

## FIXED POINT IN CAT(0) SPACES

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**Abstract:** We obtained sufficient conditions for existence of fixed points of involutions in CAT(0) spaces. Convergence results of Mann and Ishikawa iterates of weakly contractive mappings are also proved.

**Keywords:** fixed point, Lipschitzian mapping, involution, weakly contractive mappings.

### 1. Introduction and preliminaries

The interplay between the geometry of Banach spaces and fixed point theory has been very strong and fruitful. In particular, geometric properties play a key role in metric fixed point problems, see for example [11, 12], and references therein. The parallelogram law is one of fundamental property of Hilbert spaces which distinguishes them from general Banach spaces. This law is used in solving many problems in Hilbert spaces. Recently several authors have tried this idea for solving problems in Banach spaces by establishing equalities and usually inequalities analogous to the parallelogram law, see for example Goebel and Kirk [10]. Beg [2] established some inequalities in uniformly convex metric spaces analogous to the parallelogram law in Hilbert spaces and obtained some fixed point theorem. Gromov [15] introduced the notion of CAT(0) spaces. For application of these spaces in various branches of mathematics and for a vigorous discussion on these spaces, we refer to Bridson and Haefliger [5], Burago-Burago-Ivanov [7] and Beg and Abbas [3]. The results obtained in this direction were the starting point for a new mathematical field: the application of geometric theory of Banach spaces to fixed point theory. The aim of this paper is to use Bruhat and Tits [6, CN inequality] results to obtain fixed point of involutions. We also study convergence problem of Mann and Ishikawa iterates of weakly contractive mapping in CAT(0) spaces.

First we recall some basics. Let  $(X, d)$  be a metric space. A geodesic path joining  $x \in X$  to  $y \in X$  (or, more briefly, a geodesic from  $x$  to  $y$ ) is a map  $c$  from

a closed interval  $[0, l] \subset \mathbb{R}$  to  $X$  such that  $c(0) = x, c(l) = y$ , and  $d(c(t), c(t')) = |t - t'|$  for all  $t, t' \in [0, l]$ . In particular,  $c$  is an isometry and  $d(x, y) = l$ . The image of  $c$  is called a geodesic (or metric) segment joining  $x$  and  $y$ . When it is unique this geodesic segment is denoted by  $[x, y]$ . The space  $(X, d)$  is said to be a geodesic space if every two points of  $X$  are joined by a geodesic, and  $X$  is said to be uniquely geodesic if there is exactly one geodesic joining  $x$  and  $y$  for each  $x, y \in X$ . A subset  $Y \subseteq X$  is said to be convex if  $Y$  includes every geodesic segment joining any two of its points. A geodesic triangle  $\triangle(x_1, x_2, x_3)$  in a geodesic metric space  $(X, d)$  consists of three points  $x_1, x_2, x_3$  in  $X$  (the vertices of  $\triangle$ ) and a geodesic segment between each pair of vertices (the edges of  $\triangle$ ). A comparison triangle for the geodesic triangle  $\triangle(x_1, x_2, x_3)$  in  $(X, d)$  is a triangle  $\triangle(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$  in the Euclidean plane  $\mathbb{E}^2$  such that  $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$  for  $i, j \in \{1, 2, 3\}$ . A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): Let  $\triangle$  be a geodesic triangle in  $X$  and let  $\bar{\triangle}$  be a comparison triangle for  $\triangle$ . Then  $\triangle$  is said to satisfy the CAT(0) inequality if for all  $x, y \in \triangle$  and all comparison points  $\bar{x}, \bar{y} \in \bar{\triangle}$ ,

$$d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

If  $x, y_1, y_2$  are points in a CAT(0) space and if  $y_0$  is the midpoint of the segment  $[y_1, y_2]$ , then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2 \quad (\text{CN})$$

This is the (CN) inequality of Bruhat and Tits [6]. In fact (cf. [5, p. 163]), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

A metric space  $X$  is called a CAT(0) space [15] if it is geodesically connected and if every geodesic triangle in  $X$  is at least as "thin" as its comparison triangle in Euclidean plane. The complex Hilbert ball with a hyperbolic metric is a CAT(0) space, see [11, 19].

Following are some elementary facts about CAT(0) spaces, see Dhompongsa and Panyanak [8].

**Lemma 1.1.** *Let  $(X, d)$  be a CAT(0) space. Then*

- (i)  *$(X, d)$  is uniquely geodesic (see [5, pp.160]).*
- (ii) *Let  $p, x, y$  be points of  $X$ , let  $\alpha \in [0, 1]$ , and let  $m_1$  and  $m_2$  denote, respectively, the points of  $[p, x]$  and  $[p, y]$  satisfying  $d(p, m_1) = \alpha d(p, x)$  and  $d(p, m_2) = \alpha d(p, y)$ . Then*

$$d(m_1, m_2) \leq \alpha d(x, y)$$

*(see [16, Lemma 3]).*

- (iii) Let  $x, y \in X, x \neq y$  and  $z, w \in [x, y]$  such that  $d(x, z) = d(x, w)$ . Then  $z = w$ .
- (iv) Let  $x, y \in X$ . For each  $t \in [0, 1]$ , there exists a unique point  $z \in [x, y]$  such that

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y) \quad (1.1)$$

For convenience, from now on we will use the notation  $(1 - t)x \oplus ty$  for the unique point  $z$  satisfying (1.1).

**Definition 1.2.** Let  $X$  be a nonempty subset of a metric space  $X$ . A mapping  $T : X \rightarrow X$  is called  $k$ -Lipschitzian if for all  $x, y$  in  $X$ ,

$$d(Tx, Ty) \leq kd(x, y).$$

A mapping  $T : X \rightarrow X$  is called an involution if  $T^2 = I$ , where  $I$  denotes the identity map (see [12, 13]).

## 2. Fixed point of involutions

Let  $X$  be a complete CAT(0) space. Let  $T : X \rightarrow X$  be a mapping. For  $x_0 \in X$ , we define,

$$x_{n+1} = \frac{1}{2}x_n \oplus \frac{1}{2}Tx_n \quad (2.1)$$

If there exists a  $c, 0 \leq c < 1$  such that

$$d(x_{n+2}, x_{n+1}) \leq cd(x_{n+1}, x_n) \quad (2.2)$$

( $n = 0, 1, 2, \dots$ ). Then the sequence  $\{x_n\}$  converges in  $X$ . Indeed, from (2.2), it follows that  $d(x_{n+1}, x_n) \leq c^n d(x_1, x_0)$  and  $\{x_n\}$  converges to a point  $p$  (say) in  $X$ .

**Theorem 2.1.** Let  $X$  be a complete CAT(0) space. Let  $T : X \rightarrow X$  be a  $k$ -Lipschitzian involution. If  $1 \leq k < 2$  then  $T$  has a fixed point in  $X$ .

**Proof.** For arbitrary  $x_0 \in X$ , define inductively a sequence  $\{x_n\} \subset X$  by

$$x_{n+1} = \frac{1}{2}x_n \oplus \frac{1}{2}Tx_n,$$

for  $n = 0, 1, 2, \dots$ .

Now,

$$\begin{aligned} d(x_{n+1}, x_n) &= d\left(\frac{1}{2}x_n \oplus \frac{1}{2}Tx_n, x_n\right) \\ &\leq \frac{1}{2}d(Tx_n, x_n), \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, Tx_{n+1}) &= d\left(\frac{1}{2}x_n \oplus \frac{1}{2}Tx_n, T\left(\frac{1}{2}x_n \oplus \frac{1}{2}Tx_n\right)\right) \\ &\leq \frac{1}{2}\left[d\left(x_n, T\left(\frac{1}{2}x_n \oplus \frac{1}{2}Tx_n\right)\right) + d\left(Tx_n, T\left(\frac{1}{2}x_n \oplus \frac{1}{2}Tx_n\right)\right)\right] \\ &= \frac{1}{2}\left[d\left(T^2x_n, T\left(\frac{1}{2}x_n \oplus \frac{1}{2}Tx_n\right)\right) \right. \\ &\quad \left. + d\left(Tx_n, T\left(\frac{1}{2}x_n \oplus \frac{1}{2}Tx_n\right)\right)\right] \\ &\leq \frac{k}{2}\left[d\left(Tx_n, \frac{1}{2}x_n \oplus \frac{1}{2}Tx_n\right) + d\left(x_n, \frac{1}{2}x_n \oplus \frac{1}{2}Tx_n\right)\right] \\ &\leq \frac{k}{2}d(Tx_n, x_n). \end{aligned}$$

Where by assumption  $\frac{k}{2} < 1$ . It implies that

$$d(x_{n+1}, x_n) \leq \frac{1}{2}\left(\frac{k}{2}\right)^n d(Tx_0, x_0).$$

If further implies that  $\{x_n\}$  is a convergent sequence. Let  $\lim_{n \rightarrow \infty} x_n = x$  (say). Then,

$$\begin{aligned} d(Tx, x) &\leq d(Tx, Tx_n) + d(Tx_n, x_n) + d(x_n, x) \\ &\leq (1+k)d(x, x_n) + \left(\frac{k}{2}\right)^n d(Tx_0, x_0) \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \end{aligned}$$

Hence  $Tx = x$ . ■

**Remark 2.2.** As an immediate corollary to Theorem 2.1 we have Goebel [9].

**Theorem 2.3.** *Let  $X$  be a complete  $CAT(0)$ . Let  $T : X \longrightarrow X$  be a  $k$ -Lipschitzian involution. If  $0 \leq k < \sqrt{5}$ , then  $T$  has a fixed point.*

**Proof.** For any  $x \in X$ , let  $u = \frac{1}{2}x \oplus \frac{1}{2}Tx$ . Then, using [8, Lemma 2.5], we obtain

$$\begin{aligned}
 [d(u, Tu)]^2 &= \left[ d\left(Tu, \frac{1}{2}x \oplus \frac{1}{2}Tx\right) \right]^2 \\
 &\leq \frac{1}{2} \left( [d(Tu, x)]^2 + [d(Tu, Tx)]^2 - \frac{1}{2} [d(x, Tx)]^2 \right) \\
 &= \frac{1}{2} \left( [d(Tu, T^2x)]^2 + [d(Tu, Tx)]^2 - \frac{1}{2} [d(x, Tx)]^2 \right) \\
 &\leq \frac{1}{2} \left( [kd(Tx, u)]^2 + [kd(u, x)]^2 - \frac{1}{2} [d(x, Tx)]^2 \right) \\
 &= \frac{1}{2} \left( \left[ kd\left(Tx, \frac{1}{2}x \oplus \frac{1}{2}Tx\right) \right]^2 \right. \\
 &\quad \left. + \left[ kd\left(\frac{1}{2}x \oplus \frac{1}{2}Tx, x\right) \right]^2 - \frac{1}{2} [d(x, Tx)]^2 \right) \\
 &\leq \left( \frac{k^2 - 1}{4} \right) [d(x, Tx)]^2,
 \end{aligned}$$

where by assumption  $\frac{k^2 - 1}{4} < 1$ . For arbitrary  $x_0 \in X$ , defining inductively a sequence  $\{x_n\} \subset X$  by  $x_{n+1} = \frac{1}{2}x_n \oplus \frac{1}{2}Tx_n$ ,  $n = 0, 1, 2, \dots$ . Theorem 2.1 implies that this sequence is convergent, let  $\lim_{n \rightarrow \infty} x_n = x$ . Then  $Tx = x$ . ■

**Theorem 2.4.** *Let  $X$  be a CAT(0) space. If  $T : X \rightarrow X$  satisfies for every  $x, y$  in  $X$ ,*

$$(i) \quad d(T^2x, T^2y) \leq d(x, y),$$

and

$$(ii) \quad d(Tx, Ty) \leq k d(x, y),$$

with  $0 \leq k < \sqrt{5}$ , then  $T$  has a fixed point in  $X$ .

**Proof.** By Theorem 2.3,  $K_1 = \{x \in X : T^2x = x\}$  is nonempty and closed. Also  $K_1$  is convex. To prove this fact let  $x_1, x_2 \in K_1$  and  $t \in (0, 1)$ , then we have

$$\begin{aligned}
 d(x_1, x_2) &\leq d(x_1, T^2((1-t)x_1 \oplus tx_2)) + d(T^2((1-t)x_1 \oplus tx_2), x_2) \\
 &= d(T^2x_1, T^2((1-t)x_1 \oplus tx_2)) + d(T^2((1-t)x_1 \oplus tx_2), T^2x_2) \\
 &\leq d(x_1, (1-t)x_1 \oplus tx_2) + d((1-t)x_1 \oplus tx_2, x_2) \\
 &\leq td(x_1, x_2) + (1-t)d(x_1, x_2) = d(x_1, x_2).
 \end{aligned}$$

It implies that

$$\begin{aligned}
 d(x_1, T^2((1-t)x_1 \oplus tx_2)) &+ d(T^2((1-t)x_1 \oplus tx_2), x_2) \\
 &= d(x_1, ((1-t)x_1 \oplus tx_2)) + d((1-t)x_1 \oplus tx_2, x_2) = d(x_1, x_2).
 \end{aligned}$$

Since

$$d(T^2x, T^2y) \leq d(x, y), T^2x_1 = x_1 \quad \text{and} \quad T^2x_2 = x_2,$$

therefore

$$d(x_1, T^2((1-t)x_1 \oplus tx_2)) = d(x_1, (1-t)x_1 \oplus tx_2),$$

which implies that

$$T^2((1-t)x_1 \oplus tx_2) = (1-t)x_1 \oplus tx_2.$$

It further implies that  $K_1$  is convex. Moreover  $T(K_1) = K_1$  and  $T^2 = I$  on  $K_1$ . Hence Theorem 2.3 implies that  $T$  has a fixed point in  $X$ . ■

**Remark 2.5.** Gornicki and Pupka [14] obtained some fixed point results for Lipschitzian 2-rotative mappings in framework of complete metric spaces of hyperbolic type. The Lipschitzian constant  $k$  in Theorems 2.1 and 2.3 can be viewed as an improved estimate of its counterpart appeared in Theorem 3 of [14].

### 3. Convergence results for weakly contractive mappings

The concept of  $\Delta$ -convergence in general metric spaces was coined by Lim [18]. Kirk and Panyanak [17] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Dhompongsa and Panyanak [8] continued to work in this direction. Their results involved Mann and Ishikawa iteration schemes involving nonexpansive mapping. In this section, we establish different iterative schemes for weakly contractive maps and it is proved that if the iterative sequence is convergent, it will converge to the fixed point of the map defined in the frame work of CAT(0) spaces.

**Definition 3.1 ([1]).** Let  $X$  be a complete metric space. A mapping  $T : X \longrightarrow X$  is said to be weakly contractive if for each  $x, y \in X$ , we have,

$$d(T(x), T(y)) \leq d(x, y) - \Psi(d(x, y)),$$

where  $\Psi : [0, \infty) \longrightarrow [0, \infty)$  is a continuous and non decreasing such that  $\Psi$  is positive on  $(0, \infty)$ ,  $\Psi(0) = 0$ , and  $\lim_{t \rightarrow \infty} \Psi(t) = \infty$ .

**Remark 3.2.** If  $\Psi(t) = (1-k)t$  for a constant  $k \in (0, 1)$ , then the weakly contractive map becomes a contraction mapping and it has a unique fixed point by Banach contraction principle. Weakly contractive maps lie between those which satisfy Banach contraction principle and contractive maps. Weakly contractive maps also satisfy the definition of Boyd and Wong [4]. It is also known [20] that a complete metric space has a fixed point property for weakly contractive maps. Moreover the fixed point in this case is also unique.

**Theorem 3.3.** Let  $X$  be a complete CAT(0) space  $X$  and  $T$  be a weakly contractive self map on  $X$ . Let  $x_0 \in X$ , define,  $x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_nTx_n$ ,  $n \geq 0$ , where  $0 \leq \alpha_n \leq 1$  and  $\sum \alpha_n = \infty$ . Then,  $\lim_{n \rightarrow \infty} d(x_n, p) = 0$ , where  $p$  is the unique fixed point of  $T$ .

**Proof.** The existence of unique fixed point of  $T$  follows from [20]. Consider,

$$\begin{aligned}
 d(x_{n+1}, p) &= d((1 - \alpha_n)x_n \oplus \alpha_n T x_n, p) \\
 &\leq \alpha_n d(x_n, p) + (1 - \alpha_n) d(T(x_n), T(p)) \\
 &\leq \alpha_n d(x_n, p) + (1 - \alpha_n) [d(x_n, p) - \Psi(d(x_n, p))] \\
 &\leq d(x_n, p).
 \end{aligned}$$

So,  $\{d(x_n, p)\}$  is a non-negative non-increasing sequence of real numbers which converges to the real number  $q$ . Suppose  $q > 0$ . Obviously,  $q \leq d(x_n, p)$ . Now, for any fixed positive integer  $n_0$  we have,

$$\sum_{n=n_0}^{\infty} \alpha_n \Psi(q) \leq \sum_{n=n_0}^{\infty} \alpha_n \Psi(d(x_n, p)) \leq \sum_{n=n_0}^{\infty} [d(x_n, p) - d(x_{n+1}, p)] \leq d(x_{n_0}, p).$$

Which contradicts  $\sum \alpha_n = \infty$ . Hence the result follows. ■

**Theorem 3.4.** *Let  $X$  be a complete metric space  $X$  and  $T$  be a weakly contractive self map on  $X$ . Let  $x_0 \in X$ , define,*

$$\begin{aligned}
 x_{n+1} &= (1 - \alpha_n)x_n \oplus \alpha_n T y_n, & \text{and} \\
 y_n &= (1 - \beta_n)x_n \oplus \beta_n T x_n, & n \geq 0
 \end{aligned}$$

where  $0 \leq \alpha_n, \beta_n \leq 1$  for all  $n \in N$ , and  $\sum \alpha_n \beta_n = \infty$ . Then  $\lim_{n \rightarrow \infty} d(x_n, p) = 0$ , where  $p$  is the unique fixed point of  $T$ .

**Proof.** The existence of unique fixed point  $p$  (say) of  $T$  follows from [20]. Consider

$$\begin{aligned}
 d(x_{n+1}, p) &= d((1 - \alpha_n)x_n \oplus \alpha_n T y_n, p) \\
 &\leq \alpha_n d(x_n, p) + (1 - \alpha_n) d(T(y_n), T(p)) \\
 &\leq \alpha_n d(x_n, p) + (1 - \alpha_n) [d(y_n, p) - \Psi(d(y_n, p))] \\
 &\leq \alpha_n d(x_n, p) + (1 - \alpha_n) [d((1 - \beta_n)x_n \oplus \beta_n T x_n, p) \\
 &\quad - (1 - \alpha_n) \Psi(d((1 - \beta_n)x_n \oplus \beta_n T x_n, p))] \\
 &\leq \alpha_n d(x_n, p) + (1 - \alpha_n) [\beta_n d(x_n, p) + (1 - \beta_n) d(T(x_n), T(p))] \\
 &\quad - (1 - \alpha_n) \Psi(d(y_n, p)) \\
 &\leq \alpha_n d(x_n, p) + (1 - \alpha_n) \beta_n d(x_n, p) + (1 - \alpha_n)(1 - \beta_n) d(x_n, p) \\
 &\quad - (1 - \alpha_n)(1 - \beta_n) \Psi(d(x_n, p)) - (1 - \alpha_n) \Psi(d(y_n, p)) \\
 &\leq d(x_n, p) - (1 - \alpha_n)(1 - \beta_n) \Psi(d(x_n, p)) - (1 - \alpha_n) \Psi(d(y_n, p)) \\
 &\leq d(x_n, p).
 \end{aligned}$$

So,  $\{d(x_n, p)\}$  is a non-negative non-increasing sequence of real numbers which converges to the real number  $q$ . Suppose  $q > 0$ . Obviously,  $q \leq d(x_n, p)$ . Now, for

any fixed positive integer  $n_0$  we have,

$$\begin{aligned} \sum_{n=n_0}^{\infty} (1 - \alpha_n)(1 - \beta_n)\Psi(q) &\leq \sum_{n=n_0}^{\infty} (1 - \alpha_n)(1 - \beta_n)\Psi(d(x_n, p)) \\ &\leq \sum_{n=n_0}^{\infty} d(x_n, p) - d(x_{n+1}, p) \\ &\leq d(x_{n_0}, p). \end{aligned}$$

Which contradicts that  $\sum \alpha_n \beta_n = \infty$ , hence, the result follows. ■

**Acknowledgements.** Authors are grateful to the editor and referees for their valuable suggestions and critical remarks for improving this paper. We also acknowledge with thanks Higher Education Commission of Pakistan research grant: 20-918/R&D/07.

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**Received:** 13 October 2010; **revised:** 1 June 2012