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ASYMPTOTICS OF KEIPER-LI COEFFICIENTS

Juan Arias de Reyna

Abstract: We show that the Riemann Hypothesis is equivalent to the assertion $(y_m) \in \ell_2$ where y_m is defined by

$$\lambda_m = \frac{1}{2}(\log m + \gamma - \log(2\pi) - 1) + y_m,$$

and $m\lambda_m$ represents the numbers in Xian-Jin Li's criterion. This confirms and further sharpens a conjecture of J. B. Keiper.

We also present some other hypotheses equivalent to the Riemann Hypothesis.

Keywords: Riemann Hypothesis, Keiper-Li coefficients, zeta function.

1. Introduction

Keiper [7] introduced several power series[1]:

$$2\xi(s) = \sum_{j=0}^{\infty} \alpha_j (s-1)^j, \qquad \frac{\xi'(1/s)}{\xi(1/s)} = \sum_{k=0}^{\infty} \tau_k (1-s)^k, \tag{1}$$

$$\log\bigl(2\xi(1/s)\bigr) = \sum_{k=0}^{\infty} \lambda_k (1-s)^k, \qquad \frac{\xi'(s)}{\xi(s)} = \sum_{k=0}^{\infty} \sigma_{k+1} (1-s)^k, \tag{2}$$

where

$$\xi(s) = \frac{s(s-1)}{2}\pi^{-s/2}\Gamma\Bigl(\frac{s}{2}\Bigr)\zeta(s). \tag{3}$$

He demonstrated that if the Riemann Hypothesis is true then $\lambda_n \geqslant 0$ for all $n \geqslant 0$. Later Xian-Jin Li [8] proved that indeed the Riemann Hypothesis is equivalent to $\lambda_n \geqslant 0$ for all $n \geqslant 0$. Keiper also claims, by assuming the Riemann Hypothesis together with an additional hypothesis about the vertical distribution of the zeros, that

$$\lambda_m \approx \frac{\log m}{2} - \frac{\log(2\pi) + 1 - \gamma}{2}.$$

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[1] Keiper [7] defined the numbers λ_m as in (2). Later Li considered $m\lambda_m$ and called them λ_m. We will follow the notation of Keiper which is more convenient for our purposes.

We will show here that in fact the Riemann Hypothesis is equivalent to the assertion that the sequence (y_m), implicitly defined by

$$\lambda_m = \frac{\log m}{2} - \frac{\log(2\pi)+1-\gamma}{2} + y_m$$

is in ℓ_2.

Our result also reinforces the result of Voros [13], which stated that $ny_n = \mathrm{o}(n)$ is equivalent to the Riemann Hypothesis.

We define the numbers A_n by means of the power series expansion

$$\log\{(s-1)\zeta(s)\} = \sum_{n=0}^{\infty} A_n \Bigl(1-\frac{1}{s}\Bigr)^n, \qquad \Re s > 1. \tag{4}$$

These numbers are closely related to y_n above. We will also prove that the Riemann Hypothesis is equivalent to the assertion $(A_n) \in \ell_2$.

2. Some results of Keiper

Keiper [7] established many relations between coefficients of expansions (1). For example [7, equation (27)],

$$\lambda_k = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j}\binom{k-1}{j-1}\sigma_j \qquad \text{for } \ k \geqslant 1. \tag{5}$$

Keiper also gave formulas for some of these coefficients in terms of the non-trivial zeros of $\zeta(s)$:

$$\lambda_m = \frac{1}{m}\sum_{\varrho}\Bigl[1-\Bigl(\frac{\varrho}{\varrho-1}\Bigr)^m\Bigr] \qquad \text{and} \qquad \sigma_k = \sum_{\varrho}\frac{1}{(1-\varrho)^k} = \sum_{\varrho}\frac{1}{\varrho^k} \tag{6}$$

where ϱ runs through the non-trivial zeros of $\zeta(s)$, and where the terms of the sums that correspond to ϱ and $\overline{\varrho}$ must be paired. It is known [4, p. 67] that

$$\sigma_1 = \sum_{\varrho}\frac{1}{\varrho} = 1 + \frac{\gamma-\log\pi}{2} - \log 2. \tag{7}$$

Keiper related the Riemann Hypothesis with the behaviour of some of these coefficients. In particular he proved that the Riemann Hypothesis implies that all $\lambda_k \geqslant 0$. Later Xian-Jin Li [8] proved that conversely $\lambda_k \geqslant 0$ implies the Riemann Hypothesis.

Here we quote Keiper [7, p. 769]: *In fact, if we assume the Riemann hypothesis, and further that the zeros are very evenly distributed, we can show that*

$$\lambda_m \approx \frac{\log m}{2} - \frac{\log(2\pi)+1-\gamma}{2}. \tag{8}$$

This comes from the fact [4, p. 132] that the number of zeros ρ in the critical strip with $0 < \mathrm{Im}\rho < T$ *is*

$$N(T) \sim \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} \tag{9}$$

and (6)*. Note that this asymptotic conjecture is much stronger than the Riemann hypothesis. Even the coefficient of the* $\log m$ *(not to mention the constant term) could be altered by a slight preference of the zeros to cluster at, or avoid, the points* $1/2 + 2i\tan((2k+1)\pi/(2m))$*.*

We will show that, in fact, (8) is a consequence of the Riemann Hypothesis. To this end, we need no additional hypothesis. We will try to explain what the reason for Keiper's opinion may be.

3. The behaviour of λ_m

In the following argument we assume the Riemann Hypothesis. For each non-trivial zero $\varrho = 1/2 + i\gamma$ with $\gamma > 0$ we define the angle

$$\theta = \arctan\frac{1}{2\gamma}.$$

It is easy to see that

$$\frac{\varrho}{\varrho - 1} = e^{-2i\theta}$$

hence

$$\lambda_m = \frac{1}{m}\sum_{\varrho}\Big[1 - \Big(\frac{\varrho}{\varrho-1}\Big)^m\Big] = \frac{2}{m}\sum_{\gamma>0}\Re(1 - e^{-2im\theta}) = \frac{4}{m}\sum_{\gamma>0}\sin^2(m\theta).$$

Therefore, (8) may also be written as

$$\frac{4}{m}\sum_{\gamma>0}\sin^2(m\theta) \approx \frac{\log m}{2} - \frac{\log(2\pi) + 1 - \gamma}{2}. \tag{10}$$

This may well be the reason for Keiper's opinion. The above sum is related to the distribution of the numbers $m\theta$ with respect to the integral multiples of π. Therefore, (10) appears to impose a condition on the vertical distribution of the zeros. The following theorem proves that this is not true.

Theorem 3.1. *Let* (γ_j) *be a non-decreasing sequence of positive real numbers, distributed such that*

$$N(t) = \#\{j \in \mathbb{N} : \gamma_j \leqslant t\} = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + S(t)$$

where $S(t) = \mathrm{O}(\log t)$*. Defining* $\theta_j = \arctan\frac{1}{2\gamma_j}$*, we then have*

$$\frac{4}{m}\sum_{j=1}^{\infty}\sin^2(m\theta_j) = \frac{\log m}{2} - \frac{\log(2\pi) + 1 - \gamma}{2} + y_m$$

where $(y_m) \in \ell_2$*.*

Proof. We write the sum as

$$J(m) := \frac{4}{m}\sum_{j=1}^{\infty}\sin^2(m\theta_j) = \frac{4}{m}\int_0^{+\infty}\sin^2 m\Big(\arctan\frac{1}{2t}\Big)\,dN(t).$$

Since $dN(t) = (2\pi)^{-1}\log(t/2\pi)\,dt + dS(t)$, we obtain

$$\begin{aligned} J(m) = {}& \frac{2}{\pi m}\int_0^{+\infty}\sin^2 m\Big(\arctan\frac{1}{2t}\Big)\log\frac{t}{2\pi}\,dt \\ &+\frac{4}{m}\int_0^{+\infty}\sin^2 m\Big(\arctan\frac{1}{2t}\Big)\,dS(t) := J_1(m) + J_2(m). \end{aligned}$$

In the first integral $J_1(m)$, we put $y = \arctan 1/2t$

$$\begin{aligned} J_1(m) &:= \frac{2}{\pi m}\int_0^{+\infty}\sin^2 m\Big(\arctan\frac{1}{2t}\Big)\log\frac{t}{2\pi}\,dt \\ &= -\frac{2}{\pi m}\int_0^{\pi/2}\sin^2(my)\log(4\pi\tan y)\frac{dy}{2\sin^2 y} \\ &= -\frac{1}{\pi m}\int_0^{\pi/2}\frac{\sin^2(my)}{y^2}\frac{y^2}{\sin^2 y}\Big\{\log y + \log\frac{4\pi\tan y}{y}\Big\}\,dy \\ &= -\frac{1}{\pi m}\int_0^{\pi/2}\frac{\sin^2(my)}{y^2}\log(4\pi y)\,dy - \frac{1}{\pi m}\int_0^{\pi/2}\frac{\sin^2(my)}{y^2}g(y)\,dy \end{aligned}$$

where

$$g(y) := \Big(\frac{y^2}{\sin^2 y} - 1\Big)\log(4\pi y) + \frac{y^2}{\sin^2 y}\log\frac{\tan y}{y}.$$

Since $g(y)/y^2$ is a continuous function on $(0,\pi/2)$, and $g(y)/y^2$ is $\mathrm{O}(\log y)$ for $y\to 0^+$, and $\mathrm{O}(\log(\pi/2 - y))$ for $y\to\pi/2^-$, we have that $g(y)/y^2$ is an integrable function and

$$\frac{1}{\pi m}\int_0^{\pi/2}\frac{\sin^2(my)}{y^2}g(y)\,dy = \mathrm{O}(1/m).$$

We thus have

$$\begin{aligned} J_1(m) &= -\frac{1}{\pi m}\int_0^{\pi/2}\frac{\sin^2(my)}{y^2}\log(4\pi y)\,dy + \mathrm{O}(1/m) \\ &= \frac{1}{\pi}\log\frac{m}{4\pi}\int_0^{\pi m/2}\frac{\sin^2 x}{x^2}\,dx - \frac{1}{\pi}\int_0^{\pi m/2}\frac{\sin^2 x}{x^2}\log x\,dx + \mathrm{O}(1/m) \\ &= \frac{1}{\pi}\log\frac{m}{4\pi}\int_0^{+\infty}\frac{\sin^2 x}{x^2}\,dx - \frac{1}{\pi}\int_0^{\infty}\frac{\sin^2 x}{x^2}\log x\,dx + \mathrm{O}(\log m/m). \end{aligned}$$

Since it is known (see [5, p. 446: 3.821 9, and p. 599: 4.423 3]) that

$$\int_0^{+\infty}\frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2} \qquad\text{and}\qquad \int_0^{\infty}\frac{\sin^2 x}{x^2}\log x\,dx = \frac{\pi(1-\gamma-\log 2)}{2}$$

it follows that

$$J_1(m) = \frac{\log m}{2} - \frac{\log(2\pi) + 1 - \gamma}{2} + \mathrm{O}(\log m/m).$$

Now we consider the second integral

$$J_2(m) := \frac{4}{m} \int_0^{+\infty} \sin^2 m\Big(\arctan \frac{1}{2t}\Big)\, dS(t).$$

We apply summation by parts. Observe that, by definition, for $0 < t < \gamma_1$, we have $S(t) = -(t/2\pi)\log(t/2\pi) + (t/2\pi)$, $S(0) = 0$, and by hypothesis $S(t) = \mathrm{O}(\log t)$ for $t \to +\infty$, therefore

$$\begin{aligned} J_2(m) &= 8 \int_0^{+\infty} S(t) \sin m\Big(\arctan \frac{1}{2t}\Big) \cos m\Big(\arctan \frac{1}{2t}\Big) \frac{2\, dt}{1+4t^2} \\ &= 8 \int_0^{+\infty} \sin 2m\Big(\arctan \frac{1}{2t}\Big) \frac{S(t)\, dt}{1+4t^2} \\ &= 4 \int_0^{\pi/2} \sin(2my) S\Big(\frac{1}{2\tan y}\Big)\, dy. \end{aligned}$$

The function $S(1/(2\tan y)) \in \mathcal{L}^2(0, \pi/2)$ since

$$\int_0^{\pi/2} S\Big(\frac{1}{2\tan y}\Big)^2 dy = \int_0^{\infty} S(x)^2 \frac{2\, dx}{1+4x^2} < +\infty$$

and it follows that the sequence $(J_2(m))_m$ is in ℓ_2, which completes the proof. ■

Since, by assuming the Riemann Hypothesis, the ordinates of the zeros γ_n satisfy the hypothesis of Theorem 3.1 (see Titchmarsh [12, Theorem 9.4]) we obtain:

Corollary 3.1. *By assuming the Riemann Hypothesis we have*

$$\lambda_m = \frac{\log m}{2} - \frac{\log(2\pi) + 1 - \gamma}{2} + y_m$$

where $(y_m) \in \ell_2$.

4. An equivalent form of the Riemann Hypothesis

As is the norm, we denote s as a variable in the half plane $\Re(s) > 1/2$. We will also use another variable, z, connected to s by a fractional linear transformation

$$z = \frac{s-1}{s}, \qquad s = \frac{1}{1-z}, \qquad s - 1 = \frac{z}{1-z}. \tag{11}$$

This transformation maps the half plane $\Re(s) > 1/2$ onto the unit disc $|z| < 1$. The point $s = 1$ transforms into $z = 0$. Hence, any holomorphic function on the half plane can be represented by a power series in z with a radius of convergence $\geqslant 1$.

Theorem 4.1. *The coefficients A_m and λ_m, with $m \geqslant 1$, satisfy the following relation*

$$A_m = \lambda_m + \frac{\gamma+\log(4\pi)}{2} - \frac{1}{m} - \frac{1}{m}\sum_{j=2}^{m}(-1)^j\binom{m}{j}\Bigl(1-\frac{1}{2^j}\Bigr)\zeta(j).$$

(*When $m=1$, the sum must be taken as equal to* 0.)

Proof. In Keiper [7, (46)] we find

$$\log\{(s-1)\zeta(s)\} = \gamma(s-1) - \sum_{j=2}^{\infty}\frac{1}{j}\Bigl[\sigma_j + \sum_{k=2}^{\infty}(2k-1)^{-j}\Bigr](1-s)^j.$$

We define the numbers M_j by

$$\log\{(s-1)\zeta(s)\} = \sum_{j=1}^{\infty} M_j(s-1)^j.$$

Since $s-1 = z(1-z)^{-1}$ we have

$$\log\{(s-1)\zeta(s)\} = \sum_{j=1}^{\infty} M_j\Bigl(\frac{z}{1-z}\Bigr)^j = \sum_{1\leqslant j,\ 0\leqslant k} M_j\binom{j+k-1}{k}z^{j+k}$$

therefore

$$\log\{(s-1)\zeta(s)\} = \sum_{m=1}^{\infty} z^m\Biggl(\sum_{j=1}^{m} M_j\binom{m-1}{j-1}\Biggr) := \sum_{m=1}^{\infty} A_m z^m.$$

Hence

$$A_m = \gamma + \sum_{j=2}^{m}\binom{m-1}{j-1}\frac{(-1)^{j+1}}{j}\Bigl[\sigma_j + \sum_{k=2}^{\infty}(2k-1)^{-j}\Bigr].$$

Now, by (5), we obtain

$$A_m = \gamma + \lambda_m - \sigma_1 + \sum_{j=2}^{m}\binom{m-1}{j-1}\frac{(-1)^{j+1}}{j}\sum_{k=2}^{\infty}(2k-1)^{-j} \tag{12}$$

where the sum may be expressed in terms of the zeta function

$$\begin{aligned}\sum_{j=2}^{m}\binom{m-1}{j-1}\frac{(-1)^{j+1}}{j}\sum_{k=2}^{\infty}(2k-1)^{-j} &= -\frac{1}{m}\sum_{j=2}^{m}(-1)^j\binom{m}{j}\Bigl\{\Bigl(1-\frac{1}{2^j}\Bigr)\zeta(j)-1\Bigr\}\\ &= \frac{m-1}{m} - \frac{1}{m}\sum_{j=2}^{m}(-1)^j\binom{m}{j}\Bigl(1-\frac{1}{2^j}\Bigr)\zeta(j).\end{aligned}$$

Substituting this into (12), we obtain

$$A_m = \gamma + \lambda_m - \sigma_1 + \frac{m-1}{m} - \frac{1}{m}\sum_{j=2}^{m}(-1)^j\binom{m}{j}\Big(1-\frac{1}{2^j}\Big)\zeta(j).$$

Finally, our Theorem is obtained when the value of σ_1 given in (7) is substituted into this. ■

The behaviour of the sum appearing here is clarified in the following Lemma:

Lemma 4.1. *The numbers*

$$I_m := \int_0^{+\infty} L_m(t)\Big(\frac{e^{-t}}{t^2} + \frac{e^{-t}}{t} - \frac{\cosh t}{(\sinh t)^2}\Big)\,dt \tag{13}$$

where the $L_m(t) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{t^j}{j!}$ *denote the usual Laguerre polynomials, are given explicitly by*

$$I_m = m\Big(\log 2 + \gamma + \sum_{j=2}^{m-1}\frac{1}{j}\Big) - 2\sum_{j=2}^{m}(-1)^j\binom{m}{j}\Big(1-\frac{1}{2^j}\Big)\zeta(j), \qquad m>1. \tag{14}$$

Moreover, the sequence (I_m) *is rapidly decreasing, that is, for all* $k>0$, *we have* $\lim_{m\to\infty} m^k I_m = 0$.

Proof. Define

$$S_m := \frac{1}{m}\sum_{j=2}^{m}(-1)^j\binom{m}{j}\Big(1-\frac{1}{2^j}\Big)\zeta(j).$$

It is well known (see [10, p. 103 (50)]) that

$$\Big(1-\frac{1}{2^s}\Big)\zeta(s) = \frac{1}{2\Gamma(s)}\int_0^{+\infty}\frac{t^{s-1}}{\sinh t}\,dt, \qquad \Re(s)>1$$

and hence[2]

$$\Big(1-\frac{1}{2^s}\Big)\zeta(s) = \frac{1}{2\Gamma(s)}\int_0^{+\infty} t^{s-1}\Big(\frac{1}{\sinh t} - \frac{e^{-t}}{t}\Big)\,dt + \frac{1}{2(s-1)}. \tag{16}$$

[2]From equation (16) and by taking limits when $s\to 1$, it is easily deduced that

$$\int_0^{+\infty}\Big(\frac{1}{\sinh t} - \frac{e^{-t}}{t}\Big)\,dt = \gamma + \log 2 \tag{15}$$

which will be required in a later step.

Therefore

$$\begin{aligned}
S_m &= \frac{1}{m}\sum_{j=2}^{m}(-1)^j\binom{m}{j}\frac{1}{2(j-1)}\\
&\quad+\frac{1}{2m}\int_0^{+\infty}\sum_{j=2}^{m}(-1)^j\binom{m}{j}\frac{t^{j-1}}{(j-1)!}\Bigl(\frac{1}{\sinh t}-\frac{e^{-t}}{t}\Bigr)\,dt\\
&=\frac{1}{m}\sum_{j=2}^{m}(-1)^j\binom{m}{j}\frac{1}{2(j-1)}+\frac{1}{2m}\int_0^{+\infty}(L_m'(t)+m)\Bigl(\frac{1}{\sinh t}-\frac{e^{-t}}{t}\Bigr)\,dt\\
&=\frac{\gamma+\log 2}{2}+\frac{1}{m}\sum_{j=2}^{m}(-1)^j\binom{m}{j}\frac{1}{2(j-1)}+\frac{1}{2m}\int_0^{+\infty}L_m'(t)\Bigl(\frac{1}{\sinh t}-\frac{e^{-t}}{t}\Bigr)\,dt.
\end{aligned}$$

Integrating this by parts we find that

$$\begin{aligned}
S_m = \frac{\gamma+\log 2}{2}&+\frac{1}{m}\sum_{j=2}^{m}(-1)^j\binom{m}{j}\frac{1}{2(j-1)}\\
&-\frac{L_m(0)}{2m}-\frac{1}{2m}\int_0^{+\infty}L_m(t)\Bigl(\frac{e^{-t}}{t^2}+\frac{e^{-t}}{t}-\frac{\cosh t}{(\sinh t)^2}\Bigr)\,dt.
\end{aligned}$$

Now, since $L_m(0)=1$ and[3]

$$\frac{1}{m}\sum_{j=2}^{m}(-1)^j\binom{m}{j}\frac{1}{2(j-1)}=\frac{1}{2}\sum_{j=2}^{m}\frac{1}{j}$$

we obtain

$$\begin{aligned}
S_m &= \frac{\gamma+\log 2}{2}-\frac{1}{2m}+\frac{1}{2}\sum_{j=2}^{m}\frac{1}{j}-\frac{1}{2m}\int_0^{+\infty}L_m(t)\Bigl(\frac{e^{-t}}{t^2}+\frac{e^{-t}}{t}-\frac{\cosh t}{(\sinh t)^2}\Bigr)\,dt\\
&=\frac{\gamma+\log 2}{2}-\frac{1}{2m}+\frac{1}{2}\sum_{j=2}^{m}\frac{1}{j}-\frac{I_m}{2m}.
\end{aligned}$$

This completes the proof of (14).

Given an infinitely differentiable function $f\colon[0,+\infty)\to\mathbb{C}$ which tends to 0 rapidly at infinity, together with its derivatives of all orders, it is known that the integrals $a_n := \int_0^\infty f(t)L_n(t)e^{-t/2}\,dt$ form a rapidly decreasing sequence [3, Theorem 2.5]. Hence, in order to prove that $\lim_{m\to\infty} m^k I_m = 0$, we only need to show that the function $f(t)=e^{t/2}\Bigl(\frac{e^{-t}}{t^2}+\frac{e^{-t}}{t}-\frac{\cosh t}{(\sinh t)^2}\Bigr)$ has these properties. This is easy to verify: The singularity on $t=0$ is removable. The function $f(t)$ is analytic on a strip of constant width along the real axis. Since $\lim_{t\to\infty} t^n f(t)=0$ along this strip for all n, these bounds can be extended, by Cauchy's Theorem, to the derivatives. ∎

[3] This can be shown rather easily and is essentially formula 51 of Section 4.2.2: Summations of the form $\sum a_k\binom{b_k}{c_k}$ of the Tables [9, vol I, p. 612].

Theorem 4.2. *There exists $(x_m) \in \ell_2$ such that for every natural number m*

$$A_m = \lambda_m - \frac{\log m}{2} + \frac{\log(2\pi) + 1 - \gamma}{2} + x_m.$$

Proof. By substituting the result of Lemma 4.1 in the equation of Theorem 4.1, we obtain

$$A_m = \lambda_m - \frac{\log m}{2} + \frac{1 - \gamma + \log(2\pi)}{2} + \frac{1}{2}\Big\{\gamma + \log m - \sum_{j=1}^{m} \frac{1}{j}\Big\} - \frac{1}{2m} + \frac{I_m}{2m}. \tag{17}$$

To verify that (x_m), defined by

$$x_m = \frac{1}{2}\Big\{\gamma + \log m - \sum_{j=1}^{m} \frac{1}{j}\Big\} - \frac{1}{2m} + \frac{I_m}{2m},$$

is in ℓ_2, we only need to check whether I_m is bounded. This is true since (I_m) is a rapidly decreasing sequence. However, this can also be easily derived from the fact that $|e^{-x/2} L_n(x)| \leqslant 1$ (see Szegö [11, p. 164, eq. (7.21.3)]). ■

Theorem 4.3. *The Riemann Hypothesis is equivalent to the assertion that the sequence (A_n) is in ℓ_2.*

Proof. If we assume $(A_n) \in \ell_2$, then the power series (4) has radius of convergence 1. It follows that the function $\log\big\{(s-1)\zeta(s)\big\}$ is analytic on $\Re(s) > \frac{1}{2}$, and hence the Riemann Hypothesis is true.

Now, if we assume the Riemann Hypothesis, then by combining Theorem 4.2 with Corollary 3.1, we obtain $A_n = x_n + y_n$ with (x_n) and (y_n) in ℓ_2. Hence, (A_n) must also be in ℓ_2. ■

5. The connection of the Keiper-Li coefficients with primes

Proposition 5.1. *Let $f\colon (1, +\infty) \to \mathbb{C}$ be a measurable complex function such that $f(x)x^{-2}(\log x)^n \in \mathcal{L}^1(1, +\infty)$ for every rational integer $n \geqslant 0$. Suppose that*

$$F(s) = s \int_1^{+\infty} f(x) x^{-s-1}\, dx, \qquad \Re(s) > 1,$$

extends to an analytic function on a neighbourhood of $s = 1$. Then, on a neighbourhood of $z = 0$ we have the Taylor expansion

$$F\left(\frac{1}{1-z}\right) = \sum_{n=0}^{\infty} z^n \int_0^{+\infty} f(e^t) L_n(t) e^{-t}\, dt.$$

Proof. From the hypothesis for $\Re(s) > 1$, we have

$$F^{(n)}(s) = (-1)^{n-1} n \int_1^{+\infty} f(x)(\log x)^{n-1} x^{-s} \frac{dx}{x} + (-1)^n s \int_1^{+\infty} f(x)(\log x)^n x^{-s} \frac{dx}{x}.$$

Since $f(x)x^{-2}(\log x)^n \in \mathcal{L}^1(1, +\infty)$ and the function extends analytically to $s = 1$, the dominated convergence theorem gives us

$$F^{(n)}(1) = (-1)^{n-1} n \int_1^{+\infty} \frac{f(x)}{x^2} (\log x)^{n-1}\, dx + (-1)^n \int_1^{+\infty} \frac{f(x)}{x^2} (\log x)^n\, dx.$$

Since F is analytic on $s = 1$, on a neighbourhood of $s = 1$ we have

$$F(s) = \sum_{n=0}^{\infty} \frac{F^{(n)}(1)}{n!} (s-1)^n.$$

Let z be connected to s by the equations (11), therefore

$$F\Big(\frac{1}{1-z}\Big) = \sum_{n=0}^{\infty} \frac{F^{(n)}(1)}{n!} \Big(\frac{z}{1-z}\Big)^n = \sum_{n=0}^{\infty} \frac{F^{(n)}(1)}{n!} \sum_{k=0}^{\infty} \binom{-n}{k} z^n (-z)^k$$
$$= \sum_{m=0}^{\infty} z^m \sum_{n=0}^{m} (-1)^{m-n} \frac{F^{(n)}(1)}{n!} \binom{-n}{m-n}.$$

(The double series is absolutely convergent for $|z| < \varepsilon$, and the reordering is justified.)

Thus the required coefficient is equal to

$$\sum_{n=0}^{m} (-1)^{m-n} \frac{F^{(n)}(1)}{n!} \binom{-n}{m-n} = \int_1^{+\infty} \frac{f(x)}{x^2} \Big(\sum_{n=1}^{m} (-1)^{m-1} \binom{-n}{m-n} \frac{(\log x)^{n-1}}{(n-1)!} + \sum_{n=0}^{m} (-1)^m \binom{-n}{m-n} \frac{(\log x)^n}{n!} \Big)\, dx$$
$$= \int_1^{+\infty} \frac{f(x)}{x^2} \Big(\sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{(\log x)^n}{n!} \Big)\, dx$$
$$= \int_1^{+\infty} \frac{f(x)}{x^2} L_m(\log x)\, dx.$$

■

From elementary Number Theory we recall the following usual notations (see [6, (15), (16)] and [12, pp. 3, 370])

$$\Pi(x) = \sum_{p^m \leqslant x} \frac{1}{m} = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots, \tag{18}$$

$$\psi(x) = \sum_{p^m \leqslant x} \log p = \sum_{n \leqslant x} \Lambda(n), \qquad \text{and} \qquad M(x) = \sum_{n \leqslant x} \mu(n) \tag{19}$$

where $\mu(n)$ is the Möbius function and $\Lambda(n)$ von Mangoldt's function.

Theorem 5.1. *For all $n \geqslant 0$ we have*

$$A_n = \int_0^{+\infty} \bigl\{\Pi(e^t) - \mathrm{Li}(e^t)\bigr\} L_n(t) e^{-t}\,dt. \tag{20}$$

Proof. We apply Proposition 5.1 to the well-known equality (see [1])

$$\log\bigl\{(s-1)\zeta(s)\bigr\} = s \int_0^{+\infty} \bigl\{\Pi(x) - \mathrm{Li}(x)\bigr\} x^{-s-1}\,dx \tag{21}$$

The estimate of the remainder in the Prime Number Theorem $|\Pi(x) - \mathrm{Li}(x)| \leqslant Cxe^{-c\sqrt{\log x}}$ implies that $f(x) = |\Pi(x) - \mathrm{Li}(x)|$ satisfies the hypothesis of Proposition 5.1. ∎

The above relation enables one to compute, to a high precision, the values of the integrals in (20), since the A_n can be computed. High precision values of the integrals

$$B_n = \int_0^{+\infty} \bigl\{\psi(e^t) - e^t\bigr\} L_n(t) e^{-t}\,dt, \qquad C_n = \int_0^{+\infty} M(e^t) L_n(t) e^{-t}\,dt.$$

can be computed in the same way for low values of n.

Exact values of some of the integrals can also be obtained, for example, it is not difficult to show that

$$A_0 = 0 = \int_0^{+\infty} \bigl\{\Pi(e^t) - \mathrm{Li}(e^t)\bigr\} e^{-t}\,dt, \tag{22}$$

$$A_1 = \gamma = \int_0^{+\infty} \bigl\{\Pi(e^t) - \mathrm{Li}(e^t)\bigr\} (1-t) e^{-t}\,dt, \tag{23}$$

where γ is Euler's constant. Subsequent integrals can be computed in terms of the Stieltjes gamma constants.

Corollary 5.1. *The Riemann Hypothesis is equivalent to the function $\frac{\Pi(x)-\mathrm{Li}(x)}{x}$ being in $\mathcal{L}^2(1, +\infty)$.*

Proof. From Theorem 4.3, the Riemann Hypothesis is equivalent to $\sum_{n=0}^{\infty} A_n^2 < +\infty$. According to (20), this is equivalent to the function $\{\Pi(e^t) - \mathrm{Li}(e^t))e^{-t/2}$ being in $\mathcal{L}^2(0, +\infty)$. A change of variables shows that this is equivalent to $\frac{\Pi(x)-\mathrm{Li}(x)}{x}$ being an element of $\mathcal{L}^2(1, +\infty)$. ∎

The above Corollary is independent proof of the main result in [1].

A relation may now be obtained between the coefficients of Keiper-Li and the primes.

Theorem 5.2.

$$\begin{aligned}\lambda_m = \frac{\log m}{2} - \frac{1-\gamma+\log(2\pi)}{2} + \int_0^{+\infty} \bigl\{\Pi(e^t) - \mathrm{Li}(e^t)\bigr\}\, L_m(t) e^{-t}\, dt \\ - \frac{1}{2}\Bigl\{\gamma + \log m - \sum_{j=1}^{m}\frac{1}{j}\Bigr\} + \frac{1}{2m} - \frac{I_m}{2m}.\end{aligned} \tag{24}$$

6. Some final remarks

If the Riemann Hypothesis were false, then the power series (4) would have a radius of convergence strictly less than one and it would follow that $A_n = \Omega(r^n)$ with $r > 1$. However, as stated by Keiper [7, p. 769]: *the failure of the Riemann hypothesis (if such is the case) would be rather difficult to observe in the growth of the numerical values of the coefficients τ_k, since k would be* extremely *large before a (large) ρ which is off the critical line would yield $|\rho/(\rho-1)|^k$ large.* This line of reasoning also applies to our power series.

A plot of A_m is nevertheless interesting (see Figures 1–4). Understandably it is very similar to a plot of

$$m\Bigl\{\lambda_m - \Bigl(\frac{\log m}{2} - \frac{\log(2\pi)+1-\gamma}{2}\Bigr)\Bigr\}$$

as given by Keiper.

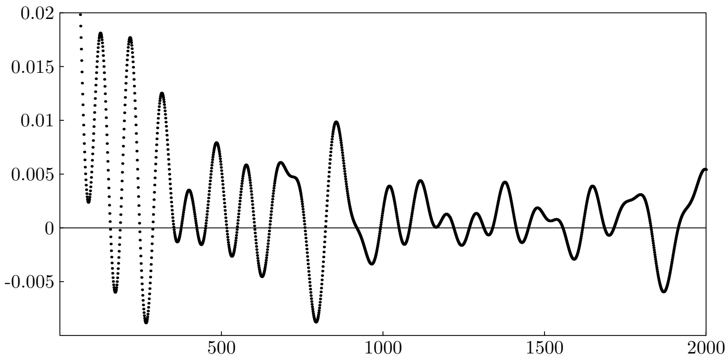

Figure 1: Values of the coefficients A_m

The coefficients λ_n were defined by Keiper (2). They can be expressed in terms of the zeros of $\zeta(s)$ (see (5) and (6)). They are also directly related to the prime numbers as we have proved in (24). As we have seen, all the terms in the right hand side of (24) are well understood except for the integrals depending on the primes. These integrals are equal to the coefficients A_n in the expansion (4). Our (24) is also connected with a formula given by Bombieri and Lagarias

$$\begin{aligned} n\lambda(n) = -\sum_{j=1}^{n}\binom{n}{j}\frac{1}{(j-1)!}\lim_{\varepsilon\to 0}\Bigl\{\sum_{m\leqslant 1/\varepsilon}\frac{\Lambda(m)(\log m)^{j-1}}{m} - \frac{1}{j}(\log(1/\varepsilon)^j\Bigr\} \\ + 1 - (\log 4\pi+\gamma)\frac{n}{2} - \sum_{j=2}^{n}(-1)^{j-1}\binom{n}{j}(1-2^{-j})\zeta(j). \end{aligned} \tag{25}$$

All these relations are unconditionally true, but the behaviour of the λ_n (or A_n) are related to the Riemann Hypothesis.

First we have Li's criterion: the RH is equivalent to the positivity of the λ_n. As shown by Bombieri and Lagarias [2] this is related to the first formula in (6). They get a general criterion for a multiset of complex numbers to lie in the half plane $\Re(s) \leqslant \frac{1}{2}$. In this way Li's criterion may be extended to more general L functions.

Not only the positivity of the λ_n are equivalent to RH, the asymptotic behaviour of these numbers is also related to it. This is easily seen from the definition (5) of the λ_n. The power series $\sum \lambda_k(1-s)^k$ has radius of convergence $\leqslant 1$ and it is equal to 1 if and only if the RH is true. It is clear that the radius of convergence only depends on the asymptotic behaviour of the λ_n.

If the RH is false there is some $r > 1$ such that for an infinite number of n we will have $|\lambda_n| \geqslant r^n$. On the other hand, as Voros [13] has pointed out and we have shown the λ_n grow only as $\log n$ if the RH is true.

In fact the behaviour of λ_n depends on that of A_n. In [1] we defined a real number J (difficult to compute) such that the RH is true if and only if

$$\sum_{n=0}^{\infty} A_n^2 = \int_0^{\infty}\{\Pi(e^t) - \mathrm{Li}(e^t)\}^2 e^{-t}\,dt = J. \tag{26}$$

Hence, we even know the value that $\sum_{n=0}^{\infty} A_n^2$ would take in case the RH were true. If the RH would be false this sum would be infinite.

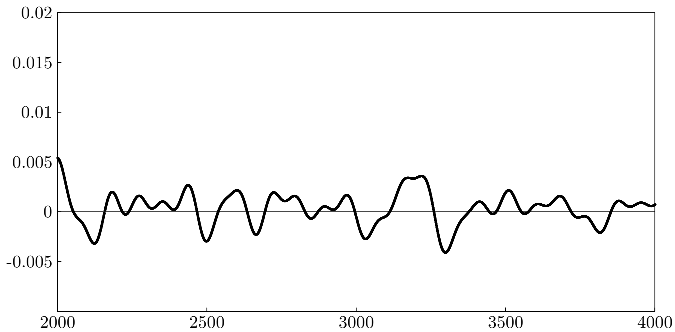

Figure 2: Values of the coefficients A_m

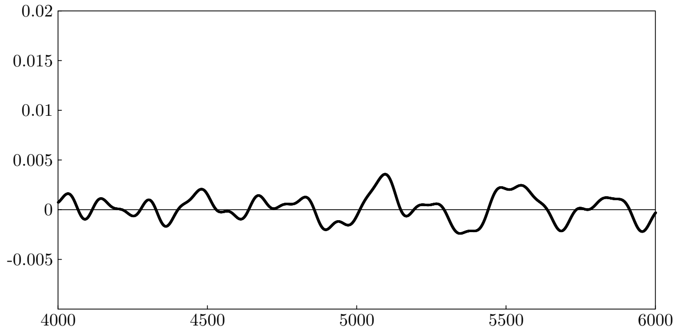

Figure 3: Values of the coefficients A_m

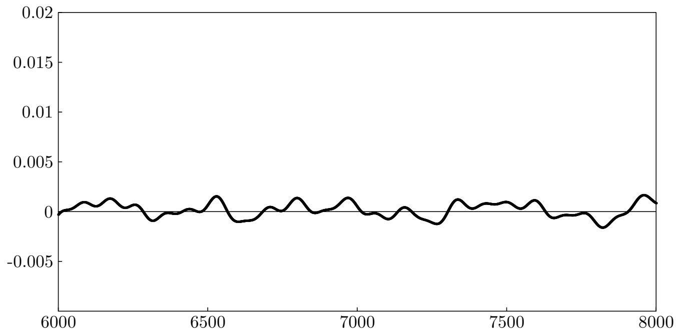

Figure 4: Values of the coefficients A_m

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Address: Juan Arias de Reyna: Facultad de Matemáticas, Universidad de Sevilla, Apdo. 1160, 41080-Sevilla, Spain.

E-mail: arias@us.es

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