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ON SOME COMPLEX EXPLICIT FORMULÆ CONNECTED WITH DIRICHLET COEFFICIENTS OF INVERSES OF SPECIAL TYPE L-FUNCTIONS FROM THE SELBERG CLASS

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Abstract: We obtain, by means of the technique introduced in by J. Kaczorowski, a meromorphic continuation and the functional equation for the function $m(F,\cdot)$, where F is from the Selberg class with a functional equation expressible with exactly one Γ function.

Keywords: coefficients of L-functions, Selberg class.

1. Introduction

Let S^{Γ} denote the subset of the Selberg class [9] consisting of the functions with a functional equation expressible with exactly one Γ function. That is a function $F \in \mathrm{S}^{\Gamma}$ satisfies the following five axioms ($s = \sigma + it$ here and futher on)

1. (Dirichlet series) For $\sigma > 1$, F is an absolutely convergent Dirichlet series
$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} \ .$$
2. (Analytic continuation) For some $m \geqslant 0$, $(s-1)^m F(s)$ is an entire function of finite order.
3. (Functional equation) F satisfies a functional equation of the form
$$\Phi_F(s) = \omega \overline{\Phi}_F(1-s)$$
where
$$\Phi_F(s) = \mathrm{Q}^s \Gamma(\lambda s + \mu) F(s)$$
with $\mathrm{Q} > 0$, $\lambda > 0$, $\Re\mu \geqslant 0$ and $|\omega| = 1$.
4. (Ramanujan hypothesis) For every $\varepsilon > 0$, $a_F(n) \ll_{\varepsilon} n^{\varepsilon}$.

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5. (Euler product) For $\sigma > 1$
$$F(s) = \prod_p F_p(s)$$
where
$$\log F_p(s) := \sum_{m=1}^{\infty} \frac{b(p^m)}{p^{ms}} \tag{1.1}$$
and $b(n) \ll n^\theta$ for some $\theta < \frac{1}{2}$.

The known invariants of functions from the Selberg class S, the degree, the ξ-invariant, the parity and the shift, may be written as
$$d_F = 2\lambda, \qquad \xi_F + 1 = 2\mu, \qquad \eta_F + 1 = 2\Re\mu \qquad \text{and} \qquad \theta_F = 2\Im\mu$$
for such F.

We note that, although the data in the functional equation in S are, in general, not unique, see for example Section 4 of Vignéras [14], Section 2 of Conrey--Ghosh [4], Section 3 of Kaczorowski [9] and Kaczorowski-Perelli [12], they are unique in the special case of the functional equation from S^Γ as a immediate consequence of a simple form of invariants given above. Throughout this paper we fix $F \in \mathrm{S}^\Gamma$ and data Q, λ, μ, ω.

We denote by $\mu_F(n)$ the Dirichlet convolution inverse of $a_F(n)$, i.e. we formally have
$$\frac{1}{F(\sigma+it)} = \sum_{n=1}^{\infty} \frac{\mu_F(n)}{n^{\sigma+it}} \ . \tag{1.2}$$

From [11, Lemma 1] it follows that for every $\varepsilon > 0$ there exists $M = M(\varepsilon)$ such that $\mu_F(n) \ll_\varepsilon n^\varepsilon$ for $(n, M) = 1$. By this estimation it follows that
$$\prod_{(p,M)>1} F_p(s) \frac{1}{F(s)} = \sum_{\substack{n=1\\(n,M)=1}}^{\infty} \frac{\mu_F(n)}{n^s} \tag{1.3}$$
converges absolutely and uniformly for $\sigma \geqslant 1+\varepsilon$ for every $\varepsilon > 0$. Using axiom (5) one obtains
$$\mu_F(p^m) \ll p^{m\theta} \sum_{k=1}^{m} \frac{1}{k!}\binom{m-1}{k-1} \ll p^{m\theta} e^{2\sqrt{m}}, \qquad m \geqslant 1.$$
Hence the Dirichlet series
$$\frac{1}{F_p(s)} = \sum_{m=0}^{\infty} \mu_F(p^m) p^{-ms}$$
converges absolutely and uniformly on compact sets for $\sigma > \theta$. As a consequence we obtain the absolute and uniform convergence of the whole series (1.2) in the half-plane $\sigma \geqslant 1+\varepsilon$ for every $\varepsilon > 0$.

For brevity of notation we put

$$\varkappa_F := \begin{cases} -\frac{\eta_F+1}{2d_F} & \text{if } \eta_F > -1 \\ -\frac{1}{d_F} & \text{if } \eta_F = -1 \ . \end{cases}$$

Then, for z from the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$, the function $m(F,z)$ is defined as follows:

$$m(F,z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{e^{sz}}{F(s)}\,ds, \tag{1.4}$$

where $F \in \mathrm{S}^{\Gamma}$. The path of integration consists of the half-line $s = \varkappa_F + it$, $\infty > t \geqslant 0$, the smooth arc $\mathcal{A}$ on the upper half-plane joining points $\varkappa_F$ and $3/2$ separating possible real zeros of $F\overline{F}$ from the zeros above the real line, and the half-line $s = 3/2 + it$, $0 \leqslant t < \infty$. Since from axiom (3) and the Stirling formula it easily follows that $1/F(s)$ is bounded on $\mathcal{C}$, the integral converges absolutely and uniformly on compact subsets of $\mathbb{H}$, and hence represents a holomorphic function on this half-plane. To formulate the main result of this paper we need two auxiliary functions

$$R(F,z) = \sum_{\substack{F(\beta)=0\\ 0\leqslant\beta\leqslant 1}} \operatorname*{Res}_{s=\beta} \frac{e^{sz}}{F(s)} \ , \tag{1.5}$$

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k!\,\Gamma(k+\nu+1)}, \tag{1.6}$$

where $J_\nu(z)$ denotes the familiar Bessel function of the first kind of order $\nu \in \mathbb{R}$ [8, formula (2), p. 4] that we only use for $z \neq 0$, choosing the standard real branch on the positive part of the real axis. As usual, δ_a^b denotes the Kronecker delta. We also use the notation $\overline{m}(F,z) := \overline{m(F,\overline{z})}$.

Theorem 1. *Let $F \in \mathrm{S}^{\Gamma}$. Then $m(F,\cdot)$ has a meromorphic continuation to $\mathbb{C}$ with simple poles at the points $z = \log n$, $\mu_F(n) \neq 0$, $n \in \mathbb{N}$, and residues*

$$\operatorname*{Res}_{z=\log n} m(F,z) = -\frac{\mu_F(n)}{2\pi i} \ .$$

Moreover, it satisfies the following functional equation

$$\begin{aligned} m(F,z) + \overline{m}(\overline{F},z) = &-\frac{2\overline{\omega}}{d_F \mathrm{Q}^{1+2i\frac{\theta_F}{d_F}}} e^{-i\frac{\theta_F}{d_F}z} \sum_{n=1}^{\infty} \frac{\overline{\mu_F(n)}}{n^{1+i\frac{\theta_F}{d_F}}} \\ &\times \left(\left(\mathrm{Q}^2 n e^z\right)^{\frac12-\frac{1}{d_F}} J_{\frac12 d_F+\eta_F}\left(2\left(\mathrm{Q}^2 n e^z\right)^{-\frac{1}{d_F}}\right) - \delta_{-1}^{\eta_F}\frac{1}{\Gamma\left(\frac12 d_F\right)}\right) \\ &- R(F,z). \end{aligned} \tag{1.7}$$

This theorem generalises a result of K. Bartz [2] since the Riemann zeta function belongs to S^{Γ}. It also generalises a result of A. Łydka [13, Theorem 1.3] since by the results contained in [3, 5, 6] the function $L(s+\frac{1}{2}, E)$ belongs to S^{Γ}, where $L(s,E)$ denotes the global L-function of an elliptic curve over $\mathbb{Q}$.

In fact the class S^{Γ} contains many more functions. Let χ be a primitive, non principal Dirichlet character. Then for every $\theta \in \mathbb{R}$ the Dirichlet L-function $L(s+i\theta,\chi)$ belongs to S^{Γ}. Let f be a normalised newform of weight k and level N, i.e. $f \in \mathbf{S}_k^{new}(N)$, such that f is a common eigenvector for all Hecke operators T_p. Then the associated L-function $L(f, s+\frac{k-1}{2})$ belongs to S^{Γ} [5, 6, 9].

Neither the complete structure of the Selberg class S, nor even the structure of S^{Γ} is known, although many conjectures are formulated [9, 12]. We note here that our result is completely independent of those conjectures.

Let us explicitly state here that the function $m(F,\cdot)$ is just a tool aimed at proving Ω and $\Omega_{\pm}$ results for the summatory functions of the function μ_F. So far this aim was achieved for the summatory function of the function μ_ζ i.e. the classical arithmetic Möbius function [10, Theorem 1]. Therefore our research is primarily motivated by the arithmetical nature of the elements of the Selberg class and the main result of this paper is just a step towards obtaining Ω results for the summatory function of μ_F where $F \in \mathrm{S}^{\Gamma}$.

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2. Auxiliary results

First we state some technical lemmas.

Lemma 1. *Let $F \in \mathrm{S}^{\Gamma}$ and let $\rho = \beta + i\gamma$ run through non-trivial zeros of the function F. Then for $|t| > 1$ we have the following formulæ*

$$\frac{F'}{F}(s) = \sum_{|t-\gamma|\leqslant 1} \frac{1}{s-\rho} + \mathrm{O}_F(\log t) \tag{2.1}$$

and

$$\log F(s) = \sum_{|t-\gamma|\leqslant 1} \log(s-\rho) + \mathrm{O}_F(\log t), \tag{2.2}$$

uniformly for $-1 \leqslant \sigma \leqslant 2$, where the implied constants depend only on F (cf. [1, Lemma 2.4]) and $-\pi < \Im \log(s-\rho) < \pi$.

The proof of Lemma 1 follows, *mutatis mutandis*, by the argument in the proof of Theorem 9.6 (B) [15]. As a corollary we have

$$\log F(\sigma+it) \ll_{\varepsilon,F} \log\left(|t|+2\right), \qquad \text{as} \ \ |t| \to \infty \tag{2.3}$$

for every $\varepsilon > 0$, in the strip $1+\varepsilon \leqslant \sigma \leqslant 2$.

For brevity of notation we put

$$\upsilon_F := \frac{|\theta_F|}{d_F} + 1 \, .$$

Then we have

Lemma 2. *Let* $z = x + iy$, $y > 0$, $s = Re^{i\varphi}$, $R \sin\varphi \geqslant \upsilon_F$, $R|\cos\varphi| \geqslant \frac{1}{2}|\varkappa_F|$, *where* $\frac{\pi}{2} < \varphi < \pi$ *and let* $F \in \mathrm{S}^{\Gamma}$. *Then for* $R \geqslant R_0(x, y)$ *we have*

$$\left| \frac{e^{sz}}{F(s)} \right| \leqslant e^{-y\frac{R}{2}}. \tag{2.4}$$

Proof. Using the asymmetric form of the functional equation for $F \in \mathrm{S}^{\Gamma}$

$$F(s) = \omega \frac{1}{h_F(s)} \overline{F}(1 - s), \tag{2.5}$$

where

$$h_F(s) = \mathrm{Q}^{2s-1} \frac{\Gamma(\lambda s + \mu)}{\Gamma(\lambda(1 - s) + \overline{\mu})} \tag{2.6}$$

we obtain

$$\log \left| \frac{e^{sz}}{F(s)} \right| = \Re(sz) - \log \left| \overline{F}(1 - s) \right| + \log |h_F(s)| \, .$$

Since $\Re(1 - s) = 1 + R|\cos\varphi| \geqslant 1 + \frac{1}{2}|\varkappa_F|$, by (2.3) we have $\log \left|\overline{F}(1 - s)\right| \ll_{\varkappa_F} \log R$. Since $R \sin\varphi \geqslant \upsilon_F$, we have

$$\log |\sin(\pi(\lambda s + \mu))| = \frac{d_F}{2} \pi R \sin\varphi + \mathrm{O}(1). \tag{2.7}$$

Using the well-known formula

$$\Gamma(s)\Gamma(1 - s) = \frac{\pi}{\sin(\pi s)}$$

and the Stirling formula we estimate

$$\begin{aligned} \log |h_F(s)| &= (2R\cos\varphi)\log \mathrm{Q} + (d_F R \cos\varphi) \log\left(\frac{1}{2} d_F R\right) \\ &\quad + d_F R \left(\varphi - \frac{3}{2}\pi\right) \sin\varphi - d_F R \cos\varphi + \mathrm{O}(\log R). \end{aligned} \tag{2.8}$$

Consequently

$$\log \left| \frac{e^{sz}}{F(s)} \right| = d_F R \log\left(\frac{d_F}{2} R\right) \cos\varphi + R f(\varphi, x, y) + \mathrm{O}(\log R), \tag{2.9}$$

where

$$f(\varphi, x, y) := (x + 2\log \mathrm{Q} - d_F)\cos\varphi + \left(-y + d_F\left(\varphi - \frac{3}{2}\pi\right)\right)\sin\varphi.$$

Since

$$f\left(\frac{\pi}{2},x,y\right)=-\left(y+d_F\pi\right)$$

and

$$\frac{\partial f}{\partial\varphi}\left(\varphi,x,y\right)\ll_{x,y}1,\quad \frac{\pi}{2}<\varphi<\pi,$$

we have for $\frac{\pi}{2}<\varphi\leqslant\frac{\pi}{2}+1/\sqrt{\log R}$

$$f(\varphi,x,y)=-\left(y+d_F2\pi\right)+\mathrm{O}_{x,y}\left(\frac{1}{\sqrt{\log R}}\right).$$

Hence, for such φ and sufficiently large R, we have

$$\log\left|\frac{e^{sz}}{F(s)}\right|\leqslant -y\frac{R}{2}.$$

For $\frac{\pi}{2}+1/\sqrt{\log R}\leqslant\varphi\leqslant\pi$ we have $|\cos\varphi|\gg 1/\sqrt{\log R}$ and hence using (2.9) we have

$$\log\left|\frac{e^{sz}}{F(s)}\right|=-d_FR\log\left(\frac{d_F}{2}R\right)|\cos\varphi|+\mathrm{O}_{x,y}(R)\leqslant -y\frac{R}{2}$$

for sufficiently large R, and the lemma follows. ■

3. Proof of Theorem 1

We split the proof of the theorem into two parts. First we prove that function $m(F,\cdot)$ has a meromorphic continuation to the whole complex plane, then we show the functional equation.

Using Lemma 2 we can shift the path of integration in (1.4) as follows:

$$\begin{aligned} m(F,z)&=\frac{1}{2\pi i}\left(\int_{\mathcal{D}}+\int_{\mathcal{A}}+\int_{\frac{3}{2}}^{\frac{3}{2}+i\infty}\right)\frac{e^{sz}}{F(s)}ds\\ &=: m_{\mathcal{D}}(F,z)+m_{\mathcal{A}}(F,z)+m_{\mathcal{L}}(F,z)\end{aligned}\tag{3.1}$$

where $\mathcal{D}$ consists of the half-line $s=\sigma+iv_F$, $-\infty<\sigma\leqslant\varkappa_F$ and the vertical line segment $[\varkappa_F+iv_F,\ \varkappa_F]$, $\mathcal{A}$ is the arc part of $\mathcal{C}$ and $\mathcal{L}=[3/2,3/2+i\infty)$. For $s=\sigma+iv_F$ with $\sigma\leqslant\varkappa_F$ and $z=x+iy$ we have

$$|e^{sz}|=e^{\sigma x-v_Fy}$$

and using (2.9)

$$\left|\frac{1}{F(\sigma+iv_F)}\right|\ll e^{-c|\sigma|\log(|\sigma|+2)}$$

for a positive c depending only on F. Hence $m_{\mathcal{D}}(F,\cdot)$ is an entire function. Since $\mathcal{A}$ is compact and omits zeros of F it follows that the function $m_{\mathcal{A}}(F,z)$ is also entire. Let $\Im(z) > 0$. Since the series $1/F\left(\frac{3}{2}+it\right) = \sum_{n=1}^{\infty} \mu_F(n) n^{-\frac{3}{2}-it}$ converges absolutely and uniformly for $0 \leqslant t < \infty$, and

$$\sum_{n=1}^{\infty} \int_0^{\infty} \left| \mu_F(n) e^{(z-\log n)\left(\frac{3}{2}+it\right)} \right| |dt| \leqslant e^{\frac{3}{2}x} \sum_{n=1}^{\infty} |\mu_F(n)|\, n^{-\frac{3}{2}} \int_0^{\infty} e^{-yt} dt \ll_{F,x} \frac{1}{y} \ll 1, \quad (3.2)$$

therefore in $m_{\mathcal{L}}(F,\cdot)$ we can interchange the order of summation and integration obtaining

$$m_{\mathcal{L}}(F,z) = \sum_{n=1}^{\infty} \mu_F(n) \frac{1}{2\pi i} \int_{\frac{3}{2}}^{\frac{3}{2}+i\infty} e^{(z-\log n)s} ds.$$

We have

$$m_{\mathcal{L}}(F,z) = -\frac{e^{\frac{3}{2}z}}{2\pi i} m_0(F,z),$$

where

$$m_0(F,z) = \sum_{n=1}^{\infty} \frac{\mu_F(n)}{n^{3/2}} \frac{1}{z-\log n}. \quad (3.3)$$

Because (3.3) is uniformly convergent on any compact subset of $\mathbb{C} \setminus \{z = \log n \mid \mu_F(n) \neq 0,\ n \in \mathbb{N}\}$ we obtain a meromorphic continuation of $m_{\mathcal{L}}(F,z)$ and, consequently, $m(F,z)$ to the whole complex plane. The only singularities are those generated by $m_0(F,z)$ i.e. simple poles at $\log n$, $n \in \mathbb{N}$, $\mu_F(n) \neq 0$, with residues

$$\operatorname*{Res}_{z=\log n} m(F,z) = -\frac{\mu_F(n)}{2\pi i}.$$

Let us now consider $\overline{m}(\overline{F},z)$, where $\Im(z) < 0$. Changing the variable $s \mapsto \overline{s}$ in (1.4), we have

$$\overline{m}(\overline{F},z) = \frac{1}{2\pi i} \int_{-\overline{\mathcal{C}}} \frac{e^{sz}}{F(s)} ds,$$

where $\overline{\mathcal{C}}$ denotes the contour conjugate to $\mathcal{C}$ and the minus sign indicates the reversed orientation. As in the first part of the proof, we replace the half-line $[\varkappa_F, \varkappa_F + i\infty)$, by the contour $-\overline{\mathcal{D}}$ consisting of the vertical line segment $[\varkappa_F, \varkappa_F - i\upsilon_F]$ and the half line $s = \sigma - i\upsilon_F$, $0 \geqslant \sigma > -\infty$. Therefore we have as in (3.1) that

$$\begin{aligned} \overline{m}(\overline{F},z) &= \frac{1}{2\pi i}\left(\int_{-\overline{\mathcal{D}}} + \int_{-\overline{\mathcal{A}}} + \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}} \right) \frac{e^{sz}}{F(s)} ds \\ &= m_{-\overline{\mathcal{D}}}(F,z) + m_{-\overline{\mathcal{A}}}(F,z) + \frac{e^{\frac{3}{2}z}}{2\pi i} m_0(F,z). \end{aligned} \quad (3.4)$$

and the equality extends to $z \in \mathbb{C}$ by analytic continuation. From (3.1) and (3.4) we obtain for $z \in \mathbb{C} \setminus \{\log n \mid \mu_F(n) \neq 0,\ n \in \mathbb{N}\}$

$$m(F,z) + \overline{m}(\overline{F},z) = \frac{1}{2\pi i}\int_{\mathcal{E}} \frac{e^{sz}}{F(s)}ds + \frac{1}{2\pi i}\int_{\mathcal{A}_2} \frac{e^{sz}}{F(s)}ds, \tag{3.5}$$

where $\mathcal{E}$ is the path consisting of $(-\infty + i\upsilon_F, \varkappa_F + i\upsilon_F]$, $[\varkappa_F + i\upsilon_F, \varkappa_F - i\upsilon_F]$ and $[\varkappa_F - i\upsilon_F, -\infty - i\upsilon_F)$ and $\mathcal{A}_2 = \mathcal{A} \cup -\overline{\mathcal{A}}$ is a closed loop. Since $\mathcal{A}$ separates the real zeros of $F\overline{F}$ from the zeros above the real line, there are no points inside the loop $\mathcal{A}_2$, apart from the interval $[0,1]$, where $e^{z\cdot}/F(\cdot)$ could have singularity. Computing residues and noting that the orientation of $\mathcal{A}_2$ is clockwise, we obtain

$$\frac{1}{2\pi i}\int_{\mathcal{A}_2} \frac{e^{sz}}{F(s)}ds = -R(F,z).$$

By (2.8) we have

$$\int_{\varkappa_F}^{-\infty} \sum_{n=1}^{\infty} \left|\frac{\mu_F(n)}{n^{1-s}}\right| |h_F(\sigma \pm i\upsilon_F)| \left|e^{(\sigma \pm i\upsilon_F)z}\right| |d\sigma| \\ \ll \int_{\varkappa_F}^{-\infty} e^{-c_1|\sigma|} e^{-|\sigma|x \mp y\upsilon_F} |d\sigma| \ll 1, \tag{3.6}$$

where $c_1 > 0$. By the functional equation (2.5), the expansion of $1/\overline{F}(1-s)$ into the absolutely and uniformly convergent Dirichlet series, and by the estimation (3.6) we obtain

$$\frac{1}{2\pi i}\int_{\mathcal{E}} \frac{e^{sz}}{F(s)}ds = \frac{\overline{\omega}}{\mathrm{Q}} \sum_{n=1}^{\infty} \frac{\overline{\mu_F}(n)}{n} \frac{1}{2\pi i}\int_{\mathcal{E}} \frac{\Gamma(\lambda s + \mu)}{\Gamma(\lambda(1-s) + \overline{\mu})} \left(\mathrm{Q}^2 n e^z\right)^s ds.$$

Under the substitution $\lambda s \mapsto s$, we have

$$\frac{1}{2\pi i}\int_{\mathcal{E}} \frac{\Gamma(\lambda s + \mu)}{\Gamma(\lambda(1-s) + \overline{\mu})} \left(\mathrm{Q}^2 n e^z\right)^s ds \\ = \frac{2}{d_F}\frac{1}{2\pi i}\int_{\lambda\mathcal{E}} \frac{\Gamma(s+\mu)}{\Gamma(\lambda + \overline{\mu} - s)} \left(\left(\mathrm{Q}^2 n e^z\right)^{\frac{2}{d_F}}\right)^s ds.$$

Evaluating the last integral by means of [7, formulæ (9), p. 205 & (3), p. 211] we obtain

$$\frac{1}{2\pi i}\int_{\mathcal{E}} \frac{\Gamma(\lambda s + \mu)}{\Gamma(\lambda(1-s) + \overline{\mu})} \left(\mathrm{Q}^2 n e^z\right)^s ds \\ = -\frac{2}{d_F}\left(\mathrm{Q}^2 n e^z\right)^{-i\frac{\theta_F}{d_F}} \left(\left(\mathrm{Q}^2 n e^z\right)^{\frac{1}{2}-\frac{1}{d_F}} J_{\frac{1}{2}d_F + \eta_F}\left(2\left(\mathrm{Q}^2 n e^z\right)^{-\frac{1}{d_F}}\right) - \delta_{-1}^{\eta_F}\frac{1}{\Gamma\left(\frac{1}{2}d_F\right)}\right)$$

and the theorem follows.

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