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A PNT EQUIVALENCE FOR BEURLING NUMBERS

Harold G. Diamond, Wen-Bin Zhang

Abstract: In classical prime number theory, several relations are considered to be equivalent to the Prime Number Theorem. For Beurling generalized numbers, some auxiliary conditions may be needed to deduce one relation from another one. We show conditions under which the Beurling analog of the sharp version of Mertens' sum formula does or does not hold.

Keywords: Beurling generalized numbers, sharp Mertens formula, Prime Number Theorem, equivalent relations.

1. Introduction

Several asymptotic formulas of classical prime number theory are considered to be "equivalent" to the Prime Number Theorem (PNT) in the sense that they are deducible from one another by quite simple real variable arguments ([2], §5.2; [5], §8.1). We use this phrase as a convenient grouping of results, recognizing that after the discovery of elementary proofs of the PNT, it does not have a logical basis. We are going to examine one of these relations in the context of Beurling generalized (g-) numbers.

A Beurling g-prime system $\mathcal{P}$ is an unbounded real sequence $p_1 \leqslant p_2 \leqslant \dots$ with $p_1 > 1$. The multiplicative semigroup $\mathcal{N}$ generated by $\mathcal{P}$ and 1 is called the associated sequence of g-integers. We do not assume $\mathcal{N} \subset \mathbb{Z}$ or that unique prime factorization holds. If a g-integer occurs in more than one way, we count it with appropriate multiplicity.

As in the classical case, we have counting functions

$$\begin{aligned} N(x) &= N_{\mathcal{P}}(x) = \#\{\mathcal{N} \cap [1, x]\} \\ \pi(x) &= \pi_{\mathcal{P}}(x) = \#\{\mathcal{P} \cap [1, x]\} \\ \Pi(x) &= \Pi_{\mathcal{P}}(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \dots \end{aligned}$$

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$$\psi(x) = \psi_{\mathcal{P}}(x) = \int_1^x \log u \, d\Pi(u).$$

Our goal is to show conditions under which the analog of the "sharp Mertens relation"

$$\int_1^x \frac{d\psi(t)}{t} = \sum_{n \leqslant x} \frac{\Lambda(n)}{n} = \log x - \gamma + o(1)$$

($\gamma =$ Euler's constant) does or does not hold for Beurling g-numbers.

2. Necessary conditions for sharp Mertens

The g-number formula we are investigating is

$$\psi_1(x) := \int_1^x \frac{d\psi(t)}{t} = \log x + c_1 + o(1). \tag{2.1}$$

In general, $c_1 \neq -\gamma$, as we can see by making a g-number system containing all the classical primes along with one additional g-prime. In such a system, (2.1) holds with $c_1 > -\gamma$.

One relation between (2.1) and the PNT is true unconditionally:

Proposition 2.1. *If (2.1) holds for a g-number system, then so does the PNT.*

Proof. We verify the PNT in the form $\psi(x) \sim x$ by integration by parts:

$$\begin{aligned}\psi(x) &= \int_1^x t \, d\psi_1(t) = x\psi_1(x) - \int_1^x \psi_1(t) \, dt \\ &= x \log x + c_1 x + o(x) - \int_1^x \{\log t + c_1 + o(1)\} \, dt \\ &= x + o(x).\end{aligned}$$

■

Next we show that in fact (2.1) implies somewhat more than the PNT.

Proposition 2.2. *Suppose that a sharp Mertens-type relation holds for a g-number system. Then $f(x) := \psi(x) - x = o(x)$ and moreover*

$$\int_1^x f(t) t^{-2} \, dt$$

converges to a finite limit as $x \to \infty$.

Proof. By Proposition 2.1, the PNT holds and thus $f(x) = o(x)$.

Again apply integration by parts, this time to $\int d\psi(t)/t$, to get

$$\begin{aligned}\psi_1(x) &= \frac{\psi(x)}{x} + \int_1^x \frac{\psi(t)}{t^2} dt \\ &= 1 + o(1) + \log x + \int_1^x f(t) t^{-2} \, dt.\end{aligned}$$

Since $\psi_1(x) - \log x$ has a limit as $x \to \infty$, so does the integral. ■

The last proposition shows that proof of (2.1) requires more than the PNT alone. Note for later use that divergence of $\int_1^\infty f(t)t^{-2}\,dt$ could occur for f too large, for example $f(x) \gg x/\log x$, or for certain oscillatory functions.

3. Sufficient conditions for sharp Mertens

Theorem 3.1. *Suppose that the PNT holds and that for some $c > 0$*

$$|N(x) - cx| \leqslant xD(x), \qquad x \geqslant 1, \tag{3.1}$$

where D is right continous, monotone decreasing, and satisfies

$$\int_1^\infty x^{-1}D(x)\,dx < \infty. \tag{3.2}$$

Then (2.1) *holds as well.*

Here are two interesting special cases of the theorem.

Corollary 3.1. *Suppose that the hypotheses of the theorem are satisfied with*

$$D(x) := \max_{y \geqslant x} \frac{|N(y) - cy|}{y}.$$

Then (2.1) *holds.*

Corollary 3.2. *Suppose that the hypotheses of the theorem are satisfied with $D(x) := C\log^{-\gamma}(x+1)$ with $C > 0$ and $\gamma > 1$. Then* (2.1) *holds.*

The theorem is proved using the following variant of Axer's Theorem ([1], Lemma 5.7; [5], Theorem 8.1). A sequences formulation of this result appears as Exercise 7 in §8.1.1 of [5].

Lemma 3.1. *Let A and B be right continuous functions supported on $[1, \infty)$ that are locally of bounded variation. Suppose that $|A(t)| \leqslant tD(t)$ where $D\!\downarrow$ and $\int_1^\infty t^{-1}D(t)\,dt < \infty$. Also, assume that $B(x) = o(x)$ and its variation function satisfies $B_v(x) = O(x)$. Then, with $*$ denoting multiplicative convolution, $\int_{1-}^x dA * dB = o(x)$.*

Sketch of a proof of the lemma. We first deduce $D(x) = o(\log^{-1} x)$ from the integrability of $t^{-1}D(t)$.

Then we use the elementary approach to Dirichlet's divisor problem to write

$$\begin{aligned}\int_{1-}^x dA * dB &= \int_{1-}^{x/M} A\Big(\frac{x}{t}\Big)\,dB(t) + \int_{1-}^M B\left(\frac{x}{t}\right)\,dA(t) - A(M)B\left(\frac{x}{M}\right) \\ &= J_1 + J_2 - J_3, \text{ say.}\end{aligned}$$

To estimate $|J_1|$, we note that

$$
\begin{aligned}
|J_1| &\leqslant \int_{1-}^{x/M} \frac{x}{t} D\left(\frac{x}{t}\right) dB_v(t) \\
&\leqslant B_v\left(\frac{x}{M}\right) MD(M) + x \int_{1-}^{x/M} B_v(t) t^{-2} D\left(\frac{x}{t}\right) dt
\end{aligned}
$$

since $D(x/t)$ is an increasing function of t. Hence $|J_1|$ is an arbitrarily small multiple of x, provided that M is taken sufficiently large.

With M fixed, the o-condition for B insures that $|B(x/t)|$ is an arbitrarily small multiple of x/t for $t \leqslant M$, provided that x is sufficiently large. Also, the integration extends over a fixed interval. We find

$$
|J_2| \leqslant \int_{1-}^{M} \left|B\left(\frac{x}{t}\right)\right| dA_v(t) = o(x).
$$

Finally, the bounds found for $|A|$ and $|B|$ guarantee that $|J_3|$ also is $o(x)$. Together, the three estimates yield $\int_{1-}^{x} dA * dB = o(x)$. ∎

Proof of the theorem. We start with the Chebyshev identity ([4], Theorem 2.6)

$$
\int_1^x \log t \, dN(t) = \int_1^x dN * d\psi
$$

for g-numbers. The left side, integrated by parts, becomes

$$
N(x)\log x - \int_1^x \frac{N(y)}{y} dy = cx\log x - cx + o(x), \tag{3.3}
$$

since

$$
\int_1^x y^{-1} N(y)\, dy = c(x-1) + \theta \int_1^x D(y)\, dy,
$$

where $|\theta| \leqslant 1$ and the last integral is $o(x/\log x)$.

For dA a measure on $[1, \infty)$ and δ_1 Dirac point mass at 1, we have

$$
\begin{aligned}
\int_{1-}^{x} (\delta_1 + du) * dA &= \iint_{st \leqslant x} (\delta_1 + ds)\, dA(t) \\
&= \int_{1-}^{x} \int_{1-}^{x/t} (\delta_1 + ds) dA(t) = \int_{1-}^{x} \frac{x}{t}\, dA(t).
\end{aligned}
$$

Thus, upon adding and subtracting $c(\delta_1 + dt)$ on the right side of the Chebyshev formula, it becomes

$$
\int_1^x \frac{cx}{t}\, d\psi(t) + \int_1^x (dN - c\,\delta_1 - c\, dt) * d\psi. \tag{3.4}
$$

Now we show that

$$I := \int_1^x (dN - c\,\delta_1 - c\,dt) * d\psi = c'x + o(x)$$

for some constant c'. We rewrite I as $I_1 + I_2$, with

$$I_1 = \int_{1-}^x (dN - c\,\delta_1 - c\,dt) * (\delta_1 + dt)$$

and

$$I_2 = \int_{1-}^x (dN - c\,\delta_1 - c\,dt) * (d\psi - \delta_1 - dt).$$

We have

$$\begin{aligned} I_1 &= \int_{1-}^x \frac{x}{t}\,(dN(t) - c\,\delta_1 - c\,dt) \\ &= N(x) - cx + x\int_1^x \frac{N(t) - ct}{t^2}\,dt \\ &= \theta x D(x) + x(c' + o(1)) = c'x + o(x), \end{aligned}$$

where

$$c' = \int_1^\infty \frac{N(t) - ct}{t^2}\,dt.$$

The integral is convergent, since it is dominated by $\int_1^\infty t^{-1}D(t)\,dt$.

Next, we claim that $I_2 = o(x)$. For $x \geqslant 1$, take

$$A(x) := N(x) - cx, \qquad B(x) := \psi(x) - x.$$

We have

$$|A(x)| \leqslant xD(x),$$

with D satisfying the conditions of the lemma. Also, $B(x) = o(x)$ by the PNT and $B_v(x) = \psi(x) + x \ll x$. It follows from the lemma that $I_2 = o(x)$.

Combining (3.3) and (3.4) with the approximation of I, we get

$$cx\log x - cx + o(x) = cx\psi_1(x) + c'x + o(x),$$

which is equivalent to (2.1). ■

4. Some technology

In the next section we construct a *continuous* g-number example, as Beurling did [3] to show the optimality of his PNT result for g-primes. In place of a weighted prime counting measure, we use a positive mass distribution, again called $d\Pi$, having support in $[1, \infty)$ and no point mass at 1. The g-integer counting measure dN is connected to $d\Pi$ by the relation

$$dN = \delta_1 + d\Pi + \frac{1}{2!} d\Pi * d\Pi + \frac{1}{3!} d\Pi * d\Pi * d\Pi + \cdots =: \exp^* d\Pi \,.$$

The $\exp^*$ operator is a map from the additive group of measures on $[1, \infty)$ to the (convolution) multiplicative group of measures on the same space. Convergence is in the sense of uniform convergence on compact sets. This operator is discussed in some detail in [4] and in Chapters 2 and 3 of [2], but we set out a few facts about it here.

For dA and dB measures on $[1, \infty)$, we have

$$\exp^*\{dA + dB\} = \{\exp^* dA\} * \{\exp^* dB\}.$$

Also, if T^c is the operator on measures defined by $\{T^c\, dA\}(t) := t^c\, dA(t)$, then, as an easy consequence of the definition of convolution,

$$T^{-1}\{dA * dB\} = \{T^{-1} dA\} * \{T^{-1} dB\},$$

and by applying T^{-1} to each term in the series for $\exp^*$, we get

$$T^{-1} \exp^*\{dA\} = \exp^*\{T^{-1} dA\}.$$

Our construction depends on the following two connected relations.

Lemma 4.1. *Let*

$$d\lambda(u) := \frac{1 - u^{-1}}{\log u}\, du, \qquad u > 1.$$

We have

$$\int_1^\infty u^{-s}\, d\lambda(u) = \log \frac{s}{s-1}, \qquad \Re s > 1, \tag{4.1}$$

and

$$\delta_1 + du = \exp^*\{d\lambda(u)\}. \tag{4.2}$$

Proof. Let $F(s)$ denote the integral on the left side of (4.1). Then

$$F'(s) = -\int_1^\infty u^{-s}\, (1 - u^{-1})\, du = -\frac{1}{s-1} + \frac{1}{s} = \left\{\log \frac{s}{s-1}\right\}'.$$

Thus F agrees with $\log s/(s-1)$ to within a constant. The constant is 0 since

$$\lim_{s \to +\infty} F(s) = 0 = \lim_{s \to +\infty} \log \frac{s}{s-1}\,.$$

For (4.2), we have for $\Re s > 1$

$$\begin{aligned}\int_{1-}^{\infty} u^{-s}\, \exp^*\{d\lambda(u)\} &= \int_{1-}^{\infty} u^{-s}\, \{\delta_1 + d\lambda(u) + \frac{1}{2!} d\lambda * d\lambda(u) + \dots\} \\ &= 1 + \int_{1}^{\infty} u^{-s}\, d\lambda(u) + \frac{1}{2!}\Big\{\int_{1}^{\infty} u^{-s}\, d\lambda(u)\Big\}^2 + \dots \\ &= 1 + \log\frac{s}{s-1} + \frac{1}{2!}\Big\{\log\frac{s}{s-1}\Big\}^2 + \dots \\ &= \exp\log\frac{s}{s-1} = \frac{s}{s-1} = \int_{1-}^{\infty} u^{-s}\{\delta_1 + du\}\,.\end{aligned}$$

By the identity theorem for Mellin transforms, the measures in the first and last integrals must be the same. ■

5. Example

We observed in Proposition 2.1 that the PNT was needed in order to establish the sharp Mertens formula (2.1). Proposition 2.2 showed that the PNT by itself was not sufficient for this purpose. The theorem and its corollaries provided conditions that give (2.1). Are those conditions excessive? In particular, could we perhaps prove the theorem under the weaker hypothesis $N(x) = cx + O(x/\log x)$? The following example shows that the answer to the last question is No.

Example 5.1. PNT and $N(x) = cx + O(x/\log x) \not\Rightarrow$ (2.1). Take

$$d\Pi(u) := \frac{1-u^{-1}}{\log u}du + \Big(\frac{1-u^{-1}}{\log u}\Big)^2 du. \tag{5.1}$$

A. This prime density satisfies the PNT, since for $x > e$,

$$\Pi(x) = c + \int_{e}^{x} \Big\{\frac{du}{\log u} + O\Big(\frac{1}{\log^2 u}\Big)\Big\}\, du = \frac{x}{\log x} + O\Big(\frac{x}{\log^2 x}\Big)\,.$$

B. To see the failure of a sharp Mertens relation, consider

$$\psi(x) = \int_{1}^{x} L\, d\Pi(u) = \int_{1}^{x} (1-u^{-1})\, du + \int_{1}^{x} \frac{(1-u^{-1})^2}{\log u}\, du = x + f(x)$$

with

$$f(x) := \int_{1}^{x} \frac{(1-u^{-1})^2}{\log u}\, du - \log x\, - 1 \sim x/\log x.$$

By the contrapositive form of Proposition 2.2, (2.1) does not hold for this g-number system.

C. It remains to show that $N(x) = cx + O(x/\log x)$. We have

$$\begin{aligned}
N(x) &= \int_{1-}^{x} \exp^* \Big\{ \frac{1-u^{-1}}{\log u} du + \Big(\frac{1-u^{-1}}{\log u}\Big)^2 du \Big\} \\
&= \int_{1-}^{x} (\delta_1 + du) * \exp^* \Big\{ \Big(\frac{1-u^{-1}}{\log u}\Big)^2 du \Big\} \\
&= \int_{1-}^{x} \frac{x}{u} \exp^* \Big\{ \Big(\frac{1-u^{-1}}{\log u}\Big)^2 du \Big\} = x \int_{1-}^{x} \exp^* \Big\{ \Big(\frac{1-u^{-1}}{\log u}\Big)^2 \frac{du}{u} \Big\}.
\end{aligned}$$

We approximate the last integral by Mellin inversion. We have

$$\zeta(s) := \int_{1-}^{\infty} u^{-s} \exp^* \Big\{ \Big(\frac{1-u^{-1}}{\log u}\Big)^2 \frac{du}{u} \Big\} = \exp \int_{1}^{\infty} u^{-s} \Big(\frac{1-u^{-1}}{\log u}\Big)^2 \frac{du}{u},$$

by the argument used in proving (4.2). To analyze the integral, differentiate

$$\log \zeta(s) = \int_{1}^{\infty} u^{-s-1} \Big(\frac{1-u^{-1}}{\log u}\Big)^2 du\,.$$

Using Lemma 4.1, we get

$$\begin{aligned}
\frac{-\zeta'}{\zeta}(s) &= \int_{1}^{\infty} u^{-s-1}\, \frac{1-2u^{-1}+u^{-2}}{\log u}\, du \\
&= \int_{1}^{\infty} u^{-s-1}\, \frac{1-u^{-1}}{\log u}\, du - \int_{1}^{\infty} u^{-s-2}\, \frac{1-u^{-1}}{\log u}\, du \\
&= \log \frac{s+1}{s} - \log \frac{s+2}{s+1} = 2\log(s+1) - \log s - \log(s+2).
\end{aligned}$$

Integration yields

$$\log \zeta(s) = s \log s + (s+2)\log(s+2) - 2(s+1)\log(s+1) + c.$$

We have $c = 0$, since the rest of the preceding expression tends to 0 as $s \to +\infty$. We thus obtain the explicit representation

$$\zeta(s) = \frac{s^s (s+2)^{s+2}}{(s+1)^{2s+2}}\,.$$

Now we apply the Mellin inversion formula. For any $a > 0$,

$$\frac{N(x)}{x} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s)\, x^s\, \frac{ds}{s}\,.$$

The zeta function is analytic on the cut complex plane with the negative axis and the origin removed. We shift the integration contour to consist of two half-lines, two line segments and a small circle:

$$\begin{aligned}
&\mathcal{C}_1 : \ -1/2 - i\infty \text{ to } -1/2 + 0i-,\\
&\mathcal{C}_2 : \ -1/2 + 0i- \ \text{ to } -\epsilon + 0i-,\\
&\mathcal{C}_3 : \ \epsilon e^{i\theta}, \ -\pi < \theta < \pi,\\
&\mathcal{C}_4 : \ -\epsilon + 0i+ \ \text{ to } -1/2 + 0i+,\\
&\mathcal{C}_5 : \ -1/2 + 0i+ \ \text{ to } -1/2 + i\infty,
\end{aligned}$$

and write

$$N(x)/x = I_1 + I_2(\epsilon) + I_3(\epsilon) + I_4(\epsilon) + I_5,$$

with I_j denoting the Mellin integral taken over $\mathcal{C}_j$.

The main contribution to $N(x)/x$ comes from $\mathcal{C}_3$. For $s = \epsilon e^{i\theta}$ with $\epsilon \to 0$,

$$s^s = \exp\{(\epsilon\cos\theta + i\epsilon\sin\theta)(\log\epsilon + i\theta)\} = \exp\{O(\epsilon|\log\epsilon|)\} \to 1.$$

Thus $\zeta(s)x^s \to 4$ uniformly on $\mathcal{C}_3$ as $\epsilon \to 0$, and

$$\lim_{\epsilon\to 0} I_3(\epsilon) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} 4i\,d\theta = 4.$$

The next contribution comes from $\mathcal{C}_2$ and $\mathcal{C}_4$. $\mathcal{C}_2$ is traversed in the positive direction with $\arg s = -\pi$, while $\mathcal{C}_4$ is traversed in the opposite direction and has $\arg s = \pi$. For $0 < t < 1/2$, we have

$$\zeta(te^{\pm\pi i}) = 4\exp\{-t(\log t \pm \pi i)\}(1 + O(t)) = 4\{1 + O(t|\log t|)\}\exp\{\mp\pi i t\},$$

and thus

$$\begin{aligned}
\lim_{\epsilon\to 0}\{I_2(\epsilon) + I_4(\epsilon)\} &= \frac{-4}{2\pi i}\int_0^{1/2}\{1 + O(t|\log t|)\}\big(e^{\pi i t} - e^{-\pi i t}\big)\,x^{-t}\,\frac{dt}{t}\\
&= -4\int_0^{1/2}\{1 + O(t|\log t|)\}\frac{\sin(\pi t)}{\pi t}e^{-t\log x}\,dt\\
&= -4\int_0^{1/2}\{1 + O(t|\log t|)\}e^{-t\log x}\,dt.
\end{aligned}$$

Now, with $u := t\log x$,

$$\begin{aligned}
\int_0^{\infty} t|\log t|\,e^{-t\log x}\,dt &\leqslant \frac{1}{\log^2 x}\int_0^{\infty} u(|\log u| + \log\log x)\,e^{-u}\,du\\
&\ll (\log\log x)/(\log^2 x),
\end{aligned}$$

and thus

$$\lim_{\epsilon\to 0}\{I_2(\epsilon) + I_4(\epsilon)\} = \frac{-4}{\log x} + O\Big(\frac{\log\log x}{\log^2 x}\Big).$$

Finally, on $\mathcal{C}_1 \cup \mathcal{C}_5$ we have $s = -1/2 + it$ with $t \neq 0$, and here, by a small calculation, $\zeta(s) = 1 + O(1/\{|t|+1\})$. Thus

$$I_1 + I_5 = \frac{x^{-1/2}}{2\pi i} \int_{-\infty}^{\infty} \Big\{1 + O\Big(\frac{1}{|t|+1}\Big)\Big\} \frac{x^{it}\,dt}{-1/2+it}\,.$$

We have

$$\int_{-\infty}^{\infty} O\Big(\frac{1}{|t|+1}\Big) \frac{dt}{|-1/2+it|} = O(1)$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x^{it}\,dt}{-1/2+it} &= O(1) + \int_{|t|>1} \frac{x^{it}}{it}\big\{1 + O\big(|t|^{-1}\big)\big\}\,dt \\ &= O(1) + \int_{1}^{\infty} 2\sin(t\log x)\,\frac{dt}{t} = O(1).\end{aligned}$$

We find that $I_1 + I_5 \ll x^{-1/2}$.

Together, the five integrals give

$$N(x) = 4x - \frac{4x}{\log x} + O\Big(\frac{x\log\log x}{\log^2 x}\Big),$$

that is, $|N(x) - cx| \ll x/\log x$ holds for this g-number system.

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Addresses: Harold G. Diamond and Wen-Bin Zhang: Dept. of Math., Univ. of Illinois, 1409 W. Green St., Urbana IL 61801 USA.

E-mail: diamond@math.uiuc.edu, cheungmanping@yahoo.com

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