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PARTIAL SUMS OF THE MÖBIUS FUNCTION IN ARITHMETIC PROGRESSIONS ASSUMING GRH

Karin Halupczok, Benjamin Suger

Abstract: We consider Mertens' function in arithmetic progression,

$$M(x,q,a) := \sum_{\substack{n\leqslant x \\ n\equiv a \bmod q}} \mu(n).$$

Assuming the generalized Riemann hypothesis (GRH), we show that the bound

$$M(x,q,a) \ll_\varepsilon \sqrt{x}\exp\Big((\log x)^{3/5}(\log\log x)^{16/5+\varepsilon}\Big)$$

holds uniform for all $q \leqslant \exp\Big(\frac{\log 2}{2}\Big\lfloor(\log x)^{3/5}(\log\log x)^{11/5}\Big\rfloor\Big)$, $\gcd(a,q)=1$ and all $\varepsilon>0$. The implicit constant is depending only on ε. For the proof, a former method of K. Soundararajan is extended to L-series.

Keywords: Möbius function, Mertens' function, GRH.

1. Introduction

Mertens' function is defined by

$$M(x) := \sum_{n\leqslant x}\mu(n).$$

It is well known that $M(x) = \mathrm{o}(x^{1/2+\varepsilon})$ is equivalent to Riemann's hypothesis.

When assuming Riemann's hypothesis for ζ, one can give even sharper bounds for $M(x)$, see [3], [8], [4], [7], [5]:

In [4], H. Maier and H. L. Montgomery proved the bound

$$M(x) \ll x^{1/2}\exp\Big(c(\log x)^{39/61}\Big) \text{ for a } c>0.$$

In [7], K. Soundararajan improved the bound by showing

$$M(x) \ll x^{1/2}\exp\Big((\log x)^{1/2}(\log\log x)^{14}\Big).$$

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In [5], A. de Roton und M. Balazard refine the result of K. Soundararajan and show

$$M(x) \ll_\varepsilon \sqrt{x}\exp\Big((\log x)^{1/2}(\log\log x)^{5/2+\varepsilon}\Big),$$

which is the best bound up to date.

In this paper we generalize the method of K. Soundararajan to provide a bound for Mertens' function in arithmetic progression,

$$M(x,q,a) := \sum_{\substack{n\leqslant x\\ n\equiv a \bmod q}} \mu(n).$$

Note that the trivial bound is $\leqslant x/q$, so bounds smaller than $x^{1/2+\varepsilon}$ are nontrivial if $q \leqslant x^{1/2-\varepsilon}$.

We adapt the method of K. Soundararajan resp. the modification of A. de Roton and M. Balazard in such a way, that it remains applicable for Dirichlet L-series. We obtain the following nontrivial upper bound assuming Riemann's hypothesis for all Dirichlet L-series $L(s,\chi)$ with χ mod q and all moduli q in question (GRH for short):

Theorem 1. *Assuming GRH, the bound*

$$M(x,q,a) \ll_\varepsilon \sqrt{x}\exp\Big((\log x)^{3/5}(\log\log x)^{16/5+\varepsilon}\Big)$$

holds uniform for all $q \leqslant \exp\Big(\frac{\log 2}{2}\Big\lfloor(\log x)^{3/5}(\log\log x)^{11/5}\Big\rfloor\Big)$, $\gcd(a,q)=1$ *and all* $\varepsilon>0$ *with an implicit constant depending only on* ε.

With this theorem, we extend the results of [7] resp. [5] to a Siegel-Walfisz-type result. The obtained bound is weaker than the one of [7] resp. [5], but still sharper than the one of [4].

The method we use is as follows. We expand the Möbius sum $M(x,q,a)$ using Dirichlet characters,

$$\begin{aligned} M(x,q,a) &= \frac{1}{\varphi(q)}\sum_{\chi(q)}\bar{\chi}(a)\sum_{n\leqslant x}\chi(n)\mu(n)\\ &= \frac{1}{\varphi(q)}\sum_{\chi(q)}\bar{\chi}(a)A(x,\chi,q) + \mathrm{O}(\log x), \end{aligned}$$

using Perron's formula with integrals

$$A(x,q,\chi) = \frac{1}{2\pi i}\int_{1+1/(\log x)-i2^K}^{1+1/(\log x)+i2^K} \frac{x^s}{L(s,\chi)s}ds, \qquad K := \Big\lfloor\frac{\log x}{\log 2}\Big\rfloor.$$

With this, bounds for $L(s,\chi)$ are needed. Considering the principal character χ_0 mod q, the formula

$$L(s,\chi_0) = \zeta(s)\prod_{p|q}\Big(1-\frac{1}{p^s}\Big)$$

shows that already the sharper bound of [7]/[5] applies (see the proof of Lemma 3). So the main work is to consider nonprincipal characters.

Like in [7]/[5], the main steps are then some propositions aiming to bound $L(s,\chi)$ to obtain an upper estimate for $A(x,q,\chi)$. They are given in Sections 7 and 8 and are resulting from the propositions in the former Sections 2 and 4, 5, 6.

Most of these propositions are stated for primitive characters. If necessary, results for nonprimitive characters $\chi \neq \chi_0$ are derived by reduction to a primitive character that induces χ.

The main idea in [7], namely the concept of V-typical ordinates, is extended to a version which allows one to work also with L-series. We give the adapted definition in Section 3.

As one important step, we show in Section 4 that there *are* actually V-typical ordinates, see Proposition 8.

In Section 5, it is shown that short intervals containing an unusual number of ordinates of nontrivial L-zeros mod q do not appear too often, even uniformly for all q up to the given bound (Proposition 9), so the V-untypical ordinates are small in number (Proposition 10). In the case of ζ, this has been the breakthrough in Soundararajan's paper [7].

The resulting bound and the range for q in Theorem 1 is then obtained by optimizing the bounds for $A(x,q,\chi)$ in Section 9. The elementary Proposition 20 plays an intrinsic rôle for this.

A remark on notations used in this paper: We mark all Propositions that assume the generalized Riemann hypothesis by the symbol (GRH). We stress that all implicit constants are absolute unless otherwise indicated.

2. List of tools

In this section, we give a collection of the tools used in the proof.

The first proposition gives an approximation of the characteristic function of a given interval:

Proposition 1. *Let $h>0$, $\Delta \geqslant 1$. Let $\mathbf{1}_{[-h,h]}$ be the characteristic function of the interval $[-h,h]$.*

There are even, entire functions F_+ and F_- depending on h and Δ, being real on the real axis and such that the following properties hold:

1. $\forall u \in \mathbb{R}:\ F_-(u) \leqslant \mathbf{1}_{[-h,h]}(u) \leqslant F_+(u)$,
2. $\int\limits_{-\infty}^{\infty} |F_\pm(u) - \mathbf{1}_{[-h,h]}(u)|du = 1/\Delta$ *and* $\hat{F}_\pm(0) = 2h \pm 1/\Delta$,
3. $\hat{F}_\pm$ *is realvalued and even, and we have* $\hat{F}_\pm(x) = 0$ *for all* $|x| \geqslant \Delta$ *and* $|x\hat{F}_\pm(x)| \leqslant 2$ *for all* $x \in \mathbb{R}$,

4. *for* $z \in \mathbb{C}$ *with* $|z| \geqslant \max\{2h, 1\}$ *we have*

$$|F_{\pm}(z)| \ll \frac{\exp(2\pi|\Im z|\Delta)}{(\Delta|z|)^2}.$$

The proof uses Beurling's Approximation of the signum function

$$\operatorname{sgn}(x) := \begin{cases} x/|x|, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Let $K(z) := \left(\frac{\sin(\pi z)}{\pi z}\right)^2$ and $H(z) = K(z)\Big(\sum_{n=-\infty}^{\infty} \frac{\operatorname{sgn}(n)}{(z-n)^2} + \frac{2}{z}\Big)$, then it can be shown that the functions

$$F_{\pm}(z) := \frac{1}{2}(H(\Delta(z+h)) \pm K(\Delta(z+h)) + H(\Delta(h-z)) \pm K(\Delta(h-z)))$$

have the properties asserted in Proposition 1. This can be seen as in [6] and [9], see also [5], we just give the proof of part 4. in more detail:

For this, let $z = x + iy$ with $x, y \in \mathbb{R}$ and $|z| \geqslant \max\{2h, 1\}$. Since $\mathrm{F}_{\pm}$ are even, consider only nonnegative x. Using $\sin(z) \ll e^{|\Im(z)|}$ and $\Im(\Delta(z+h)) = -\Im(\Delta(h-z)) = \Delta\Im(z)$, we get the desired bound for $K(\Delta(z+h)) \pm K(\Delta(h-z))$ since $|z \pm h| = |z|\left|1 \pm \frac{h}{z}\right| \geqslant |z|\left(1 - \frac{h}{|z|}\right) \geqslant \frac{1}{2}|z|$.

To estimate $H(\Delta(z+h)) + H(\Delta(h-z))$ we use the identities

$$\left(\frac{\pi}{\sin(\pi z)}\right)^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}, \text{ converging on every compact subset of } \mathbb{C} \setminus \mathbb{Z}, \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{1}{(z+n)(z+n+1)} = \frac{1}{z}, \text{ converging absolutely for } z \in \mathbb{C} \setminus -\mathbb{N}_0. \tag{2}$$

Consider $H(\Delta(z+h))$ and $H(\Delta(h-z))$ separately. By (1), we have

$$\begin{aligned} H(\Delta(z+h)) &= \left(\frac{\sin(\pi\Delta(z+h))}{\pi}\right)^2 \left(\sum_{n=-\infty}^{\infty} \frac{\operatorname{sgn}(n)}{(\Delta(z+h)-n)^2} + \frac{2}{\Delta(z+h)}\right) \\ &= 1 + \left(\frac{\sin(\pi\Delta(z+h))}{\pi}\right)^2 \\ &\quad \times \left(-2\sum_{n=1}^{\infty} \frac{1}{(\Delta(z+h)+n)^2} - \frac{1}{(\Delta(z+h))^2} + \frac{2}{\Delta(z+h)}\right), \end{aligned}$$

and (2) gives for the negative of the last term in large brackets the expression

$$\sum_{n=0}^{\infty}\left(\frac{1}{(\Delta(z+h)+n)^2}+\frac{1}{(\Delta(z+h)+n+1)^2}\right)$$
$$-\sum_{n=0}^{\infty}\frac{2}{(\Delta(z+h)+n)(\Delta(z+h)+n+1)}$$
$$=\sum_{n=0}^{\infty}\left(\frac{1}{(\Delta(z+h)+n)}-\frac{1}{(\Delta(z+h)+n+1)}\right)^2$$
$$=\sum_{n=0}^{\infty}\frac{1}{(\Delta(z+h)+n)^2(\Delta(z+h)+n+1)^2}$$
$$\leqslant\frac{1}{(\Delta(x+h+|y|))^2}\sum_{n=0}^{\infty}\frac{1}{(\Delta(x+h+|y|)+n)(\Delta(x+h+|y|)+n+1)}$$
$$=\frac{1}{(\Delta(x+h+|y|))^3}\ll\frac{1}{|\Delta(z+h)|^3}\ll\frac{1}{|\Delta z|^3}.$$

Analogously, we get

$$H(\Delta(h-z))=\left(\frac{\sin(\pi\Delta(h-z))}{\pi}\right)^2\left(\sum_{-\infty}^{\infty}\frac{\operatorname{sgn}(n)}{(\Delta(h-z)-n)^2}+\frac{2}{\Delta(h-z)}\right)$$
$$=-1+\left(\frac{\sin(\pi\Delta(z-h))}{\pi}\right)^2$$
$$\times\left(\frac{1}{(\Delta(z-h))^2}+2\sum_{1}^{\infty}\frac{1}{(\Delta(z-h)+n)^2}-\frac{2}{\Delta(z-h)}\right).$$

If $\Re(z)>h$, the treatment of the last term in large brackets is as before.

So let $\Re(z)\leqslant h$. Due to $|z|\geqslant 2h$, we have $|y|=|\Im(z)|>h$, so $z\notin\mathbb{R}$ and $|\Im(z)|\geqslant|\Re(z)|$. Again (2) gives for the last term in large brackets the expression

$$\sum_{n=0}^{\infty}\frac{1}{(\Delta(z-h)+n)^2(\Delta(z-h)+n+1)^2}$$
$$\ll\sum_{0\leqslant n\leqslant\Delta h}\frac{1}{|(\Delta(x-h)+n|+\Delta|y|)^2(|\Delta(x-h)+n+1|+\Delta|y|)^2}$$
$$+\sum_{n>\Delta h}\frac{1}{|(\Delta(x-h)+n|+\Delta|y|)^2(|\Delta(x-h)+n+1|+\Delta|y|)^2}$$
$$\ll\frac{\max\{\Delta h,1\}}{|\Delta y|^4}+\sum_{n=0}^{\infty}\frac{1}{(\Delta|y|+n)^2(\Delta|y|+n+1)^2}\ll\frac{1}{|\Delta y|^3}\ll\frac{1}{|\Delta z|^3}.$$

Summing up we obtain

$$H(\Delta(z+h))+H(\Delta(h-z))\ll\frac{e^{2\pi\Delta|\Im(z)|}}{(\Delta|z|)^3}$$

and the desired bound for $|z|\geqslant\max\{2h,1\}$. ∎

We will make use of the following explicit formula for the functions $F_\pm$.

Proposition 2. *(GRH) Let χ be a primitive character mod q. Let $t > 0$, $\Delta \geqslant 1$, $h > 0$, and $F_\pm$ the functions from Proposition 1. Then we have*

$$\sum_{\rho=\frac{1}{2}+i\gamma} F_\pm(\gamma - t) = \frac{1}{2\pi}\hat{F}_\pm(0)\log\frac{q}{\pi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} F_\pm(u-t)\Re\frac{\Gamma'}{\Gamma}\left(\frac{\frac{1}{2}+iu+\mathsf{a}}{2}\right)du$$
$$-\frac{1}{\pi}\Re\sum_{n\in\mathbb{N}}\frac{\Lambda(n)\chi(n)}{n^{\frac{1}{2}+it}}\hat{F}_\pm\left(\frac{\log n}{2\pi}\right).$$

Here the sum on the left hand side runs through all zeros of $L(s,\chi)$ in the strip $0 \leqslant \sigma \leqslant 1$ with relevant multiplicity, and where we have set

$$\mathsf{a} := \mathsf{a}(\chi) := \begin{cases} 0, & \text{if } \chi(-1) = 1, \\ 1, & \text{if } \chi(-1) = -1. \end{cases} \tag{3}$$

The proof can be established in the same way as Theorem 5.12, p. 108, in the book [2] of Iwaniec and Kowalski. It uses the Mellin transform, the explicit formula for $\frac{L'}{L}(s,\chi)$ and the residue theorem, where one has to take care of the trivial zero of $L(s,\chi)$ at $s = 0$ if $\chi(-1) = 1$.

An estimate of the integral in Proposition 2 gives the next proposition:

Proposition 3. *Let $t \geqslant 25$, $\Delta \geqslant 1$, $0 < h \leqslant \sqrt{t}$, $F_\pm$ as in Proposition 1, χ a character mod q. Then it holds that*

$$\int_{-\infty}^{\infty} F_\pm(u-t)\Re\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{\mathsf{a}+it}{2}\right)du = \left(2h \pm \frac{1}{\Delta}\right)\log\frac{t}{2} + \mathrm{O}(1),$$

where a is defined in (3).

The proof can be obtained as in [1]. It uses Stirling's formula and the properties of $F_\pm$ from Proposition 1 after splitting the integral at $t - 4\sqrt{t}$ and $t + 4\sqrt{t}$.

We make also use of the following result of Maier and Montgomery in [4] concerning moments of Dirichlet polynomials:

Proposition 4. *Consider a Dirichlet polynomial $P(s) = \sum_{p\leqslant N} a(p)p^{-s}$. For $T \geqslant 3$ and $\alpha \in \mathbb{R}$ let $s_1, ..., s_R \in \mathbb{C}$ with $1 \leqslant |\Im(s_i - s_j)| \leqslant T$ for $i \neq j$, and $\Re s_i \geqslant \alpha$ for $1 \leqslant i \leqslant R$.*

Then, for every positive integer k with $N^k \leqslant T$, it holds that

$$\sum_{r=1}^{R}|P(s_r)|^{2k} \ll T(\log T)^2 k!\Big(\sum_{p\leqslant N}|a(p)|^2 p^{-2\alpha}\Big)^k.$$

Our result relies further on the estimate in the following proposition.

Proposition 5. *Let* $T \geqslant e^{e^{33}}$, $(\log\log T)^2 \leqslant V \leqslant \frac{\log T}{\log\log T}$, $\eta = \frac{1}{\log V}$ *and* $k = \left\lfloor \frac{2V}{3(1+\eta)} \right\rfloor$.

Then we have

$$k\left(\log(k\log\log T) - 2\log(\eta V)\right) \leqslant -\frac{2}{3}V\log\frac{V}{\log\log T} + \frac{4}{3}V\log\log V + \frac{2}{3}V.$$

The proof is completely analogous to the elementary proof in [5], there Proposition 14 on page 11 and 12.

Now using Proposition 2, we can give an upper and lower bound for the number of zeros in a certain region around ordinate t.

Proposition 6. *(GRH) Let* $t \geqslant 25$, $\Delta \geqslant 2$, $0 < h \leqslant \sqrt{t}$ *and* χ *be a primitive character mod* q. *Then*

$$-\frac{\log(qt)}{2\pi\Delta} - \frac{1}{\pi}\Re\sum_{p\leqslant e^{2\pi\Delta}}\frac{\chi(p)\log(p)}{p^{\frac{1}{2}+it}}\hat{F}_-\left(\frac{\log p}{2\pi}\right) + \mathrm{O}(\log\Delta)$$
$$\leqslant N(t+h,\chi) - N(t-h,\chi) - \frac{h}{\pi}\log\frac{qt}{2\pi}$$

and

$$N(t+h,\chi) - N(t-h,\chi) - \frac{h}{\pi}\log\frac{qt}{2\pi}$$
$$\leqslant \frac{\log(qt)}{2\pi\Delta} - \frac{1}{\pi}\Re\sum_{p\leqslant e^{2\pi\Delta}}\frac{\chi(p)\log(p)}{p^{\frac{1}{2}+it}}\hat{F}_+\left(\frac{\log p}{2\pi}\right) + \mathrm{O}(\log\Delta).$$

Proof. We only show the upper bound, the lower bound estimate can be done in a complete analogous way.

We use the functions of Proposition 1 and the results from Propositions 2 and 3, we see analogously to [5] (there Proposition 15 from page 12 on):

$$N(t+h,\chi) - N(t-h,\chi) \leqslant \left(2h + \frac{1}{\Delta}\right)\frac{1}{2\pi}\log\frac{qt}{2\pi}$$
$$+ \mathrm{O}(1) - \frac{1}{\pi}\Re\sum_{n\leqslant e^{2\pi\Delta}}\frac{\Lambda(n)\chi(n)}{n^{\frac{1}{2}+it}}\hat{F}_+\left(\frac{\log n}{2\pi}\right).$$

Here

$$\frac{1}{\pi}\Re\sum_{n\leqslant e^{2\pi\Delta}}\frac{\Lambda(n)\chi(n)}{n^{\frac{1}{2}+it}}\hat{F}_+\left(\frac{\log n}{2\pi}\right) = \frac{1}{\pi}\Re\sum_{p\leqslant e^{2\pi\Delta}}\frac{\log p\ \chi(p)}{p^{\frac{1}{2}+it}}\hat{F}_+\left(\frac{\log p}{2\pi}\right)$$
$$+ \frac{1}{\pi}\Re\sum_{p\leqslant e^{\pi\Delta}}\frac{\log p\ \chi(p)^2}{p^{1+2it}}\hat{F}_+\left(\frac{\log p}{\pi}\right) + \mathrm{O}(1)$$
$$= \frac{1}{\pi}\Re\sum_{p\leqslant e^{2\pi\Delta}}\frac{\log p\ \chi(p)}{p^{\frac{1}{2}+it}}\hat{F}_+\left(\frac{\log p}{2\pi}\right) + \mathrm{O}(\log\Delta),$$

and this finishes the proof. ∎

3. V-typical ordinates

The method of Soundararajan in [7] relies on the notion of V-typical ordinates. We modify this definition for our purposes and define $V_{(\delta,\chi,\mathrm{q})}$-typical ordinates as follows.

Definition 1 ($\boldsymbol{V_{(\delta,\chi,q)}}$-typical). *Let $q \in \mathbb{N}$ and χ a character mod q. If χ is nonprincipal, let it be induced by χ_1 mod q_1, let $T > e$ and $0 < \delta \leqslant 1$. Let $V \in \left[(\log\log T)^2, \dfrac{\log T}{\log\log T}\right]$.*

An ordinate $t \in [T, 2T]$ is called $\underline{V_{(\delta,\chi,q)}\text{-typical}}$ of order T, if the following properties hold:

(i) $\forall \sigma \geqslant \frac{1}{2}$: $\left|\sum_{n\leqslant x} \dfrac{\chi_1(n)\Lambda(n)}{n^{\sigma+it}\log n}\dfrac{\log\left(\frac{x}{n}\right)}{\log x}\right| \leqslant 2V$ *with* $x = T^{\frac{1}{V}}$,

(ii) $\forall t' \in (t-1, t+1)$: $N(t'+h,\chi) - N(t'-h,\chi) \leqslant (1+\delta)V$ *with* $h = \dfrac{\delta\pi V}{\log(q_1 T)}$ *and* $[t'-h, t'+h] \subseteq [t-1, t+1]$,

(iii) $\forall t' \in (t-1, t+1)$: $N(t'+h,\chi) - N(t'-h,\chi) \leqslant V$ *with* $h = \dfrac{\pi V}{\log V \log(q_1 T)}$ *and* $[t'-h, t'+h] \subseteq [t-1, t+1]$.

If at least one of the three properties does not hold, we call t a $\underline{V_{(\delta,\chi,q)}\text{-untypical}}$ ordinate of order T.

In what follows, the meaning of χ, q and δ is often clear from the context, then we will write simply V-typical instead of $V_{(\delta,\chi,q)}$-typical of order T.

4. $\boldsymbol{V}$ such that all $\boldsymbol{t \in [T, 2T]}$ are $\boldsymbol{V}$-typical

Proposition 7. *Let t be sufficiently large and let $0 < h \leqslant \sqrt{t}$, let χ be a primitive character mod q. Then*

$$\left|N(t+h,\chi) - N(t-h,\chi) - \frac{h}{\pi}\log\frac{qt}{2\pi}\right| \leqslant \frac{\log(qt)}{2\log\log(qt)} + \left(\frac{1}{2} + \mathrm{o}(1)\right)\frac{\log(qt)\log\log\log(qt)}{(\log\log(qt))^2} \qquad \text{for } t \to \infty.$$

Proof. As in [5], we estimate the sum of Proposition 6 as follows:

$$\left|\frac{1}{\pi}\Re\sum_{p\leqslant e^{2\pi\Delta}} \frac{\log p\,\chi(p)}{p^{\frac{1}{2}+it}}\hat{F}_+\left(\frac{\log p}{2\pi}\right)\right| \ll \sum_{p\leqslant e^{2\pi\Delta}} \frac{1}{\sqrt{p}} \ll \frac{e^{\pi\Delta}}{\Delta}. \tag{4}$$

Now set $\Delta = \frac{1}{\pi} \log \frac{\log(qt)}{\log\log(qt)}$. By estimate (4), we obtain

$$\begin{aligned}
\Big| N(t+h,\chi) &- N(t-h,\chi) - \frac{h}{\pi}\log\frac{qt}{2\pi}\Big| \\
&\leqslant \frac{\log(qt)}{2(\log\log(qt) - \log\log\log(qt))} + \mathrm{O}\left(\frac{\frac{\log(qt)}{\log\log(qt)}}{\log\log(qt) - \log\log\log(qt)}\right) \\
&= \frac{\log(qt)}{2\log\log(qt)} \sum_{k=0}^{\infty} \left(\frac{\log\log\log(qt)}{\log\log(qt)}\right)^k + \mathrm{O}\left(\frac{\log(qt)}{(\log\log(qt))^2}\right) \\
&= \frac{\log(qt)}{2\log\log(qt)} + \frac{\log(qt)\log\log\log(qt)}{2(\log\log(qt))^2}(1+\mathrm{o}(1))
\end{aligned}$$

with an o(1)-term not depending on q, more precise, it is $\mathrm{O}((\log\log\log t)^{-1})$. ■

Proposition 8. *Let χ be a character mod q, q_1 be the conductor of χ and $0 < \delta \leqslant 1$. Further let T be sufficiently large, at least $T \geqslant \max\{q^2, e^{e^9}\}$, and let V be such that*

$$\frac{3}{4} + \frac{\log\log\log T}{\log\log T} \leqslant V \frac{\log\log T}{\log T} \leqslant 1$$

holds. Then all ordinates $t \in [T, 2T]$ are V-typical of order T.

As a consequence of this proposition, we conclude that V-typical ordinates exist.

Proof. We have to verify properties (i), (ii) and (iii) from Definition 1.

Ad (i): Let $f(u) := \sum_{2\leqslant n\leqslant u} \frac{\Lambda(n)\chi_1(n)}{\sqrt{n}\log n}$, $u \geqslant 2$. Then (see [5], page 16):

$$|f(u)| \leqslant \sum_{2\leqslant n\leqslant u} \frac{\Lambda(n)}{\sqrt{n}\log n} \ll \frac{\sqrt{u}}{\log u},$$

and from this we obtain

$$\sum_{n\leqslant x} \frac{\Lambda(n)\chi_1(n)}{\sqrt{n}\log n} \log\frac{x}{n} = \int_1^x \frac{f(u)}{u} du \ll \frac{\sqrt{x}}{\log x}.$$

Since $x = T^{\frac{1}{V}} \leqslant T^{\frac{4\log\log T}{3\log T}} \leqslant (\log T)^2$, we have

$$\left| \sum_{n\leqslant x} \frac{\chi_1(n)\Lambda(n)}{n^{\sigma+it}\log n} \frac{\log\left(\frac{x}{n}\right)}{\log x} \right| \ll \frac{\sqrt{x}}{(\log x)^2} \ll \frac{\log T}{(\log\log T)^2} = \mathrm{o}(V).$$

Ad (ii): Let $t' \in [t-1, t+1]$ and $h = \frac{\delta\pi V}{\log(q_1 T)}$. Since $h = \frac{\delta\pi V}{\log(q_1 T)} \leqslant \pi V \leqslant \log T < \sqrt{T}$, we can apply Proposition 7 on the primitive character χ_1 mod q_1

that induces χ. We obtain, using $q^2 \leqslant T$, that

$$\begin{aligned}
N(t'+h,\chi) - N(t'-h,\chi)&\\
&\leqslant \frac{h}{\pi}\log\frac{q_1t'}{2\pi} + \frac{\log(q_1t')}{2\log\log(q_1t')} + \Big(\frac{1}{2}+\mathrm{o}(1)\Big)\frac{\log(q_1t')\log\log\log(q_1t')}{(\log\log(q_1t'))^2}\\
&\leqslant \frac{h}{\pi}\log\frac{q_1T}{\pi} + \frac{\log(2qT)}{2\log\log T} + \Big(\frac{1}{2}+\mathrm{o}(1)\Big)\frac{\log(2qT)\log\log\log T}{(\log\log T)^2}\\
&\leqslant \delta V + \frac{\log T^{3/2}}{2\log\log T} + + \Big(\frac{1}{2}+\mathrm{o}(1)\Big)\frac{\log T^{3/2}\log\log\log T}{(\log\log T)^2}\\
&= \delta V + \frac{3\log T}{4\log\log T} + \Big(\frac{3}{4}+\mathrm{o}(1)\Big)\frac{\log T\log\log\log T}{(\log\log T)^2}\\
&\leqslant \delta V + \frac{3\log T}{4\log\log T} + \frac{\log T\log\log\log T}{(\log\log T)^2}\\
&\leqslant (1+\delta)V.
\end{aligned}$$

Ad (iii): Let $t' \in [t-1, t+1]$ and $h = \frac{\pi V}{\log V \log(q_1 T)}$, then

$$\begin{aligned}
N(t+h,\chi) - N(t-h,\chi)&\\
&\leqslant \frac{h}{\pi}\log\frac{q_1t'}{2\pi} + \frac{\log(q_1t')}{2\log\log(q_1t')}\\
&\quad + \Big(\frac{1}{2}+\mathrm{o}(1)\Big)\frac{\log(q_1t')\log\log\log(q_1t')}{(\log\log(q_1t'))^2} \text{ by Prop. 7}\\
&\leqslant \frac{V}{\log V} + \frac{3\log T}{4\log\log T}\\
&\quad + \Big(\frac{3}{4}+\mathrm{o}(1)\Big)\frac{\log T\log\log\log T}{(\log\log T)^2} \text{ analogously to (ii)}\\
&= \frac{3\log T}{4\log\log T} + \Big(\frac{3}{4}+\mathrm{o}(1)\Big)\frac{\log T\log\log\log T}{(\log\log T)^2}\\
&\leqslant \frac{3\log T}{4\log\log T} + \frac{\log T\log\log\log T}{(\log\log T)^2} \leqslant V. \qquad \blacksquare
\end{aligned}$$

5. The number of V-untypical, well separated ordinates

Proposition 9. *Let $\chi \neq \chi_0$ be a character mod q and q_1 be the conductor of χ. Further let*

1. *T be large, at least $T \geqslant q^2$,*
2. *$0 < h \leqslant \sqrt{T}$,*
3. *$(\log\log T)^2 \leqslant V \leqslant \dfrac{\log T}{\log\log T}$,*
4. *$T \leqslant t_1 < t_2 < \cdots < t_R \leqslant 2T$ and $t_{r+1} - t_r \geqslant 1$ for $1 \leqslant r < R$,*
5. *$N(t_r+h,\chi) - N(t_r-h,\chi) - \dfrac{h}{\pi}\log\dfrac{q_1t_r}{2\pi} \geqslant V + \mathrm{O}(1)$ for $1 \leqslant r \leqslant R$.*

Then

$$R \ll T \exp\Big(-\frac{2}{3}V \log \frac{V}{\log\log T} + \frac{4}{3}V \log\log V + \mathrm{O}(V)\Big).$$

Proof. If $q_1 = q$, then χ is primitive. If $q_1 < q$, then χ is induced by a primitive character χ_1 mod q_1, and we have

$$N(t,\chi) = N(t,\chi_1).$$

Therefore we can apply the results from Proposition 6 for χ_1 and q_1. By the estimate from Proposition 6 we obtain

$$\begin{aligned} V + \mathrm{O}(1) &\leqslant N(t_r + h, \chi_1) - N(t_r - h, \chi_1) - \frac{h}{\pi} \log \frac{q_1 t_r}{2\pi} \\ &\leqslant \frac{\log(2qT)}{2\pi\Delta} + \left| \frac{1}{\pi} \sum_{p \leqslant e^{2\pi\Delta}} \frac{\chi(p) \log p}{p^{\frac{1}{2}+it_r}} \hat{\mathrm{F}}_+ \left(\frac{\log p}{2\pi}\right) \right| + \mathrm{O}(\log \Delta), \qquad \Delta \geqslant 2. \end{aligned}$$

If we define $a(p) := \frac{\chi(p)\log p}{\pi} \hat{F}_+\left(\frac{\log p}{2\pi}\right)$, we have:

$$\left| \sum_{p \leqslant e^{2\pi\Delta}} \frac{a(p)}{p^{\frac{1}{2}+it_r}} \right| \geqslant V - \frac{\log(2qT)}{2\pi\Delta} + \mathrm{O}(\log\Delta) + \mathrm{O}(1),$$

where $|a(p)| \leqslant 4$ holds by Proposition 1.

Let

$$\eta = \frac{1}{\log V} \qquad \text{and} \qquad \Delta = \frac{(1+\eta)\log(qT)}{2\pi V}.$$

Then we have

$$\exp(2\pi\Delta) = (qT)^{\frac{1+\eta}{V}} \leqslant T^{\frac{3(1+\eta)}{2V}} \qquad \text{since} \qquad q \leqslant \sqrt{T},$$

hence

$$\log \Delta \ll \log\log T \leqslant \sqrt{V}.$$

We obtain

$$\begin{aligned} V - \frac{\log(2qT)}{2\pi\Delta} + \mathrm{O}(\log\Delta) + \mathrm{O}(1) &= V - \frac{V\log(2qT)}{(1+\eta)\log(qT)} + \mathrm{O}\left(\sqrt{V}\right) \\ &\geqslant \frac{\eta V}{1+\eta} - \frac{\log 2}{(1+\eta)\log\log T} + \mathrm{O}\left(\sqrt{V}\right) \geqslant \frac{1}{2}\eta V. \end{aligned}$$

So we have

$$\left| \sum_{p \leqslant e^{2\pi\Delta}} \frac{a(p)}{p^{\frac{1}{2}+it_r}} \right| \geqslant \frac{1}{2}\eta V \qquad \text{for} \ \ 1 \leqslant r \leqslant R.$$

Let $k \in \mathbb{N}$ with $k \leqslant \left\lfloor \frac{2V}{3(1+\eta)} \right\rfloor$. Then we can apply Proposition 4 with $N = (qT)^{(1+\eta)/V}$ since $(qT)^{k\frac{1+\eta}{V}} \leqslant T^{k\frac{3(1+\eta)}{2V}} \leqslant T$ for $q^2 \leqslant T$.

Raising to the $2k$-th power and summing over all $r = 1, \dots, R$, applying Proposition 4 for $\alpha = \frac{1}{2}$ and $N = \left\lfloor (qT)^{\frac{1+\eta}{V}} \right\rfloor$, we obtain analogously to [5] (page 15):

$$R\Big(\frac{\eta V}{2}\Big)^{2k} \leqslant \sum_{r=1}^{R} \Bigg| \sum_{p \leqslant (qT)^{\frac{1+\eta}{V}}} \frac{a(p)}{p^{\frac{1}{2}+it_r}} \Bigg|^{2k} \ll T(\log T)^2 (Ck \log\log T)^k$$

with an absolute constant $C > 0$. So we have by now

$$R \ll T(\log T)^2 (4C)^k \Big(\frac{k \log\log T}{\eta^2 V^2}\Big)^k.$$

Now set $k = \lfloor \frac{2V}{3(1+\eta)} \rfloor$, and we obtain by Proposition 5:

$$\Big(\frac{k \log\log T}{\eta^2 V^2}\Big)^k \leqslant \exp\Big(-\frac{2}{3} V \log \frac{V}{\log\log T} + \frac{4}{3} V \log\log V + \frac{2}{3} V \Big).$$

With

$$(\log T)^2 (4C)^k = \exp(\mathrm{O}(V)), \qquad \text{see [5]},$$

we get the assertion with an absolute O-constant. ■

Proposition 10. *(GRH) Let χ be a character mod q with conductor q_1. Further let T be large, let*

$$2(\log\log T)^2 \leqslant V \leqslant \frac{\log T}{\log\log T},$$

and let $T \leqslant t_1 < t_2 < \cdots < t_R \leqslant 2T$ be V-untypical ordinates with $t_{r+1} - t_r \geqslant 1$ for all $1 \leqslant r < R$. Then

$$R \ll T \exp\Big(-\frac{2}{3} V \log \frac{V}{\log\log T} + \frac{4}{3} V \log\log V + \mathrm{O}(V) \Big)$$

with an O-constant independent of q and χ.

Proof. If t is a V-untypical ordinate, then at least one of the criteria of Definition 1 is false. For each criterion that is hurt, we give estimates for the corresponding number R_1, R_2 and R_3 of such well-separated ordinates being counted in the Proposition.

If criterion (i) is false for t_r, then there exists a $\sigma_r \geqslant \frac{1}{2}$ such that

$$\Bigg| \sum_{n \leqslant x} \frac{\Lambda(n)\chi_1(n)}{n^{\sigma_r + it_r} \log n} \frac{\log \frac{x}{n}}{\log x} \Bigg| > 2V,$$

note here that $x = T^{\frac{1}{V}}$.

The size of the sum over $n = p^\alpha$ with $\alpha \geqslant 2$ is

$$\Big|\sum_{\substack{n=p^\alpha\leqslant x\\ \alpha\geqslant 2}} \frac{\Lambda(n)\chi_1(n)}{n^{\sigma_r+it_r}\log n}\frac{\log\frac{x}{n}}{\log x}\Big| \leqslant \sum_{p\leqslant\sqrt{x}}\frac{1}{p} + \sum_{\substack{p^\alpha\leqslant x\\ \alpha\geqslant 3}}\frac{1}{p^{\frac{\alpha}{2}}}$$
$$\ll \log\log x \ll \log\log T \ll \sqrt{V}.$$

So if we count the ordinates t_r with

$$\Big|\sum_{p\leqslant x}\frac{\chi_1(p)}{p^{\sigma_r+it_r}}\frac{\log\frac{x}{p}}{\log x}\Big| \geqslant V,$$

where again $x = T^{\frac{1}{V}}$, we get an upper bound for R_1.

Now we apply Proposition 4 of Maier and Montgomery, we obtain

$$R_1 V^{2k} \leqslant \sum_{r\leqslant R}\Big|\sum_{p\leqslant x}\frac{\chi_1(p)}{p^{\sigma_r+it_r}}\frac{\log\frac{x}{p}}{\log x}\Big|^{2k} \ll T(\log T)^2 k!\left(\sum_{p\leqslant x}\frac{\log^2\frac{x}{p}}{p\log^2 x}\right)^k,$$

where $x^k \leqslant T$ holds for every $k \leqslant V$.

Now

$$\sum_{p\leqslant x}\frac{\log^2\frac{x}{p}}{p\log^2 x} \leqslant \sum_{p\leqslant x}\frac{1}{p} \ll \log\log x \leqslant \log\log T.$$

As in [5], we obtain with $k = \lfloor V\rfloor$:

$$R_1 \ll T(\log T)^2\Big(\frac{Ck\log\log T}{V^2}\Big)^k = T\exp\Big(-V\log\frac{V}{\log\log T} + \mathrm{O}(V)\Big).$$

Now let (ii) be false, i. e. for t_r there exists a t_r' with $|t_r - t_r'| \leqslant 1$ and

$$N\Big(t_r' + \frac{\pi\delta V}{\log(q_1 T)},\chi\Big) - N\Big(t_r' - \frac{\pi\delta V}{\log(q_1 T)},\chi\Big) > (1+\delta)V.$$

With

$$\delta V = \frac{\delta V}{\log(q_1 T)}\log\left(\frac{q_1 t_r'}{2\pi}\right) + \mathrm{o}(1) \qquad \text{for } \ T\to\infty$$

we obtain

$$N\Big(t_r' + \frac{\pi\delta V}{\log(q_1 T)},\chi\Big) - N\Big(t_r' - \frac{\pi\delta V}{\log(q_1 T)},\chi\Big) - \frac{\delta V}{\log(q_1 T)}\log\Big(\frac{q_1 t_r'}{2\pi}\Big) \geqslant V + \mathrm{O}(1).$$

Now we can apply Proposition 9, if the t_r' have a sufficiently large distance from another. So instead of the sequence t_r' being induced from t_r for $1 \leqslant r \leqslant R_2$, consider the three subsequences $t_{3s+\ell}'$ with $\ell \in \{1,2,3\}$, $0 \leqslant s \leqslant \big\lfloor\frac{R_2-\ell}{3}\big\rfloor$, they have the property $t'_{3(s+1)+\ell} - t'_{3s+\ell} \geqslant 1$. We can apply Proposition 9 on any of the three subsequences and obtain

$$R_2 \leqslant 3\Big(\Big\lfloor \frac{R_2}{3} \Big\rfloor + 1\Big) + 2 \ll T \exp\Big(-\frac{2}{3} V \log\Big(\frac{V}{\log\log T}\Big) + \frac{4}{3} V \log\log V + \mathrm{O}(V)\Big).$$

For R_3 we obtain, analogously as in [5], the same bound with a similar calculation. ■

6. Logarithmic derivative of $L(s,\chi)$

In this section, we consider only primitive characters.

Proposition 11. *Let χ be a primitive character mod q, T be sufficiently large, $\frac{1}{2} \leqslant \sigma \leqslant 2$, $T \leqslant t \leqslant 2T$ and $L(\sigma + it, \chi) \neq 0$. Then*

$$\Re \frac{L'}{L}(\sigma + it, \chi) = F(\sigma + it, \chi) - \frac{1}{2} \log(qT) + \mathrm{O}(1),$$

where $F(s,\chi) := \sum_{\rho} \Re \frac{1}{s-\rho}$ and the sum runs through all nontrivial zeros of $L(s,\chi)$.

Proof. We use the formula

$$\frac{L'}{L}(s,\chi) = -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathsf{a}}{2}\right) + B(\chi) + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

that holds for primitive characters, where $\Re B(\chi) = -\sum_{\rho} \Re(\frac{1}{\rho})$ and the sum runs through all nontrivial zeros ρ of $L(s,\chi)$. By Stirling's formula we obtain

$$\begin{aligned}
\Re\frac{L'}{L}(\sigma+it,\chi) &= -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{\sigma+it+\mathsf{a}}{2}\right) + \Re B(\chi) \\
&\quad + \sum_{\rho}\Re\left(\frac{1}{\sigma+it-\rho} + \frac{1}{\rho}\right) \\
&= -\frac{1}{2}\log q - \frac{1}{2}\log|\sigma+it+\mathsf{a}| + F(\sigma+it,\chi) \\
&\quad + \mathrm{O}(|\sigma+it+\mathsf{a}|^{-1}) + \mathrm{O}(1) \\
&= F(\sigma+it,\chi) - \frac{1}{2}\log(qT) + \mathrm{O}(1).
\end{aligned}$$
■

Proposition 12. *Let χ be a primitive character mod q. Let $x \geqslant 1$, and consider $z \in \mathbb{C}$ that is not a pole of $\frac{L'}{L}(z,\chi)$. Then*

$$\begin{aligned}
\sum_{n\leqslant x}\frac{\chi(n)\Lambda(n)}{n^z}\log\left(\frac{x}{n}\right) &= -\frac{L'}{L}(z,\chi)\log x - \left(\frac{L'}{L}\right)'(z,\chi) \\
&\quad - \sum_{\rho}\frac{x^{\rho-z}}{(\rho-z)^2} - \sum_{n\geqslant 0}\frac{x^{-2n-\mathsf{a}-z}}{(z+2n+\mathsf{a})^2}.
\end{aligned}$$

Proof. Since

$$\frac{L'}{L}(s,\chi) \ll \log(q|s|) \qquad \text{for } \Re s \leqslant -\frac{1}{2} \quad \text{and} \quad |s+m| > \frac{1}{4} \quad \text{for all } m \in \mathbb{N},$$

the proof works analogously to [5], where the term coming from the pole at $s=1$ is removed and the sum over the trivial zeros has been adjusted. ■

Estimating the last sum analogously to [5], we obtain:

Proposition 13. *Let χ be a primitive character mod q, $T \geqslant 1$ and $1 \leqslant x \leqslant T$. Let $z \in \mathbb{C}$, $\Re z \geqslant 0$, $T \leqslant \Im z \leqslant 2T$, and let z be not a pole of $\frac{L'}{L}(z,\chi)$.*

Then

$$\sum_{n \leqslant x} \frac{\chi(n)\Lambda(n)}{n^z} \log\Big(\frac{x}{n}\Big) = -\frac{L'}{L}(z,\chi)\log x - \Big(\frac{L'}{L}\Big)'(z,\chi) - \sum_{\rho} \frac{x^{\rho-z}}{(\rho-z)^2} + \mathrm{O}(T^{-1}). \tag{5}$$

7. Lower bound for $\log|L(s,\chi)|$

With the aid of V-typical ordinates, we estimate $\log L(s,\chi)$ from below.

Proposition 14 (GRH). *Let χ be a nonprincipal character mod q induced by χ_1 mod q_1. Let T be sufficiently large and $T \leqslant t \leqslant 2T$.*

Then for all $\frac{1}{2} \leqslant \sigma \leqslant 2$ and $2 \leqslant x \leqslant T$ it holds that

$$\log|L(\sigma+it,\chi)| \geqslant \Re\Big(\sum_{n\leqslant x} \frac{\Lambda(n)\chi_1(n)}{n^{\sigma+it}\log n} \frac{\log\frac{x}{n}}{\log x}\Big) - \Big(1 + \frac{x^{\frac{1}{2}-\sigma}}{(\sigma-\frac{1}{2})\log x}\Big)\frac{F(\sigma+it,\chi)}{\log x} + \mathrm{O}\Big(\sqrt{\frac{\log q}{\log\log q}}\Big),$$

where F is the function from Proposition 11.

Proof. At first, let χ be primitive. By integrating equation (5) from $z = \sigma+it$ to $z = 2+it$, we obtain analogously to [5]:

$$\log|L(\sigma+it,\chi)| \geqslant \Re\Big(\sum_{n\leqslant x} \frac{\Lambda(n)\chi(n)}{n^{\sigma+it}\log n} \frac{\log\frac{x}{n}}{\log x}\Big) - \Big(1 + \frac{x^{\frac{1}{2}-\sigma}}{(\sigma-\frac{1}{2})\log x}\Big)\frac{F(\sigma+it,\chi)}{\log x} + \mathrm{O}(1).$$

Now let χ mod q be not primitive and induced by the primitive character χ_1 mod q_1.

Then we have

$$L(s,\chi) = L(s,\chi_1)\prod_{p|q}\Big(1 - \frac{\chi_1(p)}{p^s}\Big). \tag{6}$$

We obtain with equation (6):

$$\begin{aligned}\log\left|L(s,\chi)\right| &= \log|L(s,\chi_1)| + \sum_{p|q}\log\left|1-\frac{\chi_1(p)}{p^s}\right|\\ &\geqslant \Re\Big(\sum_{n\leqslant x}\frac{\Lambda(n)\chi_1(n)}{n^{\sigma+it}}\frac{\log\frac{x}{n}}{\log n\,\log x}\Big) - \Big(1+\frac{x^{\frac12-\sigma}}{(\sigma-\frac12)\log x}\Big)\frac{F(\sigma+it,\chi_1)}{\log x}\\ &\quad + \mathrm{O}(1) + \sum_{p|q}\log\left|1-\frac{\chi(p)}{p^s}\right|.\end{aligned}$$

For the last sum we get

$$\sum_{p|q}\log\left|1-\frac{\chi(p)}{p^s}\right| \leqslant \sum_{p|q}\frac{1}{p^{1/2}} \leqslant \sum_{j=1}^{2\log q}\frac{1}{p_j^{1/2}} \ll \sqrt{\frac{\log q}{\log\log q}}. \tag{7}$$

From equation (6) we see further that

$$F(s,\chi) = F(s,\chi_1),$$

so we get the stated bound. ■

Now we would like to give an estimate for $L(s,\chi)$ in the interval $\Re(s)\in\left(\frac12,2\right)$. For this, we split the interval at $\frac12+\frac{V}{\log T}$ and give a bound for each part. This is done in the next two propositions.

Proposition 15 (GRH). *Let χ be a nonprincipal character mod q, and further let T be sufficiently large, at least $T\geqslant q$, let $V\in[(\log\log T)^2,\frac{\log T}{\log\log T}]$ and let $t\in[T,2T]$ be $V_{\delta,\chi,q}$-typical of order T.*

Then it holds for $\frac12+\frac{V}{\log T}\leqslant\sigma\leqslant 2$, that

$$\log|L(\sigma+it,\chi)| \geqslant f_{\delta,q}(V,\sigma+it),$$

where $f_{\delta,q}:\mathbb{R}\times\mathbb{C}\to\mathbb{R}$, $f_{\delta,q}(V,\sigma+it)=\mathrm{O}\left(\frac{V}{\delta}+\sqrt{\frac{\log q}{\log\log q}}\right)$.

Proof. In Proposition 14 we set $x=T^{\frac1V}$. Then $2\leqslant x\leqslant T$, and since $\frac12+\frac{V}{\log T}\leqslant\sigma$, we have

$$\frac{x^{\frac12-\sigma}}{(\sigma-\frac12)\log x} \leqslant \frac{\exp(-V\frac{\log x}{\log T})}{V\frac{\log x}{\log T}} = e^{-1}\leqslant 1.$$

Applying now Proposition 14, we obtain:

$$\log|L(\sigma+it,\chi)| \geqslant -2V - 2\frac{V}{\log T}F(\sigma+it,\chi) + \mathrm{O}\left(\sqrt{\frac{\log q}{\log\log q}}\right) =: f_{\delta,q}(V,\sigma+it),$$

since t is V-typical.

We aim to majorize $F(\sigma+it,\chi)$ independent from q and χ. As in [5], we divide the region of the zero-ordinates in two parts as follows.

(i) γ with $\frac{2\pi n\delta V}{\log(q_1 T)} \leqslant |t-\gamma| \leqslant \frac{2\pi(n+1)\delta V}{\log(q_1 T)}$ for $0 \leqslant n \leqslant N = \left\lfloor \frac{\log(q_1 T)}{4\pi\delta V} \right\rfloor$,

(ii) γ with $\{\gamma \,:\, |\gamma - t| \geqslant \frac{1}{2}\}$, where q_1 denotes the conductor of χ mod q.

Consider the set of γ from (i):

$$\begin{aligned}\sum_{\gamma \text{ from (i)}} \Re\frac{1}{\sigma+it-\frac{1}{2}-i\gamma} &= 2\sum_{\gamma \text{ from (i)}} \frac{(\sigma-\frac{1}{2})}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} \\ &\leqslant 2(1+\delta)V\sum_{n=0}^{N} \frac{(\sigma-\frac{1}{2})}{(\sigma-\frac{1}{2})^2+\left(\frac{2\pi n\delta V}{\log(q_1 T)}\right)^2} \qquad \text{since } t \text{ is } V\text{-typical, (ii),} \\ &\leqslant 4V\Big(\frac{1}{\sigma-\frac{1}{2}}+\frac{\log(q_1 T)}{4\delta V}\Big),\end{aligned}$$

since for $a, c \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ we have $\sum_{n=0}^{N} \frac{a}{a^2+(cn)^2} \leqslant \frac{1}{a} + \frac{\pi}{2c}$, see [5] Prop. 6, and we continue with

$$\leqslant 4\log(q_1 T) + \frac{\log(q_1 T)}{\delta} \leqslant 5\frac{\log(qT)}{\delta}.$$

For the sum over γ with (ii) we work with the known formula

$$\sum_{\rho\in\mathcal{N}(\chi)} \frac{1}{1+(t-\Im(\rho))^2} \ll \log(q(2+|t|)) \tag{8}$$

holding for primitive characters mod q. Since $\mathcal{N}(\chi) = \mathcal{N}(\chi_1)$ if χ mod q is induced by χ_1 mod $q_1 \leqslant q$, we can use this formula also in the case of a nonprimitive character mod q.

For $0 \leqslant \sigma - \frac{1}{2} \leqslant \frac{3}{2}$ and $|t-\gamma| \geqslant \frac{1}{2}$ we have

$$\frac{\sigma-\frac{1}{2}}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} \leqslant \frac{8}{1+(t-\gamma)^2}, \tag{9}$$

therefore we can estimate the sum over γ with (ii) using (9) and (8) as follows:

$$\begin{aligned}\sum_{|t-\gamma|\geqslant\frac{1}{2}} \Re\Big(\frac{1}{\sigma+it-\frac{1}{2}-i\gamma}\Big) &= \sum_{|t-\gamma|\geqslant\frac{1}{2}} \frac{\sigma-\frac{1}{2}}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} \\ &\leqslant \sum_{|t-\gamma|\geqslant\frac{1}{2}} \frac{8}{1+(t-\gamma)^2} \leqslant \sum_{\rho\in\mathcal{N}(\chi)} \frac{8}{1+(t-\Im(\rho))^2} \\ &\ll \log(qt).\end{aligned}$$

Now consider $g(x) := \frac{\log(qx)}{\log x}$, we see that $g(x)$ is monotonously decreasing for $x > 1$, and so for $x \geqslant q$ we have $g(x) \leqslant g(q) = 2$.

We resume the two results for the regions (i) and (ii) as follows:

$$\Big|2\frac{V}{\log T}F(\sigma+it,\chi)\Big| \ll \frac{\log(qT)}{\log T}\frac{V}{\delta} \ll \frac{V}{\delta} \qquad \text{since} \;\; q \leqslant T,$$

which gives the asserted bound for $f_{\delta,q}(V,\sigma+it)$. ■

Proposition 16 (GRH). *Let χ be a character mod q, let T be sufficiently large, $V \in [(\log\log T)^2, \frac{\log T}{\log\log T}]$ and $t \in [T,2T]$ be V-typical (of order T).*

Then we have for all $\frac{1}{2} < \sigma \leqslant \sigma_0 = \frac{1}{2} + \frac{V}{\log T}$:

$$\begin{aligned}\log|L(\sigma+it,\chi)| \geqslant \log|L(\sigma_0+it,\chi)| - V\log\frac{\sigma_0-\frac{1}{2}}{\sigma-\frac{1}{2}} \\ - 2(1+\delta)V\log\log V + \mathrm{O}\Big(\frac{V}{\delta^2}+\sqrt{\frac{\log q}{\log\log q}}\Big).\end{aligned}$$

Proof. Consider at first a primitive character χ mod q, i. e. $q_1 = q$. We work as in [5], p. 8, and get:

$$\log|L(\sigma_0+it,\chi)| - \log|L(\sigma+it,\chi)| \leqslant \frac{1}{2}\sum_{\gamma}\log\frac{(\sigma_0-\frac{1}{2})^2+(t-\gamma)^2}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}.$$

In order to estimate the sum, we divide the set of γ in three subsets such that we can make use of the fact that t is a V-typical ordinate.

The division of the γ is as follows.

(a) γ with $|t-\gamma| \leqslant \frac{\pi V}{\log V\log(qT)}$,

(b) γ with $\big(2\pi\delta n + \frac{\pi}{\log V}\big)\frac{V}{\log(qT)} \leqslant |t-\gamma| \leqslant \big(2\pi\delta(n+1) + \frac{\pi}{\log V}\big)\frac{V}{\log(qT)}$ $\big(0 \leqslant n \leqslant N = \lfloor\frac{\log(qT)}{4\pi\delta V}\rfloor\big)$,

(c) γ with $\{\gamma : |t-\gamma| \geqslant \frac{1}{2}\}$.

Since $\sigma \leqslant \sigma_0$, we have

$$\frac{(\sigma_0-\frac{1}{2})^2+(t-\gamma)^2}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} \leqslant \frac{(\sigma_0-\frac{1}{2})^2}{(\sigma-\frac{1}{2})^2}.$$

For the γ from (a) we use property (iii) from the definition of V-typical and obtain

$$\begin{aligned}\frac{1}{2}\sum_{|t-\gamma|\leqslant\frac{\pi V}{\log V\log(qT)}}\log\frac{(\sigma_0-\frac{1}{2})^2+(t-\gamma)^2}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} &\leqslant \frac{1}{2}\sum_{|t-\gamma|\leqslant\frac{\pi V}{\log V\log(qT)}}\log\frac{(\sigma_0-\frac{1}{2})^2}{(\sigma-\frac{1}{2})^2} \\ &\leqslant V\log\frac{\sigma_0-\frac{1}{2}}{\sigma-\frac{1}{2}}.\end{aligned}$$

We use the fact that $\frac{(\sigma_0-\frac{1}{2})^2+(t-\gamma)^2}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}$ is decreasing in $|t-\gamma|$. With this, we estimate the set of γ in (b) using property (ii) in the definition of V-typical. For the γ with (c) we use the general zero estimate for $L(s,\chi)$ and obtain in the same way as in [5]:

$$\frac{1}{2}\sum_{\gamma\text{'s in (b)}} \log\frac{(\sigma_0-\frac{1}{2})^2+(t-\gamma)^2}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} \leqslant 2(1+\delta)V\log\log V + \mathrm{O}\left(\frac{V}{\delta^2}\right)$$

and

$$\frac{1}{2}\sum_{|\gamma-t|\geqslant\frac{1}{2}} \log\frac{(\sigma_0-\frac{1}{2})^2+(t-\gamma)^2}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} \ll \frac{V}{\log\log T}.$$

This gives the assertion for primitive characters.

Now if χ is not primitive mod q and induced by the primitive character χ_1 mod q_1, we use equation (7) and obtain

$$\begin{aligned}
\log|L(\sigma+it,\chi)| &= \log|L(\sigma+it,\chi_1)| + \mathrm{O}\Big(\sqrt{\frac{\log q}{\log\log q}}\Big)\\
&\geqslant \log|L(\sigma_0+it,\chi_1)| - V\log\frac{\sigma_0-\frac{1}{2}}{\sigma-\frac{1}{2}} - 2(1+\delta)V\log\log V\\
&\quad + \mathrm{O}(\frac{V}{\delta^2}) + \mathrm{O}\Big(\sqrt{\frac{\log q}{\log\log q}}\Big)\\
&= \log|L(\sigma_0+it,\chi))| + -V\log\frac{\sigma_0-\frac{1}{2}}{\sigma-\frac{1}{2}} - 2(1+\delta)V\log\log V\\
&\quad + \mathrm{O}\Big(\frac{V}{\delta^2} + \sqrt{\frac{\log q}{\log\log q}}\Big).
\end{aligned}$$

■

At the end of this section we combine the results from propositions 8, 15 and 16. With these, we obtain a lower bound for the whole stripe $\Re(s) \in \big(\frac{1}{2}, 2\big)$.

Proposition 17 (GRH). *Let χ be a character mod q, $|t|$ be sufficiently large, at least $|t| \geqslant q$, and $\frac{1}{2} < \sigma \leqslant 2$. Then*

$$\log|L(\sigma+it,\chi)| \geqslant -\frac{\log|t|}{\log\log|t|}\log\frac{1}{(\sigma-\frac{1}{2})} - 3\frac{\log|t|\log\log\log|t|}{\log\log|t|}.$$

Proof. As in [5], we choose

$$V = \frac{\log|t|}{\log\log|t|} \text{ and } \delta = \frac{1}{2},$$

note that then $\mathrm{O}\left(\frac{V}{\delta^2} + \sqrt{\frac{\log q}{\log\log q}}\right) = \mathrm{O}(V)$. ■

By now, we gave estimates for $L(s,\chi)$ in a region for sufficiently large $\Im(s)$. We also need an estimate for $L(s,\chi)$ in the remaining region, which we give in the next Proposition.

Proposition 18 (GRH). *Let x be large, $c > 0$. Further let $T_0(x) := T_0 := 2^{\lfloor (\log x)^{3/5}(\log\log x)^c \rfloor}$, and $\sigma = \frac{1}{2} + \frac{1}{\log x}$. Then there exists a $C > 0$, such that for all $|t| \leqslant T_0$, $q \leqslant \sqrt{T_0}$ and a nonprincipal character χ mod q we have*

$$|L(\sigma + it, \chi)| \geqslant T_0^{-C \log\log x}.$$

Proof. At first, let χ be a primitive character mod q, and $q \leqslant \sqrt{T_0}$. By the explicit formula for the logarithmic derivation of L we obtain

$$\int_{\sigma+it}^{2+it} \frac{L'}{L}(s+it,\chi)ds = \int_{\sigma+it}^{2+it} \left(\sum_{\substack{\rho \in \mathcal{N}(\chi) \\ |\Im(s) - \Im(\rho)| \leqslant 1}} \frac{1}{s-\rho} + \mathrm{O}(\log(q(2 + |\Im(s)|))) \right) ds,$$

hence

$$\begin{aligned} \log L(2+it,\chi) - \log L(\sigma+it,\chi) = &\sum_{\substack{\rho \in \mathcal{N}(\chi) \\ |t - \Im(\rho)| \leqslant 1}} \log(2 + it - \rho) \\ &- \sum_{\substack{\rho \in \mathcal{N}(\chi) \\ |t - \Im(\rho)| \leqslant 1}} \log(\sigma + it - \rho) + \mathrm{O}(\log(q(2+|t|))). \end{aligned}$$

Considering the real parts, it follows that

$$\begin{aligned} \log |L(\sigma+it,\chi)|^{-1} = &\sum_{\substack{\rho \in \mathcal{N}(\chi) \\ |t - \Im(\rho)| \leqslant 1}} \log \left| \frac{3}{2} + i(t - \Im(\rho)) \right| \\ &+ \sum_{\substack{\rho \in \mathcal{N}(\chi) \\ |t - \Im(\rho)| \leqslant 1}} \log \frac{1}{|\sigma + it - \rho|} + \mathrm{O}(\log(q(2+|t|))). \end{aligned}$$

To give an estimate of the first sum, we have

$$\left| \frac{3}{2} + i(t - \Im(\rho)) \right| \leqslant \frac{5}{2} \text{ for } |t - \Im(\rho)| \leqslant 1,$$

hence

$$\sum_{|t - \Im(\rho)| \leqslant 1} \log \left| \frac{3}{2} + i(t - \Im(\rho)) \right| \ll \log(qt),$$

and to give an estimate of the second sum, we have

$$|\sigma + it - \rho|^{-1} = \left| \frac{1}{\log x} + i(t - \Im(\rho)) \right|^{-1} \leqslant \log x,$$

hence

$$\sum_{|t - \Im(\rho)| \leqslant 1} \log \frac{1}{|\sigma + it - \rho|} \ll \log(qt) \log\log\log x.$$

Therefore we obtain

$$\log |L(\sigma+it,\chi)|^{-1} \ll \log(qt)\log\log x.$$

If we note that $t \leqslant T_0$ and $q \leqslant \sqrt{T_0}$, we obtain

$$\log |L(\sigma+it,\chi)|^{-1} \ll \log T_0 \log\log x.$$

This gives the assertion for primitive characters.

Now let χ be a nonprimitive character mod q and induced by χ_1 mod q_1. We conclude:

$$\begin{aligned}\log |L(\sigma+it,\chi)|^{-1} &= \log |L(\sigma+it,\chi_1)|^{-1} - \sum_{p|q} \log \left|1-\frac{\chi(p)}{p^s}\right| \\ &= \log |L(\sigma+it,\chi_1)|^{-1} + \mathrm{O}\Big(\sqrt{\frac{\log T_0}{\log\log T_0}}\Big) \\ &\ll \log T_0 \log\log x \Big(1+\mathrm{O}\Big(\frac{1}{\sqrt{\log T_0 \log\log T_0}\log\log x}\Big)\Big) \\ &\ll \log T_0 \log\log x. \qquad \blacksquare\end{aligned}$$

8. Majorant of $\left|x^z L(z,\chi)^{-1}\right|$

In this section we give a majorant of $\left|x^z L(z,\chi)^{-1}\right|$ for certain z. It is a consequence of Propositions 15 and 16.

Proposition 19 (GRH). *Let χ be a character mod q. Further let t be sufficiently large (at least $t \geqslant q$), $x \geqslant t$, $V' \in \left[(\log\log t)^2, \frac{\log(t/2)}{\log\log(t/2)}\right]$, $V \geqslant V'$, t be V'-typical of order T'.*

Then for $V' \leqslant (\Re z - \frac{1}{2})\log x \leqslant V$, $|\Im z| = t$, we have

$$\begin{aligned}&\left|x^z L(z,\chi)^{-1}\right| \\ &\quad\leqslant \sqrt{x}\exp\left(V\log\frac{\log x}{\log t} + 2(1+\delta)V\log\log V + \mathrm{O}\Big(V\delta^{-2} + \sqrt{\frac{\log x}{\log\log x}}\Big)\right).\end{aligned}$$

Proof. By taking notion of the changed error term, everything remains as in [5], see Proposition 22 there. ■

9. Upper bound for $M(x,q,a)$

We need some preliminaries for the proof of the theorem.

For a character χ mod q, let

$$A(x,\chi,q) := \frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-i2^K}^{1+\frac{1}{\log x}+i2^K} \frac{x^s}{L(s,\chi)s}ds, \qquad \text{where } K := \left[\frac{\log x}{\log 2}\right],$$

and by Perron's formula we have:

$$M(x,q,a)=\frac{1}{\varphi(q)}\sum_{\chi(q)}\overline{\chi}(a)A(x,\chi,q)+\mathrm{O}(\log x). \tag{10}$$

We aim to give a good upper bound for $A(x,\chi,q)$.

Further we assume w.l.o.g., that $x\geqslant q^2$, as otherwise we can estimate trivially.

Now we give some definitions being valid during this section.

Definition 2.

$$K:=\left\lceil\frac{\log x}{\log 2}\right\rceil,\qquad \kappa:=\left\lfloor(\log x)^{3/5}(\log\log x)^{c}\right\rfloor,$$
$$T_k:=2^k\qquad \textit{for}\ \ \kappa\leqslant k\leqslant K,\qquad \textit{so}\ \ q^2\leqslant T_\kappa\leqslant T_k.$$

For k with $\kappa\leqslant k<K$ and for $n\in\mathbb{N}\cap[T_k,2T_k)$, we define the integer V_n to be the smallest integer in the interval $\left[(\log\log T_k)^2+1,\frac{\log T_k}{\log\log T_k}\right]$, such that all points in $[n,n+1]$ are V_n-typical ordinates of order T_k. The existence of these V_n is obtained by Proposition 8.

Lemma 1. *Let $x\geqslant 2$, $c>1$, $q\in\mathbb{N}$ and $1<q\leqslant 2^{\kappa/2}$. Further let χ be a nonprincipal character mod q and $\delta\in(0,1]$. Then*

$$\frac{A(x,\chi,q)}{\sqrt{x}}\ll_\delta \exp\left((\log x)^{3/5}(\log\log x)^{c+1+\delta}\right)+B(x,\chi,q),$$

where

$$B(x,\chi,q)=\sum_{n=T_\kappa}^{T_K-1}\frac{1}{n}\exp\left(V_n\log\left(\frac{\log x}{\log n}\right)+2(1+2\delta)V_n\log\log V_n+D\sqrt{\frac{\log x}{\log\log x}}\right)$$

with an absolute constant $D>0$.

Proof. We choose the following path of integration $S(x,\chi,q)$, we describe it for the upper half plane $\Im(z)\geqslant 0$, it passes out analogously in the lower half plane.

1. A vertical segment $\left[\frac{1}{2}+\frac{1}{\log x},\frac{1}{2}+\frac{1}{\log x}+iT_\kappa\right]$.
2. Further vertical segments $\left[\frac{1}{2}+\frac{V_n}{\log x}+in,\frac{1}{2}+\frac{V_n}{\log x}+i(n+1)\right]$.
3. A horizontal segment $\left[\frac{1}{2}+\frac{1}{\log x}+iT_\kappa,\frac{1}{2}+\frac{V_{T_\kappa}}{\log x}+iT_\kappa\right]$.
4. Additional horizontal segments for $T_\kappa\leqslant n\leqslant T_K-2$, namely

$$\left[\frac{1}{2}+\frac{V_n}{\log x}+i(n+1),\frac{1}{2}+\frac{V_{n+1}}{\log x}+i(n+1)\right].$$

5. The last horizontal segment $\left[\frac{1}{2}+\frac{V_{T_K-1}}{\log x}+iT_K, 1+\frac{1}{\log x}+iT_K\right]$.

Hence

$$|A(x,\chi,q)| = \frac{1}{2\pi}\left|\int\limits_{S(x,\chi,q)} \frac{x^s}{L(s,\chi)s}ds\right|.$$

We consider just the first segment more accurate, the others can be estimated analogously to [5]:

Ad 1.:

$$\begin{aligned}
\frac{1}{2\pi}\left|\int\limits_{\substack{S(x,\chi,q)\\ |\Im(z)|\leqslant T_\kappa}} \frac{x^s}{L(s,\chi)s}ds\right| &\leqslant \frac{1}{2\pi}x^{\frac{1}{2}+\frac{1}{\log x}}\int\limits_{-T_\kappa}^{T_\kappa}\left|L\Big(\frac{1}{2}+\frac{1}{\log x}+it,\chi\Big)\right|^{-1}\frac{dt}{\sqrt{\frac{1}{4}+t^2}}\\
&\leqslant \frac{e}{2\pi}\sqrt{x}\max_{|t|\leqslant T_\kappa}\left|L\Big(\frac{1}{2}+\frac{1}{\log x}+it,\chi\Big)\right|^{-1}\int\limits_{-T_\kappa}^{T_\kappa}\frac{dt}{\sqrt{\frac{1}{4}+t^2}}\\
&\leqslant \sqrt{x}\max_{|t|\leqslant T_\kappa}\left|L\Big(\frac{1}{2}+\frac{1}{\log x}+it,\chi\Big)\right|^{-1}\int\limits_{0}^{T_\kappa}\frac{dt}{\sqrt{\frac{1}{4}+t^2}}\\
&\leqslant 2\sqrt{x}\max_{|t|\leqslant T_\kappa}\left|L\Big(\frac{1}{2}+\frac{1}{\log x}+it,\chi\Big)\right|^{-1}\log T_\kappa\\
&\ll \sqrt{x}\,(\log T_\kappa)\,T_\kappa^{C\log\log x} \qquad \text{by Prop. 18}\\
&\leqslant \sqrt{x}T_\kappa^{C_1\log\log x} \qquad \text{with } C_1 = C+1.
\end{aligned}$$

Ad 2.:

$$\begin{aligned}
\frac{1}{2\pi}\left|\int\limits_{\frac{1}{2}+\frac{V_n}{\log x}+in}^{\frac{1}{2}+\frac{V_n}{\log x}+i(n+1)} \frac{x^s}{L(s,\chi)s}ds\right| &\leqslant \frac{1}{2\pi n}\max_{\substack{z\in\{\frac{1}{2}+\frac{V_n}{\log x}+it;\\ t\in[n,n+1]\}}}\left|x^z L(z,\chi)^{-1}\right| \qquad \text{as } |s|\geqslant|n|\\
&\leqslant \frac{1}{n}\sqrt{x}\exp\Big(V_n\log\Big(\frac{\log x}{\log n}\Big)+2(1+\delta)V_n\log\log V_n\\
&\quad + D\Big(\frac{V_n}{\delta^2}+\sqrt{\frac{\log x}{\log\log x}}\Big)\Big),
\end{aligned}$$

where $D>0$ is an absolute constant, see Proposition 19.

Ad 3.:

$$\frac{1}{2\pi}\left|\int\limits_{\frac{1}{2}+\frac{1}{\log x}+iT_\kappa}^{\frac{1}{2}+\frac{V_{T_\kappa}}{\log x}+iT_\kappa} \frac{x^s}{L(s,\chi)s}ds\right| \leqslant \sqrt{x}T_\kappa^3 \qquad \text{by Prop. 17.}$$

Ad 4.: Here we use Proposition 19 for n with $T_\kappa \leqslant n \leqslant T_K - 2$:

$$\Bigg| \int\limits_{\frac{1}{2}+\frac{V_n}{\log x}+i(n+1)}^{\frac{1}{2}+\frac{V_{n+1}}{\log x}+i(n+1)} \frac{x^s}{L(s,\chi)s} ds \Bigg|$$

$$\leqslant \frac{1}{n}\sqrt{x}\exp\Big(V_n \log\Big(\frac{\log x}{\log n}\Big) + 2(1+\delta)V_n \log\log V_n + D\Big(\frac{V_n}{\delta^2} + \sqrt{\frac{\log x}{\log\log x}}\Big)\Big)$$

$$+ \frac{1}{n+1}\sqrt{x}\exp\Big(V_{n+1}\log\Big(\frac{\log x}{\log(n+1)}\Big) + 2(1+\delta)V_{n+1}\log\log V_{n+1}$$

$$+ D\Big(\frac{V_{n+1}}{\delta^2} + \sqrt{\frac{\log x}{\log\log x}}\Big)\Big).$$

Ad 5.: We obtain using Proposition 15:

$$\frac{1}{2\pi}\Bigg| \int\limits_{\frac{1}{2}+\frac{V_{T_K-1}}{\log x}+iT_K}^{1+\frac{1}{\log x}+iT_K} \frac{x^s}{L(s,\chi)s} ds \Bigg| \ll_\delta \sqrt{x}. \qquad \blacksquare$$

The following proposition is similar to Proposition 23 in [5], the modification here is necessary, but the proof works analogously.

Proposition 20. *Let $A, C > 0$ and let $A \geqslant 4C^4 + 1$, then for $V > e^{3C/2}$ it holds that*

$$AV - \frac{2}{3}V\log V + CV\log\log V \leqslant e^{3A/2}\Big(\frac{3}{2}A\Big)^{3C/2}.$$

Lemma 2. *Under the conditions of Lemma 1 we have*

$$B(x,\chi,q) \ll_\delta \exp\Big((\log x)^{3/5}(\log\log x)^{13/2-3c/2+8\delta}\Big).$$

Proof. We define for $\kappa \leqslant k < K$:

$$B(T_k, x, \chi, q) := \sum_{T_k \leqslant n < 2T_k} \frac{1}{n}\exp\Big(V_n\log\Big(\frac{\log x}{\log n}\Big) + 2(1+2\delta)V_n\log\log V_n\Big),$$

then

$$B(x,\chi,q) \leqslant K \max_{\kappa\leqslant k<K} B(T_k,x,\chi,q)\exp\Big(D\sqrt{\frac{\log x}{\log\log x}}\Big)$$

$$\ll \log x \max_{\kappa\leqslant k<K} B(T_k,x,\chi,q)\exp\Big(D\sqrt{\frac{\log x}{\log\log x}}\Big),$$

so it remains to estimate $B(T_k, x, \chi, q)$.

To simplify the notation, we write now $T_k = T$, $a(T) := (\log\log T)^2$, $b(T) := \frac{\log T}{\log\log T}$ and $\mathcal{V}(V,T) := \{n \in \mathbb{N};\ T \leqslant n < 2T, V_n = V\}$.

We sort the summands corresponding to the values of the V_n:

$$\begin{aligned} B(T,x,\chi,q) &= \sum_{\substack{V\in\mathbb{N}\\ a(T)\leqslant V\leqslant b(T)}} \sum_{\substack{T\leqslant n<2T\\ V_n=V}} \frac{1}{n}\exp\Big(V\log\Big(\frac{\log x}{\log n}\Big) + 2(1+2\delta)V\log\log V\Big) \\ &\leqslant \frac{1}{T} \sum_{\substack{V\in\mathbb{N}\\ a(T)\leqslant V\leqslant b(T)}} \exp\Big(V\log\Big(\frac{\log x}{\log T}\Big) + 2(1+2\delta)V\log\log V\Big)\operatorname{card}\mathcal{V}(V,T). \end{aligned} \tag{11}$$

Now we split the sum over V. For $V \leqslant 2a(T)+1$ we use the trivial estimate

$$\operatorname{card}\{n\in\mathbb{N};\ T\leqslant n<2T,\ V_n=V\} \leqslant T. \tag{12}$$

Then we estimate the corresponding sum for this part:

$$\begin{aligned} \frac{1}{T} \sum_{\substack{V\in\mathbb{N}\\ a(T)\leqslant V\leqslant 2a(T)+1}} \exp\Big(V\log\Big(\frac{\log x}{\log T}\Big) + 2(1+2\delta)V\log\log V\Big)\operatorname{card}\mathcal{V}(V,T) \\ = \exp\Big(\mathrm{O}((\log\log x)^3)\Big). \end{aligned} \tag{13}$$

Now consider $V\in\mathbb{N}$ with $2a(T)+1 < V \leqslant b(T)$, we split

$$\begin{aligned} \mathcal{V}(V,T) &= \{n\equiv 0 \bmod 2;\ n\in\mathcal{V}(V,T)\} \cup \{n\equiv 1 \bmod 2;\ n\in\mathcal{V}(V,T)\} \\ &=: \mathcal{V}_0(V,T)\cup\mathcal{V}_1(V,T). \end{aligned}$$

Consider a number $n\in\mathcal{V}(V,T)$ for a fixed V with $2a(T)+1 < V \leqslant b(T)$. Since $V_n = V$ is the smallest integer such that all $t\in[n,n+1]$ are V_n-typical of order T, there exists at least one $t_n\in[n,n+1]$ being (V_n-1)-untypical of order T.

So choose for any $n\in\mathcal{V}(V,T)$ a $t_n\in[n,n+1]$ being $(V-1)$-untypical. This assignment gives a bijection between $\mathcal{V}(V,T)$ and the set

$$\mathcal{U}(V,T) := \{t_n;\ n\in\mathcal{V}(V,T), t_n\in[n,n+1] \text{ and } t_n \text{ is } (V-1)\text{-untypical}\}$$

of $(V-1)$-untypical ordinates. Hence the cardinalities of both sets are equal, and in $\mathcal{U}(V,T)$ all elements are $(V-1)$-untypical of order T.

Further we define for $h\in\{0,1\}$ the set

$$\mathcal{U}_h(V,T) := \{t_n\in\mathcal{U}(V,T);\ n\in\mathcal{V}_h(V,T)\}.$$

For $t_n\neq t_m$ with $t_n, t_m\in\mathcal{U}_h(V,T)$ we have $|t_n-t_m|\geqslant 1$: If w.l.o.g. $n<m$, then $t_m-t_n\geqslant m-(n+1)\geqslant 1$ since $t_n\in[n,n+1]$, $t_m\in[m,m+1]$ and $n\equiv m \bmod 2$. So the sets $\mathcal{U}_h(V,T)$ are sets of well distanced $(V-1)$-untypical ordinates in the sense of Proposition 10.

Since $\operatorname{card}\mathcal{V}(V,T) = \operatorname{card}\mathcal{U}(V,T) = \operatorname{card}\mathcal{U}_0(V,T) + \operatorname{card}\mathcal{U}_1(V,T)$, we can estimate the cardinality measure of the set $\mathcal{V}(U,T)$ using Proposition 10, we obtain

$$\begin{aligned}\operatorname{card}\mathcal{V}(V,T) \ll T\exp\Big(&-\frac{2}{3}(V-1)\log\Big(\frac{V-1}{\log\log T}\Big)\\ &+\frac{4}{3}(V-1)\log\log(V-1)+\mathrm{O}(V)\Big)\\ &\ll_\delta T\exp\Big(-\frac{2}{3}V\log\Big(\frac{V}{\log\log T}\Big)+\Big(\frac{4}{3}+\delta\Big)V\log\log V\Big).\end{aligned} \tag{14}$$

This leads to the following result:

$$\begin{aligned}B(T,x,\chi,q) \leqslant \exp&\Big(\mathrm{O}((\log\log x)^3)\Big)\\ &+\sum_{\substack{V\in\mathbb{N}\\ 2a(T)+1\leqslant V\leqslant b(T)}}\frac{1}{T}\exp\Big(V\log\Big(\frac{\log x}{\log T}\Big)\\ &+2(1+2\delta)V\log\log V\Big)\operatorname{card}\mathcal{V}(V,T) \qquad \text{by (11) and (13)}\\ \ll_\delta \exp&\Big(\mathrm{O}((\log\log x)^3)\Big)\\ &+\sum_{\substack{V\in\mathbb{N}\\ 2a(T)+1\leqslant V\leqslant b(T)}}\exp\Big(V\log\Big(\frac{\log x\,(\log\log T)^{2/3}}{\log T}\Big)\\ &-\frac{2}{3}V\log V+\Big(\frac{10}{3}+5\delta\Big)V\log\log V\Big) \qquad (15)\\ \ll_\delta \exp&\Big(\mathrm{O}((\log\log x)^3)\Big)\\ &+\sum_{\substack{V\in\mathbb{N}\\ 2a(T)+1\leqslant V\leqslant b(T)}}\exp\Big(V\log\Big(\frac{\log x\ \log\log T}{\log T}\Big)\\ &-\frac{2}{3}V\log V+\Big(\frac{10}{3}+5\delta\Big)V\log\log V\Big), \qquad (16)\end{aligned}$$

where in (15) the implicit constant in the estimate depends on δ since we used equation (14).

In order to majorize the last sum (16), we use Proposition 20 with the following parameters:

$$A := \log\Big(\frac{\log x\ \log\log T}{\log T}\Big) \qquad \text{and} \qquad C := \frac{10}{3}+5\delta.$$

(Then $A \geqslant 4C^4+1$ and $V > e^{3C/2}$ hold if x is large enough.)

We obtain

$$\sum_{\substack{V\in\mathbb{N}\\ 2a(T)+1\leqslant V\leqslant b(T)}} \exp\Big(V\log\Big(\frac{\log x\ \log\log T}{\log T}\Big)-\frac{2}{3}V\log V+\Big(\frac{10}{3}+5\delta\Big)V\log\log V\Big)$$
$$\leqslant\ \frac{\log T}{\log\log T}\exp\Big(\Big(\log x\frac{\log\log T}{\log T}\Big)^{3/2}\Big(\frac{3}{2}\log\Big(\log x\frac{\log\log T}{\log T}\Big)\Big)^{5+15\delta/2}\Big). \tag{17}$$

Since

$$\frac{\log\log T}{\log T}=\frac{\log\log T_k}{\log T_k}\leqslant\frac{\log\log T_\kappa}{\log T_\kappa}\ll\frac{\log\log x}{(\log x)^{3/5}(\log\log x)^c}\leqslant(\log x)^{-3/5},$$

we have

$$\Big(\log x\frac{\log\log T}{\log T}\Big)^{3/2}\leqslant\Big((\log x)^{2/5}(\log\log x)^{1-c}\Big)^{3/2}=(\log x)^{3/5}(\log\log x)^{3/2-3c/2},$$

and as $c\geqslant 1$, we obtain further

$$\Big(\frac{3}{2}\log\Big(\log x\frac{\log\log T}{\log T}\Big)\Big)^{5+15\delta/2}\leqslant(\log\log x)^{5+15\delta/2}.$$

Using these estimates, we continue the estimation of (17) with

$$\begin{aligned}
&\leqslant\exp\Big(\log\log x+(\log x)^{3/5}(\log\log x)^{3/2-3c/2+5+15\delta/2}\Big)\\
&=\exp\Big((\log x)^{3/5}(\log\log x)^{13/2-3c/2+15\delta/2}+\log\log x\Big)\\
&\ll_\delta\exp\Big((\log x)^{3/5}(\log\log x)^{13/2-3c/2+8\delta}\Big).
\end{aligned}$$

Now we resume everything including the term $\exp\Big(D\sqrt{\frac{\log x}{\log\log x}}\Big)$ again, we obtain

$$B(x,\chi,q)\ll_\delta\exp\Big((\log x)^{3/5}(\log\log x)^{13/2-3c/2+8\delta}\Big)\exp\Big((D+1)\sqrt{\frac{\log x}{\log\log x}}\Big),$$

and using the estimate

$$\begin{aligned}
&(\log x)^{3/5}(\log\log x)^{13/2-3c/2+8\delta}+(D+1)\sqrt{\frac{\log x}{\log\log x}}\\
&\qquad\ll(\log x)^{3/5}(\log\log x)^{13/2-3c/2+8\delta}\Big(1+\log(x)^{-1/10}(\log\log x)^{3c/2}\Big)\\
&\qquad\ll(\log x)^{\frac{3}{5}}(\log\log x)^{13/2-3c/2+8\delta},
\end{aligned}$$

we obtain finally

$$B(x,\chi,q)\ll_\delta\exp\Big((\log x)^{3/5}(\log\log x)^{13/2-3c/2+8\delta}\Big). \qquad\blacksquare$$

Now we still have to consider the principal character mod q, for this we use the result of the zeta-function.

Lemma 3. *Let $q \in \mathbb{N}$, $x \geqslant q > 1$, then we have for the principal character χ_0 mod q the estimate*

$$A(x, \chi_0, q) \ll_\delta \sqrt{x} \exp\Big((\log x)^{1/2} (\log\log x)^{5/2+4\delta}\Big).$$

Proof. Due to the formula

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \Big(1 - \frac{1}{p^s}\Big),$$

we use the estimate for the zeta-integral. So we estimate the product $\Big|\prod_{p|q}(1-p^{-s})^{-1}\Big|$ for $\sigma \geqslant \frac{1}{2}$.

For this, consider the logarithm of the product and include the series expansion of the logarithm:

$$\begin{aligned}
\Big|\sum_{p|q} -\log\Big(1-\frac{1}{p^s}\Big)\Big| &= \Big|\sum_{p|q} -\sum_{k\in\mathbb{N}} (-1)^{k+1} \frac{(-p^{-s})^k}{k}\Big| = \Big|\sum_{p|q}\sum_{k\in\mathbb{N}} (-1)^{2k+2} \frac{1}{kp^{ks}}\Big| \\
&\leqslant \sum_{p|q}\sum_{k\in\mathbb{N}} \frac{1}{kp^{k/2}} = \sum_{p|q} \frac{1}{p^{1/2}} + \frac{1}{2}\sum_{p|q}\frac{1}{p} + \sum_{p|q}\sum_{k>2} \frac{1}{kp^{k/2}} \\
&\leqslant \sum_{i=1}^{2\log q} \frac{1}{p_i^{1/2}} + \frac{1}{2}\sum_{p\leqslant q}\frac{1}{p} + \mathrm{O}(1) \\
&\ll \sqrt{\frac{\log q}{\log\log q}} + \log\log q + \mathrm{O}(1).
\end{aligned}$$

We conclude

$$|L(s,\chi_0)|^{-1} \ll |\zeta(s)|^{-1} \exp\left(D\sqrt{\frac{\log q}{\log\log q}}\right)$$

for an absolute constant $D > 0$.

Since $\sqrt{\frac{\log q}{\log\log q}}$ is monotonic increasing in q, we have for $x \geqslant q$

$$L(s,\chi_0)^{-1} \ll \zeta(s)^{-1} \exp\left(D\sqrt{\frac{\log x}{\log\log x}}\right).$$

Now the additional term $\sqrt{\frac{\log x}{\log\log x}}$ does not disturb the magnitude of the exponent in the final result, since we have

$$\begin{aligned}\left|\int\limits_{S(x,\chi,q)} L(z,\chi_0)^{-1}\frac{x^z}{z}dz\right| &\ll \int\limits_{S(x,\chi,q)} \left|\zeta(z)^{-1}\frac{x^z}{z}\right| dz \exp\left(D\sqrt{\frac{\log x}{\log\log x}}\right)\\ &\ll_\delta \sqrt{x}\exp\left((\log x)^{1/2}(\log\log x)^{5/2+4\delta} + D\sqrt{\frac{\log x}{\log\log x}}\right)\\ &\ll \sqrt{x}\exp\left((\log x)^{1/2}(\log\log x)^{5/2+4\delta}\right),\end{aligned}$$

where we have set $c = \frac{5}{2} + 3\delta$ in the estimate at the end of the paper of [5]. ■

Proof of Theorem 1. Let $q > 2$, since for $q = 2$ there is only the principal character and we can use then the sharper result from Lemma 3.

We use equation (10), Lemma 1 and Lemma 2 and set $c = \frac{11}{5} + \frac{16}{5}\delta$, together with Lemma 3 we obtain

$$\begin{aligned}\left|M(x,a,q)\right| &\leqslant \frac{1}{\varphi(q)}\sum_{\chi(q)}\left|\sum_{n\leqslant x}\chi(n)\mu(n)\right| = \frac{1}{\varphi(q)}\sum_{\chi(q)}\left|A(x,\chi,q)\right| + \mathrm{O}(\log x)\\ &= \frac{1}{\varphi(q)}|A(x,\chi_0,q)| + \frac{1}{\varphi(q)}\sum_{\substack{\chi(q)\\ \chi\neq\chi_0}}\left|A(x,\chi,q)\right| + \mathrm{O}(\log x)\\ &\ll_\delta \frac{1}{\varphi(q)}\sqrt{x}\exp\left((\log x)^{1/2}(\log\log x)^{5/2+4\delta}\right)\\ &+ \frac{\varphi(q)-1}{\varphi(q)}\sqrt{x}\exp\left((\log x)^{3/5}(\log\log x)^{16/5+16\delta/5}\right)\\ &\ll \sqrt{x}\exp\left((\log x)^{3/5}(\log\log x)^{16/5+16\delta/5}\right).\end{aligned}$$

Since $\delta \in (0,1]$ can be chosen arbitrary, we get the assertion with the choice $\delta = \frac{5}{16}\varepsilon$. ■

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Addresses: Karin Halupczok: Westfälische Wilhelms-Universität Münster, Mathematisches Institut, Einsteinstr. 62, D-48149 Münster, Germany;
Benjamin Suger: Albert-Ludwigs-Universität Freiburg, Institut für Informatik, Georges--Köhler-Allee 79, D-79110 Freiburg, Germany.

E-mail: karin.halupczok@uni-muenster.de, suger@informatik.uni-freiburg.de

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