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DIFFERENTIABILITY OF STRONGLY PARACONVEX VECTOR-VALUED FUNCTIONS

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Dedicated to the memory of Susanne Dierolf

Abstract: In the paper the notion of strongly $\alpha(\cdot)$-K-paraconvex functions is introduced. It is shown that a strongly $\alpha(\cdot)$-K-paraconvex function defined on a convex set contained in a Banach space X with values in $\mathbb{R}^n$ is:

(a) Fréchet differentiable on a dense G_δ-set provided X is an Asplund space,

(b) Gateaux differentiable on a dense G_δ-set provided X is separable.

Keywords: strongly $\alpha(\cdot)$-K-paraconvexity, Gateaux and Fréchet differentiability.

1. Introduction. Properties of K-convex functions

Let X, Y be linear spaces. Let $K \subset Y$ be a convex cone. We recall (see for example Jahn (1986), (2004), Pallaschke-Rolewicz (1997)) that a function $f(\cdot)$ mapping a convex set $Q \subset X$ into Y is called K-convex if

$$f\big(tx + (1-t)y\big) \leqslant_K tf(x) + (1-t)f(y).$$

In other words a function $f(\cdot)$ mapping a convex set $Q \subset X$ into Y will be called K-convex if

$$tf(x) + (1-t)f(y) \in f\big(tx + (1-t)y\big) + K. \tag{1.1}$$

As a trivial consequence of (1.1) we obtain

Proposition 1.1. *Let X, Y be linear spaces. Let $K, K_1 \subset Y$ be two convex cones. If $K \subset K_1$, then each K-convex function $f(\cdot)$ mapping a convex set $Q \subset X$ into Y is also K_1-convex.*

Proposition 1.2. *Let X, Y be linear spaces. Let $K \subset Y$ be a convex pointed cone (i.e. $K \cap (-K) = \{0\}$). If functions $f(\cdot)$ and $-f(\cdot)$ mapping a convex set in X into Y are K-convex than there are affine on its domain.*

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Proof. By definition

$$f\bigl(tx+(1-t)y\bigr) \leqslant_K tf(x)+(1-t)f(y),$$

and

$$-f\bigl(tx+(1-t)y\bigr) \leqslant_K -\bigl[tf(x)+(1-t)f(y)\bigr].$$

In other words

$$f\bigl(tx+(1-t)y\bigr)-[tf(x)+(1-t)f(y)] \in K,$$

and

$$f\bigl(tx+(1-t)y\bigr)-[tf(x)+(1-t)f(y)] \in (-K).$$

Since the cone K is pointed,

$$f\bigl(tx+(1-t)y\bigr)-[tf(x)+(1-t)f(y)] = 0. \tag{1.2}$$

We put

$$y_0 = f(0), \tag{1.3}$$

and

$$f_0(x) = f(x) - y_0. \tag{1.4}$$

By (1.3) and (1.4) we get

$$0 = f_0\bigl(tx+(1-t)0\bigr)-[tf_0(x)+(1-t)f_0(0)] = f_0(tx)-tf_0(x) = 0,$$

i.e., the function $f(\cdot)$ is homogeneous.

Putting $t=1/2$ in (1.2) we get

$$f_0\left(\frac{1}{2}x+\frac{1}{2}y\right)-\left[\frac{1}{2}f_0(x)+\frac{1}{2}f_0(y)\right] = 0.$$

This, together with homogeneity implies that $f_0(\cdot)$ is linear. Thus $f(\cdot)$ is affine. ■

Proposition 1.3. *Let X be a linear space. Let $K \subset \mathbb{R}^n$ be a closed convex pointed cone. Let a function $f(\cdot)$ mapping a convex set $Q \subset X$ into $\mathbb{R}^n$ be K-convex. Then there are n linearly independent functionals $\{\ell_1, \ell_2, ..., \ell_n\}$ defined on $\mathbb{R}^n$ such that the functions $\{\ell_1(f(\cdot)), \ell_2(f(\cdot)), ..., \ell_n(f(\cdot)\}$ are convex.*

Proof. Since $K \subset \mathbb{R}^n$ there are n linearly independent functionals $\{\ell_1, \ell_2, ..., \ell_n\}$ defined on $\mathbb{R}^n$ such that

$$K \subset K_1 = \{x \in R^n : \ell_1(x) \geqslant 0,\ \ell_2(x) \geqslant 0, ..., \ell_n(x) \geqslant 0\}.$$

Thus by Proposition 1.1 the function $f(\cdot)$ is K_1-convex. It implies that the functions $\{\ell_1(f(\cdot)), \ell_2(f(\cdot)), ..., \ell_n(f(\cdot)\}$ are convex. ■

2. Definition of strongly K-paraconvex functions

In this section we introduce the notion of strongly $\alpha(\cdot)$-k-paraconvex functions (compare Rolewicz (2000)). Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $K \subset Y$ be a closed convex pointed cone. Let k belong to the relative interior of K, $k \in Int_r K$.

Let $\alpha(\cdot)$ be a nondecreasing function mapping the interval $[0, +\infty)$ into itself such that

$$\lim_{t \downarrow 0} \frac{\alpha(t)}{t} = 0.$$

Let a continuous function $f(\cdot)$ be defined on an open convex subset $\Omega \subset X$ and having values in Y. We say that the function $f(\cdot)$ is strongly $\alpha(\cdot)$-k-paraconvex if there is $C \geqslant 0$ such that for all $x, y \in \Omega$ and $0 \leqslant t \leqslant 1$ we have

$$f\big(tx + (1-t)y\big) \leqslant_K tf(x) + (1-t)f(y) + C \min[t, (1-t)]\alpha(\|x-y\|_X)k.$$

We say that a continuous function $f(\cdot)$ defined on an open convex subset $\Omega \subset X$ and having values in Y is strongly $\alpha(\cdot)$-K-paraconvex if it is strongly $\alpha(\cdot)$-k-paraconvex for all $k \in Int_r K$.

Proposition 2.1. *Let X, Y be Banach spaces. Let $K \subset Y$ be a convex pointed cone. Let $k_0 \in Int_r K$. Then each strongly $\alpha(\cdot)$-k_0-paraconvex function $f(\cdot)$ mapping a convex set $Q \subset X$ into Y is strongly $\alpha(\cdot)$-K-paraconvex .*

This is based on the following

Lemma 2.1. *Let Y be a linear space. Let $K \subset Y$ be a convex pointed cone. Let $h, k \in Int_r K$. Then there is $a > 0$ such that*

$$k \leqslant_K ah.$$

Proof. Let Y_0 be linear space generated by elements h, k. Let $K_0 = K \cap Y_0$. Y_0 is a two dimensional space and K_0 is a two dimensional pointed cone. Thus there are vectors ℓ and r such that the cone $Int_r K_0$ can be represent in the form $K_0 = \{a\ell + br : a, b > 0\}$. Since $h, k \in Int_r K$, there are positive numbers $\alpha_\ell^h, \beta_\ell^h, \alpha_r^k, \beta_r^k$ such that

$$h = \alpha_\ell^h \ell + \beta_r^h r,$$

$$k = \alpha_\ell^k \ell + \beta_r^k r.$$

It is easy to see that any $a \geqslant \max[\frac{\alpha_\ell^k \ell}{\alpha_\ell^h}, \frac{\beta_r^k}{\beta_\ell^h}]$ satisfies the requested inequality. ∎

Proof of Proposition 2.1. By Lemma 2.1, for each $k \in Int_r K$ there is $a > 0$ such that $k \leqslant ak_0$. It immediately implies the thesis. ∎

By Proposition 2.1 we trivially obtain

Example 2.1. Let X, Y be Banach spaces. Let $K \subset Y$ be a closed convex pointed cone. Let $k_0 \in Int_r K$. Then each strongly $\alpha(\cdot)$-k_0-paraconvex function $f(\cdot)$ mapping a convex set $Q \subset X$ into Y is strongly $\alpha(\cdot)$-K-paraconvex.

In similar way as for K-convex function (see Proposition 1.3) we can show

Proposition 2.2. *Let X be a Banach space. Let $K \subset \mathbb{R}^n$ be a closed convex pointed cone. Let a function $f(\cdot)$ mapping a convex set $Q \subset X$ into $\mathbb{R}^n$ be strongly $\alpha(\cdot)$-K-paraconvex . Then there are n linearly independent functionals $\{\ell_1, \ell_2, ..., \ell_n\}$ defined on $\mathbb{R}^n$ such that the functions $\{\ell_1(f(\cdot)), \ell_2(f(\cdot)), ..., \ell_n(f(\cdot)\}$ are strongly $\alpha(\cdot)$-paraconvex.*

3. Mazur and Asplund theorems for strongly $\alpha(\cdot)$-K-paraconvex vector-valued functions.

As an obvious consequence of Proposition 2.2 we get the following generalization of Mazur (1933) and Asplund (1968) theorems.

Theorem 3.1. *Let Ω_X be an open convex set in a real Banach space $(X, \|\cdot\|_X)$. Let K be a convex closed pointed cone in $\mathbb{R}^n$ with any norm $\|\cdot\|$. Let $f(\cdot)$ be a strongly $\alpha(\cdot)$-K-paraconvex function defined on Ω_X with values in $\mathbb{R}^n$. Then the function $f(\cdot)$ is:*

(a) *Fréchet differentiable on a dense G_δ-set provided X is an Asplund space,*
(b) *Gateaux differentiable on dense G_δ-set provided X is separable.*

Proof. By Proposition 2.2 there exist n linearly independent functionals $\{\ell_1, \ell_2, ..., \ell_n\}$ defined on $\mathbb{R}^n$ such that the functions $\{\ell_1(f(\cdot)), \ell_2(f(\cdot)), ..., \ell_n(f(\cdot)\}$ are strongly $\alpha(\cdot)$-paraconvex. Thus by Rolewicz (2005), (2006), there is a dense G_δ-set Ω such that all functions $\{\ell_1(f(\cdot)), \ell_2(f(\cdot)), ..., \ell_n(f(\cdot)\}$ are

(a) Fréchet differentiable on Ω provided X is an Asplund space,
(b) Gateaux differentiable on Ω provided X is separable.

Since in $\mathbb{R}^n$ all norms are equivalent (in particular they are equivalent to the norm $\|\cdot\|_\infty^\ell = \max\{|\ell_1(\cdot)|, |\ell_2(\cdot)|, ..., |\ell_n(\cdot)|\}$) we obtain the theorem. ■

Problem 4. *Does Theorem 3.1 hold for infinite dimensional spaces Y?*

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