+270ab+152b^2}{77a+124b}, & \text{if } 2a\geqslant b>\frac{11}{8}a,\\ c\geqslant\frac{37a^2+95ab+54b^2}{22a+47b}, & \text{if } 4a\geqslant b>2a\end{cases} \tag{3.28}$$

Now Corollary 1.2 follows from (3,27), (3.28) and Theorem 2 at once. ■

4. The proof of Theorem 3

The exponential totient function $\phi^{(e)}(n)$ is multiplicative and for each prime power p^a one has $\phi^{(e)}(p^a)=\phi(a)$, where ϕ is the Euler function. Hence for $\Re s>1$, by Euler product we get

$$\sum_{n=1}^{\infty}\frac{\phi^{(e)}(n)}{n^s}=\prod_p\left(1+\sum_{m=1}^{\infty}\frac{\phi^{(e)}(m)}{p^{ms}}\right) \tag{4.1}$$

Applying the product representation of Riemann zeta-function

$$\zeta(s)=\prod_p(1+p^{-s}+p^{-2s}+p^{-3s}+\ldots)=\prod_p(1-p^{-s})^{-1}, \Re s>1, \tag{4.2}$$

we have for $\Re s>1$

$$\zeta(s)\zeta(3s)\zeta^2(5s)\zeta^4(7s)=\prod_p(1-p^{-s})^{-1}(1-p^{-3s})^{-1}(1-p^{-5s})^{-2}(1-p^{-7s})^{-4}. \tag{4.3}$$

Let

$$\begin{aligned} f_{\phi^{(e)}}(z) &:= 1+\sum_{m=1}^{\infty}\phi^{(e)}(m)z^m \\ &= 1+z+z^2+2z^3+2z^4+4z^5+2z^6+6z^7+4z^8+6z^9 \\ &\quad +4z^{10}+\sum_{m=11}^{\infty}q_r(m)z^m. \end{aligned} \tag{4.4}$$

By a simple calculation one get for $|z| < 1$

$$(1-z)(1-z^3)(1-z^5)^2(1-z^7)^4 = 1 - z - z^3 + z^4 - 2z^5 + 2z^6 - 4z^7 + 6z^8 - 2z^9 + 5z^{10} + \ldots + z^{42},$$

$$f_{\phi^{(e)}}(z)(1-z)(1-z^3)(1-z^5)^2(1-z^7)^4 = 1 - 3z^6 - 4z^8 + 4z^9 - 9z^{10} + \sum_{m=11}^{\infty} c_m z^m,$$

and

$$\left(1 + z^6 + z^{12} + \ldots\right)^3 \left(1 + z^8 + z^{16} + \ldots\right)^4 = 1 + 3z^6 + 4z^8 + 6z^{12} + \ldots.$$

From the above two relations, we easily obtain for $|z| < 1$

$$f_{\phi^{(e)}}(z)(1-z)(1-z^3)(1-z^5)^2(1-z^7)^4(1-z^6)^{-3}(1-z^8)^{-4} = 1 + 4z^9 + \sum_{m=10}^{\infty} C_m z^m. \quad (4.5)$$

From (4.1)–(4.5) we get that

$$\Phi^{(e)}(s) := \sum_{n=1}^{\infty} \frac{\phi^{(e)}(n)}{n^s} = \frac{\zeta(s)\zeta(3s)\zeta^2(5s)\zeta^4(7s)}{\zeta^3(6s)\zeta^4(8s)} V(s), \qquad \Re s > 1, \quad (4.6)$$

where $V(s)$ is absolutely convergent for $\Re s > \frac{1}{9}$.

Now the asymptotic formula (1.15) follows from (4.6) and (1.13) via the well-known convolution method.

The Ω-estimate (1.16) follows from Theorem 2 of Küleitner and Nowak [17] (or by Balasubramanian, Ramachandra and Subbarao's method in [2]). This completes the proof of Theorem 3.

References

[1] R.C. Baker, *The square-free divisor problem II*, Quart. J. Oxford **47**(2) (1996), 133–146.

[2] R. Balasubramanian, K. Ramachandra and M.V. Subbarao, *On the error function in the asymptotic formula for the counting function of k-full numbers*, Acta Arith. **50** (1988), 107–118.

[3] Xiaodong Cao, *On the general divisor problem* (in Chinese), Acta Math. Sin. **36**(5) (1993), 644–653.

[4] Xiaodong Cao, *The distribution of square-full integers*, Period. Math. Hung. **34**(3) (1997), 169–175.

[5] Xiaodong Cao, Wenguang Zhai, *Some arithmetical functions involving exponential divisors*, Journal of Integer Sequences **13** (2010), Article 10.3.7.

[6] E. Fouvry, H. Iwaniec, *Exponential sums with monomials*, J. Number Theory **33**(3) (1989), 311–333 .
[7] D.R. Heath-Brown, *The Pjateckii-Sapiro prime number theorem*, J. Number Theory **16** (1983), 242–266.
[8] M.N. Huxley, *Area, Lattice Points, and Exponential Sums*, Oxford University Press, New York, 1996.
[9] M.N. Huxley, *Exponential sums and lattice points III*, Proc. London Math. Soc. **87**(3) (2003), 591–609.
[10] A. Ivić, *The Riemann Zeta-Function*, John. Wiley & Sons, 1985.
[11] A. Ivić, *The general divisor problem*, J. Number Theory **27** (1987), 73–91.
[12] A. Ivić, E. Krätzel, M. Kühleitner, W.G. Nowak, *Lattice points in large regions and related arithmetic functions: recent developments in a very classic topic* (English summary), Elementare und analytische Zahlentheorie, 89–128, Schr.Wiss. Ges. Johann Wolfgang Goethe Univ.Frankfurt am Main, 20, Franz Steiner Verlag Stuttgart, Stuttgart, 2006.
[13] I. Kátai, M.V. Subbarao, *On the distribution of exponential divisors*, Ann. Univ. Sci. Budapest. Sect. Comput. **22** (2003), 161–180.
[14] E. Krätzel, *Lattice Points*, Kluwer, Dordrecht-Boston-London, 1988.
[15] E. Krätzel, *The four-dimensional divisor problem*, Unif. Distrib. Theory **3**(1) (2008), 93–103.
[16] E. Krätzel, *New estimates in the four-dimensional divisor problem with applications*, Acta Math. Hungar. **126**(3) (2010), 258–278.
[17] M. Küleitner, W.G. Nowak, *An omega theorem for a class of arithmetic functions*, Math. Nachr. **165** (1994), 79–98.
[18] A. Mercier, W.G. Nowak, *Problème des diviseurs pour des valeurs polynômiales II*, Ann. Sci. Math. Québec **16**(1) (1992), 81–94.
[19] Y.-F.S. Pétermann, J. Wu, *On the sum of exponential divisors of an integer*, Acta Math. Hungar. **77**(1-2) (1997), 159–175.
[20] Y.-F.S. Pétermann, *Arithmetical functions involving exponential divisors: Note on two papers by L. Tóth*, Ann. Univ. Sci. Budapest. Sect. Comput. **32** (2010), 143–149.
[21] J. Sándor, *On an exponential totient function*, Sudia Univ. Babes-Bolyai Math. **41** (1996), 91–94.
[22] E.G. Straus and M.V. Subbarao, *On exponential divisors*, Duke Math. J. **41** (1974), 465–471.
[23] P. Sargos and J. Wu, *Mutiple exponential sums with monomials and their applications in number theory*, Acta Math. Hungar. **87**(4) (2000), 333–354.
[24] M.V. Subbarao, *On some arithmetic convolutions*, in The Theory of arithmetic functions, Lecture Notes in Mathematics, no. 251, 247–271, Springer, 1972.
[25] L. Tóth, *On certain arithmetic functions involving exponential divisors*, Ann. Univ. Sci. Budapest. Sect. Comput. **24** (2004), 285–294.
[26] L. Tóth, *On certain arithmetical functions involving exponential divisors II*, Ann. Univ. Sci. Budapest. Sect. Comput. **27** (2007), 155–166.
[27] L. Tóth, *An order result for the exponential divisor function*, Publ. Math. Debrecen **71**(1-2) (2007), 165–171.

[28] M. Vogts, *Many-dimensional generalized divisor problems*, Math. Nachr. **124** (1985), 103–121.

[29] Jie Wu, *Problème de diviseurs exponentiels et entiers exponentiellement sans facteur carré*, J. Théor. Nombres Bordeaux **7**(1) (1995), 133–141.

[30] Jie Wu, *On the distribution of square-full integers*, Arch. Math. (Basel) **77**(3) (2001), 233–240.

[31] Wenguang Zhai and Xiaodong Cao, *On the mean square of the error term for the asymmetric two-dimensional divisor problem (I)*, Monatsh. Math. **159** (2010), 185–209.

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