Functiones et Approximatio
48.2 (2013), 167–181
doi: 10.7169/facm/2013.48.2.1

THE METRICAL THEORY OF SIMULTANEOUSLY SMALL LINEAR FORMS

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Abstract: In this paper we investigate the metrical structure of the set of all points $X \in \mathbb{R}^n$ which satisfy a simultaneously small system of Diophantine inequalities for infinitely many integer vectors. We establish the complete metric theory for the given system which implies a general Khintchine–Groshev type theorem, as well as its Hausdorff measure generalization. The latter includes the original dimension results obtained in [5] as special cases.

Keywords: Diophantine approximation, Khintchine type theorems, system linear forms, Hausdorff measure.

1. Introduction

Notation. For two real quantities a and b we will write $a \ll b$ if there exists a constant $C > 0$ such that $a \leqslant Cb$. If $a \ll b$ and $b \ll a$ we write $a \asymp b$ and a and b are said to be comparable. For a set $A \subset \mathbb{R}^k$, $|A|_k$ is the k-dimensional Lebesgue measure of the set A.

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a real positive decreasing function with $\psi(r) \to 0$ as $r \to \infty$. Such a function will be referred to as an *approximation* function. An $m \times n$ matrix $X = (x_{ij}) \in \mathbb{R}^{mn}$ is said to be *ψ–approximable* if the system of inequalities

$$|q_1 x_{1i} + q_2 x_{2i} + \cdots + q_m x_{mi}| \leqslant \psi(|\mathbf{q}|) \qquad \text{for} \ \ (1 \leqslant i \leqslant n),$$

is satisfied for infinitely many $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$. Here and throughout $|\mathbf{q}|$ will denote the supremum norm of the vector $\mathbf{q}$. Specifically, $|\mathbf{q}| = \max\{|q_1|, |q_2|, \dots, |q_m|\}$. The system $q_1 x_{1i} + q_2 x_{2i} + \cdots + q_m x_{mi}$ of n linear forms in m variables $q_1, q_2, \dots, q_m$ will be written more concisely as $\mathbf{q}X$, where the matrix X is regarded as a point in $\mathbb{R}^{mn}$. It is easily verified that ψ-approximability is not affected under translation by integer vectors and we can therefore restrict attention to the unit cube

The first author was partially supported by The Higher Education Commission (HEC) of Pakistan – grant ref: 3-6-25/HEC/Overseas/2006/782.

2010 Mathematics Subject Classification: primary: 11J83; secondary: 11J13, 11K60

$\mathbb{I}^{mn} := [-\frac{1}{2}, \frac{1}{2}]^{mn}$. The set of ψ–approximable points in $\mathbb{I}^{mn}$ will be denoted by $W_0(m,n;\psi)$;

$$W_0(m,n;\psi) := \{X \in \mathbb{I}^{mn} : |\mathbf{q}X| < \psi(|\mathbf{q}|) \text{ for i.m. } \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}\},$$

where 'i.m.' is to be read as 'infinitely many'. In the case when $\psi(r) = r^{-\tau}$ for some $\tau > 0$ we shall write $W_0\,(m,n;\tau)$ instead of $W_0\,(m,n;\psi)$.

It is worth relating the above formulation to the set of ψ-approximable matrices as studied in classical Diophantine approximation. In such a setting, the metric structure of the lim sup set

$$W(m,n;\psi) = \{X \in \mathbb{I}^{mn} : \|\mathbf{q}X\| < \psi(|\mathbf{q}|) \text{ for i.m. } \ \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}\},$$

where $\|x\|$ denotes the distance of x to the nearest integer vector, is a central problem and the theory is well established, see for example [1]. The set $W_0\,(m,n;\psi)$ is therefore an analogue of $W(m,n;\psi)$ with $|\cdot|$ replacing $\|\cdot\|$. The aim of this paper is to obtain the complete metric theory for the set $W_0(m,n;\psi)$.

It is readily verified that $W_0(1,n;\psi) = \{0\}$ as any $x = (x_1, x_2, \ldots, x_n) \in W_0(1,n;\psi)$ must satisfy the inequality $|qx_j| < \psi(q)$ infinitely often. As $\psi(q) \to 0$ as $q \to \infty$ this is only possible if $x_j = 0$ for all $j = 1, 2, \ldots, n$. Thus when $m = 1$ the set $W_0(1,n;\psi)$ is a singleton and has both zero measure and dimension. We will therefore assume that $m \geqslant 2$.

Before giving the main results of this paper we include a brief review of some of the work done previously on the measure theoretic structure of $W_0(m,n;\psi)$. The first result is due to Dickinson [5].

Theorem (D). *When $\tau > \frac{m}{n} - 1$ and $m \geqslant 2$*

$$\dim(W_0(m,n;\tau)) = (m-1)n + \frac{m}{\tau+1},$$

and when $0 < \tau \leqslant \frac{m}{n} - 1$,

$$\dim(W_0(m,n;\tau)) = mn.$$

Theorem (D) does not hold for the cases $m \leqslant n$ if $\tau < \dfrac{m}{m-1} - 1$. As a consequence of Theorem 2 below we have corrected this mistake. To the best of our knowledge the only other result is due to Kemble [12] who established a Khintchine-Groshev type theorem for $W_0(m,1;\psi)$ under various conditions on the approximating function. We shall remove these conditions and prove the precise analogue of the Khintchine-Groshev theorem for $W_0(m,n;\psi)$.

Finally, it is worth mentioning the set is not only of number theoretic interest but appears naturally in operator theory, see [6] for further details.

2. Statement of main results

The results of this paper depend crucially on the choice of m and n and the statements of the main results of this paper are split into two cases; one when $m > n$ and the other when $m \leqslant n$.

In what follows $\mathcal{H}^f$ will denote f-dimensional Hausdorff measure, which will be defined fully in §3.1, and for any approximating function ψ we define Ψ to be the function $\Psi(r) := \frac{\psi(r)}{r}$.

Theorem 1. *Let $m > n$ and ψ be an approximating function. Let f be a dimension function such that $r^{-mn}f(r)$ is non-increasing and $r^{-(m-1)n}f(r)$ is increasing. Then*

$$\mathcal{H}^f\left(W_0\left(m,n;\psi\right)\right) = \begin{cases} 0, & \text{if } \sum\limits_{\mathrm{r}=1}^{\infty} \mathrm{f}(\Psi(\mathrm{r}))\Psi(\mathrm{r})^{-(\mathrm{m}-1)\mathrm{n}}\mathrm{r}^{\mathrm{m}-1} < \infty, \\ \mathcal{H}^f(\mathbb{I}^{mn}), & \text{if } \sum\limits_{\mathrm{r}=1}^{\infty} \mathrm{f}(\Psi(\mathrm{r}))\Psi(\mathrm{r})^{-(\mathrm{m}-1)\mathrm{n}}\mathrm{r}^{\mathrm{m}-1} = \infty. \end{cases}$$

Note that if the dimension function f is such that $r^{-mn}f(r) \to \infty$ as $r \to 0$ then $\mathcal{H}^f(\mathbb{I}^{mn}) = \infty$ and Theorem 1 is the analogue of the classical result of Jarník (see [11]).

If we now set $f(r) = r^s$ for $s > 0$, Theorem 1 reduces to the following s-dimensional Hausdorff measure statement.

Corollary 1. *Let $m > n$ and ψ be an approximating function. Let s be such that $(m-1)n < s \leqslant mn$. Then,*

$$\mathcal{H}^s\left(W_0\left(m,n;\psi\right)\right) = \begin{cases} 0, & \textit{if } \sum\limits_{r=1}^{\infty} \Psi(r)^{s-(m-1)n}r^{m-1} < \infty, \\ \mathcal{H}^s(\mathbb{I}^{mn}), & \textit{if } \sum\limits_{r=1}^{\infty} \Psi(r)^{s-(m-1)n}r^{m-1} = \infty. \end{cases}$$

Corollary 1 is more discriminating than the Hausdorff dimension result of Dickinson. Indeed, Dickinson's result (for the cases when $m > n$) can be deduced from it.

In the case when $f(r) = r^{mn}$, the Hausdorff f-measure $\mathcal{H}^f$ is simply Lebesgue measure supported on $\mathbb{I}^{mn}$ and Theorem 1 implies a natural analogue of the Khintchine–Groshev theorem for $W_0\left(m,n;\psi\right)$.

Corollary 2. *Let $m > n$ and ψ be an approximating function, then*

$$|W_0\left(m,n;\psi\right)|_{mn} = \begin{cases} 0, & \textit{if } \sum\limits_{r=1}^{\infty} \Psi(r)^n r^{m-1} < \infty, \\ 1, & \textit{if } \sum\limits_{r=1}^{\infty} \Psi(r)^n r^{m-1} = \infty. \end{cases}$$

Theorem 1 establishes the metric theory for $W_0\left(m,n;\psi\right)$ when $m > n$. For the cases when $m \leqslant n$ the problem becomes more complicated due to the fact that the set $W_0(m,n;\psi)$ can be shown to lie in a manifold of dimension at most $(m-1)(n+1)$. This results in some changes to the statement of Theorem 1 in the cases when $m \leqslant n$.

Firstly consider the case when $m = n$ and take any $X \in W_0(m,m;\psi)$. We claim that the column vectors of X are linearly dependent. To see why assume to the contrary, that they are linearly independent. Since $X \in W_0(m,m;\psi)$ there exists infinitely many $\mathbf{q}$ such that

$$|\mathbf{q}X| < \psi(|\mathbf{q}).$$

As X is assumed to have linearly independent column vectors, it follows that X is invertible and

$$1 \leqslant |\mathbf{q}| = |\mathbf{q}XX^{-1}| \leqslant C(X)\psi(\mathbf{q}) \to 0 \qquad \text{as } \ |\mathbf{q}| \to \infty$$

for some constant $C(X)$ depending only on X. This is clearly impossible and we must have the column vectors of X being linearly dependent, as claimed. Thus any such X must lie in the hypersurface Γ where

$$\Gamma := \{Y \in \mathbb{I}^{m^2} : \det(Y) = 0\}.$$

As det is a multinomial, it follows that Γ is a co-dimension 1 hypersurface in $\mathbb{I}^{m^2}$.

When $m < n$ the argument is more involved. From the above argument for the case when $m = n$, every minor of X of order m is 0. It follows that the rank of X is at most $m-1$ and the rows of X must therefore be linearly dependent. Consider first the case when the rank is $m-1$ and assume without loss of generality that the first $m-1$ rows are linearly independent. Denoting the j^{th} row by $R^{(j)}$, we have

$$R^{(m)} = \sum_{j=1}^{m-1} \lambda_j R^{(j)}.$$

As $(m-1)(n+1) = (m-1)n + m - 1$, the total number of independent variables required to specify X is $(m-1)(n+1)$. When the rank of X is strictly less than m the total number of variables needed to specify X is

$$rn + (m-1)(m-r) \leqslant (m-1)(n+1)$$

where $r < m-1$ is the rank of X.

In conclusion, when $m \leqslant n$ the set $W_0(m,n;\psi)$ lies in the intersection of the different algebraic hyper-surfaces defined above and this set, Γ, has co-dimension $m-1$. In light of this remark an upper bound for $\dim W_0\,(m,n;\psi)$ follows immediately,

$$\dim W_0(m,n;\psi) \leqslant (m-1)(n+1).$$

Theorem 2. *Let $m \leqslant n$ and ψ be an approximating function. Let f and $r^{-(n-m+1)(m-1)}f(r)$ be dimension functions. Assume that $r^{-(m-1)(n+1)}f(r)$ is non-increasing and $r^{-(m-1)n}f(r)$ is increasing. Then $\mathcal{H}^f(W_0(m,n;\psi)) = 0$ whenever*

$$\sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n} r^{m-1} < \infty.$$

If

$$\sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n} r^{m-1} = \infty,$$

then

$$\mathcal{H}^f(W_0(m,n;\psi)) = \begin{cases} \infty, & \text{if } r^{-(m-1)(n+1)} f(r) \to \infty \text{ as } r \to 0, \\ K, & \text{if } r^{-(m-1)(n+1)} f(r) \to C \text{ as } r \to 0, \end{cases}$$

where $C > 0$ *is some fixed constant and* $0 < K < \infty$.

It is worth noting that for a dimension function f which satisfies $r^{-(m-1)(n+1)} f(r) \to C > 0$ as $r \to 0$, the measure $\mathcal{H}^f$ is comparable to standard $(m-1)(n+1)$-dimensional Lebesgue measure. In the case when $f(r) = r^{(m-1)(n+1)}$, we obtain the following analogue of the Khintchine-Groshev theorem.

Corollary 3. *Let* $m \leqslant n$ *and* ψ *be an approximating function and assume that the conditions of Theorem 2 hold for the dimension function* $f(r) := r^{(m-1)(n+1)}$. *Then*

$$|W_0(m,n;\psi)|_{(m-1)(n+1)} = \begin{cases} 0, & \text{if } \sum\limits_{r=1}^{\infty} \psi(r)^{m-1} < \infty, \\ K & \text{if } \sum\limits_{r=1}^{\infty} \psi(r)^{m-1} = \infty. \end{cases}$$

where $0 < K < \infty$.

As above, if we set $f(r) = r^s$ we obtain the $m \leqslant n$ analogue of Corollary 1.

Corollary 4. *Let* $m \leqslant n$ *and* ψ *be an approximating function. Let* s *be such that* $(m-1)n < s \leqslant (m-1)(n+1)$. *Then,*

$$\mathcal{H}^s(W_0(m,n;\psi)) = \begin{cases} 0, & \text{if } \sum_{r=1}^{\infty} \Psi(r)^{s-(m-1)n} r^{m-1} < \infty, \\ \mathcal{H}^s(\Gamma), & \text{if } \sum_{r=1}^{\infty} \Psi(r)^{s-(m-1)n} r^{m-1} = \infty \\ & \text{and } s < (m-1)(n+1), \\ K, & \text{if } \sum_{r=1}^{\infty} \Psi(r)^{s-(m-1)n} r^{m-1} = \infty \\ & \text{and } s = (m-1)(n+1), \end{cases}$$

where $0 < K < \infty$. *Also, if*

$$\inf\left\{ s : \sum_{r=1}^{\infty} \Psi(r)^{s-(m-1)n} r^{m-1} < \infty \right\} \leqslant (m-1)(n+1)$$

then

$$\dim W_0(m,n;\psi) = \inf\left\{ s : \sum_{r=1}^{\infty} \Psi(r)^{s-(m-1)n} r^{m-1} < \infty \right\}.$$

On the other hand, if $\inf\left\{ s : \sum_{r=1}^{\infty} \Psi(r)^{s-(m-1)n} r^{m-1} < \infty \right\} > (m-1)(n+1)$ *then*

$$\dim W_0(m,n;\psi) = (m-1)(n+1).$$

When $m \leqslant n$ the above corollary not only gives Hausdorff dimension of $W_0(m,n;\tau)$ but also that the Hausdorff measure at the critical exponent is infinity. The Corollary below corrects the mistake in [5].

Corollary 5. *For $m \leqslant n$,*

$$\dim W_0(m,n;\tau) = \begin{cases} (m-1)n + \frac{m}{\tau+1}, & \text{if } \tau > \frac{m}{m-1} - 1, \\ (m-1)(n+1), & \text{if } \tau \leqslant \frac{m}{m-1} - 1. \end{cases}$$

Remark. In the cases when $m \leqslant n$, the complementary set to $W_0(m,n;\psi)$ corresponds to the set of badly approximable systems of linear forms. In line with the classical theory one would expect this set to be of maximal Hausdorff dimension. This is indeed the case and the reader is referred to [9, 10] for full details of the proofs.

The paper is organized as follows. In Section 3, we give the definitions of Hausdorff measure and ubiquity, which is the main tool for proving Theorem 1, in a manner appropriate to the setting of this paper. Section 3 also includes the statement of the 'Slicing' lemma (Lemma 1) which is used to prove Theorem 2. The paper continues with the proof of Theorem 1 in § 4. As is common when proving such 'zero-full' results the proof is split into two parts; the convergence case and the divergence case. We conclude the paper with the proof of Theorem 2.

3. Basic definitions and auxiliary results

In this section we give the definitions of some fundamental concepts along with some auxiliary results which are needed for the proofs of Theorems 1 and 2.

3.1. Hausdorff measure and dimension

Below we give a brief introduction to Hausdorff f-measure and dimension. For further details see [7].

A *dimension function* is an increasing, continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(r) \to 0$ as $r \to 0$. Let $X \subset \mathbb{R}^n$ and $\rho > 0$. A ρ-cover of X is a countable collection $\{B_i\}_{i=1}^{\infty}$ of balls in $\mathbb{R}^n$ with diameters $\operatorname{diam}(B_i) \leqslant \rho$ satisfying $X \subset \bigcup_{i=1}^{\infty} B_i$. Define

$$\mathcal{H}^f_\rho(X) = \inf_C \left\{ \sum_{i=1}^{\infty} f(\operatorname{diam}(B_i)) : B_i \in C \right\}$$

where the infimum is taken over all possible ρ-covers C of X. The *Hausdorff f-measure* of X is defined to be

$$\mathcal{H}^f(X) = \lim_{\rho \to 0} \mathcal{H}^f_\rho(X).$$

In the particular case when $f(r) := r^s$ $(s > 0)$, we write $\mathcal{H}^s(X)$ for $\mathcal{H}^f(X)$ and the measure is referred to as s-dimensional Hausdorff measure. The *Hausdorff dimension* of a set X, denoted by $\dim(X)$, is defined to be

$$\dim(X) := \inf\{s \in \mathbb{R}^+ \ : \ \mathcal{H}^s(X) = 0\} = \sup\{s \in \mathbb{R}^+ \ : \ \mathcal{H}^s(X) = \infty\}.$$

At the critical exponent $s = \dim X$ the quantity $\mathcal{H}^s(X)$ can be zero, infinite or strictly positive and finite. In the latter case; i.e. when $0 < \mathcal{H}^s(X) < \infty$, the set X is said to be an s-set.

3.2. Ubiquitous systems

We now describe the main tool used in proving the divergence case of Theorem 1; the idea of a locally ubiquitous system. The set-up presented below is tailored towards the current problem. The full notion of ubiquity is more general and details can be found in [1] and [3].

Let $\Re = \{R_{\mathbf{q}} : \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}\}$ be the family of subsets $R_{\mathbf{q}} := \{X \in \mathbb{I}^{mn} : \mathbf{q}X = \mathbf{0}\}$. The sets $R_{\mathbf{q}}$ will be referred to as *resonant sets*. Let the function $\beta : \mathbb{Z}^m \setminus \{\mathbf{0}\} \to \mathbb{R}^+ : \mathbf{q} \to |\mathbf{q}|$ attach a *weight* to the resonant set $R_{\mathbf{q}}$. Given an approximating function ψ and $R_{\mathbf{q}}$, let

$$\Delta\left(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)\right) := \{X \in \mathbb{I}^{mn} : \operatorname{dist}(X, R_{\mathbf{q}}) \leqslant \Psi(|\mathbf{q}|)\}$$

where $\operatorname{dist}(X, R_{\mathbf{q}}) := \inf\{|X - Y| : Y \in R_{\mathbf{q}}\}$. Thus $\Delta\left(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)\right)$ is a Ψ-neighbourhood of $R_{\mathbf{q}}$. Notice that in the case when the resonant sets are points the sets $\Delta\left(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)\right)$ are balls centred at the points $\mathbf{q}$.

Let

$$\Lambda(m,n;\psi) = \{X \in \mathbb{I}^{mn} : X \in \Delta\left(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)\right) \text{for i.m. } \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}\}.$$

The set $\Lambda(m,n;\psi)$ is a 'limsup' set. It is the set of points in $\mathbb{I}^{mn}$ which lie in infinitely many of the sets $\Delta\left(R_q, \Psi(|\mathbf{q}|)\right)$. This is apparent if we restate $\Lambda(m,n;\psi)$ in a manner which emphasises its limsup structure. For any $t \in \mathbb{N}$ define

$$\Delta(\psi, t) := \bigcup_{2^{t-1} \leqslant |\mathbf{q}| \leqslant 2^t} \Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)). \tag{1}$$

It follows that

$$\Lambda(m,n;\psi) = \limsup_{t \to \infty} \Delta(\psi, t) = \bigcap_{N=1}^{\infty} \bigcup_{t=N}^{\infty} \Delta(\psi, t). \tag{2}$$

We now move onto the formal definition of a locally ubiquitous system. As stated above the definition given is in a simplified form suitable for the problem at hand. In the more abstract setting as developed in [1], there are specific conditions that the measure on the ambient space needs to satisfy. These conditions are not stated below as they hold trivially for Lebesgue measure, the measure on our ambient space $\mathbb{I}^{mn}$, and stating the conditions would complicate the discussion somewhat. Nevertheless the reader should be aware that in the more abstract notion of ubiquity these extra conditions exist and need to be established.

Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a function with $\rho(r) \to 0$ as $r \to \infty$ and let

$$\hat{\Delta}(\rho, t) := \bigcup_{\mathbf{q} \in J(t)} \Delta\left(R_{\mathbf{q}}, \rho\left(2^t\right)\right)$$

where $J(t)$ is defined to be the set

$$J(t) := \left\{\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\} : |\mathbf{q}| \leqslant 2^t\right\}.$$

Definition 1. *Let $B := B(X, r)$ be an arbitrary ball with centre $X \in \mathbb{I}^{mn}$ and $r \leqslant r_o$. Suppose there exists a function ρ and an absolute constant $\kappa > 0$ such that*

$$|B \cap \hat{\Delta}(\rho, t)|_{mn} \geqslant \kappa |B|_{mn} \qquad \text{for } \; t \geqslant t_o(B).$$

Then $\Re$ is said to be a locally ubiquitous *system relative to ρ.*

Loosely speaking the definition of local ubiquity says that the set $\hat{\Delta}(\rho, t)$ locally approximates the underlying space $\mathbb{I}^{mn}$ in terms of the Lebesgue measure. The function ρ, will be referred to as the *ubiquity function.* The actual value of the constant κ in the above definition is irrelevant. It is the existence of such a constant that is important. In practice the local ubiquity of a system can be established using standard arguments concerning the distribution of the resonant sets in $\mathbb{I}^{mn}$ from which the function ρ arises naturally. Clearly if $|\hat{\Delta}(\rho, t)|_{mn} \to 1$ as $t \to \infty$ then $\Re$ is locally ubiquitous.

A final definition is needed before we state a simplified version of Theorem 1 from [3], which will allow us to prove the divergence part of Theorem 1. A function f will be said to be *2-regular* if there exists a positive constant $\lambda < 1$ such that for t sufficiently large

$$f(2^{t+1}) \leqslant \lambda f(2^t).$$

Theorem 3. *Suppose that $\Re$ is locally ubiquitous relative ρ and ψ is an approximation function. Let f be a dimension function such that $r^{-\delta} f(r)$ is non-increasing. Furthermore suppose that $r^{-\gamma} f(r)$ is increasing and ρ is 2-regular. Then*

$$\mathcal{H}^f(W_0(m, n; \psi)) = \mathcal{H}^f(\mathbb{I}^{mn}) \qquad \text{if} \quad \sum_{n=1}^{\infty} \frac{f(\Psi(2^t))\Psi(2^t)^{-\gamma}}{\rho(2^t)^{\delta-\gamma}} = \infty.$$

3.3. Slicing

We now state a result which is the key ingredient in the proof of Theorem 2. The result was used in [2] to prove the Hausdorff measure version of W. M. Schmidt's inhomogeneous linear forms theorem in metric number theory. The authors refer to the technique as "slicing". We will merely state the result. For a more detailed discussion and proof see [2].

Before we state the theorem, it is necessary to introduce a little notation. Suppose that V is a linear subspace of $\mathbb{I}^k$, $V^\perp$ will be used to denote the linear subspace of $\mathbb{I}^k$ orthogonal to V. Further, $V + a := \{v + a : v \in V\}$ for $a \in V^\perp$.

Lemma 1. *Let $l, k \in \mathbb{N}$ be such that $l \leqslant k$ and let f and $g : r \to r^{-l} f(r)$ be dimension functions. Let $B \subset \mathbb{I}^k$ be a* Borel *set and let V be a $(k-l)$–dimensional linear subspace of $\mathbb{I}^k$. If, for a subset S of $V^\perp$ of positive $\mathcal{H}^l$ measure,*

$$\mathcal{H}^g\left(B \cap (V+b)\right) = \infty \qquad \text{for all} \;\; b \in S,$$

then $\mathcal{H}^f(B) = \infty$.

We are now in a position to begin the proofs of Theorems 1 and 2.

4. Proof of Theorem 1

As stated above, the proof of Theorem 1 is split into two parts; the convergence case and the divergence case. We begin with the convergence case.

4.1. The convergence case

Recall that in the statement of Theorem 1 we assumed that $m > n$ and we imposed some conditions on the dimension function f. As it turns out these conditions are not needed for the convergence case and we can prove a less restrictive result which has the added benefit of also implying the convergence case of Theorem 2.

Theorem 4. *Let ψ be an approximating function and let f be a dimension function. If*

$$\sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n} r^{m-1} < \infty,$$

then

$$\mathcal{H}^f\left(W_0\left(m,n;\psi\right)\right) = 0.$$

Obviously Theorem 4 implies the convergence cases of Theorems 1 and 2.

Proof. To prove Theorem 4 we make use of the natural cover of $W_0\left(m,n;\psi\right)$ given by Equations (1) and (2). It follows almost immediately that for each $N \in \mathbb{N}$ the family

$$\left\{ \bigcup_{R_{\mathbf{q}} : |\mathbf{q}| = r} \Delta\left(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)\right) : r = N, N+1, \dots \right\}$$

is a cover for the set $W_0\left(m,n;\psi\right)$. That is

$$W_0\left(m,n;\psi\right) \subset \bigcup_{r > N} \bigcup_{|\mathbf{q}| = r} \Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|))$$

for any $N \in \mathbb{N}$.

Now, for each resonant set $R_{\mathbf{q}}$, let $\Delta(\mathbf{q})$ be a collection of mn-dimensional closed hypercubes C with disjoint interiors and side length $\Psi(|\mathbf{q}|)$ such that

$$C \bigcap \bigcup_{|\mathbf{q}| = r} \Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)) \neq \emptyset$$

and

$$\Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)) \subset \bigcup_{C \in \Delta(q)} C.$$

Then

$$\#\Delta(\mathbf{q}) \ll (\Psi(|\mathbf{q}|))^{-(m-1)n}$$

where # denotes cardinality. Note that

$$W_0\,(m,n;\psi) \subset \bigcup_{r>N} \bigcup_{|\mathbf{q}|=r} \Delta\,(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)) \subset \bigcup_{r>N} \bigcup_{\mathbf{q}:|\mathbf{q}|=r} \bigcup_{C \in \Delta(\mathbf{q})} C.$$

It follows that

$$\begin{aligned}\mathcal{H}^f\,(W_0\,(m,n;\psi)) &\leqslant \sum_{r>N} \sum_{\Delta(q):|\mathbf{q}|=r} \sum_{C \in \Delta(\mathbf{q})} f(\Psi(|\mathbf{q}|)) \\ &\ll \sum_{r>N} r^{m-1} f\,(\Psi(r))\,\Psi(r)^{-(m-1)n} \to 0 \qquad \text{as} \;\; N \to \infty,\end{aligned}$$

and therefore $\mathcal{H}^f(W_0\,(m,n;\psi)) = 0$, as required. ■

4.2. The divergence case

When $m > n$, the divergence part of Theorem 1 relies on the notion of ubiquity, as defined above, and primarily Theorem 3. To use ubiquity we must show that $\Re$ is locally ubiquitous with respect to some suitable ubiquity function ρ.

In order to establish ubiquity two technical lemmas are needed. The first result is due to Dickinson [5] and is an analogue of Dirichlet's theorem.

Lemma 2. *For each $X \in \mathbb{I}^{mn}$ and any $N \in \mathbb{N}$, there exists a non-zero integer vector $\mathbf{q}$ in $\mathbb{Z}^m$ with $|\mathbf{q}| \leqslant N$ such that*

$$|\mathbf{q}X| < mN^{-\frac{m}{n}+1}.$$

The second lemma is essentially a slight modification of another result of Dickinson from the same paper. The key difference being the introduction of a function ω instead of log. As the changes needed to prove the second result are minimal we merely state the result here and refer the reader to the original proof as found in [5].

Lemma 3. *Let ω be a positive real increasing function such that $\frac{1}{\omega(t)} \to 0$ as $t \to \infty$ and there exists $C > 1$ such that for t sufficiently large $\omega\,(2t) < C\omega\,(t)$. Then the family $(\Re, \beta)$ is locally ubiquitous with respect to the function $\rho : \mathbb{N} \to \mathbb{R}^+$ where $\rho(t) = mN^{-\frac{m}{n}}\omega(N)$.*

To apply Theorem 3, set $\delta = mn$ and $\gamma = (m-1)n$. The sum in Theorem 3 becomes

$$\sum_{t=1}^{\infty} f(\Psi(2^t))\Psi(2^t)^{-(m-1)n}(2^t)^m\omega(t)^{-n}$$

which is comparable to

$$\sum_{r=1}^{\infty} f\left(\Psi\left(r\right)\right)\Psi\left(r\right)^{-(m-1)n} r^{m-1}\omega\left(r\right)^{-n}. \tag{3}$$

To obtain the precise statement of Theorem 1 we need to remove the ω factor from equation (3).

Firstly, note that if the sum in equation (3) diverges then so does the sum

$$\sum_{r=1}^{\infty} f\left(\Psi\left(r\right)\right)\Psi\left(r\right)^{-(m-1)n} r^{m-1} \tag{4}$$

On the other hand, if the sum in equation (4) diverges, then there exists a strictly increasing sequence of positive integers $\{r_i\}_{i\in\mathbb{N}}$ such that

$$\sum_{r_{i-1}\leqslant r\leqslant r_i}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n}r^{m-1} > 1$$

and $r_i > 2r_{i-1}$. Define ω to be the step function $\omega(r) = i^{\frac{1}{n}}$ for $r_{i-1} \leqslant r \leqslant r_i$ and ω satisfies the required properties. This function was first introduced in [4]. With ω defined as above, the convergence or otherwise of

$$\sum_{r=1}^{\infty} f\left(\Psi\left(r\right)\right)\Psi\left(r\right)^{-(m-1)n} r^{m-1}\omega\left(r\right)^{-n}$$

coincides with that of

$$\sum_{r=1}^{\infty} f\left(\Psi\left(r\right)\right)\Psi\left(r\right)^{-(m-1)n} r^{m-1}$$

and Theorem 1 is proved.

5. Proof of Theorem 2

In view of Theorem 4 we need only prove the divergence part of Theorem 2. The proof will be split into two sub-cases. The first, which we refer to as the "infinite measure" case, is for dimension functions f with $r^{-(m-1)(n+1)}f(r) \to \infty$. The second case corresponds to f which satisfy $r^{-(m-1)(n+1)}f(r) \to C$ for some constant $C > 0$. In the latter case the measure is comparable to $(m-1)(n+1)$-dimensional Lebesgue measure. We call this the "finite measure" case. Before we begin the proof of Theorem 2 recall that $W_0(m,n;\psi)$ lies in a manifold (hypersurface) Γ of dimension at most $(m-1)(n+1)$ if $m \leqslant n$.

Following [5] and the argument leading to Theorem 2, we restrict ourself to $X \in W_0\left(m,n;\psi\right)$ which has rank at most $m-1$. It can be readily verified that the set of X with a lower rank is of strictly lower dimension. Let

$$\widehat{W_0}\left(m,n;\psi\right) := \left\{X \in W_0\left(m,n;\psi\right) : X \text{ is of rank } m-1\right\},$$

and let A be the set of points of the form

$$\left(X^{(1)}, X^{(2)}, \ldots, X^{(m-1)}, \sum_{j=1}^{m-1} a_j^{(1)} X^{(j)}, \ldots, \sum_{j=1}^{m-1} a_j^{(n-m+1)} X^{(j)}\right),$$

with

$$\left(X^{(1)}, X^{(2)}, \ldots, X^{(m-1)}\right) \in \widehat{W}_0(m, m-1; \psi)$$

and $a_j^{(i)} \in \left(-\frac{1}{m-1}, \frac{1}{m-1}\right)$ for $1 \leqslant i \leqslant (n-m+1)$.

It is straightforward to show that

$$A \subseteq W_0(m, n; \psi).$$

We now define the function

$$\eta : \widehat{W}_0(m, m-1, \psi) \times \left(-\frac{1}{m-1}, \frac{1}{m-1}\right)^{(n-(m-1))(m-1)} \to A$$

by

$$\begin{aligned} &\eta\left(X^{(1)}, X^{(2)}, \ldots, X^{(m-1)}, a_1^1, \ldots, a_{m-1}^1, \ldots, a_1^{(n-(m-1))}, \ldots, a_{m-1}^{(n-(m-1))}\right) \\ &\quad = \left(X^{(1)}, X^{(2)}, \ldots, X^{(m-1)}, \sum_{j=1}^{m-1} a_j^{(1)} X^{(j)}, \ldots, \sum_{j=1}^{m-1} a_j^{(n-m+1)} X^{(j)}\right). \end{aligned}$$

As in [5] it can be shown that η is an embedding and its range is diffeomorphic to A. This in turn implies that η is (locally) bi-Lipschitz.

5.1. The infinite measure case

As mentioned above the proof of Theorem 2 is split into two parts. In this section we concentrate on the infinite measure case which can be deduced from the following lemma.

Lemma 4. *Let $m \leqslant n$ and ψ be an approximating function. Let f and $g : r \to r^{-(n-(m-1))(m-1)} f(r)$ be dimension functions with $r^{-(m-1)(n+1)} f(r) \to \infty$ as $r \to 0$. Further, let $r^{-m(m-1)} g(r)$ be non-increasing and $r^{-(m-1)^2} g(r)$ be increasing. If*

$$\sum_{r=1}^{\infty} f(\Psi(r)) \Psi(r)^{-(m-1)n} r^{m-1} = \infty,$$

then

$$\mathcal{H}^f(A) = \infty.$$

Proof. As η is bi-Lipschitz, we have that

$$\begin{aligned}\mathcal{H}^f(A) &= \mathcal{H}^f\left(\eta\left(\widehat{W}_0(m,m-1,\psi)\times I^{(n-(m-1))(m-1)}\right)\right)\\ &\asymp \mathcal{H}^f\left(\widehat{W}_0(m,m-1,\psi)\times I^{(n-(m-1))(m-1)}\right),\end{aligned}$$

where $I=\left(-\frac{1}{m-1},\frac{1}{m-1}\right)\cap\mathbb{I}$. The proof relies on the slicing technique of Lemma 1. Let

$$B := \widehat{W}_0(m,m-1;\psi)\times I^{(n-(m-1))(m-1)} \subseteq \mathbb{I}^{(m-1)(n+1)}$$

and V be the space $\mathbb{I}^{m(m-1)}\times\{0\}^{(m-1)(n+1-m)}$. As $\widehat{W}_0(m,m-1;\psi)$ is a lim sup set, B is a Borel set. Let $S:=\{0\}^{m(m-1)}\times I^{(n+1-m)(m-1)}$. Clearly S is a subset of $V^\perp$ and further it has positive $\mathcal{H}^{(n-(m-1))(m-1)}$-measure. Now, for each $b\in S$

$$\begin{aligned}\mathcal{H}^g\left(B\cap(V+b)\right) &= \mathcal{H}^g\left((\widehat{W}_0(m,m-1;\psi)\times I^{(n-(m-1))(m-1)})\cap(V+b)\right)\\ &= \mathcal{H}^g(\widehat{W}_0(m,m-1;\psi)\times\{0\}^{(n+1-m)(m-1)}+b)\\ &\asymp \mathcal{H}^g\left(\widehat{W}_0(m,m-1;\psi)\times\{0\}^{(n+1-m)(m-1)}\right) \overset{Theorem\ 1}{=} \infty,\end{aligned}$$

if

$$\sum_{r=1}^{\infty} g(\Psi(r))\Psi(r)^{-(m-1)^2}r^{m-1}=\infty.$$

Applying Lemma 1 and using the relation between f and g, it follows that $\mathcal{H}^f(A)=\infty$ if

$$\sum_{r=1}^{\infty} g(\Psi(r))\Psi(r)^{-(m-1)^2}r^{m-1}=\sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{(m-1)n}r^{m-1}=\infty,$$

as required. ■

As $A\subseteq W_0(m,n;\psi)$, $\mathcal{H}^f(A)=\infty$ implies that $\mathcal{H}^f(W_0(m,n;\psi))=\infty$. This completes the proof of Theorem 2.

5.2. Finite measure case

We now come to the case where $r^{-(m-1)(n+1)}f(r)\to C$ as $r\to 0$ and $C>0$ is finite. In this case, $\mathcal{H}^f$ is comparable to $(m-1)(n+1)$-dimensional Lebesgue measure and

$$\sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n}r^{m-1}=\sum_{r=1}^{\infty}\psi^{m-1}(r).$$

We begin with the following lemma. The proof of which can be found in [8] and is a straightforward adaptation of Corollary 2.4 from [7].

Lemma 5. *Suppose that $L \subset \mathbb{R}^l$, $M \subset \mathbb{R}^k$ and $\eta : L \to M$ is a bi-Lipschitz transformation. Let f be a dimension function. Then for any $C \subseteq L$, $\mathcal{H}^f(C) \asymp \mathcal{H}^f(\eta(C))$.*

In applying Lemma 5, we first note that

$$\left|\widehat{W}_0(m, m-1; \psi) \times I^{(n-(m-1))(m-1)}\right|_{(m-1)(n+1)} \asymp |A|_{(m-1)(n+1)}.$$

Corollary 2 implies that $|\widehat{W}_0(m, m-1; \psi)|_{m(m-1)} = 1$. This result coupled with an application of Fubini's Theorem implies that the $(m-1)(n+1)$-dimensional Lebesgue measure of A is positive and finite. Using Lemma 5 we can conclude that the $(m-1)(n+1)$-dimensional Lebesgue measure of $W_0\,(m, n; \psi)$ is positive and finite, as required.

Acknowledgements. We would like to thank Sanju Velani for first suggesting this problem as part of the first author's PhD research project and for his many helpful comments. Thanks also go to Victor Beresnevitch for many illuminating discussions on the technicalities of slicing. MH would like to thank his wife, Naghmana Tehseen, for her love and support. Finally, we would like to thank the referee whose detailed comments on the paper helped us no end.

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Received: 21 June 2012