Functiones et Approximatio
44.2 (2011), 307–315

ON ASYMPTOTICS OF ENTROPY OF A CLASS OF ANALYTIC FUNCTIONS

Vyacheslav Zakharyuta

Dedicated to the memory of Susanne Dierolf

Abstract: Let (K, D) be a compact subset of an open set D on a Stein manifold Ω of dimension n, $H^\infty(D)$ the Banach space of all bounded and analytic in D functions endowed with the uniform norm, and A_K^D be a compact subset in the space of continuous functions $C(K)$ consisted of all restrictions of functions from the unit ball $\mathbb{B}_{H^\infty(D)}$. In 1950s Kolmogorov raised the problem of a strict asymptotics ([K1, K2, KT]) of an entropy of this class of analytic functions: $\mathcal{H}_\varepsilon\left(A_K^D\right) \sim \tau\left(\ln\frac{1}{\varepsilon}\right)^{n+1}$, $\varepsilon \to 0$, with a constant τ. The main result of this paper, which generalizes and strengthens the Levin's and Tikhomirov's result in [LT], shows that this asymptotics is equivalent to the asymptotics for the widths (Kolmogorov diameters): $\ln d_k\left(A_K^D\right) \sim -\sigma k^{1/n}$, $k \to \infty$, with the constant $\sigma = \left(\frac{2}{\tau(n+1)}\right)^{1/n}$. This result makes it possible to get a positive solution of the above entropy problem by applying recent results [Z2] on the asymptotics for the widths $d_k\left(A_K^D\right)$.

Keywords: Entropy and widths asymptotics, spaces of analytic functions, Kolmogorov problem, Bedford Taylor capacity of a condenser.

1. Introduction

The ε-*entropy* of a set A in a metric space $X = (X, \rho)$ is defined by the formula: $\mathcal{H}_\varepsilon(A) = \mathcal{H}_\varepsilon(A, X) := \ln N_\varepsilon(A, X)$, where $N_\varepsilon(A, X)$ is the smallest integer N such that A can be covered by N sets of diameter not greater than 2ε (we assume that $\mathcal{H}_\varepsilon(A) = +\infty$ if there is no finite covering of that sort).

Let K be a compact subset of an open set D on a Stein manifold Ω of dimension n, $H^\infty(D)$ the Banach space of all bounded and analytic in D functions endowed with the uniform norm, and A_K^D be a compact subset in the space of continuous functions $C(K)$ consisting of all restrictions of functions from the unit ball $\mathbb{B}_{H^\infty(D)}$. Hereafter, if it is not mentioned specially, we assume that *the restriction operator* $R : H^\infty(D) \to C(K)$ is *injective*, so one can set that $A_K^D = \mathbb{B}_{H^\infty(D)}$. For the sake of brevity, any pair (K, D) satisfying the above conditions will be called a condenser on a Stein manifold Ω.

2010 Mathematics Subject Classification: primary: 28D20, 47B06, 32A07; secondary: 32U35, 32U20

In 1950s Kolmogorov raised the problem of a strict asymptotics ([K1, K2, KT])

$$\mathcal{H}_\varepsilon\left(A_K^D\right) \sim \tau\left(\ln\frac{1}{\varepsilon}\right)^{n+1}, \qquad \varepsilon \to 0, \tag{1}$$

with a constant τ (it was already known that the weak asymptotics $\mathcal{H}_\varepsilon\left(A_K^D\right) \asymp \left(\ln\frac{1}{\varepsilon}\right)^{n+1}$, $\varepsilon \to 0$, holds for good enough condensers (K, D) [K2]).

For a set A in a Banach space X the *Kolmogorov diameters (*or *widths) of* A *with respect to the unit ball* $\mathbb{B}_X$ of the space X are the numbers (see, e.g., [M, T]):

$$d_k(A) = d_k(A, \mathbb{B}_X) := \inf_{L\in\mathcal{L}_k} \sup_{x\in A} \inf_{y\in L} \|x-y\|_X, \qquad k = 0, 1, \ldots, \tag{2}$$

where $\mathcal{L}_k$ is the set of all vector subspaces of X of dimension $\leqslant k$.

It is proved in [LT], which is an appendix to the posthumous paper of V. D. Erokhin [E], that the asymptotics (1) for $n = 1$ follows from the asymptotics

$$\ln d_k\left(A_K^D\right) \sim -\frac{k}{\tau}, \qquad k \to \infty. \tag{3}$$

Our main goal is to prove here the following assertion, which generalizes and strengthens this result.

Theorem 1. *Let (K, D) be a condenser on a Stein manifold Ω of dimension n. The asymptotics (1) holds if and only if the asymptotics*

$$\ln d_k\left(A_K^D\right) \sim -\sigma k^{1/n}, \qquad k \to \infty \tag{4}$$

takes place with the constant $\sigma = \left(\frac{2}{\tau(n+1)}\right)^{1/n}$.

This result will be drawn from a more general Theorem 5, which is proved in Section 3. Its proof is based on 1) Mityagin's result on an estimate of entropy from below in Theorem 4 [M] (see Lemma 2 below) and 2) Theorem 4 in this paper on an estimate of entropy from above. The latter is obtained by a modification of the proof from [LT], which develops the proof of the estimate from above in Theorem 4, [M] (see Lemmas 1 and 2 in [LT] and Lemma 3 below). Using Theorem 1 and applying the results on the asymptotics (4), we discuss the original Kolmogorov problem on the asymptotics (1) in Section 4.

Notation:

- The relation $f(t) \approx g(t)$, $t \nearrow \infty$, means that for every $\varepsilon > 0$ there is $T > 0$ such that $f(t) \leqslant g\left(t^{1+\varepsilon}\right)$ and $g(t) \leqslant f\left(t^{1+\varepsilon}\right)$ for $t > T$.
- Given a positive sequence $a = (a_k)$ its *counting function* is

$$m_a(t) := \sharp\{k : a_k \leqslant t\}, \qquad t > 0.$$

- Given a set A in a metric space $X = (X, \rho)$ its α-extension is the set

$$[A]_\alpha = \{x \in X : \inf\{\rho(x, y) : y \in A\} \leqslant \alpha\}, \qquad \alpha \geqslant 0.$$

- For a couple of linear topological spaces $X \hookrightarrow Y$ means a continuous linear imbedding with dense image.

2. Interestimates between entropy and widths

The following lemma is an easy adaptation of Mityagin's result (contained in Theorem 4 from [M]) to the case of complex Banach spaces. Notice that for special A and X those estimates may be considerably better: see, for instance, [M], Theorem 3 and its corollaries, where $X = l_p$ and A is an l_p-ellipsoid.

Lemma 2 ([M, Theorem 4]). *Let A be an absolutely convex set in a complex Banach space X. Then*

$$2\int_0^{\frac{1}{2\varepsilon}} \frac{m_c(t)}{t}\,dt \leqslant \mathcal{H}_\varepsilon(A,X) \leqslant 2m_a\left(\frac{8}{\varepsilon}\right)\ln\frac{8\left(d_0(A,\mathbb{B}_X)+\varepsilon\right)}{\varepsilon}, \tag{5}$$

where $c = (c_j) = \left(\frac{j}{d_{j-1}(A,\mathbb{B}_X)}\right)_{j\in\mathbb{N}}$ *and* $a = (a_k) = \left(1/d_{k-1}(A,\mathbb{B}_X)\right)$.

Below (Theorem 4) we show, modifying the technique from [LT], that the right inequality (5) can be refined.

The next lemma is a slight modification of Lemmas 1 and 2 from [LT] (see p. 127 there); the proof below, basically the same as in [LT], develops the proof of the right-hand inequality (5) from [M].

Lemma 3. *Let A be a compact set in a complex Banach space X. Then for all positive ε, δ and nonnegative α we have an inequality*

$$\mathcal{H}_{\varepsilon+\alpha}\left([A]_\alpha, X\right) \leqslant \mathcal{H}_{\varepsilon+\alpha+\delta}\left([A]_{\varepsilon+\alpha}, X\right) + 2m_a\left(\frac{2}{\varepsilon}\right)\ln\left(\frac{8(\varepsilon+\alpha+\delta)}{\varepsilon}\right), \tag{6}$$

where $a = \left(\frac{1}{d_{j-1}(A,\mathbb{B}_X)}\right)$.

Proof. Given $\varepsilon > 0$ set $m = m_a\left(\frac{2}{\varepsilon}\right)$. Then $d_m(A,\mathbb{B}_X) < \varepsilon/2$, hence there exists a complex subspace L of a dimension $m = m_a\left(\frac{2}{\varepsilon}\right)$ such that

$$\sup_{x\in A}\inf_{y\in L}\{\|x-y\|\} < \frac{\varepsilon}{2}.$$

Set $F := \cup_{x\in A}\{z \in L : \|x-z\| \leqslant \varepsilon/2\}$ and take a set $\{z_l : l = 1,\dots,M\} \subset F$ with the largest M such that $\|z_l - z_k\| \geqslant \varepsilon/2$, $k \neq l$. If S is a set such that $F \subset [S]_{\varepsilon/2}$ then $[A]_\alpha \subset [S]_{\alpha+\varepsilon}$. Therefore, applying Lemma 6 from [M], we obtain

$$N_{\varepsilon+\alpha}\left([A]_\alpha, X\right) \leqslant N_{\frac{\varepsilon}{2}}(F,X) \leqslant M. \tag{7}$$

The balls $z_l + \frac{\varepsilon}{8}\mathbb{B}_X \cap L$ are pairwise disjoint and contained in $[A]_\varepsilon \cap L \subset [A]_{\varepsilon+\alpha} \cap L$. Hence,

$$M\left(\frac{\varepsilon}{8}\right)^{2m}\mathcal{V}(\mathbb{B}_X\cap L) \leqslant \mathcal{V}\left([A]_{\varepsilon+\alpha}\cap L\right), \tag{8}$$

Here $\mathcal{V}(E)$ stays for the Euclidean volume of a set E in the m-dimensional complex space L. On the other hand,

$$[A]_{\varepsilon+\alpha} \cap L \subset \bigcup_{k=1}^{N} (w_k + (\varepsilon+\alpha+\delta)\mathbb{B}_X \cap L)$$

for some finite set $\{w_k,\ k=1,\dots,N\} \subset [A]_{\varepsilon+\alpha} \cap L$ with

$$N = N_{\varepsilon+\alpha+\delta}\left([A]_{\varepsilon+\alpha} \cap L, X \cap L\right) \leqslant N_{\varepsilon+\alpha+\delta}\left([A]_{\varepsilon+\alpha}, X\right).$$

Hence,

$$\mathcal{V}\left([A]_{\varepsilon+\alpha} \cap L\right) \leqslant N_{\varepsilon+\alpha+\delta}\left([A]_{\varepsilon+\alpha}, X\right)(\varepsilon+\alpha+\delta)^{2m}\,\mathcal{V}(\mathbb{B}_X \cap L).$$

Combining this inequality with (7) and (8), we obtain an estimate

$$N_{\varepsilon+\alpha}\left([A]_{\alpha}, X\right) \leqslant N_{\varepsilon+\alpha+\delta}\left([A]_{\varepsilon+\alpha}, X\right)\left(\frac{8(\varepsilon+\alpha+\delta)}{\varepsilon}\right)^{2m}$$

and, after taking the logarithm, the inequality (6). ■

Theorem 4. *Let A be a compact absolutely convex set in a complex Banach space X. Then there exists a constant $M > 0$ such that*

$$\mathcal{H}_\varepsilon(A, X) \lesssim 2\int_0^{\frac{M}{\varepsilon}} \frac{m_a(t)}{t}\,dt, \qquad \varepsilon \searrow 0, \tag{9}$$

where $a = (a_k) = (1/d_{k-1}(A, \mathbb{B}_X))$.

Proof. Consider $0 < \varepsilon_s < \varepsilon_{s-1} < \ldots < \varepsilon_1 < \varepsilon_0$ with $\varepsilon_0 \geqslant d_0(A, \mathbb{B}_X) = \frac{\operatorname{diam} A}{2}$ and apply repeatedly Lemma 3, taking first $\alpha = 0,\ \varepsilon = \varepsilon_s,\ \delta = \varepsilon_{s-1}$, then $\alpha = \varepsilon_s,\ \varepsilon = \varepsilon_{s-1},\ \delta = \varepsilon_{s-2}$, then $\alpha = \varepsilon_s + \varepsilon_{s-1},\ \varepsilon = \varepsilon_{s-2},\ \delta = \varepsilon_{s-3}$ and so on finishing with $\alpha = \varepsilon_s + \ldots + \varepsilon_2,\ \varepsilon = \varepsilon_1,\ \delta = \varepsilon_0$. Since

$$\mathcal{H}_{\varepsilon_s+\varepsilon_{s-1}+\ldots+\varepsilon_1+\varepsilon_0}\left([A]_{\varepsilon_s+\varepsilon_{s-1}+\cdots+\varepsilon_1}, X\right) = \ln 1 = 0,$$

finally we obtain an estimate

$$\mathcal{H}_\varepsilon(A, X) \leqslant 2\sum_{j=1}^{s} m_a\left(\frac{2}{\varepsilon_j}\right)\ln\left(\frac{8(\varepsilon_s+\varepsilon_{s-1}+\ldots+\varepsilon_{j-1})}{\varepsilon_j}\right) \tag{10}$$

which is analogous to the inequality (20) in [LT]. Consider any integer-valued function $s = s(t) \nearrow \infty$ such that $s(t) = o(\ln t)$ and set $\varepsilon_j := \varepsilon^{j/s},\ t_j := 1/\varepsilon_j = t^{j/s},\ j = 1,\dots,s+1$; $\varepsilon_0 = 1/t_0 \geqslant d_0(A, \mathbb{B}_X)$. Taking into account that

$$\gamma(t) := \ln\frac{\varepsilon_{j-1}}{\varepsilon_j} = \ln t_j - \ln t_{j-1} = \frac{\ln t}{s(t)} \to \infty \qquad \text{as} \quad t \to \infty,$$

we have, due to $s(t) = o(\ln t)$, that $\varepsilon_s + \varepsilon_{s-1} + \ldots + \varepsilon_{j-1} \leqslant \frac{\varepsilon_{j-1}}{1-\varepsilon^{1/s}} \leqslant 2\varepsilon_{j-1}$ for sufficiently small ε. Hence the estimate (10) can be rewritten in a form:

$$\mathcal{H}_{\frac{1}{t}}(A, X) \lesssim 2\left(1 + \frac{16}{\gamma(t)}\right) \sum_{j=1}^{s} m_a(2t_j)(\ln t_j - \ln t_{j-1})$$
$$\lesssim 2 \sum_{j=1}^{s} m_a(2t_j)(\ln t_{j+1} - \ln t_j) \leqslant 2 \int_0^{t_{s+1}} m_a(2\tau)\ d\ln\tau$$

with $t \longrightarrow \infty$. The last inequality is true, because, due to monotonicity of the integrand, the sum in the left term of the inequality is a lower integral sum for the Stieltjes integral $\int_{t_1}^{t_{s+1}} m_a(2\tau)\ d\ln\tau$. Let $I(t) := \int_0^t m_a(\tau)\ d\ln\tau = \int_0^t \frac{m_a(\tau)}{\tau}\ d\tau$ (remember that the function m_a vanishes on the interval $(0, 1/d_0)$). Then

$$\mathcal{H}_{\frac{1}{t}}(A, X) \lesssim 2I(2t_{s+1}) = 2I(2t\exp(\gamma(t))), \qquad t \to \infty. \tag{11}$$

Since the function $\gamma(t)$ in the above considerations can be taken arbitrarily slow, there exists a constant $M > 0$ such that

$$\mathcal{H}_{\frac{1}{t}}(A, X) \lesssim 2I(Mt) \qquad \text{as} \quad t \to \infty. \tag{12}$$

Indeed, suppose the contrary that there is a sequence $t_k \uparrow \infty$ and $q > 1$ such that $\mathcal{H}_{\frac{1}{t_k}}(A, X) > q2I\left(2^{k+1}\ t_k\right)$, $k \in \mathbb{N}$. This assumption leads to a contradiction, since the relation (11) fails for the continuous function $\gamma(t)$ which equals $k\ln 2$ at the point t_k, $k \in \mathbb{N}$ and is linear on each interval (t_{k-1}, t_k) (hence for any integer-valued function which is slower than $\gamma(t)$). Thus (12) is true with some constant M, so (9) is proved. ■

3. Equivalence of the entropy and widths asymptotics

Notice that for *non-decreasing* sequences $a = (a_k)$, $b = (b_k)$ the asymptotic formula $\ln a_k \sim \ln b_k$, $k \to \infty$ is equivalent to the relation $m_b(t) \approx m_a(t)$, $t \to \infty$.

Theorem 5. *Let $X_1 \hookrightarrow X_0$ be a couple of complex Banach spaces with linear compact dense imbedding and $0 < \alpha < \infty$. Then the asymptotics*

$$\mathcal{H}_\varepsilon(\mathbb{B}_{X_1}, X_0) \sim \tau\ \left(\ln\frac{1}{\varepsilon}\right)^{\alpha+1}, \qquad \varepsilon \to 0 \tag{13}$$

is equivalent to the asymptotics

$$-\ln d_{k-1}(\mathbb{B}_{X_1}, \mathbb{B}_{X_0}) \sim \sigma\ k^{1/\alpha}, \qquad k \to \infty \tag{14}$$

with $\sigma = \left(\frac{2}{(\alpha+1)\tau}\right)^{1/\alpha}$.

Proof. Let $a = \{a_k\} := \left\{\frac{1}{d_{k-1}}\right\}$. Let us suppose that the asymptotics (14) holds. Then $m_a(t) \approx \left(\frac{\ln t}{\sigma}\right)^\alpha$ or, what is the same in our case, $m_a(t) \sim \left(\frac{\ln t}{\sigma}\right)^\alpha$. On the other hand, $\ln c_k = \ln k + \ln a_k \sim \ln a_k$, hence $m_c(t) \approx m_a(t) \sim \left(\frac{\ln t}{\sigma}\right)^\alpha$. Putting this asymptotics into (5) and (9), we obtain, by integrating the asymptotic inequalities, the asymptotics (13).

Let us suppose that (13) takes place. Then, setting $\varepsilon = 1/s$, from Lemma 2 and Theorem 5 we obtain asymptotic estimates

$$2\int_0^s \frac{m_c(t)}{t}\, dt \lesssim \tau\ (\ln s)^{\alpha+1} \lesssim 2\int_0^s \frac{m_a(t)}{t}\, dt, \qquad s \to \infty. \tag{15}$$

It is easy to see that $m_c\left(\frac{s}{2}\right) \leqslant 2\int_{\frac{s}{2}}^s \frac{m_c(t)}{t}\, dt \lesssim \tau\ (\ln s)^{\alpha+1}$, hence we have $m_c(s) \lesssim \tau\ (\ln 2s)^{\alpha+1} \lesssim \tau\ (\ln s)^{\alpha+1}$, which implies an asymptotic inequality $\ln c_k \gtrsim \left(\frac{k}{\tau}\right)^{1/(\alpha+1)}$ as $k \to \infty$. Therefore $\ln k = o(\ln c_k)$ and

$$\ln c_k = \ln k + \ln a_k \sim \ln a_k. \tag{16}$$

By change of variables $u = \ln t$, $v = \ln s$ in (15) we obtain

$$2\int_0^v m_c(e^u)\ du \lesssim \tau v^{\alpha+1} \lesssim 2\int_0^v m_a(e^u)\ du, \qquad v \to \infty. \tag{17}$$

It follows from (16) that for each $\varepsilon > 0$ there is $u_0 > 0$ such that

$$m_a(e^u) \leqslant m_c\left(e^{u(1+\varepsilon)}\right), \qquad u \geqslant u_0.$$

Therefore, setting $C(\varepsilon) = 2\int_0^{u_0} m_a(e^u)\ du$, by (17), we have

$$\begin{aligned} 2\int_0^v m_a(e^u)\ du &\leqslant C(\varepsilon) + 2\int_0^v m_c\left(e^{u(1+\varepsilon)}\right)\ du \leqslant C(\varepsilon) + 2\int_0^{v(1+\varepsilon)} \frac{m_c(e^u)}{1+\varepsilon}\ du \\ &\lesssim C(\varepsilon) + \tau(1+\varepsilon)^\alpha v^{\alpha+1} \lesssim \tau(1+\varepsilon)^\alpha v^{\alpha+1}, \qquad v \to \infty. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $2\int_0^v m_a(e^u)\ du \lesssim \tau v^{\alpha+1},\ v \to \infty$. Combining this with (17), we obtain $2\int_0^v m_a(e^u)\ du \sim \tau v^{\alpha+1},\ v \to \infty$. Since the integrand satisfies the Tauberian condition of non-decreasing, one can differentiate this asymptotics (see, e.g., [dB]), so that

$$2m_a(e^v) \sim \tau(\alpha+1)v^\alpha, \qquad v \to \infty$$

Going back to the variable $s = e^v$, we obtain

$$m_a(s) \sim \frac{\tau(\alpha+1)}{2}\ (\ln s)^\alpha, \qquad s \to \infty,$$

which implies that $-\ln d_{k-1} \sim \sigma\ k^{1/\alpha},\ k \to \infty$, with $\sigma = \left(\frac{2}{(\alpha+1)\tau}\right)^{1/\alpha}$. ∎

4. Entropy asymptotics for some class of analytic functions (Kolmogorov problem)

Here we discuss the problem of the asymptotics (1) in the light of Theorem 1 and the results on the asymptotics (4) in [Z2], to where we send the reader for the history of the problem (see also a survey [Z3]).

First we need to introduce some definitions.

Definition 6. *The Green pluripotential of a condenser* (K, D) *on a Stein manifold* Ω *is the function*

$$\omega(z) = \omega(D, K; z) := \limsup_{\zeta \to z} \sup \{u(\zeta) : u \in \mathcal{P}(K, D)\}, \tag{18}$$

where $\mathcal{P}(K, D)$ *is the class of all functions* u *plurisubharmonic in* D *and such that* $u|_K \leqslant 0$ *and* $u(\zeta) \leqslant 1$ *in* D.

Definition 7. *A condenser* (K, D) *on a Stein manifold* Ω *is called pluriregular if*

(i) *its Green pluripotential* ω *vanishes on* K *and* $\omega(z_j) \to 1$ *for each sequence* $\{z_j\} \subset D$ *without limit points in* D*, shortly,* $\lim\limits_{z \longrightarrow \partial D} \omega(z) = 0$*;*

(ii) $K = \widehat{K}_D$ *and* D *has no component disjoint with* K.

It is known that for a pluriregular condenser (K, D) the function (18) is continuous in D [Z1]. Bedford and Taylor [BT2] (see also, Sadullaev [S]) introduced a capacity which, for a pluriregular condenser (K, D), has the form

$$C(K, D) := \int_K (dd^c \omega(z))^n, \tag{19}$$

here the complex Monge-Ampére operator $u \longrightarrow (dd^c u)^n$ associates to any function $u \in Psh(D) \cap L^\infty_{loc}(D)$ some non-negative Borel measure; in particular, the measure $(dd^c \omega(z))^n$ is supported by K (for details see [BT1, BT2]). It is convenient to introduce also the *pluricapacity* $\tau(K, D) = \frac{1}{(2\pi)^n} C(K, D)$, which differs from the capacity (19) by a natural factor so that it coincides with the Green capacity in the case $n = 1$.

Definition 8. *A couple of Banach spaces* (X_0, X_1)*, such that*

$$X_1 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow X_0, \tag{20}$$

is called admissible for a condenser (K, D) *if for each couple of Banach spaces* (Y_0, Y_1) *such that*

$$X_1 \hookrightarrow Y_0 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow Y_1 \hookrightarrow X_0$$

we have $\ln d_k(\mathbb{B}_{X_1}, \mathbb{B}_{X_0}) \sim \ln d_k(\mathbb{B}_{Y_1}, \mathbb{B}_{Y_0})$ *as* $k \to \infty$.

An admissible couple of Banach spaces (hence, Hilbert spaces) exists for any pluriregular condenser (K, D) (see, e.g., [Z2]).

Theorem 9. *Let* (K, D) *be a pluriregular condenser on a Stein manifold* Ω, $\dim \Omega = n$. *Then the following statements are equivalent:*

(a) *the couple* $(H_\infty(D), AC(K))$ *is admissible for the condenser* (K, D);

(b) *the asymptotics (4) holds with the constant* $\sigma = \left(\frac{n!}{\tau(K,D)}\right)^{1/n}$;

(c) *the asymptotics (1) holds with the constant* $\tau = \frac{2\tau(K,D)}{(n+1)!}$.

The equivalence of (a) and (b) has been proved in [Z2], Theorem 1.5 and Corollary 1.7 (notice that one of important steps in their proofs is the recent result on approximation of the pluripotential $\omega(z) - 1$ by multipolar pluricomplex Green functions [N1, N2, P]; for more details see [Z2]). The equivalence of (b) and (c) follows from Theorem 5. So any concrete result on the asymptotics (4) one can translate to a result on the asymptotics (1) and vice versa. In particular, applying [Z2], Corollary 9.1, we obtain

Theorem 10. *Let us suppose that* (K, D) *is a pluriregular condenser on a Stein manifold* Ω, $\dim \Omega = n$, *such that* D *is strictly pluriregular, i.e. there is a continuous plurisubharmonic function* $u(z)$ *in some open set* $G \Supset D$ *such that* $D = \{z \in G : u(z) < 0\}$. *Then the asymptotics (1) takes place with the constant* $\tau = \frac{2\tau(K,D)}{(n+1)!}$.

The translation of other width asymptotics assertions from [Z2] into the results on the entropy asymptotics are left to readers.

References

[BT1] E. Bedford, A. B. Taylor, *The Dirichlet Problem for a complex Monge-Ampére equation,* Inventions Math. **37** (1976), 1–44.

[BT2] E. Bedford, A. B. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. **149** (1982), 1–40.

[dB] N. G. de Bruijn, *Asymptotic methods in analysis,* North-Holland, Amsterdam, 1961.

[E] V. D. Erokhin, *On the best approximation of functions analytically extendable from a given continuum into a given domain*, Uspehi Matem. Nauk **14** (1968), 91–132.

[K1] A. N. Kolmogorov, *Estimates of the minimal number of elements of* ε*-nets in various functional spaces an their application to the problem on the representation of functions of several variables with superpositions of functions of lesser number of variables*, Uspehi Matem. Nauk **10** (1955), 192–193.

[K2] A. N. Kolmogorov, *Asymptotic characteristics of certain totally bounded metric spaces*, Doklady AN SSSR **108** (1956), 585–589.

[KT] A. N. Kolmogorov, V. M. Tikhomirov, ε*-entropy and* ε*-capacity of sets in functional spaces*, Uspehi Matem. Nauk **14** (1959), 3–86.

[LT] A. L. Levin, V. M. Tikhomirov, *On theorem of V. D. Erokhin,* Russian Math. Surveys **23** (1968), 119–132.

[M] B. S. Mityagin, *Approximative dimension and bases in nuclear spaces*, Russian Math. Survey **16** (1963), 59–127.

[N1] S. Nivoche, *Sur une conjecture de Zahariuta et un probléme de Kolmogorov,* C. R. Acad. Sci. Paris, Sér. I Math. **333** (2001), 839–843.

[N2] S. Nivoche, *Proof of the conjecture of Zahariuta concerning a problem of Kolmogorov on the ε-entropy*, Invent. Math. **158** (2004), 413–450.

[P] E. Poletsky, *Approximation of plurisubharmonic functions by multipole Green functions,* Trans. Amer. Math. Soc. **335** (2003), 1579–1591.

[S] A. Sadullaev, *Plurisubharmonic measures and capacity on complex manifolds*, Uspehi Matem. Nauk **36** (1981), 53–105.

[T] V. M. Tikhomirov, *Some Problems of Approximation Theory,* MGU, Moscow, 1976.

[Z1] V. P. Zakharyuta, *Extremal plurisubharmonic functions, Hilbert scales, and the isomorphism of spaces of analytic functions of several variables,* I, II. Teor. Funkciĭ, Funkcional. Anal. i Priložen. **19** (1974), 133-157; ibid. **21** (1974), 65–83.

[Z2] V. Zakharyuta, *Kolmogorov problem on widths asymptotics and pluripotential theory*, Contemporary Mathematics **481** (2009), 171-196.

[Z3] V. Zakharyuta, *Extendible bases and Kolmogorov problem on asymptotics of entropy and widths of some class of analytic functions*, Annales de la Faculte des Sciences de Toulouse, to appear.

Address: Vyacheslav Zakharyuta: Sabanci University of Istanbul, 34956, Istanbul, Turkey.

E-mail: zaha@sabanciuniv.edu

Received: 12 July 2010; **revised:** 30 January 2011